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HOMOGENIZATION OF THE FIRST INITIAL BOUNDARY-VALUE PROBLEM FOR PARABOLIC SYSTEMS: OPERATOR ERROR ESTIMATES

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Abstract. Let $\mathcal{O} \subset \mathbb{R}^d$ be a bounded domain of class $C^{1,1}$. In $L_2(\mathcal{O};\mathbb{C}^n)$, a selfadjoint matrix second order elliptic differential operator $B_{D,\varepsilon}$, $0<\varepsilon\le 1$, is considered with the Dirichlet boundary condition. The principal part of the operator is given in a factorized form. The operator involves first and zero order terms. The operator $B_{D,\varepsilon}$ is positive definite; its coefficients are periodic and depend on $\mathbf{x}/\varepsilon$. The behavior of the operator exponential $e^{-B_{D,\varepsilon}t}$, $t>0$, is studied as $\varepsilon\to 0$. Approximations for the exponential $e^{-B_{D,\varepsilon}t}$ are obtained in the operator norm on $L_2(\mathcal{O};\mathbb{C}^n)$ and in the norm of operators acting from $L_2(\mathcal{O};\mathbb{C}^n)$ to the Sobolev space $H^1(\mathcal{O};\mathbb{C}^n)$. The results are applied to homogenization of solutions of the first initial boundary-value problem for parabolic systems.

Introduction

The paper concerns homogenization theory of periodic differential operators (DO's). We mention books on homogenization: [BaPa, BeLPap, ZhKO, Sa].

0.1. Statement of the problem. Let $\Gamma \subset \mathbb{R}^d$ be a lattice, and let Ω be the elementary cell of the lattice Γ. For a Γ-periodic function ψ in $\mathbb{R}^d$, we denote $\psi^\varepsilon(\mathbf{x}) := \psi(\mathbf{x}/\varepsilon)$, where $\varepsilon>0$, and $\overline{\psi} := |\Omega|^{-1}\int_\Omega \psi(\mathbf{x})\,d\mathbf{x}$.

Let $\mathcal{O} \subset \mathbb{R}^d$ be a bounded domain of class $C^{1,1}$. In $L_2(\mathcal{O};\mathbb{C}^n)$, we study a selfadjoint matrix strongly elliptic second order DO $B_{D,\varepsilon}$, $0<\varepsilon\le 1$, with the Dirichlet boundary condition. The principal part of the operator $B_{D,\varepsilon}$ is given in a factorized form $A_\varepsilon = b(\mathbf{D})^* g^\varepsilon(\mathbf{x}) b(\mathbf{D})$, where $b(\mathbf{D})$ is a homogeneous first order matrix DO, and $g(\mathbf{x})$ is a Γ-periodic, bounded, and positive definite matrix-valued function in $\mathbb{R}^d$. (The precise assumptions on $b(\mathbf{D})$ and $g(\mathbf{x})$ are given below in Subsection 1.3.) The operator $B_{D,\varepsilon}$ is given by the differential expression

$$B_\varepsilon = b(\mathbf{D})^* g^\varepsilon(\mathbf{x}) b(\mathbf{D}) + \sum_{j=1}^d \bigl(a_j^\varepsilon(\mathbf{x}) D_j + D_j a_j^\varepsilon(\mathbf{x})^*\bigr) + Q^\varepsilon(\mathbf{x}) + \lambda Q_0^\varepsilon(\mathbf{x}) \tag{0.1}$$

with the Dirichlet condition on $\partial\mathcal{O}$. Here the $a_j(\mathbf{x})$, $j=1,\dots,d$, and $Q(\mathbf{x})$ are Γ-periodic matrix-valued functions, in general, unbounded; a Γ-periodic matrix-valued function $Q_0(\mathbf{x})$ is such that $Q_0(\mathbf{x})>0$ and $Q_0, Q_0^{-1} \in L_\infty$. The constant λ is chosen so that the operator $B_{D,\varepsilon}$ is positive definite. (The precise assumptions on the coefficients are given below in Subsection 1.4.)

