Алгебра и анализ
Том 29 (2017), № 4

St. Petersburg Math. J.
Vol. 29 (2018), No. 4, Pages 657–681
http://dx.doi.org/10.1090/spmj/1511
Article electronically published on June 1, 2018

SIGNAL RECOVERY VIA TV-TYPE ENERGIES

M. FUCHS, J. MÜLLER, AND C. TIETZ

Dedicated to the memory of Stefan Hildebrandt

Abstract. One-dimensional variants are considered of the classical first order total variation denoising model introduced by Rudin, Osher, and Fatemi. This study is based on previous work of the authors on various denoising and inpainting problems in image analysis, where variational methods in arbitrary dimensions were applied. More than being just a special case, the one-dimensional setting makes it possible to study regularity properties of minimizers by more subtle methods that do not have correspondences in higher dimensions. In particular, quite strong regularity results are obtained for a class of data functions that contains many of the standard examples from signal processing such as rectangle or triangle signals as a special case. The analysis of the related Euler–Lagrange equation, which turns out to be a second order two-point boundary value problem with Neumann conditions, by ODE methods completes the picture of this investigation.

§1. Introduction

Since the publication of the seminal paper [1] of Rudin, Osher, and Fatemi in 1992, total variation based denoising and inpainting methods have proved to be very effective when dealing with two- or higher-dimensional noisy data such as digital images, which nowadays has become their main field of application. However, their one-dimensional counterparts in signal processing seem to find usage as well, mainly in connection with the recovery of piecewise constant data as it is frequently encountered in many practical sciences such as geophysics or biophysics (cf. [2] and the Introduction in [3]), whereas in, e.g., [4], TV-models were applied to the filtering of gravitational wave signals. Apart from the variety of possible applications, our interest in the one-dimensional case primarily stems from our previous work on TV-based variational problems in image analysis. In [5–11], variants of the classical TV-functional have been studied in any dimension replacing the regularization term by a convex functional of linear growth, which approximates the TV-seminorm and in addition has suitable ellipticity properties that make the considered models more reachable to analytical techniques. When trying to improve our results in the one-dimensional setting, we found ourselves surprised that, first, this is not a consequence of completely elementary arguments, and second, there are certain features of the corresponding solutions that do not seem to have analogs in arbitrary dimensions. In this context, we would also like to mention the papers [12, 13] and [14] where similar considerations were applied to study the classical TV-model as well as its generalizations towards functionals that involve higher derivatives in one dimension.

We proceed with a precise formulation of our setting and results. Let $f \in L^\infty(0,1)$ be a given function, which represents an observed signal (possibly corrupted by an additive

2010 *Mathematics Subject Classification.* Primary 26A45; Secondary 49J05, 49J45, 34B15.

Key words and phrases. Total variation, signal denoising, variational problems in one independent variable, linear growth, existence and regularity of solutions.

The authors thank Michael Bildhauer for many stimulating discussions.

Gaussian noise). We shall always assume that $0 \le f \le 1$ a.e. on $(0,1)$. For a given density function $F\colon \mathbb{R} \to [0,\infty)$ of linear growth, we consider the following minimization problem:

$$J[u] := \int_0^1 F(\dot u)\,dt + \frac{\lambda}{2}\int_0^1 (u-f)^2\,dt \to \min\,. \tag{1.1}$$

Here, $\int dt$ is Lebesgue's integral in one dimension, $\dot u := \frac{d}{dt}u$ denotes the (weak) derivative of a function $u\colon (0,1) \to \mathbb{R}$, and $\lambda > 0$ is a regularization parameter, which controls the balance between the smoothing and the data-fitting effect resulting from the minimization of the first and the second integral, respectively. We impose the following mild conditions on our energy density F:

$$F \in C^2(\mathbb{R}), \quad F(-p) = F(p), \quad F(0) = 0, \tag{F1}$$

$$|F'(p)| \le \nu_1, \tag{F2}$$

$$F(p) \ge \nu_2|p| - \nu_3, \quad \text{and} \tag{F3}$$

$$F''(p) > 0 \tag{F4}$$

for all $p \in \mathbb{R}$ and for some constants $\nu_1, \nu_2 > 0$, $\nu_3 \in \mathbb{R}$. Note that from (F1) and (F2) it follows $F(p) \le \nu_1|p|$ for all $p \in \mathbb{R}$. Moreover it should be obvious that the condition $F(0) = 0$ is imposed only for notational simplicity. Examples of a reasonable choice of F are given by the regularized TV-density, $F_\epsilon(p) := \sqrt{\epsilon^2 + p^2} - \epsilon$ for some $\epsilon > 0$ or $F(p) := \Phi_\mu(|p|)$, where Φ_μ denotes the standard example of the so-called μ-elliptic density, i.e., for a given ellipticity parameter $\mu > 1$ we consider

$$\Phi_\mu(r) := \int_0^r \int_0^s (1+t)^{-\mu}\,dt\,ds, \quad r \ge 0,$$

and observe the formulas

$$\begin{cases} \Phi_\mu(r) = \frac{1}{\mu-1}r + \frac{1}{\mu-1}\frac{1}{\mu-2}(r+1)^{-\mu+2} - \frac{1}{\mu-1}\frac{1}{\mu-2}, & \mu \ne 2, \\ \Phi_2(r) = r - \ln(1+r), & r \ge 0. \end{cases} \tag{1.2}$$

It is easily confirmed that $F(p) := \Phi_\mu(|p|)$, $p \in \mathbb{R}$, satisfies the condition of μ-ellipticity

$$F''(p) \ge \frac{c_1}{(1+|p|)^\mu} \tag{F5}$$

for a constant $c_1 > 0$ as well as (F1)-(F4). We remark that we have

$$\lim_{\mu\to\infty} (\mu-1)\Phi_\mu(|p|) = |p|,$$

which underlines that $\Phi_\mu(|p|)$ is a good candidate for approximating the TV-density (see, e.g., [5–7] or [11]). Next, we introduce the positive number

$$\lambda_\infty = \lambda_\infty(F) := \lim_{p\to\infty} F'(p). \tag{1.3}$$

This value will turn out to be a sort of a natural threshold in the investigation of the regularity properties for minimizers of problem (1.1).

Example 1.1. For F_ϵ it is immediate that $\lambda_\infty(F_\epsilon) = 1$ independently of ϵ, whereas for $F = \Phi_\mu$ we have

$$\lambda_\infty(\Phi_\mu) = \frac{1}{\mu-1}.$$

Before giving a *résumé* of our results concerning problem (1.1), we have to add some comments on functions and related spaces. For a general overview on one-dimensional variational problems and a synopsis of the related function spaces, we refer to [15]. For $1 \le p \le \infty$ and $m \in \mathbb{N}$, we denote by $W^{m,p}_{(\mathrm{loc})}(0,1)$ the standard Sobolev space on the interval $(0,1)$ of (locally) m-times weakly in $L^p(0,1)$ differentiable functions equipped with the norm $\|\cdot\|_{m,p}$. For a more detailed analysis of these spaces we refer to classical textbooks on this subject such as, e.g., [16]. We shall frequently make use of the identification $W^{m,\infty}(0,1) = C^{m-1,1}\big([0,1]\big)$, where for $0 < \alpha \le 1$, $C^{m,\alpha}(0,1)$ as usual denotes the space of m-times differentiable functions with locally Hölder continuous derivatives on $(0,1)$, and $C^{m,\alpha}\big([0,1]\big)$ has an obvious meaning. In the case where $\alpha = 1$ and $m = 0$, $C^{0,1}\big([0,1]\big) = W^{1,\infty}(0,1) =: \mathrm{Lip}(0,1)$ is the space of Lipschitz-continuous functions, where our notion makes implicit use of the fact that these functions possess a Lipschitz continuous extension to the boundary. We further would like to remark that some authors prefer to write $AC(0,1)$ in place of $W^{1,1}(0,1)$, referring to the more classical notion of "absolutely continuous" functions forming a proper subspace of $C^0\big([0,1]\big)$, see, e.g., [15, Chapter 2]. Finally, $BV(0,1)$ denotes the space of functions of bounded variation on $(0,1)$, i.e., the set of all functions $u \in L^1(0,1)$ whose distributional derivative Du can be represented by a signed Radon measure of finite total mass $\int_0^1 |Du|$. For more information concerning these spaces the reader is referred to the monographs [17] and [18].

Due to [18, Theorem 3.28, p. 136] (see also [15, Section 2.3, p. 90]), there is always a "good" representative of a BV-function u that is continuous up to a countable set of jump points $\{x_k\} \subset (0,1)$, $k \in \mathbb{N}$, i.e., in particular, the left and the right limit exist at all points. In what follows, we shall tacitly identify any BV-function with this particular representative. We note that the classical derivative of this good representative, which we denote by $\dot{u}$, exists at almost all points (see [18, Theorem 3.28, p. 136] once again). The measure Du can then be decomposed into the following sum

$$Du = \overbrace{\dot{u}\mathcal{L}^1}^{=:D^a u} + \overbrace{\sum_{k\in\mathbb{N}} h(x_k)\delta_{x_k} + D^c u}^{=:D^s u}, \tag{1.4}$$

and we have (compare [18, Corollary 3.33])

$$|Du|(0,1) = \int_0^1 |\dot{u}|\,dt + \sum_{k\in\mathbb{N}} |h(x_k)| + |D^c u|(0,1).$$

Here, $h(x_k) := \lim_{x\downarrow x_k} u(x) - \lim_{x\uparrow x_k} u(x)$ denotes the "jump-height" and δ_{x_k} is Dirac's measure at x_k. The sum $\sum_{k\in\mathbb{N}} h(x_k)\delta_{x_k}$ is named the *jump part* $D^j u$ of Du, which, together with the so-called *Cantor part* $D^c u$ forms the *singular part* $D^s u$. Furthermore, $\dot{u}\mathcal{L}^1$ is the *absolutely continuous part* $D^a u$ with respect to the measure $\mathcal{L}^1$, and $D^a u + D^s u$ is the Lebesgue decomposition of Du.

Coming back to the subject of our investigation, we put problem (1.1) in a more precise form, i.e., we consider the minimization problem

$$J[w] := \int_0^1 F(\dot{w})\,dt + \frac{\lambda}{2}\int_0^1 (w-f)^2\,dt \to \text{ min in } W^{1,1}(0,1) \tag{1.5}$$

for a density F satisfying (F1)–(F4), in particular F is of linear growth. Hence, the Sobolev space $W^{1,1}(0,1)$ is the natural domain of J. However, due to the nonreflexivity of this space, in general we cannot expect to find a solution. Following ideas in [5], we therefore pass to a relaxed version K of the above functional, which is defined for $w \in BV(0,1)$ and takes a particularly simple form in our one-dimensional setting; this means that we replace (1.5) by the problem

$$K[w] := \int_0^1 F(\dot{w})\,dt + \lambda_\infty |D^s w|(0,1) + \frac{\lambda}{2}\int_0^1 (w-f)^2\,dt \to \text{ min in } BV(0,1). \tag{1.6}$$

We would like to note that the above formula coincides with the usual notion of relaxation in BV (cf. [18, Theorem 5.47, p. 304]), because under the assumptions imposed on F the recession function $F^\infty(p) := \lim_{t\to\infty} F(tp)/t$ simplifies to $F^\infty(p) = \lambda_\infty|p|$.

From the point of view of regularity, BV-minimizers (i.e., solutions of problem (1.6)) are not very popular. However, it turns out that if we (strongly) restrict the size of the free parameter λ, it is possible to establish the existence of a unique J-minimizer u in the space $W^{1,1}(0,1)$. Part a) of the following theorem is concerned with this issue, whereas in part b) we show that the minimizer of the relaxed version (1.6) in the space $BV(0,1)$ is exactly the solution u from part a). Part c) is devoted to the regularity behavior of the J-minimizer u. Here we can prove optimal regularity, which in this context means that u is of class $C^{1,1}$ on the interval $[0,1]$. Furthermore, it turns out that u solves a Neumann-type two-point boundary value problem. Precisely, we have the following statements.