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The coefficients of the operator (0.1) oscillate rapidly for small ε. Let $\mathbf{u}_\varepsilon(\mathbf{x},t)$ be the solution of the first initial boundary-value problem

$$(0.2)\quad \begin{cases} Q_0^\varepsilon(\mathbf{x})\partial_t \mathbf{u}_\varepsilon(\mathbf{x},t) = -B_\varepsilon \mathbf{u}_\varepsilon(\mathbf{x},t), & \mathbf{x}\in\mathcal{O},\ t>0; \\ \mathbf{u}_\varepsilon(\mathbf{x},t)=0, & \mathbf{x}\in\partial\mathcal{O},\ t>0;\ \ Q_0^\varepsilon(\mathbf{x})\mathbf{u}_\varepsilon(\mathbf{x},0)=\boldsymbol{\varphi}(\mathbf{x}),\ \mathbf{x}\in\mathcal{O}, \end{cases}$$

where $\boldsymbol{\varphi}\in L_2(\mathcal{O};\mathbb{C}^n)$. We are interested in the behavior of the solution in the small period limit.

0.2. Main results. It turns out that, as $\varepsilon\to 0$, the solution $\mathbf{u}_\varepsilon(\,\cdot\,,t)$ converges in $L_2(\mathcal{O};\mathbb{C}^n)$ to the solution $\mathbf{u}_0(\,\cdot\,,t)$ of the following effective problem with constant coefficients:

$$(0.3)\quad \begin{cases} \overline{Q_0}\partial_t \mathbf{u}_0(\mathbf{x},t) = -B^0 \mathbf{u}_0(\mathbf{x},t), & \mathbf{x}\in\mathcal{O},\ t>0; \\ \mathbf{u}_0(\mathbf{x},t)=0, & \mathbf{x}\in\partial\mathcal{O},\ t>0;\quad \overline{Q_0}\mathbf{u}_0(\mathbf{x},0)=\boldsymbol{\varphi}(\mathbf{x}),\quad \mathbf{x}\in\mathcal{O}. \end{cases}$$

Here B^0 is the differential expression for the effective operator B_D^0. Our first main result is the estimate

$$(0.4)\qquad \|\mathbf{u}_\varepsilon(\,\cdot\,,t)-\mathbf{u}_0(\,\cdot\,,t)\|_{L_2(\mathcal{O})}\le C\varepsilon(t+\varepsilon^2)^{-1/2}e^{-ct}\|\boldsymbol{\varphi}\|_{L_2(\mathcal{O})},\quad t\ge 0,$$

for sufficiently small ε. For fixed time $t>0$, this estimate is of sharp order $O(\varepsilon)$. Our second main result is approximation of the solution $\mathbf{u}_\varepsilon(\,\cdot\,,t)$ in the energy norm:

$$(0.5)\qquad \|\mathbf{u}_\varepsilon(\,\cdot\,,t)-\mathbf{v}_\varepsilon(\,\cdot\,,t)\|_{H^1(\mathcal{O})}\le C(\varepsilon^{1/2}t^{-3/4}+\varepsilon t^{-1})e^{-ct}\|\boldsymbol{\varphi}\|_{L_2(\mathcal{O})},\quad t>0.$$

Here $\mathbf{v}_\varepsilon(\,\cdot\,,t)=\mathbf{u}_0(\,\cdot\,,t)+\varepsilon\mathcal{K}_D(t;\varepsilon)\boldsymbol{\varphi}(\,\cdot\,)$ is the first order approximation of the solution $\mathbf{u}_\varepsilon(\,\cdot\,,t)$. The operator $\mathcal{K}_D(t;\varepsilon)$ is a corrector. It involves rapidly oscillating factors, and so depends on ε. We have $\|\varepsilon\mathcal{K}_D(t;\varepsilon)\|_{L_2\to H^1}=O(1)$. For fixed t, estimate (0.5) is of order of $O(\varepsilon^{1/2})$ due to the influence of the boundary layer. The presence of the boundary layer is confirmed by the fact that, in a strictly interior subdomain $\mathcal{O}'\subset\mathcal{O}$, the order of the H^1-estimate can be improved:

$$\|\mathbf{u}_\varepsilon(\,\cdot\,,t)-\mathbf{v}_\varepsilon(\,\cdot\,,t)\|_{H^1(\mathcal{O}')}\le C\varepsilon(t^{-1/2}\delta^{-1}+t^{-1})e^{-ct}\|\boldsymbol{\varphi}\|_{L_2(\mathcal{O})},\quad t>0.$$