Theorem 1.1 (full regularity for small values of λ)**.** *Suppose that* $0 \le f \le 1$ *a.e. on* $[0,1]$ *and that the density* F *satisfies* (F1)–(F4)*. Next, assume that the parameter* λ *satisfies*

$$\lambda < \lambda_\infty(F) \tag{1.7}$$

with $\lambda_\infty(F)$ *defined as in* (1.3)*. Then*:

a) *Problem* (1.5) *admits a unique solution* $u \in W^{1,1}(0,1) = AC(0,1)$ *and this solution satisfies* $0 \le u(x) \le 1$ *for all* $x \in [0,1]$.

b) *The relaxation "$K \to$ min in $BV(0,1)$" has only one solution, which coincides with u from part* a).

c) *The minimizer u belongs to the class* $W^{2,\infty}(0,1) = C^{1,1}\big([0,1]\big)$ *and solves the following Neumann-type boundary value problem*:

$$\text{(BVP)}\qquad \begin{cases} \ddot{u} = \lambda \dfrac{u-f}{F''(\dot{u})} \ \ \textit{a.e. on}\ \ (0,1), \\ \dot{u}(0) = \dot{u}(1) = 0. \end{cases}$$

Remark 1.1. The bound (1.7) on the parameter λ occurs for technical reasons, because it allows us to prove a general statement on the solvability of problem (1.5). In practice, this threshold strongly depends on the data function f as well and, as numerical experiments suggest, often exceeds λ_∞. In Theorems 1.3 and 1.6 we shall determine better estimates for λ under which we can expect the solvability of (1.5), whereas Theorem 1.4 proves that the statement of Theorem 1.1 is indeed only true for a restricted range of λ.

Next we drop the bound (1.7) and pass to the relaxed variational problem (1.6).

Theorem 1.2 (partial regularity for arbitrary values of λ)**.** *Suppose that* $0 \le f \le 1$ *a.e. on* $[0,1]$*, and let the density* F *satisfy* (F1)–(F4) *as well as the additional requirement*

$$\text{(F6)}\qquad F''(p) \le c_2 \frac{1}{1+|p|}$$

for all $p \in \mathbb{R}$*, where* $c_2 > 0$ *is a constant. Moreover, let* $\lambda > 0$ *denote any number. Then*:

a) *Problem* (1.6) *admits a unique solution* $u \in BV(0,1)$ *satisfying* $0 \le u \le 1$ *a.e.*

b) *There is an open subset* $\operatorname{Reg}(u)$ *of* $(0,1)$ *such that* $u \in W^{2,\infty}_{\mathrm{loc}}(\operatorname{Reg}(u))$ *and*

$$\mathcal{L}^1\big((0,1) - \operatorname{Reg}(u)\big) = 0.$$

We can choose

$$\operatorname{Reg}(u) := \big\{s \in (0,1) \,:\, s \textit{ is a Lebesgue point of } \dot{u}\big\},$$

where $\dot{u}$ *is defined in* (1.4)*. Moreover, there are numbers* $0 < t_1 \le t_2 < 1$ *such that* $u \in C^{1,1}\big([0,t_1] \cup [t_2,1]\big)$.

c) *If there is a subinterval* $(a,b) \subset (0,1)$ *such that* $f \in W^{1,2}_{\mathrm{loc}}(a,b)$*, then* $u \in W^{1,1}(a,b) \cap W^{1,2}_{\mathrm{loc}}(a,b)$*. In case* $(a,b) = (0,1)$*, the solution* $u \in W^{1,1}(0,1) \cap W^{1,2}_{\mathrm{loc}}(0,1)$ *is* J*-minimizing in* $W^{1,1}(0,1)$*.*

Remark 1.2.

(i) Note that we need (F6) only for proving part c). Parts a) and b) remain valid without (F6).

(ii) The requirement (F6) is not as restrictive as it may appear at first sight. In particular, it is easy to confirm that for given $\epsilon > 0$ and $\mu > 1$ our examples from the Introduction, $F(p) := F_\epsilon(p)$ and $F(p) := \Phi_\mu(|p|)$, satisfy condition (F6).

Since signals in practice are usually modeled by more regular functions rather than merely through measurable ones (we have, e.g., rectangular- or "sawtooth"-like signals in mind, which are differentiable outside a small exceptional set), it is reasonable to ask to what extend these properties are reproduced by the K-minimizer u. The next theorem shows how the results of Theorem 1.2 can be improved if we assume better data.

Theorem 1.3 (regularity for special data)**.** *Suppose that the density* F *satisfies* (F1)–(F4)*, assume* $0 \le f \le 1$ *a.e. on* $[0,1]$*, and let* u *be the* K*-minimizer from Theorem* 1.2*.*

a) *Let* $t_0 \in (0,1)$ *be a point where some representative of the data function* f *is continuous. Then the good representative of* u *introduced before* (1.4) *is continuous at* t_0*.*

b) *Assume that there is an interval* $[a,b] \subset (0,1)$ *such that* $f \in \mathrm{Lip}(a,b) = W^{1,\infty}(a,b)$*. Then* $u \in C^2(a,b)$*.*

c) *Suppose* $f \in W^{1,1}(0,1)$ *and define*

$$\omega_\infty := \lim_{p\to\infty} pF'(p) - F(p) \in (0,\infty]. \tag{1.8}$$

Then, if $\lambda\big(\frac{1}{2} + \|\dot f\|_1\big) < \omega_\infty$*, it follows that* $u \in C^{1,1}\big([0,1]\big)$*.*

Corollary 1.1. *If the data function* f *is globally Lipschitz-continuous on* $[0,1]$*, then* $u \in C^2\big([0,1]\big)$*.*

Proof of Corollary 1.1. Applying Theorem 1.3 b) with a and b arbitrarily close to 0 and 1, respectively, yields $u \in C^2(0,1)$. In particular, it is therefore immediate that u satisfies the differential equation from Theorem 1.1 c)

$$\ddot u = \lambda \frac{u - f}{F''(\dot u)} \tag{1.9}$$

everywhere on $(0,1)$. Due to Theorem 1.2 b) we have $u \in C^1\big([0,t_1] \cup [t_2,1]\big)$ and therefore $\dot u$ is uniformly continuous on $[0,1]$, which means that the right-hand side of equation (1.9) belongs to the space $C^0\big([0,1]\big)$. Thus $\ddot u$ exists even at 0 and 1 and is a continuous function on $[0,1]$. □

Remark 1.3.

(i) From part a) we infer that if f is continuous on an interval $(a,b) \subset [0,1]$, then also $u \in C^0(a,b)$.

(ii) We would like to remark that part b) in particular applies to piecewise affine data functions such as triangular or rectangular signals as shown in Figure 1.

We then obtain the differentiability of the corresponding K-minimizers outside the set of jump points of the data. In particular, if the data are Lipschitz except for a countable set of jump-type discontinuities, then K attains its minimum in the space $SBV(0,1)$ of special functions of bounded variation (see [18, Chapter 4] for a definition).

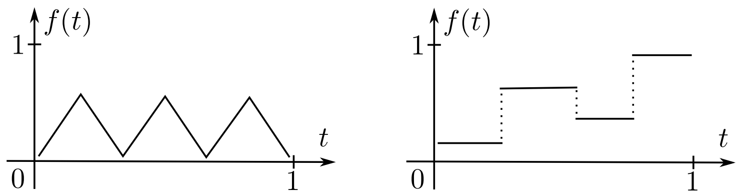

FIGURE 1. Examples of typical data functions.

(iii) The main feature of part c) of Theorem 1.3 is that, even though full $C^{1,1}$-regularity may fail to occur in general if the parameter λ exceeds λ_∞, it can still occur up to $2\lambda_\infty$ provided the oscillation of the data f is sufficiently small. If we take for example the regularized graph-length integrand as our density F, i.e., $F(p) := F_\epsilon(p) = \sqrt{\epsilon^2 + p^2} - \epsilon$, then it is easily verified that

$$\omega_\infty(F_\epsilon) = \epsilon.$$

Consequently, we get full $C^{1,1}$-regularity for all parameters λ up to the bound

$$\frac{\epsilon}{\frac{1}{2} + \|\dot{f}\|_1},$$

which might be larger than $\lambda_\infty(F_\epsilon) = 1$ provided we choose ϵ sufficiently large. If we take $F(p) = \Phi_\mu(|p|)$, then $\lambda_\infty = \frac{1}{\mu-1}$, whereas

$$\lim_{p\to\infty} p\Phi_\mu'(p) - \Phi_\mu(p) = \begin{cases} \frac{1}{\mu-1}\frac{1}{\mu-2}, & \mu > 2, \\ \infty, & 1 < \mu \le 2, \end{cases}$$

so that in particular ω_∞ is unbounded if we let μ approach 2 from above.

Next we would like to demonstrate the sharpness of our previous regularity results, in particular we want to indicate that singular (i.e., discontinuous) minimizers can occur if we pass from Lipschitz signals f studied in Theorem 1.3 (cf. also Corollary 1.1) to functions f having jumps at some interior points of the interval $[0,1]$. To be precise, we let for $\mu > 1$

$$F(p) = \Phi_\mu(|p|), \quad p \in \mathbb{R}, \tag{1.10}$$

with Φ_μ as defined in (1.2) and recall that for this density we have (compare Example 1.1)

$$\lambda_\infty = \frac{1}{\mu - 1}. \tag{1.11}$$

Moreover, we define

$$f\colon [0,1] \to [0,1],\ f(t) = \begin{cases} 0, & 0 \le t \le \frac{1}{2}, \\ 1, & \frac{1}{2} < t \le 1. \end{cases} \tag{1.12}$$

Theorem 1.4 (existence of discontinuous minimizers). *Under the assumptions* (1.10) *and* (1.12) *and with parameters $\lambda > 0$, $\mu > 1$, let $u \in BV(0,1)$ denote the unique solution of problem* (1.6) (*being of class $C^2([0,1] - \{\frac{1}{2}\})$ by Theorem* 1.3 *and an obvious modification of the proof of Corollary* 1.1). *Then, assuming that $\mu > 2$ and that λ satisfies*

$$\lambda > \frac{8}{\mu - 2}, \tag{1.13}$$

we have

$$\sup_{0\le t<1/2} u < \frac{1}{2} < \inf_{1/2<t\le 1} u,$$

which means that u has a jump discontinuity at $t = 1/2$.

Remark 1.4. From our previous works [5, 6] and [10] we see that for $\mu \in (1,2)$ and any $\lambda > 0$ this phenomenon cannot occur, i.e., the minimizer u is a regular function. Thus, the value $\mu = 2$ separates regular and irregular behavior of the solutions.

Remark 1.5. Assume that $\lambda > 0$ is fixed. Then from (1.13) it follows that we can force the minimizer u to create a jump point at $t = \frac{1}{2}$ by choosing μ sufficiently large.

Remark 1.6. By Theorem 1.1, the solution is regular provided that

$$\lambda < \lambda_\infty \overset{(1.11)}{=} \frac{1}{\mu - 1}.$$

On the other hand, inequality (1.13) states that

$$\lambda > \frac{8}{\mu - 2} = 8\left(\frac{1}{\mu - 1} + \frac{1}{(\mu - 1)(\mu - 2)}\right) = 8(\lambda_\infty + \omega_\infty),$$

which suggests that our solution is irregular whenever λ and μ are chosen in such a way that $\lambda > 8(\lambda_\infty + \omega_\infty)$ (see Corollary 1.2 below).

With respect to Theorem 1.4 and Remark 1.4, it remains to discuss the situation for the limit case of $\mu = 2$, which can be done in a very general form: it turns out that our arguments are valid for all μ-elliptic densities F with exponent $\mu \in (1,2]$ and for arbitrary measurable data f leading to $C^{1,1}$-regularity of minimizers. It should be noted that, in particular, this implies the smoothness of minimizers in case $1 < \mu < 2$ without referring to the higher-dimensional results. Precisely, we have the next claim.

Theorem 1.5 (regularity for μ-elliptic densities for $1 < \mu \le 2$)**.** *Suppose* $0 \le f \le 1$ *a.e. on* $[0,1]$ *and consider a density* F *with* (F1)–(F5)*. Moreover, fix any number* $\lambda > 0$*. Then, if*

$$\mu \in (1,2], \tag{1.14}$$

then a unique solution $u \in BV(0,1)$ *of problem* (1.6) *is of class* $C^{1,1}\big([0,1]\big)$.

From the proofs of Theorem 1.4 and Theorem 1.5 we obtain the following slightly more general result on regular or irregular behavior of minimizers avoiding the notion of μ-ellipticity (F5).

Corollary 1.2. *Let* F *satisfy* (F1)–(F4) *and define* ω_∞ *as in* (1.8).

a) *In case* $\omega_\infty = \infty$*, any solution* $u \in BV(0,1)$ *is of class* $C^{1,1}\big([0,1]\big)$ *independently of the value of* λ *and for arbitrary data* $f \in L^\infty(0,1)$, $0 \le f \le 1$ *a.e.*

b) *If* $\omega_\infty < \infty$ *and if* (F6) *is true, then there is a critical value* λ_{crit} *of the parameter* λ *such that the solution* u *of* (1.6) *with* f *defined as in* (1.12) *is discontinuous* (*exactly at* $t = \frac{1}{2}$) *provided we choose* $\lambda > \lambda_{\text{crit}}$*. Moreover,*

$$\max\{\lambda_\infty, 8\omega_\infty\} \le \lambda_{\text{crit}} \le 8(\lambda_\infty + \omega_\infty). \tag{1.15}$$

Remark 1.7. Comparing part b) of the above corollary and parts a), c) of Theorem 1.3, we would like to emphasize that the occurence of discontinuous minimizers requires discontinuous data.

By part c) of Theorem 1.1, the minimization problem (1.1) leads to the second-order Neumann problem (BVP). Conversely, we could take this equation as our starting point and examine the existence and regularity of solutions purely by methods from the theory of ordinary differential equations. In the papers [19] and [20], Thompson worked out an extensive theory for a large class of two-point boundary value problems with both continuous and measurable right-hand sides, which we could apply to our situation with the following result.

Table 1. Overview of the various regularity statements.

Data f	Density F	Bound on λ	Regularity of u	Reference
$L^\infty(0,1)$	(F1)–(F4)	$0<\lambda<\lambda_\infty$	$C^{1,1}([0,1])$	Theorem 1.1 a)
$L^\infty(0,1)$	(F1)–(F4)	$\lambda>0$	$W^{2,\infty}_{\mathrm{loc}}(\mathrm{Reg}(u))$	Theorem 1.2 b)
$W^{1,2}_{\mathrm{loc}}(a,b)$	(F1)–(F4), (F6)	$\lambda>0$	$W^{1,1}(a,b)\cap W^{1,2}_{\mathrm{loc}}(a,b)$	Theorem 1.2 c)
continuous at t_0	(F1)–(F4)	$\lambda>0$	continuous at t_0	Theorem 1.3 a)
$W^{1,1}(0,1)$	(F1)–(F4)	$\lambda(\frac{1}{2}+\|\dot{f}\|_1)<\omega_\infty$	$C^{1,1}\big([0,1]\big)$	Theorem 1.3 c)
$L^\infty(0,1)$	(F1)–(F5) $\mu\in(1,2]$	$\lambda>0$	$C^{1,1}\big([0,1]\big)$	Theorem 1.5
$L^\infty(0,1)$	(F1)–(F4) $\omega_\infty=\infty$	$\lambda>0$	$C^{1,1}\big([0,1]\big)$	Corollary 1.2 a)
$L^\infty(0,1)$	(F1)–(F5)	$0<\lambda<\lambda_\mu$	$W^{2,1}(0,1)$	Theorem 1.6

Theorem 1.6 (regularity for μ-elliptic densities and $\mu>1$ arbitrary). *Suppose* $0\le f(t)\le 1$ *a.e. on* $[0,1]$, *and let* F *satisfy* (F1)–(F3) *as well as* (F5). *If the parameter* λ *satisfies*

$$0<\lambda<\sup_{L>1} c_1\int_1^L \frac{s\,ds}{(1+s)^\mu}=:\lambda_\mu,$$

where c_1 *is as in* (F5), *then there exists* $v\in W^{2,1}(0,1)$ *satisfying* $0\le v(t)\le 1$ *for all* $t\in[0,1]$ *and solving the Neumann problem* (BVP) *a.e. on* $[0,1]$. *Furthermore, this solution coincides with the unique* K*-minimizer* u *from the space* $BV(0,1)$.

Remark 1.8. (i) The reader familiar with the theory of lower and upper solutions will recognise the above bound λ_μ as a sort of "Nagumo-condition" (see, e.g., [21]), which guarantees *a priori* bounds on the first derivative of the solution v.
(ii) If f is continuous, the differential equation implies $v\in C^2\big([0,1]\big)$.
(iii) Using the example $F(p)=\Phi_\mu(|p|)$, we would like to demonstrate how λ_μ might actually improve the bound for λ stated in (1.7) of Theorem 1.1: obviously, the integral defining λ_μ diverges for $1<\mu\le 2$ and is unbounded if μ approaches 2 from above. In combination with part (ii) of this remark, we consequently get full $C^2\big([0,1]\big)$-regularity for arbitrarily large values of the parameter λ and continuous data f if we let $\mu\downarrow 2$.

Since it is somewhat difficult to track the various regularity statements from Theorem 1.1 up to Theorem 1.6, we have summarized our main results in the form of a table. It shows the regularity of the K-minimizer u depending on the data f, the density F, and the bound on the parameter λ.

Our paper is organized as follows. In §2 we prove Theorem 1.1 and thus, the solvability of problem (1.5) and the regularity of the unique $W^{1,1}$-minimizer under a rather strong bound on the parameter $\lambda>0$. §3 is devoted to the study of the relaxed problem (1.6), where the parameter $\lambda>0$ may be chosen arbitrarily large. The subsequent section deals with a refinement of our regularity result for certain classes of "well-behaved" data. §5 is devoted to the construction of the counterexample from Theorem 1.4. Subsequently, we give the proof of Theorem 1.5, where μ-elliptic densities are considered for $\mu\in(1,2]$, and then take a closer look at the Neumann-type boundary value problem (BVP) from Theorem 1.6 in §7. Finally, we compare our results with a numerically computed example.

§2. Proof of Theorem 1.1

Proof of part a). Let us assume the validity of the hypotheses of Theorem 1.1. First, we note that problem (1.5) has at most one solution thanks to the strict convexity of the data fitting quantity $\int_0^1 (w-f)^2\,dt$ with respect to w. Next, we show that there exists at least one solution. For this, we approximate our original variational problem by a sequence of more regular problems admitting smooth solutions with appropriate convergence properties. This technique is well known from [5–7] or [11]. To become more precise, for fixed $\delta \in (0,1]$ we consider the problem

$$J_\delta[w] := \int_0^1 F_\delta(\dot w)\,dt + \frac{\lambda}{2}\int_0^1 (w-f)^2\,dt \to \min \text{ in } \ W^{1,2}(0,1), \tag{2.1}$$

where

$$F_\delta(p) := \frac{\delta}{2}|p|^2 + F(p), \quad p \in \mathbb{R}. \tag{2.2}$$

In the following lemma we state that (2.1) is uniquely solvable in $W^{1,2}(0,1)$ and in addition we summarize some useful properties of the only J_δ-minimizer u_δ. In fact, these results are well known and have been proved in a much more general setting (see, e.g., [5] and [6, 7]). $\square$

Lemma 2.1. *Problem* (2.1) *admits a unique solution* $u_\delta \in W^{1,2}(0,1)$, *for which we have*:

a) $0 \le u_\delta \le 1$ *on* $[0,1]$,
b) $u_\delta \in W^{2,2}_{\mathrm{loc}}(0,1)$ (*not necessarily uniformly in* δ),
c) $\sup_{0<\delta\le 1} \|u_\delta\|_{W^{1,1}(0,1)} < \infty$,
d) $\sup_{0<\delta\le 1} \delta \int_0^1 |\dot u_\delta|^2\,dt < \infty$.

Proof of Lemma 2.1. By the direct method, it is immediate that problem (2.1) has a unique solution $u_\delta \in W^{1,2}(0,1)$. Since $0 \le f \le 1$ a.e. on Ω, a truncation argument as was already carried out in [5], proof of Theorem 1.8 a), (we refer the reader to [22] as well) shows that $0 \le u_\delta \le 1$ on Ω, and this proves part a). For part b) we use the well-known difference quotient technique. Observing that we have the uniform estimate $J_\delta[u_\delta] \le J[0]$ we directly obtain parts c) and d) if we use the definition of J_δ and recall the linear growth of F. $\square$

Remark 2.1. Note that the results of Lemma 2.1 do not depend on the size of the parameter $\lambda > 0$.

Remark 2.2. In our particular one-dimensional case we emphasize once again that, using Sobolev's embedding $W^{2,2}_{\mathrm{loc}}(0,1) \hookrightarrow C^1(0,1)$ (see [16]), we conclude that $\dot u_\delta(t)$ exists for all $t \in (0,1)$ and is continuous.

Before starting the proof of Theorem 1.1, we recall that from the assumptions (F1)–(F4) imposed on the density F and the definition of λ_∞ (compare (1.3)), it follows that

$$\mathrm{Im}(F') = (-\lambda_\infty, \lambda_\infty). \tag{2.3}$$

Next, we fix $\lambda \in (0, \lambda_\infty)$ and observe the validity of the following lemma, which is of elementary nature but will be important during the further proof.

Lemma 2.2. *The inverse function of* $F'_\delta \colon \mathbb{R} \to \mathbb{R}$ *is uniformly* (*in* δ) *bounded on the set* $[-\lambda, \lambda]$.

Proof of Lemma 2.2. We observe that F' is an odd and strictly monotone increasing function (compare (F4)) inducing a diffeomorphism between $\mathbb{R}$ and the open interval $(-\lambda_\infty, \lambda_\infty)$. Let us write $(F')^{-1}\big([-\lambda,\lambda]\big) = [-\alpha, \alpha]$, where $F'(\alpha) = \lambda$. Next we choose

$t \in [-\lambda, \lambda]$ and assume that $(F_\delta')^{-1}(t) > \alpha$. Then, since F_δ' is strictly monotone increasing, we have

$$t > F_\delta'(\alpha) = \delta\alpha + F'(\alpha) = \delta\alpha + \lambda > \lambda,$$

which is a contradiction. The case where $(F_\delta')^{-1}(t) < -\alpha$ is treated in the same manner. Thus, the lemma is proved. □

After these preparations, we proceed with the proof of Theorem 1.1 a). First, we introduce the continuous functions

$$\sigma_\delta := F_\delta'(\dot{u}_\delta). \tag{2.4}$$

We wish to note (see, e.g., [11]) that σ_δ is the (unique) solution of the variational problem dual to (2.1) (we will come back to this later in the proof of Theorem 1.3 c)). Using (F2) together with Lemma 2.1 d), we obtain

$$\sigma_\delta \in L^2(0,1) \text{ uniformly in } \delta. \tag{2.5}$$

Next, we observe that u_δ solves the Euler equation

$$0 = \int_0^1 F_\delta'(\dot{u}_\delta)\dot{\varphi}\,dt + \lambda \int_0^1 (u_\delta - f)\varphi\,dt \tag{2.6}$$

for all $\varphi \in W^{1,2}(0,1)$. Note that, by (2.4), this equation states that σ_δ is weakly differentiable with

$$\dot{\sigma}_\delta = \lambda(u_\delta - f) \text{ a.e. on } (0,1). \tag{2.7}$$

Combining Lemma 2.1 a) with (2.5) and (2.7), we see (recall our assumption $0 \le f \le 1$ a.e. on $(0,1)$) that

$$\sigma_\delta \in W^{1,\infty}(0,1) = C^{0,1}\big([0,1]\big) \text{ uniformly in } \delta \text{ and } \|\dot{\sigma}_\delta\|_\infty \le \lambda. \tag{2.8}$$

Choosing $\varphi \in C^1\big([0,1]\big)$ in (2.6) and recalling (2.7), we get (see [23, (18.16) Theorem, p. 285] or [15, Chapter 2])

$$0 = \int_0^1 \big(\dot{\sigma}_\delta\varphi + \sigma_\delta\dot{\varphi}\big)\,dt = \int_0^1 \frac{d}{dt}\big(\sigma_\delta\varphi\big)\,dt = \sigma_\delta(1)\varphi(1) - \sigma_\delta(0)\varphi(0).$$

Thus, since $\varphi \in C^1\big([0,1]\big)$ is arbitrary, we must have

$$\sigma_\delta(0) = \sigma_\delta(1) = 0. \tag{2.9}$$

Note that (2.8) and (2.9) imply

$$\|\sigma_\delta\|_\infty \le \lambda. \tag{2.10}$$

At this point, the definition of σ_δ, (2.8), (2.9), (2.10), and Lemma 2.2 yield the existence of a constant $M > 0$ independent of δ and such that

$$\|\dot{u}_\delta\|_\infty \le M. \tag{2.11}$$

Here we have made essential use of the restriction $\lambda < \lambda_\infty$. As a consequence, there exists a function $u \in W^{1,\infty}(0,1)$ such that $u_\delta \rightrightarrows u$ uniformly as $\delta \downarrow 0$, and $\dot{u}_\delta \rightharpoonup \dot{u}$ in $L^p(0,1)$ for all finite $p > 1$ as $\delta \downarrow 0$, at least for a subsequence. Now, our goal is to show that u is J-minimal: thanks to the J_δ-minimality of u_δ it follows for all $v \in W^{1,2}(0,1)$ we have

$$J[u_\delta] \le J_\delta[u_\delta] \le J_\delta[v] \xrightarrow{\delta\downarrow 0} J[v]$$

together with

$$J[u] \le \liminf_{\delta\to 0} J[u_\delta].$$

Thus, we have $J[u] \le J[v]$ for all $v \in W^{1,2}(0,1)$ and from this we get $J[u] \le J[w]$ for all $w \in W^{1,1}(0,1)$ by approximating w with a sequence $(v_k) \subset W^{1,2}(0,1)$ in the

$W^{1,1}$-topology. This finally proves that u is a solution of problem (1.5). This proves part a).