Here $\delta=\operatorname{dist}\{\mathcal{O}';\partial\mathcal{O}\}$.

In the general case, the corrector involves a smoothing operator. We distinguish conditions under which it is possible to use a simpler corrector without any smoothing operator. Along with estimate (0.5), we obtain approximation of the flux $g^\varepsilon b(\mathbf{D})\mathbf{u}_\varepsilon(\,\cdot\,,t)$ in the L_2-norm.

The constants in estimates (0.4) and (0.5) are controlled in terms of the problem data; they do not depend on $\boldsymbol{\varphi}$. Therefore, estimates (0.4) and (0.5) can be rewritten in the uniform operator topology. In a simpler case where $Q_0(\mathbf{x})=\mathbf{1}_n$, we have

$$\left\|e^{-B_{D,\varepsilon}t}-e^{-B_D^0 t}\right\|_{L_2(\mathcal{O})\to L_2(\mathcal{O})}\le C\varepsilon(t+\varepsilon^2)^{-1/2}e^{-ct},\quad t\ge 0,$$
$$\left\|e^{-B_{D,\varepsilon}t}-e^{-B_D^0 t}-\varepsilon\mathcal{K}_D(t;\varepsilon)\right\|_{L_2(\mathcal{O})\to H^1(\mathcal{O})}\le C(\varepsilon^{1/2}t^{-3/4}+\varepsilon t^{-1})e^{-ct},\ t>0.$$

The results of such type are called *operator error estimates* in the homogenization theory.

0.3. Operator error estimates. Survey. Currently, the study of operator error estimates is an actively developing field of the homogenization theory. The interest in this subject arose in connection with the papers [BSu1, BSu2] by M. Sh. Birman and T. A. Suslina, where the operator A_ε of the form $b(\mathbf{D})^*g^\varepsilon(\mathbf{x})b(\mathbf{D})$ acting in $L_2(\mathbb{R}^d;\mathbb{C}^n)$ was studied. By the *spectral approach*, it was proved that

$$(0.6)\qquad \|(A_\varepsilon+I)^{-1}-(A^0+I)^{-1}\|_{L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)}\le C\varepsilon.$$

Here $A^0 = b(\mathbf{D})^* g^0 b(\mathbf{D})$ is an effective operator and g^0 is a constant effective matrix. Approximation for the operator $(A_\varepsilon + I)^{-1}$ in the $(L_2 \to H^1)$-norm was obtained in [BSu4]:

$$\|(A_\varepsilon + I)^{-1} - (A^0 + I)^{-1} - \varepsilon K(\varepsilon)\|_{L_2(\mathbb{R}^d) \to H^1(\mathbb{R}^d)} \le C\varepsilon. \tag{0.7}$$

Later, T. A. Suslina carried estimates (0.6) and (0.7) over to more general operator B_ε of the form (0.1) acting in $L_2(\mathbb{R}^d; \mathbb{C}^n)$. We also mention the paper [Bo] by D. I. Borisov, where the expression for the effective operator B^0 was found and the approximations (0.6), (0.7) for the resolvent were obtained. In [Bo], it was assumed that the coefficients of the operator depend not only on the rapid variable, but also on the slow variable; however, the coefficients of B_ε were assumed to be sufficiently smooth.