Proof of part b). Considering the relaxed variant K from (1.6) of the functional J, it is easy to check that K has a unique solution $\widetilde{u} \in BV(0,1)$, compare the comments given at the beginning of the proof of Theorem 1.2 a). Together with the J-minimality of u, this implies $K[\widetilde{u}] \le J[u]$ because $K[w] = J[w]$ for all functions $w \in W^{1,1}(0,1)$. To show the reverse inequality, we note that we can approximate $\widetilde{u} \in BV(0,1)$ by a sequence of smooth functions $(u_n) \subset C^\infty(0,1)$ such that (as $n \to \infty$)

$$u_n \to \widetilde{u} \ \text{ in } \ L^1(0,1) \ \text{ and } \ \int_0^1 \sqrt{1+\dot{u}_n^2}\, dt \to \int_0^1 \sqrt{1+|D\widetilde{u}|^2}$$

(see, e.g., [24, Proposition 2.3]), where $\int_0^1 \sqrt{1+|D\widetilde{u}|^2}$ denotes the total variation of the vector measure $(\mathcal{L}^1, Du)^T$. Note that for any finite $p > 1$ we even have $u_n \to \widetilde{u}$ in $L^p(0,1)$ by the BV-embedding theorem. Now it is well known that the functional K is continuous with respect to the above notion of convergence (see, e.g., [24, Proposition 2.2]), and it follows that

$$K[\widetilde{u}] = \lim_{n\to\infty} K[u_n] = \lim_{n\to\infty} J[u_n] \ge J[u].$$

Hence, $K[\widetilde{u}] = J[u]$, i.e., u is K-minimal and $u = \widetilde{u}$ due to the uniqueness of the K-minimizer. □

Proof of part c). By (2.6) and Lemma 2.1 b),

$$\ddot{u}_\delta = \lambda \frac{(u_\delta - f)}{F''_\delta(\dot{u}_\delta)} \ \text{a.e. on } (0,1),$$

whence $\dot{u}_\delta \in W^{1,\infty}(0,1)$ uniformly in δ on account of (2.11). Thus, the functions $\dot{u}_\delta$ have a unique Lipschitz extension to the boundary points 0 and 1, which in particular implies the differentiability of u_δ at 0 and 1, with values of the derivatives given by the values of the Lipschitz extension of $\dot{u}_\delta$. Thus, there is a clear meaning of $\dot{u}_\delta(0)$ and $\dot{u}_\delta(1)$. By continuity reasons, the defining equation (2.4) for σ_δ extends to the boundary points of $(0,1)$, and since F'_δ vanishes exactly at the origin, from (2.9) it follows that $\dot{u}_\delta(0) = \dot{u}_\delta(1) = 0$. Combining this with the uniform boundedness of u_δ in $C^{1,1}([0,1])$, we immediately see that $u \in C^{1,1}([0,1])$, together with the boundary condition $\dot{u}(0) = \dot{u}(1) = 0$. Furthermore, u solves the Euler equation

$$0 = \int_0^1 F'(\dot{u})\dot{\varphi}\, dt + \lambda \int_0^1 (u-f)\varphi\, dt$$

for all $\varphi \in C_0^1(0,1)$ and from this we conclude the validity of the relation

$$\frac{d}{dt} F'(\dot{u}) = \lambda(u-f) \ \text{a.e. on } (0,1).$$

Consequently, we have

$$\ddot{u} = \lambda \frac{u-f}{F''(\dot{u})} \ \text{a.e. on } \ (0,1),$$

together with $\dot{u}(0) = \dot{u}(1) = 0$, i.e., u solves the boundary value problem (BVP), which was the statement of part c). □

§3. Proof of Theorem 1.2

Let us assume the validity of the hypotheses of Theorem 1.2. We start with the following.

Proof of part a). That in fact the functional K as in (1.6) admits a unique minimizer $u \in BV(0,1)$ is straightforward in the framework of the theory of BV-functions (see, e.g., [18, Theorem 3.23, p. 132] as well as [18, Remark 5.46 and Theorem 5.47, pp. 303–304]). The justification that we have $0 \le u \le 1$ a.e. on $(0,1)$ follows by a truncation argument (see [7] in the case of pure denoising and [22]). For later purposes, we show that the minimizer u can also be obtained as the limit of the regularizing sequence introduced in Lemma 2.1, giving $0 \le u \le 1$ as a byproduct of Lemma 2.1 a): as done there, we study the problem

$$J_\delta[w] := \frac{\delta}{2}\int_0^1 |\dot{w}|^2 dx + J[w], \quad w \in W^{1,2}(0,1),$$

where, in particular, $0 \le u_\delta \le 1$ for all $t \in [0,1]$ (see Lemma 2.1, a)). Next we show that $u_\delta \to u$ in $L^1(0,1)$ and a.e. at least for a subsequence. First, by Lemma 2.1 c), there exists $\widetilde{u} \in BV(0,1)$ such that (for a subsequence) $u_\delta \to \widetilde{u}$ in $L^1(0,1)$. By lower semicontinuity we have

$$K[\widetilde{u}] \le \liminf_{\delta\downarrow 0} J[u_\delta],$$

which, by using the K-minimality of u, yields

$$K[u] \le K[\widetilde{u}] \le \liminf_{\delta\downarrow 0} J_\delta[u_\delta].$$

As in the proof of Theorem 1.1 b), we approximate the function u by a sequence of smooth functions $(u_m) \subset C^\infty(0,1)$ such that (as $m \to \infty$)

$$u_m \to u \text{ in } L^1(0,1), \quad \int_0^1 \sqrt{1+\dot{u}_m^2}\, dt \to \int_0^1 \sqrt{1+|Du|^2},$$

and observe that $u_m \to u$ in $L^p(0,1)$ for each finite $p > 1$. Since K is continuous with respect to the above notion of convergence, we obtain $K[u_m] \to K[u]$ as $m \to \infty$. By the J_δ-minimality of u_δ, this implies

$$K[u] \le K[\widetilde{u}] \le \liminf_{\delta\downarrow 0} J_\delta[u_\delta] \le \liminf_{\delta\downarrow 0} J_\delta[u_m] = J[u_m] = K[u_m].$$

Thus, after passing to the limit as $m \to \infty$, we get

$$K[u] \le K[\widetilde{u}] \le K[u],$$

which implies $u = \widetilde{u}$ by the uniqueness of the K-minimizer, whence $0 \le u \le 1$ a.e. on $(0,1)$. □

Proof of part b). With σ_δ as defined in the proof of Theorem 1.1 (see (2.4)), we recall that we have (2.7)–(2.9) at hand. Note that at this stage no bound on λ was necessary. Thus, there exists $\sigma \in W^{1,\infty}(0,1)$ with $\sigma_\delta \rightrightarrows \sigma$ as $\delta \downarrow 0$ (at least for a subsequence). Moreover

$$\begin{cases} \dot{\sigma} = \lambda(u-f) \text{ and thus } |\dot{\sigma}(t)| \le \lambda \text{ a.e.,} \\ |\sigma(t)| \le \lambda \text{ on } [0,1], \\ \sigma(0) = \sigma(1) = 0. \end{cases} \tag{3.1}$$

In accordance with [11, Theorem 1.3] (in the case of pure denoising), σ is the unique solution of the dual problem associated with (1.5), and

$$\sigma = F'(\dot{u}) \text{ a.e.,} \tag{3.2}$$

where u is the unique solution of problem (1.6) in the class $BV(0,1)$ and $\dot{u}$ in what follows denotes the Lebesgue point representative of the density of the absolutely continuous part $D^a u$ of the measure Du. Thus, there is a null set $A \subset (0,1)$ such that

$$\sigma(t) = F'(\dot{u}(t)), \quad t \in (0,1) - A, \tag{3.3}$$

(see (3.2)). Let us fix $t_0 \in (0,1) - A$. Then $|\sigma(t_0)| < \lambda_\infty$, and since σ is continuous (recall (3.1)), there exists $\epsilon > 0$ with

$$|\sigma(t)| \le \lambda_\infty - \alpha \ \text{ for all } \ t \in [t_0 - \epsilon, t_0 + \epsilon], \tag{3.4}$$

where $\alpha > 0$ is chosen appropriately. Recalling that $\sigma_\delta \rightrightarrows \sigma$, we see that, for $\delta \le \delta_\epsilon$, (3.4) yields

$$|\sigma_\delta(t)| \le \lambda_\infty - \frac{\alpha}{2} \ \text{ for all } \ t \in [t_0 - \epsilon, t_0 + \epsilon]. \tag{3.5}$$

By Lemma 2.2, $(F'_\delta)^{-1}$ is uniformly (with respect to δ) bounded on $\big[-\lambda_\infty + \frac{\alpha}{2}, \lambda_\infty - \frac{\alpha}{2}\big]$. Hence, there exists a number $L > 0$, independent of δ such that (compare (2.11))

$$\|\dot{u}_\delta\|_{L^\infty(t_0-\epsilon, t_0+\epsilon)} \le L \ \text{ for all } \ \epsilon \le \delta. \tag{3.6}$$

Since u is the L^1-limit of the sequence (u_δ) (compare the proof of part a) of this theorem), (3.6) ensures

$$u \in C^{0,1}\big([t_0 - \epsilon, t_0 + \epsilon]\big).$$

Next, using the Euler equation (2.6) for u_δ on $(t_0 - \epsilon, t_0 + \epsilon)$, we deduce the identity

$$\ddot{u}_\delta = \lambda \frac{(u_\delta - f)}{F''_\delta(\dot{u}_\delta)} \ \text{ a.e. on } \ (t_0 - \epsilon, t_0 + \epsilon),$$

which yields the existence of a number $L' > 0$ independent of δ and such that

$$\|\ddot{u}_\delta\|_{L^\infty(t_0-\epsilon, t_0+\epsilon)} \le L'. \tag{3.7}$$

From (3.7), it finally follows

$$u \in C^{1,1}\big([t_0 - \epsilon, t_0 + \epsilon]\big),$$

showing that u is of class $C^{1,1}$ in a neighborhood of a point $t \in (0,1)$ if and only if t is a Lebesgue point of $\dot{u}$. Recalling (3.1), we can conclude that (3.4) (which by the way implies (3.6) and (3.7)) is true on a suitable interval $[0, t_1]$. This can be achieved by setting $t_1 < \sup\{s \in [0,1] : |\sigma(s)| < \lambda_\infty\}$, for instance. Hence, $u \in C^{1,1}\big([0, t_1]\big)$. Using similar arguments, we can show the existence of a number t_2 such that $0 < t_1 \le t_2 < 1$ and $u \in C^{1,1}([t_2, 1])$. This proves part b) of the theorem.

Proof of part c). Our strategy is to prove $u_\delta \in W^{1,2}_{\mathrm{loc}}(a,b)$ uniformly with respect to δ. With this result at hand along with the fact that the K-minimizing function $u \in BV(0,1)$ is obtained as the limit of the sequence (u_δ), we see that $u \in BV(a,b) \cap W^{1,2}_{\mathrm{loc}}(a,b)$, so that $u \in W^{1,1}(a,b)$. First, we recall that $u_\delta \in W^{2,2}_{\mathrm{loc}}(0,1)$ (compare Lemma 2.1) and that $F'_\delta(\dot{u}_\delta)$ is of class $W^{1,2}_{\mathrm{loc}}(0,1)$ and satisfies

$$(F'_\delta(\dot{u}_\delta))' = F''_\delta(\dot{u}_\delta)\ddot{u}_\delta \ \text{ a.e. on } \ (0,1).$$

Therefore, from (2.6) we get

$$\int_0^1 F''_\delta(\dot{u}_\delta)\ddot{u}_\delta \dot{\varphi}\, dt = \lambda \int_0^1 (u_\delta - f)\dot{\varphi}\, dt \tag{3.8}$$

for all $\varphi \in C_0^\infty(0,1)$; by approximation, (3.8) remains true for the functions $\varphi \in W^{1,2}(0,1)$ that are compactly supported in $(0,1)$. Next, we fix a point $x_0 \in (a,b)$, a number $R > 0$

such that $(x_0 - 2R, x_0 + 2R) \Subset (a, b)$ and $\eta \in C_0^\infty(x_0 - 2R, x_0 + 2R)$ with $\eta \equiv 1$ on $(x_0 - R, x_0 + R)$, $0 \le \eta \le 1$ as well as $|\dot\eta| \le \frac{c}{R}$. We choose $\varphi := \eta^2 \dot u_\delta$ in (3.8) obtaining

$$\begin{aligned} I_0 &:= \int_{x_0-2R}^{x_0+2R} F_\delta''(\dot u_\delta)(\ddot u_\delta)^2 \eta^2 \, dt \\ &= -2 \int_{x_0-2R}^{x_0+2R} F_\delta''(\dot u_\delta)\ddot u_\delta \dot u_\delta \dot\eta\eta \, dt + \lambda \int_{x_0-2R}^{x_0+2R} (u_\delta - f)\dot\varphi \, dt =: I_1 + \lambda I_2. \end{aligned} \tag{3.9}$$

We start with estimating I_1 where, by using Young's inequality for fixed $\epsilon > 0$, we get

$$|I_1| \le \epsilon I_0 + c\epsilon^{-1} \int_{x_0-2R}^{x_0+2R} F_\delta''(\dot u_\delta)\dot u_\delta^2 \dot\eta^2 \, dt. \tag{3.10}$$

For I_2, integration by parts (recall that $f \in W^{1,2}_{\mathrm{loc}}(a, b)$) gives

$$I_2 = -\int_{x_0-2R}^{x_0+2R} (\dot u_\delta - \dot f)\dot u_\delta \eta^2 \, dt = -\int_{x_0-2R}^{x_0+2R} \dot u_\delta^2 \eta^2 \, dt + \int_{x_0-2R}^{x_0+2R} \dot f \dot u_\delta \eta^2 \, dt. \tag{3.11}$$

Putting together (3.10) and (3.11) and absorbing terms (we choose $\epsilon > 0$ sufficiently small), we see that (3.9) implies

$$\begin{aligned} &\int_{x_0-2R}^{x_0+2R} F_\delta''(\dot u_\delta)(\ddot u_\delta)^2 \eta^2 \, dt + \lambda \int_{x_0-2R}^{x_0+2R} \dot u_\delta^2 \eta^2 \, dt \\ &\qquad \le c \int_{x_0-2R}^{x_0+2R} F_\delta''(\dot u_\delta)\dot u_\delta^2 \dot\eta^2 \, dt + c \int_{x_0-2R}^{x_0+2R} |\dot f| \, |\dot u_\delta| \eta^2 \, dt. \end{aligned} \tag{3.12}$$

The first integral on the right-hand side of (3.12) can be handled by the uniform estimate $J_\delta[u_\delta] \le J[0]$, the linear growth of F, and condition (F6). More precisely, we get

$$\int_{x_0-2R}^{x_0+2R} F_\delta''(\dot u_\delta)\dot u_\delta^2 \dot\eta^2 \, dt \le c(R) \int_{x_0-2R}^{x_0+2R} (\delta + (1 + \dot u_\delta^2)^{-\frac{1}{2}})\dot u_\delta^2 \, dt \le c(R),$$

where $c(R)$ denotes a local constant independent of δ. To the second integral, we apply Young's inequality ($\epsilon > 0$):

$$\int_{x_0-2R}^{x_0+2R} |\dot f| \, |\dot u_\delta| \eta^2 \, dt \le \epsilon \int_{x_0-2R}^{x_0+2R} \dot u_\delta^2 \eta^2 \, dt + c\epsilon^{-1} \int_{x_0-2R}^{x_0+2R} \dot f^2 \eta^2 \, dt.$$

Absorbing terms by choosing $\epsilon > 0$ sufficiently small and using (3.12) (recall that $\eta \equiv 1$ on $(x_0 - R, x_0 + R)$ and $f \in W^{1,2}_{\mathrm{loc}}(a, b)$ once again), we obtain

$$\int_{x_0-R}^{x_0+R} F_\delta''(\dot u_\delta)(\ddot u_\delta)^2 \, dt + \lambda \int_{x_0-R}^{x_0+R} \dot u_\delta^2 \, dt \le c(f, R), \tag{3.13}$$

where $c(f, R)$ is a local constant independent of δ. This proves that

$$u_\delta \in W^{1,2}(x_0 - R, x_0 + R) \text{ uniformly with respect to } \delta.$$

Now part c) of the theorem follows from a covering argument. $\square$

Remark 3.1. From the proof of part b) we see how the singular set $\mathrm{Sing}(u) := [0, 1] - \mathrm{Reg}(u)$ can be given in terms of σ: due to (3.2), we have $|\sigma(t)| < \lambda_\infty$ at almost all points $t \in [0, 1]$ and thus, since σ is continuous, we have

$$-\lambda_\infty \le \sigma(t) \le \lambda_\infty \quad \text{for all} \quad t \in [0, 1].$$

We claim that $\mathrm{Sing}(u)$ is exactly the set of points where $|\sigma|$ attains the maximal value λ_∞, i.e.,

$$\mathrm{Sing}(u) = \big\{t \in [0, 1] \, : \, |\sigma(t)| = \lambda_\infty\big\}.$$

Indeed, let $t_0 \in [0,1]$ be a regular point of u, i.e., there is a small neighborhood $(t_0-\epsilon, t_0+\epsilon)$ of t_0 such that u is of class $C^{1,1}(t_0-\epsilon, t_0+\epsilon)$. Hence $|\dot{u}|$ is bounded on $(t_0-\epsilon, t_0+\epsilon)$ and (3.3) along with the continuity of σ implies $|\sigma(t_0)| < \lambda_\infty$. Conversely, if $s_0 \in [0,1]$ is a point where $|\sigma(s_0)| < \lambda_\infty$, the arguments after (3.3) show that s_0 is a regular point.

§4. Proof of Theorem 1.3

Proof of part a). Without loss of generality, in the following we identify f with the representative that is continuous at t_0. Moreover, we recall that we consider the "good" representative of u as specified in the Introduction around formula (1.4). Assume that the statement is false, i.e., the left and the right limit of u at t_0,

$$u^-(t_0) := \lim_{t_k \uparrow t_0} u(t_k), \quad u^+(t_0) := \lim_{t_k \downarrow t_0} u(t_k),$$

do not coincide. We may assume that

$$u^-(t_0) < f(t_0) \text{ and } u^+(t_0) \geq f(t_0), \tag{4.1}$$

and from the proof it will be clear that all the other possible cases can be treated similarly. Let $h_0 := u^+(t_0) - u^-(t_0)$ denote the jump-height at t_0. Then, from (4.1) it follows in particular that there exist numbers $\epsilon > 0$ and $0 < d < h_0$ such that

$$u(t) < f(t) - d \text{ for all } t \in [t_0-\epsilon, t_0].$$

We may further assume that u is continuous at $t_0-\epsilon$. Now we define $\widetilde{u}$ by

$$\widetilde{u}(t) := u(t) + d\chi_{[t_0-\epsilon,t_0]}(t).$$

That means that on $[t_0-\epsilon, t_0]$ we "move" u a little closer to f so that, in particular,

$$\int_0^1 (\widetilde{u}-f)^2\,dt < \int_0^1 (u-f)^2\,dt. \tag{4.2}$$

We write (cf. (1.4)) $Du = \dot{u}\mathcal{L}^1 + \sum_{k=0}^\infty h_k\delta_{x_k} + D^c u$, where $\{t_k\}_{k=0}^\infty$ is the jump-set of u. Clearly, $\widetilde{u} \in BV(0,1)$,

$$D\widetilde{u} = \dot{u}\mathcal{L}^1 + (h_0-d)\delta_{t_0} + d\delta_{t_0-\epsilon} + \sum_{k=1}^\infty h_k\delta_{x_k} + D^c u,$$

and in conclusion

$$K[\widetilde{u}] = \int_0^1 F(\dot{u})\,dt + \lambda_\infty\Big(|h_0-d| + d + \sum_{k=1}^\infty |h_k|\Big) + \lambda_\infty|D^c u|(0,1) + \frac{\lambda}{2}\int_0^1 (\widetilde{u}-f)^2\,dt.$$

Since $d < h_0$ and by (4.2), this implies

$$K[\widetilde{u}] < K[u],$$

which contradicts the minimality of u.

Proof of part b). First we notice that, by Theorem 1.2 part b), there are s_1 and s_2 in (a,b), arbitrarily close to a and b respectively such that $s_1 < s_2$ and u is $C^{1,1}$-regular in a small neighborhood of s_1 and s_2. Hence, the singular set

$$S := \mathrm{Sing}(u) \cap [s_1, s_2]$$

is a compact subset of (s_1,s_2). Moreover, by part a) of Theorem 1.3 we have $u \in C^0(a,b)$. Assume $S \neq \varnothing$. Then there exists $\overline{s} := \inf S > a$, which is an element of S itself, because the singular set is closed. In particular, $\sigma(\overline{s}) = \pm\lambda_\infty$ (cf. Remark 3.1), i.e., σ has a maximum (respectively, minimum) in $\overline{s}$, and since $\dot{\sigma} = \lambda(u - f) \in C^0(a, b)$, it follows that

$$\dot{\sigma}(\overline{s}) = 0,$$

whence

$$u(\overline{s}) = f(\overline{s}). \tag{4.3}$$

Without loss of generality we may assume that $\sigma(\overline{s}) = \lambda_\infty$. Since σ is continuous at $\overline{s}$, for any sequence $t_k \uparrow \overline{s}$ approaching $\overline{s}$ from the left we must have $\sigma(t_k) \to \lambda_\infty$ and thus, since of $\dot{u} = DF^{-1}(\sigma)$, we see that

$$\dot{u}(t_k) \to \infty \quad \text{for any sequence } t_k \uparrow \overline{s}. \tag{4.4}$$

In particular, for arbitrary $M > 0$ there exists $\epsilon > 0$ such that

$$\dot{u}(t) > M \quad \text{for} \quad t \in [\overline{s} - \epsilon, \overline{s}). \tag{4.5}$$

Now choose $M := \|\dot{f}\|_{\infty;[s_1,s_2]}$ in (4.5). Then $\frac{d}{dt}(u - f) > 0$ on $[\overline{s} - \epsilon, \overline{s})$, which is not compatible with (4.3) unless $u - f < 0$ on $[\overline{s} - \epsilon, \overline{s})$. But in this case, the differential equation

$$\ddot{u} = \lambda \frac{u - f}{F''(\dot{u})} \text{ a.e. on } [\overline{s} - \epsilon, \overline{s}) \tag{4.6}$$

implies that $\dot{u}$ is strictly monotone decreasing on $[\overline{s} - \epsilon, \overline{s})$ and thereby $\dot{u}(\overline{s} - \epsilon) \geq \dot{u}(s)$ for all $s \in [\overline{s} - \epsilon, \overline{s})$, which is inconsistent with (4.4). This shows that $\mathrm{Sing}(u) \cap (a, b) = \varnothing$ by contradiction, and hence $u \in C^1(a, b)$. Moreover, since σ is locally bounded away from λ_∞, we even have $u \in W^{2,\infty}_{\mathrm{loc}}(a, b)$. Hence, (4.6) is true at almost all points of (a, b), and by the continuity of $\dot{u}$ the right-hand side of (4.6) is continuous. Therefore, $u \in C^2(a, b)$.

Proof of part c). As has already been mentioned, the auxiliary quantity σ introduced in the proof of Theorem 1.2 has an independent meaning as the solution of the dual problem to $J \to \min$. As, e.g., in [7] or [11], we obtain the dual problem from the Lagrangian given by

$$L(v, \kappa) := \int_0^1 \kappa \dot{v}\, dt - \int_0^1 F^*(\kappa)\, dt + \underbrace{\frac{\lambda}{2} \int_0^1 (v - f)^2\, dt}_{=:\Psi(v)},$$

where $(v, \kappa) \in W^{1,1}(0, 1) \times L^\infty(0, 1)$,

$$F^*(\kappa) := \sup_{w \in L^1(0,1)} \big(\langle \kappa, w\rangle - F(w)\big)$$

is the convex conjugate, and

$$\langle \kappa, w\rangle := \int_0^1 \kappa w\, dt$$

denotes the duality product of $L^1(0, 1)$ and $L^\infty(0, 1)$. By standard results from convex analysis (see, e.g., [25, Remark 3.1, p. 56]), the functional J can be expressed in terms of the Lagrangian by

$$J[v] = \sup_{\kappa \in L^\infty(0,1)} L(v, \kappa),$$

and

$$R[\kappa] := \inf_{v \in W^{1,1}(0,1)} L(v, \kappa), \quad \kappa \in L^\infty(0, 1),$$

is called the dual functional. The dual problem consists in maximizing $R[\kappa]$ in $L^\infty(0, 1)$. Obviously, $\kappa := \sigma$ is an admissible choice, and since $\sigma \in W^{1,\infty}(0, 1)$ and $\sigma(0) = F'(\dot{u}(0)) = 0 = F'(\dot{u}(1)) = \sigma(1)$ (cf. (3.1)), we can integrate by parts and derive the following integral representation of the dual functional (cf. also [26, Theorem 9.8.1, p. 366]):

$$\begin{aligned} R[\sigma] &= \inf_{v\in W^{1,1}(0,1)} \int_0^1 \sigma\dot v\,dt - \int_0^1 F^*(\sigma)\,dt + \Psi(v) \\ &= -\int_0^1 F^*(\sigma)\,dt - \sup_{v\in W^{1,1}(0,1)} \Big(-\int_0^1 \sigma \dot v\,dt - \Psi(v)\Big) \\ &= -\int_0^1 F^*(\sigma)\,dt - \sup_{v\in W^{1,1}(0,1)} \Big(\int_0^1 \dot\sigma v\,dt - \Psi(v)\Big) = -\int_0^1 F^*(\sigma)\,dt - \Psi^*(\dot\sigma). \end{aligned}$$