To parabolic systems, the spectral approach was applied in the papers [Su1, Su2] by T. A. Suslina, where the principal term of approximation was found, and in [Su3], where an estimate with corrector was proved:

$$\left\|e^{-A_\varepsilon t} - e^{-A^0 t}\right\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \le C\varepsilon(t + \varepsilon^2)^{-1/2}, \quad t \ge 0, \tag{0.8}$$

$$\left\|e^{-A_\varepsilon t} - e^{-A^0 t} - \varepsilon \mathcal{K}(t;\varepsilon)\right\|_{L_2(\mathbb{R}^d) \to H^1(\mathbb{R}^d)} \le C\varepsilon(t^{-1/2} + t^{-1}), \quad t \ge \varepsilon^2. \tag{0.9}$$

In these estimates, the exponentially decaying function of t is absent, because the bottom of the spectra of A_ε and A^0 is zero. The exponential of the operator B_ε of the form (0.1) was studied in the paper [M] by Yu. M. Meshkova, where analogs of inequalities (0.8) and (0.9) were obtained.

A *different approach* to operator error estimates in homogenization theory was suggested by V. V. Zhikov in [Zh2]. In [Zh2, ZhPas1], estimates of the form (0.6) and (0.7) for the acoustics and elasticity operators were obtained. The "*modified method of the first order approximation*" or the "*shift method*", in the terminology of the authors, was based on analysis of the first order approximation to the solution and introduction of an additional parameter. Along with problems in $\mathbb{R}^d$, in [Zh2, ZhPas1] homogenization problems in a bounded domain $\mathcal{O} \subset \mathbb{R}^d$ with the Dirichlet or Neumann boundary conditions were studied. To parabolic equations, the shift method was applied in [ZhPas2], where analogs of estimates (0.8) and (0.9) were proved. The further results of V. V. Zhikov, S. E. Pastukhova, and their students were discussed in the recent survey [ZhPas3].

Operator error estimates for the Dirichlet and Neumann problems for second order elliptic equations in a bounded domain were studied by many authors. Apparently, the first result is due to Sh. Moskow and M. Vogelius, who proved the estimate

$$\|A_{D,\varepsilon}^{-1} - (A_D^0)^{-1}\|_{L_2(\mathcal{O}) \to L_2(\mathcal{O})} \le C\varepsilon; \tag{0.10}$$

see [MoV, Corollary 2.2]. Here the operator $A_{D,\varepsilon}$ acts in $L_2(\mathcal{O})$, where $\mathcal{O} \subset \mathbb{R}^2$, and is given by $-\operatorname{div} g^\varepsilon(\mathbf{x})\nabla$ with the Dirichlet condition on $\partial\mathcal{O}$. The matrix-valued function $g(\mathbf{x})$ is assumed to be infinitely smooth.

For arbitrary dimension, homogenization problems in a bounded domain were studied in [Zh2] and [ZhPas1]. The acoustics and elasticity operators with the Dirichlet or Neumann boundary conditions were considered without any smoothness assumptions on coefficients. The authors obtained approximation with corrector for the inverse operator in the $(L_2 \to H^1)$-norm with error estimate of order of $O(\sqrt{\varepsilon})$. The order deteriorates as compared with a similar result in $\mathbb{R}^d$; this is explained by the boundary influence. As a rough consequence, approximation of the form (0.10) with error estimate $O(\sqrt{\varepsilon})$ was deduced. Similar results for the operator $-\operatorname{div} g^\varepsilon(\mathbf{x})\nabla$ in a bounded domain $\mathcal{O} \subset \mathbb{R}^d$ with the Dirichlet or Neumann boundary conditions were obtained by G. Griso [Gr1, Gr2] with the help of the "unfolding" method. In [Gr2], for the same operator a sharp-order estimate (0.10) was proved. For elliptic systems similar results were independently obtained in [KeLiS] and in [PSu, Su5]. The further results and a detailed survey can be found in [Su6, Su7].