Next, we want to compute $\Psi^*(\dot\sigma)$. By definition, we have

$$\Psi^*(\dot\sigma) = \sup_{v\in W^{1,1}(0,1)} \Big(\langle v,\dot\sigma\rangle - \frac{\lambda}{2}\langle v-f, v-f\rangle \Big) = \sup_{v\in W^{1,1}(0,1)} \Big\langle v, \dot\sigma - \frac{\lambda}{2}v + \lambda f \Big\rangle - \frac{\lambda}{2}\langle f,f\rangle.$$

Applying Hölder's inequality, we get

$$\Big\langle v, \dot\sigma - \frac{\lambda}{2} v + \lambda f\Big\rangle \le -\frac{\lambda}{2}\|v\|_2^2 + \|\dot\sigma + \lambda f\|_2 \|v\|_2 \tag{4.7}$$

and elementary calculus shows that the right-hand side is maximal for $\|v\|_2 = \|\frac{\dot\sigma}{\lambda} + f\|_2$. An easy computation confirms that for the choice $v = \frac{\dot\sigma}{\lambda} + f$ the left-hand side of (4.7) attains this maximal value, and it follows that

$$\Psi^*(\dot\sigma) = \int_0^1 \Big(\frac{\dot\sigma}{\lambda} + f\Big)\dot\sigma\,dt - \frac{\lambda}{2}\int_0^1 \Big(\frac{\dot\sigma}{\lambda}+f\Big)^2 dt = \int_0^1 \frac{\dot\sigma^2}{2\lambda} + \dot\sigma f\,dt.$$

Thereby for $R[\sigma]$ we obtain

$$R[\sigma] = -\int_0^1 \frac{\dot\sigma^2}{2\lambda} + \dot\sigma f\,dt - \int_0^1 F^*(\sigma)\,dt. \tag{4.8}$$

Now assume that $\mathrm{Sing}(u) \neq \varnothing$. By Remark 3.1, this means that there exists at least one point $t \in [0,1]$ where $\sigma(t) = \pm\lambda_\infty$. Let $\widehat t$ denote the smallest such t. Since $\sigma(0) = 0$, we have $\widehat t > 0$, and without loss of generality we may assume that $\sigma(\widehat t) = \lambda_\infty$. Let $\varphi \in C_0^\infty\big([0,\widehat t)\big)$ be an arbitrary test function. On $[0,\widehat t)$ we have $|\sigma| < \lambda_\infty$ and since $\operatorname{spt}\varphi$ is a compact subset of $[0,\widehat t)$ (and σ is continuous) there exists $\epsilon_0 = \epsilon_0(\varphi)$ such that $|\sigma(t) + \epsilon\varphi(t)| \le \lambda_\infty - \delta$ for some $\delta > 0$ and for all $0 \le \epsilon < \epsilon_0$. By Theorem 26.4 and Corollary 26.4.1 in [27], F^* is finite and continuously differentiable on $(-\lambda_\infty, \lambda_\infty)$ (with the derivative $(F^*)' = (F')^{-1}$), whence

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} F^*(\sigma(t) + \epsilon\varphi(t)) = (F^*)'(\sigma(t))\varphi(t) \in L^1(0,\widehat t).$$

Together with (4.8) and the maximality of σ, this implies that the following Euler equation must be fulfilled for all $\varphi \in C_0^\infty(0,\widehat t)$:

$$-\int_0^1 \frac{\dot\sigma}{\lambda}\dot\varphi + f\dot\varphi\,dt - \int_0^1 (F^*)'(\sigma)\varphi\,dt = 0. \tag{4.9}$$

Since $[0,\widehat t) \subset \mathrm{Reg}(u)$ and $f \in W^{1,1}(0,1)$ by assumption, we have (see (3.1))

$$\dot\sigma = \lambda(u-f) \in W^{1,1}(0,\widehat t), \tag{4.10}$$

and therefore $\sigma \in W^{2,1}(0,1)$, so that (4.9) implies the following differential equation:

$$\frac{\ddot\sigma}{\lambda} + \dot f - (F^*)'(\sigma) = 0 \text{ a.e. on } (0,\widehat t). \tag{4.11}$$

Let $\{s_k\} \subset [0, \widehat{t})$, $k \in \mathbb{N}$, denote a sequence with $s_k \uparrow \widehat{t}$ as $k \to \infty$. Multiplying (4.11) by $\dot{\sigma}$ and integrating by parts (recall that $\dot{\sigma} \in W^{1,1}(0, \widehat{t})$) yields

$$\frac{\dot{\sigma}(s_k)^2}{2\lambda} - \frac{\dot{\sigma}(0)^2}{2\lambda} + \int_0^{s_k} \dot{f}\sigma \, dt - F^*(\sigma_{s_k}) = 0.$$

Since $\dot{\sigma}$ is bounded by λ, this implies the estimate

$$F^*(\sigma(s_k)) < \lambda \left(\frac{1}{2} + \|\dot{f}\|_1 \right) + \frac{\dot{\sigma}(s_k)^2}{2\lambda}. \tag{4.12}$$

At $\widehat{t}$, σ attains its maximum, and since it is continuously differentiable on $(0,1)$ (this follows from (3.1) in combination with the fact that u is continuous on $(0,1)$ by part a) of Theorem 1.3), it follows that

$$\frac{\dot{\sigma}(s_k)^2}{2\lambda} \to 0 \quad \text{for} \quad k \to \infty,$$

whence

$$\lim_{k\to\infty} F^*(\sigma(s_k)) \le \lambda \left(\frac{1}{2} + \|\dot{f}\|_1 \right). \tag{4.13}$$

But the following calculation shows (see also Figure 2) that the limit on the left-hand side coincides with the quantity ω_∞ from the assumptions of part c):

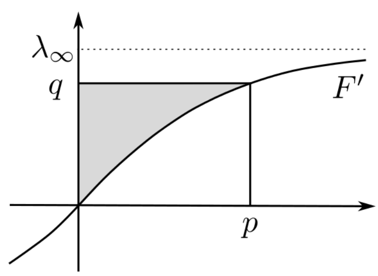

FIGURE 2. $\int_0^q (F')^{-1}(t)\, dt = pq - \int_0^p F'(t)\, dt$.

$$\begin{aligned}\lim_{q\uparrow\lambda_\infty} F^*(q) &= \lim_{q\uparrow\lambda_\infty} \int_0^q (F^*)'(t)\, dt = \lim_{q\uparrow\lambda_\infty} \int_0^q (F')^{-1}(t)\, dt \\ &= \lim_{q\uparrow\lambda_\infty} q(F')^{-1}(q) - \int_0^{(F')^{-1}(q)} F'(t)\, dt \overset{p:=(F')^{-1}(q)}{=} \lim_{p\uparrow\infty} pF'(p) - F(p).\end{aligned}$$

Hence, (4.13) contradicts our requirements on f and λ, and therefore the assumption $\mathrm{Sing}(u) \neq \varnothing$ is false. □

§5. Proof of Theorem 1.4

Suppose all the assumptions of Theorem 1.4 are fulfilled. In the sequel, we make the dependence of the minimizer on the parameter λ more explicit by denoting by u_λ the unique solution of problem (1.6) for a given $\lambda > 0$. Thanks to Theorems 1.2 and 1.3, we have the following properties:

(i) $u_\lambda \in C^2\big([0,1] - \{\frac{1}{2}\}\big)$ (cf. Theorem 1.3 b)), $0 \le u_\lambda \le 1$ a.e., and u_λ satisfies

$$\begin{cases} \ddot{u}_\lambda = \lambda \dfrac{u_\lambda}{F''(\dot{u}_\lambda)}, \quad \dot{u}_\lambda(0) = 0, \quad \text{on } [0, \frac{1}{2}) \quad (1), \\ \ddot{u}_\lambda = \lambda \dfrac{1 - u_\lambda}{F''(\dot{u}_\lambda)}, \quad \dot{u}_\lambda(1) = 0, \quad \text{on } (\frac{1}{2}, 1] \quad (2), \end{cases}$$

(ii) $\ddot{u}_\lambda \geq 0$ on $[0, \frac{1}{2})$ and hence $\dot{u}_\lambda$ increases on $[0, \frac{1}{2})$; $\ddot{u}_\lambda \leq 0$ on $(\frac{1}{2}, 1]$ and hence $\dot{u}_\lambda$ decreases on $[0, \frac{1}{2})$,

(iii) $\dot{u}_\lambda \geq 0$ on $[0, \frac{1}{2})$ (due to $\dot{u}_\lambda(0) = 0$ and (ii)), and hence u_λ increases on $[0, \frac{1}{2})$.

Furthermore, we see that the symmetry of our data f with respect to the point $(\frac{1}{2}, \frac{1}{2})$ is reproduced by u_λ:

(iv) The two continuous branches of u_λ, $u_\lambda|_{[0,\frac{1}{2})}$ and $u_\lambda|_{(\frac{1}{2},1]}$, are symmetric with respect to the point $(\frac{1}{2}, \frac{1}{2})$, i.e.,

$$u_\lambda(t) = \underbrace{1 - u_\lambda(1-t)}_{=:\widetilde{u}_\lambda(t)}, \quad t \in [0,1] - \{1/2\}.$$

Proof of (iv). We show that $K[\widetilde{u}_\lambda] = K[u_\lambda]$. Then the result follows from the uniqueness of the K-minimizer in $BV(0,1)$ (Theorem 1.2 a)). Let

$$h := \lim_{t \downarrow \frac{1}{2}} u_\lambda(u) - \lim_{t \uparrow \frac{1}{2}} u_\lambda(u)$$

denote the height of the (possible) jump of u_λ at $t = \frac{1}{2}$. Then the distributional derivative of u_λ is given by

$$Du_\lambda = D^a u_\lambda + h\delta_{1/2}$$

and, consequently,

$$\begin{aligned} K[u_\lambda] &= \int_0^{\frac{1}{2}} \dot{u}_\lambda \, dt + \int_{\frac{1}{2}}^1 \dot{u}_\lambda \, dt + \lambda_\infty |h\delta_{1/2}|(0,1) + \frac{\lambda}{2} \int_0^1 (u_\lambda - f)^2 \, dt \\ &= \int_0^{\frac{1}{2}} \dot{u}_\lambda \, dt + \int_{\frac{1}{2}}^1 \dot{u}_\lambda \, dt + \frac{|h|}{\mu - 1} + \frac{\lambda}{2} \int_0^1 (u_\lambda - f)^2 \, dt. \end{aligned}$$

For $\widetilde{u}_\lambda$ we obtain

$$\begin{aligned} K[\widetilde{u}_\lambda] &= \int_0^{\frac{1}{2}} \dot{u}_\lambda(1-t) \, dt + \int_{\frac{1}{2}}^1 \dot{u}_\lambda(1-t) \, dt + \lambda_\infty |h\delta_{1/2}|(0,1) + \frac{\lambda}{2} \int_0^1 (\widetilde{u}_\lambda - f)^2 \, dt \\ &= \int_0^{\frac{1}{2}} \dot{u}_\lambda \, dt + \int_{\frac{1}{2}}^1 \dot{u}_\lambda \, dt + \frac{|h|}{\mu - 1} + \frac{\lambda}{2} \int_0^1 (\widetilde{u}_\lambda - f)^2 \, dt, \end{aligned}$$

but clearly $\int_0^1 (\widetilde{u}_\lambda - f)^2 \, dt = \int_0^1 (u_\lambda - f)^2 \, dt$, whence $K[\widetilde{u}_\lambda] = K[u_\lambda]$. □

Finally, we note that the value of $u_\lambda(0)$ tends to zero as $\lambda \to \infty$:

(v) $$\lim_{\lambda \to \infty} u_\lambda(0) = 0.$$

Proof of (v). Since u_λ is K-minimal in $BV(0,1)$ and $f \in BV(0,1)$, we have

$$K[u_\lambda] \leq K[f] = \lambda_\infty |\delta_{1/2}|(0,1) = \lambda_\infty = \frac{1}{\mu - 1},$$

and thus, by properties (iii) and (iv),

$$\frac{\lambda}{2} u_\lambda(0)^2 = 2\frac{\lambda}{2} \int_0^{\frac{1}{2}} u_\lambda(0)^2 \, dt \leq \frac{\lambda}{2} \int_0^{\frac{1}{2}} (u_\lambda - f)^2 \, dt \leq K[u_\lambda] \leq K[f] = \frac{1}{\mu - 1},$$

so that

(5.1) $$u_\lambda(0) \leq \sqrt{\frac{2}{\lambda(\mu - 1)}} \xrightarrow{\lambda \to \infty} 0.$$ □

By property (iv), the continuity of u_λ necessarily implies $u_\lambda(1/2) = 1/2$. We can exploit this fact to prove that the minimizer develops jumps once we can show that, starting from a certain value of the parameter λ, u_λ is bounded away from $1/2$ on $[0, 1/2)$. To this end, we make use of equation (1) from property (i):

$$\begin{aligned} \ddot{u}_\lambda(t) = \lambda \frac{u_\lambda(t)}{F''(\dot{u}_\lambda(t))} \quad &\Leftrightarrow \quad F''(\dot{u}_\lambda(t))\ddot{u}_\lambda(t) = \lambda u_\lambda(t) \\ &\Leftrightarrow \quad \frac{d}{dt} F'(\dot{u}_\lambda(t))\dot{u}_\lambda(t) = \lambda u_\lambda(t)\dot{u}_\lambda(t). \end{aligned}$$

Integrating the last equation from 0 to s for some $s \in [0, \frac{1}{2})$ yields

$$\begin{aligned} &\int_0^s \frac{d}{dt} F'(\dot{u}_\lambda(t))\dot{u}_\lambda(t)\, dt = \int_0^s \lambda u_\lambda(t)\dot{u}_\lambda(t)\, dt \\ \Leftrightarrow \quad &\Big[F'(\dot{u}_\lambda(t))\dot{u}_\lambda(t)\Big]_0^s - \int_0^s \underbrace{F'(\dot{u}_\lambda(t))\ddot{u}_\lambda(t)}_{=\frac{d}{dt}F(\dot{u}_\lambda(t))}\, dt = \Big[\frac{\lambda}{2} u_\lambda(t)^2\Big]_0^s, \end{aligned}$$

and with $\dot{u}_\lambda(0) = 0$ and $F'(0) = 0$ we arrive at

$$\dot{u}_\lambda(s)F'(\dot{u}_\lambda(s)) - F(\dot{u}_\lambda(s)) = \frac{\lambda}{2}\big(u_\lambda(s)^2 - u_\lambda(0)^2\big). \tag{5.2}$$

Note that (5.2) formally corresponds to a law of conservation if we interpret (1) as the equation of motion of a particle of mass $1/\lambda$ under the influence of a time-independent exterior force.

The left-hand side of (5.2) is nonnegative by the convexity of F and therefore we get

$$u_\lambda(s) = \sqrt{u_\lambda(0)^2 + \frac{2}{\lambda}\big(\dot{u}_\lambda(s)F'(\dot{u}_\lambda(s)) - F(\dot{u}_\lambda(s))\big)} \text{ for } s \in \big[0, 1/2\big). \tag{5.3}$$

From (5.3) we see that if the left-hand side of (5.2) is bounded, then by property (iv) u_λ is bounded from below by $1/2$ if we choose λ sufficiently large. But for our density F from (1.10) we have (see Remark 1.3 (iii))

$$\lim_{p\to\infty} pF'(p) - F(p) = \begin{cases} \infty & \text{if } 1 < \mu \le 2, \\ \frac{1}{(\mu-1)(\mu-2)} & \text{if } 2 < \mu, \end{cases}$$

and for $\mu > 2$, the last equation together with (5.1) and (5.3) gives

$$u_\lambda(s) \le \sqrt{\frac{2}{\lambda(\mu-1)} + \frac{2}{\lambda(\mu-1)(\mu-2)}} = \sqrt{\frac{2}{\lambda(\mu-2)}} \quad \text{for} \quad s \in [0, 1/2)$$

which implies $\sup_{0\le s<\frac{1}{2}} u_\lambda(s) < \frac{1}{2}$ if λ satisfies (1.13). The corresponding lower bound on the infimum follows by the symmetry property (iv). $\square$

Proof of Corollary 1.2 b). We define the critical value of λ by

$$\lambda_{\text{crit}} := \sup\{\lambda : u_\lambda \text{ is continuous}\}.$$

First we note that any minimizer u_λ (independently of λ) satisfies $0 \le u_\lambda \le \frac{1}{2}$ on $[0, \frac{1}{2})$ because otherwise "cutting-off" at height $\frac{1}{2}$ would yield a BV-function for which the functional K has a strictly smaller value. Thus, (5.3) implies

$$\frac{1}{2} \ge \sqrt{u_\lambda(0)^2 + \frac{2}{\lambda}\big(\dot{u}_\lambda(s)F'(\dot{u}_\lambda(s)) - F(\dot{u}_\lambda(s))\big)} \quad \text{for} \quad s \in \big[0, 1/2\big).$$

Consequently the limit as $s \to \frac{1}{2}$ gives (remember that $\dot{u}_\lambda(s) \to \infty$ as $s \to \frac{1}{2}$ because $\lambda > \lambda_{\text{crit}}$)

$$\frac{1}{2} \ge \sqrt{u_\lambda(0)^2 + \frac{2}{\lambda}\omega_\infty},$$

whence

$$u_\lambda(0)^2 \le \frac{\lambda - 8\omega_\infty}{4\lambda}$$

and therefore $\lambda \ge 8\omega_\infty$. The upper bound on λ_{crit} follows like in the proof of Theorem 1.4 from estimate (5.1) (with general λ_∞ in place of $1/(\mu-1)$) and (5.3). □

§6. Proof of Theorem 1.5

Under the assumptions of Theorem 1.5, we set

$$\operatorname{Reg}(u) := \big\{t \in [0,1] \,:\, u \text{ is } C^{1,1} \text{ on a neighborhood of } t\big\}.$$

From Theorem 1.2 b) we deduce that $\operatorname{Sing}(u) := [0,1] - \operatorname{Reg}(u)$ is a compact subset of $(0,1)$. Assume that $\operatorname{Sing}(u) \neq \varnothing$, and let s denote the first singular point so that $u \in C^{1,1}\big([0,s)\big)$ and therefore

$$\ddot{u}F''(\dot{u}) = \lambda(u-f) \text{ a.e. on } (0,s). \tag{6.1}$$

From (6.1) we deduce (compare the derivation of (5.2)) that

$$\dot{u}(t)F'(\dot{u}(t)) - F(\dot{u}(t)) = \frac{\lambda}{2}\big(u(t)^2 - u(0)^2\big) - \int_0^t f(\tau)\dot{u}(\tau)\,d\tau \text{ for } t \in [0,s). \tag{6.2}$$

Clearly (6.2) implies ($\omega(p) := pF'(p) - F(p)$)

$$|\omega(\dot{u}(t))| \le \frac{\lambda}{2} + |Du|(0,1), \quad t \in [0,s), \tag{6.3}$$

because $0 \le u, f \le 1$ a.e. on $(0,1)$. By the convexity of F (together with $F(0)=0$), we see that $\omega \ge 0$, $\omega(0) = 0$, and moreover,

$$\omega(p) = \int_0^p \omega'(q)\,dq = \int_0^q qF''(q)\,dq.$$

Thus, ω is monotone increasing with

$$\lim_{p\to\infty} \omega(p) = \infty, \quad \lim_{p\to-\infty} \omega(p) = \infty, \tag{6.4}$$

which follows from (F5) together with the assumption (1.14). Since we assume that s is the first singular point of u, it follows that

$$\lim_{k\to\infty} |\dot{u}(t_k)| = \infty$$

for a suitable sequence $t_k \uparrow s$, because otherwise $|\sigma(s)| < \lambda_\infty$ and hence $s \in \operatorname{Reg}(u)$ (cf. Remark (3.1)). This contradicts (6.3) in view of (6.4). □

Proof of Corollary 1.2 a). The fact that $\operatorname{Sing}(u) \neq \varnothing$ follows exactly along the same lines because now we have (6.4) due to our assumption $\omega_\infty = \infty$. □

§7. Proof of Theorem 1.6

Essentially, we need to show that for $\lambda < \lambda_\mu$ the conditions of Theorem 6 on page 295 in [20] are fulfilled. Without further explanation we shall adopt the notation of that work. First of all, we notice that due to our restriction $0 \le f(t) \le 1$ we see that $\alpha(t) \equiv 0$ and $\beta(t) \equiv 1$ is a trivial lower and upper solution of (BVP), respectively, because

$$0 \ge \lambda\frac{0-f}{F''(0)} \quad \text{and} \quad 0 \le \lambda\frac{1-f}{F''(0)},$$

due to $\|f\|_\infty \le 1$ and $F'' > 0$. Second, the right-hand side of the equation (BVP) can be rewritten as

$$\Phi(t,v,\dot{v}) = \lambda\frac{v-f(t)}{F''(\dot{v})},$$

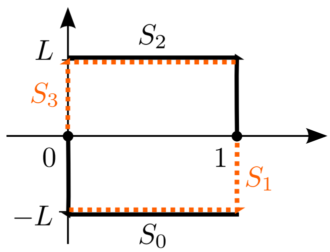

FIGURE 3. The sets S_0, S_1, S_2 and S_3.

where $\Phi(t, y, p) := \lambda \dfrac{y - f(t)}{F''(p)}$ is a Carathéodory function if f is merely measurable. Moreover, by (F5) we can estimate Φ by

$$|\Phi(t, y, p)| \le \frac{\lambda}{c_1}\big(1 + |p|\big)^{\mu},$$

and hence, letting $h(p) := \frac{\lambda}{c_1}(1 + |p|)^{\mu}$, $\bar{h}(p) \equiv 1$, $r(t) := \epsilon$ for some $\epsilon > 0$ and choosing λ in such a way that

$$\lambda < \frac{c_1}{1 + K\epsilon} \int_1^{\infty} \frac{s\, ds}{(1+s)^{\mu}}, \tag{7.1}$$

where K denotes the quantity $\sup\{s/h(s) \mid s \in [1, \infty]\}$, we find that if $L > 0$ is sufficiently large, then Φ satisfies the following Bernstein–Nagumo–Zwirner condition (compare [20, Definition 4]):

$$\begin{cases} |\Phi(t, y, p)| \le h(|p|)\bar{h}(p) + r(t) \text{ for all } (t, y) \in [0, 1] \times [0, 1] \quad \text{and} \\ \int_1^L \frac{s\, ds}{h(s)} > 1 + K\epsilon. \end{cases}$$

The boundary conditions are formulated as set conditions, i.e., $(v(0), \dot{v}(0)) \in \mathcal{J}(0)$ and $(v(1), \dot{v}(1)) \in \mathcal{J}(1)$ for some closed connected subsets $\mathcal{J}(0), \mathcal{J}(1) \subset [0, 1] \times \mathbb{R}$. In our case, we can choose

$$\mathcal{J}(0) = \mathcal{J}(1) = [0, 1] \times \{0\}$$

which corresponds to our Neumann condition. The verification that the sets $\mathcal{J}(0) = \mathcal{J}(1) := [0, 1] \times \{0\}$ are of "compatible type 1" in the sense of Definition 14 in [20] is straightforward. Let sets S_0, S_1, S_2, and S_3 be introduced as in Definition 15 in [20] (see Figure 3 below). Then

$$\mathcal{J}(0) \cap \{S_0 \cup S_2\} = \mathcal{J}(1) \cap \{S_1 \cup S_3\} = \{(0, 0), (0, 1)\} \ne \varnothing.$$

That is, all conditions of Theorem 4 are fulfilled and there is a solution $v \in W^{2,1}(0, 1)$ of (BVP) with $0 \le v(t) \le 1$ for almost all $t \in [0, 1]$. Note that letting ϵ tend to zero in (7.1) gives the postulated bound λ_{μ} for λ.