For the matrix operator of the form (0.1) with the Dirichlet condition, homogenization problems were studied by Q. Xu in [Xu1, Xu3]. The case of the Neumann boundary condition was studied in [Xu2]. However, in the papers by Q. Xu, the operator is subject to a rather restrictive condition of uniform ellipticity. Approximations of the generalized resolvent of the operator (0.1) with two-parametric error estimates were obtained in the recent paper [MSu3] (see also the brief communication [MSu4]). We focus on these results in more detail, since they are basic for us. For $\zeta \in \mathbb{C} \setminus \mathbb{R}_+$, $|\zeta| \geq 1$, and sufficiently small ε, we have

$$\big\|(B_{D,\varepsilon} - \zeta Q_0^\varepsilon)^{-1} - (B_D^0 - \zeta \overline{Q_0})^{-1}\big\|_{L_2(\mathcal{O})\to L_2(\mathcal{O})} \leq C(\phi)\varepsilon|\zeta|^{-1/2}, \tag{0.11}$$

$$\begin{aligned}\big\|(B_{D,\varepsilon} - \zeta Q_0^\varepsilon)^{-1} - (B_D^0 - \zeta \overline{Q_0})^{-1} - \varepsilon K_D(\varepsilon;\zeta)\big\|_{L_2(\mathcal{O})\to H^1(\mathcal{O})} \\ \leq C(\phi)\big(\varepsilon^{1/2}|\zeta|^{-1/4} + \varepsilon\big).\end{aligned} \tag{0.12}$$

Note that the values $C(\phi)$ are controlled explicitly in terms of the problem data and the angle $\phi = \arg\zeta$. Estimates (0.11) and (0.12) are uniform with respect to ϕ in any domain of the form $\{\zeta = |\zeta|e^{i\phi} \in \mathbb{C} : |\zeta| \geq 1, \phi_0 \leq \phi \leq 2\pi - \phi_0\}$ with arbitrarily small $\phi_0 > 0$. Moreover, in [MSu3], analogs of estimates (0.11) and (0.12) were proved for a wider domain of the spectral parameter ζ.

We proceed to discussion of parabolic problems in a bounded domain. In the two-dimensional case, some estimates of operator type for elliptic and parabolic equations were obtained in [ChKonLe]. However, in [ChKonLe] the matrix g was assumed to be C^∞-smooth, and the initial data for a parabolic equation belonged to $H^2(\mathcal{O})$. In the case of arbitrary dimension and without smoothness assumptions on coefficients, approximation for the exponential of the operator $b(\mathbf{D})^* g^\varepsilon(\mathbf{x}) b(\mathbf{D})$ (with the Dirichlet or Neumann conditions) was found in the paper [MSu1]:

$$\big\|e^{-A_{D,\varepsilon}t} - e^{-A_D^0 t}\big\|_{L_2(\mathcal{O})\to L_2(\mathcal{O})} \leq C\varepsilon(t+\varepsilon^2)^{-1/2}e^{-ct}, \quad t \geq 0,$$
$$\big\|e^{-A_{D,\varepsilon}t} - e^{-A_D^0 t} - \varepsilon\mathcal{K}_D(t;\varepsilon)\big\|_{L_2(\mathcal{O})\to H^1(\mathcal{O})} \leq C\varepsilon^{1/2}t^{-3/4}e^{-ct}, \quad t \geq \varepsilon^2.$$

The method of [MSu1] was based on employing the identity

$$e^{-A_{D,\varepsilon}t} = -\frac{1}{2\pi i}\int_\gamma e^{-\zeta t}(A_{D,\varepsilon} - \zeta I)^{-1}\,d\zeta,$$

where $\gamma \subset \mathbb{C}$ is a contour enclosing the spectrum of $A_{D,\varepsilon}$ in positive direction. This identity allowed us to deduce approximations for the operator exponential $e^{-A_{D,\varepsilon}t}$ from the corresponding approximations of the resolvent $(A_{D,\varepsilon} - \zeta I)^{-1}$ with two-parametric error estimates (with respect to ε and ζ). The required approximations for the resolvent were found in [Su7].

The operator with coefficients periodic in the space and time variables was studied by J. Geng and Z. Shen in [GeS]. In that paper operator error estimates were obtained for the equation

$$\partial_t \mathbf{u}_\varepsilon(\mathbf{x},t) = -\operatorname{div} g(\varepsilon^{-1}\mathbf{x}, \varepsilon^{-2}t)\nabla\mathbf{u}_\varepsilon(\mathbf{x},t)$$

in a bounded domain of class $C^{1,1}$. The results of [GeS] were generalized to the case of Lipschitz domains by Q. Xu and Sh. Zhou, see [XuZ].

0.4. Method. We develop the method of the paper [MSu1]. It is based upon the following representation for the solution $\mathbf{u}_\varepsilon$ of the first initial boundary-value problem

(0.2): $\mathbf{u}_\varepsilon(\,\cdot\,,t) = -\frac{1}{2\pi i}\int_\gamma e^{-\zeta t}(B_{D,\varepsilon} - \zeta Q_0^\varepsilon)^{-1}\boldsymbol{\varphi}\,d\zeta$, where $\gamma \subset \mathbb{C}$ is a suitable contour. The solution of the effective problem (0.3) admits a similar representation. Hence,

$$\mathbf{u}_\varepsilon(\,\cdot\,,t) - \mathbf{u}_0(\,\cdot\,,t) = -\frac{1}{2\pi i}\int_\gamma e^{-\zeta t}\left((B_{D,\varepsilon} - \zeta Q_0^\varepsilon)^{-1} - (B_D^0 - \zeta\overline{Q_0})^{-1}\right)\boldsymbol{\varphi}\,d\zeta. \tag{0.13}$$

Using the results of [MSu3] (estimate (0.11)), we obtain approximation of the resolvent for $\zeta \in \gamma$ and employ the representation (0.13). This leads to (0.4). Note that the dependence of the right-hand side of (0.11) on ζ for large $|\zeta|$ is important for us. Approximation with the corrector taken into account is obtained in a similar way.

0.5. Plan of the paper. The paper consists of five sections and Appendix (§§6–8). In §1, we describe the class of operators $B_{D,\varepsilon}$, introduce the effective operator B_D^0, and formulate the required results about approximation of the operator $(B_{D,\varepsilon} - \zeta Q_0^\varepsilon)^{-1}$. The main results of the paper are obtained in §2. In §3, these results are applied to homogenization of the solutions of the first initial boundary-value problem for nonhomogeneous parabolic equation. §§4, 5 are devoted to applications of the general results. In §4, a scalar elliptic operator with a singular potential of order $O(\varepsilon^{-1})$ is considered. In §5, we study an operator with a singular potential of order $O(\varepsilon^{-2})$. In Appendix (§§6–8), we prove some statements concerning removal of the smoothing operator from the corrector. The case of additional smoothness of the boundary is considered in §7; the case of a strictly interior subdomain is discussed in §8. The required properties of the oscillating factors in the corrector are obtained in §6.

0.6. Notation. Let $\mathfrak{H}$ and $\mathfrak{H}_*$ be complex separable Hilbert spaces. The symbols $(\,\cdot\,,\,\cdot\,)_{\mathfrak{H}}$ and $\|\cdot\|_{\mathfrak{H}}$ stand for the inner product and the norm in $\mathfrak{H}$; the symbol $\|\cdot\|_{\mathfrak{H}\to\mathfrak{H}_*}$ denotes the norm of a continuous linear operator acting from $\mathfrak{H}$ to $\mathfrak{H}_*$.