Now, let $v \in W^{2,1}(0, 1)$ be a solution of (BVP). We want to show that v coincides with the K-minimizer u from Theorem 1.2. The convexity of the functional J shows that for any $w \in C^{1,1}\big([0, 1]\big)$ we have

$$J[w] \ge J[v] + \langle DJ[v], w - v\rangle$$

with

$$\langle DJ[v], w - v\rangle = \int_0^1 F'(\dot{v})(\dot{w} - \dot{v})\, dt + \lambda \int_0^1 (v - f)(w - v)\, dt.$$

Since $F'(0) = 0$, we have

$$\int_0^1 F'(\dot{v})(\dot{w} - \dot{v})\, dt = \int_0^1 \frac{d}{dt}\big[F'(\dot{v})(w - v)\big]\, dt - \int_0^1 F''(\dot{v})\ddot{v}(w - v)\, dt = -\int_0^1 F''(\dot{v})\ddot{v}(w - v)\, dt.$$

By assumption, v solves (BVP) a.e. on $(0,1)$, which implies that

$$\langle DJ[v], w-v\rangle = \int_0^1 (w-v)\big[F''(\dot v)\ddot v - \lambda(v-f)\big]\,dt = 0 \tag{7.2}$$

for all $w \in C^{1,1}\big([0,1]\big)$. Thus, we get $J[v] \le J[w]$ for all $w \in C^{1,1}\big([0,1]\big)$. Now let u denote the minimizer of K in $BV(0,1)$. We can construct a sequence $u_k \in C^\infty\big([0,1]\big)$ such that

$$|Du_k|(0,1) \xrightarrow{k\to\infty} |Du|(0,1), \quad u_k \to u \text{ in } L^1(0,1),$$

and

$$\sqrt{1+|Du_k|^2}(0,1) \xrightarrow{k\to\infty} \sqrt{1+|Du|^2}(0,1).$$

To see this, consider

$$\widehat{u}\colon \mathbb{R} \to [0,1],\ \widehat{u}(t) := \begin{cases} u(0), & t \le 0,\\ u(t), & 0 \le t \le 1,\\ u(1), & t \ge 1. \end{cases} \tag{7.3}$$

Since u is of class $C^{1,1}$ near 0 and 1, it follows that $\widehat{u} \in BV_{\mathrm{loc}}(\mathbb{R})$ and

$$|D\widehat{u}|(\{0\}) = |D\widehat{u}|(\{1\}) = 0. \tag{7.4}$$

Let $\eta \in C_0^\infty(\mathbb{R})$ be a cut-off function such that $\eta \equiv 1$ on $[0,1]$, and consider a symmetric mollifier ρ_ϵ supported on the closed ball with radius $\epsilon > 0$ around 0. By the properties of mollification, $\widehat{u}_\epsilon := \rho_\epsilon * (\eta\widehat{u})$ converges to $\widehat{u}$ in $L^1(0,1)$ as $\epsilon \downarrow 0$. Moreover, by (7.4) and [18, Proposition 3.7, p. 121], we have

$$|D(\widehat{u}_\epsilon)|(0,1) \xrightarrow{\epsilon\downarrow 0} |D(\eta\widehat{u})|(0,1) = |Du|(0,1)$$

and, by similar arguments,

$$\int_0^1 \sqrt{1+|D(\widehat{u}_\epsilon)|^2}\,dt = \left|\rho_\epsilon * (\mathcal{L}^1, D(\eta\widehat{u}))^T\right|(0,1) \xrightarrow{\epsilon\downarrow 0} \sqrt{1+|Du|^2}(0,1).$$

Hence, $u_k := \rho_{1/k} * \eta\widehat{u}$ for $k \in \mathbb{N}$ has the desired properties. From Proposition 2.3 in [24] it follows that

$$J[v] \le J[u_k] = K[u_k] \xrightarrow{k\to\infty} K[u],$$

and since u is K-minimal, we conclude that

$$K[u] \le K[v] = J[v] \le K[u],$$

which means that $K[u] = K[v]$, whence $u = v$ by the uniqueness of the K-minimizer. $\square$

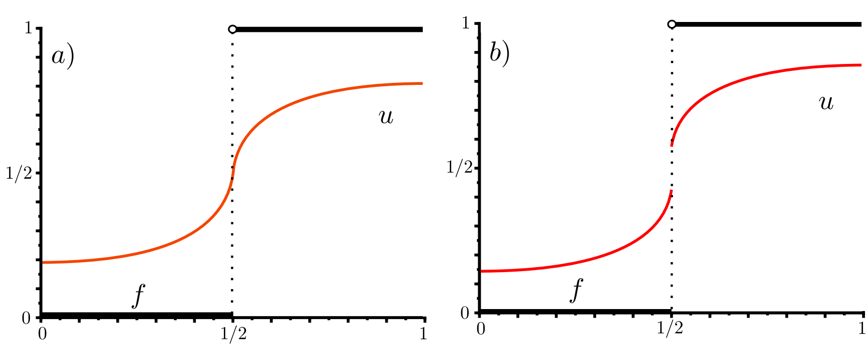

FIGURE 4. Example plots of the K-minimizer u for $\mu = 3$ and a) $\lambda = 4$, b) $\lambda = 5$.

§8. Comparison with a numerical example

In this short appendix we would like to compare the above theoretical considerations with a numerical example, which has been computed with the free software Scilab[1]. Besides giving a confirmation of our previous results, this is mainly intended to show that none of our given bounds on the parameter λ is actually sharp. In fact, we seem to obtain smooth solutions for values of λ larger than $\max\{\lambda_\infty, \omega_\infty\}$, and discontinuous minimizers can occur below the threshold $\frac{8}{\mu-2}$, which was predicted by Corollary 1.2 b). It is still an open problem to determine exact bounds, which clearly should depend on both F and f.

We choose the data f from (1.12), i.e., f is constant on $[0,1/2]$ and $(1/2,1]$ with a single jump of height 1 at $t = 1/2$, and the μ-elliptic density $F(p) = \Phi_3(|p|)$ (remember that by Theorem 1.5 there will be no singular minimizers for $\mu \le 2$, which justifies our choice $\mu = 3$). Then our K-minimizer u should be smooth for $\lambda < 8\omega_\infty = 4$. In practice, we seem to get smooth solutions up to about $\lambda < 4.16$. For $\lambda = 4.16$ the tangent of u at $t = 1/2$ becomes nearly vertical, and for $\lambda > 4.16$ the minimizer develops a jump. In Figure 4 we show exemplarily the graphs of u for $\lambda = 4$ and $\lambda = 5$. Next, we would like to note that for $\lambda = 4.16$ the value of $u(0)$ is approximately 0.183; for the bound (5.3) established in the proof of Theorem 1.4 this yields

$$u(s) \le \sqrt{0.183^2 + \frac{1}{4.16}} \approx 0.523,$$

and thus suits our previous considerations quite well.

References

[1] L. I. Rudin, S. Osher, and E. Fatemi, *Nonlinear total variation based noise removal algorithms*, Phys. D **60** (1992), no. 1-4, 259–268. MR3363401

[2] M. A. Little and N. S. Jones, *Generalized methods and solvers for noise removal from piecewise constant signals.* I. *Background theory*, Proc. Math. Phys. Eng. Sci. **467** (2011), no. 2135, 3088–3114.

[3] I. Selesnick, A. Parekh, and I. Bayram, *Convex* 1-D *total variation denoising with non-convex regularization*, IEEE Signal Process. Lett. **22** (2015), no. 2, 141–144.

[4] A. Torres, A. Marquina, J. A. Font, and J. M. Ibáñez, *Total-variation-based methods for gravitational wave denoising*, Phys. Rev. D. **90** (2014), no. 8, 084029.

[5] M. Bildhauer and M. Fuchs, *A variational approach to the denoising of images based on different variants of the TV-regularization*, Appl. Math. Optim. **66** (2012), no. 3, 331–361. MR2996430

[6] M. Bildhauer and M. Fuchs, *On some perturbations of the total variation image inpainting method. Pt.* I: *regularity theory*, Probl. Mat. Anal. **76** (2014), 39–52; English transl., J. Math. Sci. (N.Y.) **202** (2014), no. 2, 154–169. MR3391330

[7] M. Bildhauer and M. Fuchs, *On some perturbations of the total variation image inpainting method. Pt.* II: *relaxation and dual variational formulation*, Probl. Mat. Anal. **77** (2015), 3–18; English transl., J. Math. Sci. (N.Y.) **205** (2015), no. 2, 121–140. MR3391337

[8] M. Bildhauer and M. Fuchs, *Image inpainting with energies of linear growth. A collection of proposals*, Probl. Mat. Anal. **74** (2014), 45–50; English transl., J. Math. Sci. (N.Y.) **196** (2014), no. 4, 490–497. MR3391307

[9] M. Bildhauer and M. Fuchs, *On some perturbations of the total variation image inpainting method. Pt.* III: *Minimization among sets with finite perimeter*, Probl. Mat. Anal. **78** (2015), 27–30; English transl., J. Math. Sci. (N.Y.) **207** (2015), no. 2, 142–146. MR3391700

[10] M. Bildhauer, M. Fuchs, and C. Tietz, $C^{1,\alpha}$*-interior regularity for minimizers of a class of variational problems with linear growth related to image inpainting*, Algebra i Analiz **27** (2015), no. 3, 51–65; English transl., St. Petersburg Math. J. **27** (2016), no. 3, 381–392. MR3570956

[11] M. Fuchs and C. Tietz, *Existence of generalized minimizers and of dual solutions for a class of variational problems with linear growth related to image recovery*, Probl. Mat. Anal. **81** (2015), 107–120; English transl., J. Math. Sci. (N.Y.) **210** (2015), no. 4, 458–475. MR3405450

[1] `http://www.scilab.org/`

[12] K. Bredies, K. Kunisch, and T. Valkonen, *Properties of L^1-TGV^2: the one-dimensional case*, J. Math. Anal. Appl. **398** (2013), no. 1, 438–454. MR2984346
[13] D. Strong and T. Chan, *Edge-preserving and scale-dependent properties of total variation regularization*, Inverse Problems **19** (2003), no. 6, S165–S187. MR2036526
[14] K. Papafitsoros and K. Bredies, *A study of the one dimensional total generalised variation regularisation problem*, Inverse Probl. Imaging **9** (2015), no. 2, 511–550. MR3356570
[15] G. Buttazzo, M. Giaquinta, and S. Hildebrandt, *One-dimensional variational problems. An introduction*, Oxford Lecture Series in Mathematics and its Applications, vol. 15, The Clarendon Press, Oxford University Press, New York, 1998. MR1694383
[16] R. A. Adams, *Sobolev spaces*, Pure Appl. Math., vol. 65, Acad. Press, New-York–London, 1975. MR0450957 (56:9247)
[17] E. Giusti, *Minimal surfaces and functions of bounded variation*, Monographs in Mathematics, vol. 80, Birkhäuser Verlag, Basel, 1984. MR775682
[18] L. Ambrosio, N. Fusco, and D. Pallara, *Functions of bounded variation and free discontinuity problems*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 2000. MR1857292
[19] H. B. Thompson, *Second order ordinary differential equations with fully nonlinear two-point boundary conditions.* I, Pacific J. Math. **172** (1996), no. 1, 255–277. MR1379297 (976:34026)
[20] H. B. Thompson, *Second order ordinary differential equations with fully nonlinear two-point boundary conditions.* II, Pacific J. Math. **172** (1996), no. 1, 279–297. MR1379297 (976:34026)
[21] C. De Coster and P. Habets, *Two-point boundary value problems: lower and upper solutions*, Mathematics in Science and Engineering, vol. 205, Elsevier B. V., Amsterdam, 2006. MR2225284
[22] M. Bildhauer and M. Fuchs, *A geometric maximum principle for variational problems in spaces of vector valued functions of bounded variation* (English, with English and Russian summaries), Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **385** (2010), no. Kraevye Zadachi Matematicheskoĭ Fiziki i Smezhnye Voprosy Teorii Funktsiĭ. 41, 5–17, 234; English transl., J. Math. Sci. (N.Y.) **178** (2011), no. 3, 235–242. MR2749367
[23] E. Hewitt and K. Stromberg, *Real and abstract analysis. A modern treatment of the theory of functions of a real variable*, Springer-Verlag, New York, 1965. MR0188387
[24] G. Anzellotti and M. Giaquinta, *Convex functionals and partial regularity*, Arch. Rational Mech. Anal. **102** (1988), no. 3, 243–272. MR944548
[25] I. Ekeland and R. Témam, *Convex analysis and variational problems*, Corrected reprint of the 1976 English edition, Classics in Applied Mathematics, vol. 28, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1999. Translated from the French. MR1727362
[26] H. Attouch, G. Buttazzo, and G. Michaille, *Variational analysis in Sobolev and BV spaces. Applications to PDEs and optimization*, MPS/SIAM Series on Optimization, vol. 6, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA; Mathematical Programming Society (MPS), Philadelphia, PA, 2006. MR2192832
[27] R. T. Rockafellar, *Convex analysis*, Princeton Landmarks Math. Phys., Princeton Univ. Press, Princeton, 2015.

Department of Mathematics, Saarland University, P.O. Box 151150, 66041 Saarbrücken, Germany
Email address: fuchs@math.uni-sb.de

Department of Mathematics, Saarland University, P.O. Box 151150, 66041 Saarbrücken, Germany
Email address: jmueller@math.uni-sb.de

Department of Mathematics, Saarland University, P.O. Box 151150, 66041 Saarbrücken, Germany
Email address: tietz@math.uni-sb.de

Received 10/FEB/2017
Originally published in English