The set of natural numbers and the set of nonnegative integers are denoted by $\mathbb{N}$ and $\mathbb{Z}_+$, respectively. We denote $\mathbb{R}_+ := [0,\infty)$. The symbols $\langle\,\cdot\,,\,\cdot\,\rangle$ and $|\cdot|$ denote the inner product and the norm in $\mathbb{C}^n$; $\mathbf{1}_n$ is the identity $(n\times n)$-matrix. If a is an $(m\times n)$-matrix, then the symbol $|a|$ denotes the norm of a viewed as an operator from $\mathbb{C}^n$ to $\mathbb{C}^m$. If $\alpha = (\alpha_1,\dots,\alpha_d) \in \mathbb{Z}_+^d$ is a multiindex, $|\alpha|$ denotes its length: $|\alpha| = \sum_{j=1}^d \alpha_j$. For $z \in \mathbb{C}$, the complex conjugate number is denoted by z^*. (We use such a nonstandard notation because the upper bar will denote the mean value of a periodic function over the periodicity cell.) We denote $\mathbf{x} = (x_1,\dots,x_d) \in \mathbb{R}^d$, $iD_j = \partial_j = \partial/\partial x_j$, $j = 1,\dots,d$, and $\mathbf{D} = -i\nabla = (D_1,\dots,D_d)$. The L_p-classes of $\mathbb{C}^n$-valued functions in a domain $\mathcal{O} \subset \mathbb{R}^d$ are denoted by $L_p(\mathcal{O};\mathbb{C}^n)$, $1 \le p \le \infty$. The Sobolev classes of $\mathbb{C}^n$-valued functions in a domain $\mathcal{O} \subset \mathbb{R}^d$ are denoted by $H^s(\mathcal{O};\mathbb{C}^n)$. By $H_0^1(\mathcal{O};\mathbb{C}^n)$ we denote the closure of $C_0^\infty(\mathcal{O};\mathbb{C}^n)$ in $H^1(\mathcal{O};\mathbb{C}^n)$. If $n = 1$, we write simply $L_p(\mathcal{O})$, $H^s(\mathcal{O})$, etc., but sometimes, if this does not lead to confusion, we use this simple notation for the spaces of vector-valued or matrix-valued functions. The symbol $L_p((0,T);\mathfrak{H})$, $1 \le p \le \infty$, denotes the L_p-space of $\mathfrak{H}$-valued functions on the interval $(0,T)$.

Various constants in estimates are denoted by $c, C, \mathrm{C}, \mathcal{C}, \mathfrak{C}$ (probably, with indices and marks).

The main results of the present paper were announced in [MSu4].

§1. The results on homogenization of the Dirichlet problem for elliptic systems

1.1. Lattices in $\mathbb{R}^d$. Let $\Gamma \subset \mathbb{R}^d$ be a lattice generated by a basis $\mathbf{a}_1,\dots,\mathbf{a}_d \in \mathbb{R}^d$:

$$\Gamma = \Big\{\mathbf{a} \in \mathbb{R}^d \,:\, \mathbf{a} = \sum_{j=1}^d \nu_j \mathbf{a}_j, \nu_j \in \mathbb{Z}\Big\},$$

and let Ω be the elementary cell of the lattice Γ:

$$\Omega = \Big\{ \mathbf{x} \in \mathbb{R}^d \,:\, \mathbf{x} = \sum_{j=1}^{d} \tau_j \mathbf{a}_j, -\frac{1}{2} < \tau_j < \frac{1}{2} \Big\}.$$

By $|\Omega|$ we denote the Lebesgue measure of the cell Ω: $|\Omega| = \operatorname{meas}\Omega$. We put $2r_1 := \operatorname{diam}\Omega$.

Let $\widetilde{H}^1(\Omega)$ denote the subspace of functions in $H^1(\Omega)$ whose Γ-periodic extension to $\mathbb{R}^d$ belongs to $H^1_{\mathrm{loc}}(\mathbb{R}^d)$. If $\Phi(\mathbf{x})$ is a Γ-periodic matrix-valued function in $\mathbb{R}^d$, we put $\Phi^\varepsilon(\mathbf{x}) := \Phi(\mathbf{x}/\varepsilon)$, $\varepsilon > 0$; $\overline{\Phi} := |\Omega|^{-1} \int_\Omega \Phi(\mathbf{x})\, d\mathbf{x}$, $\underline{\Phi} := \big(|\Omega|^{-1} \int_\Omega \Phi(\mathbf{x})^{-1}\, d\mathbf{x} \big)^{-1}$. Here, in the definition of $\overline{\Phi}$ it is assumed that $\Phi \in L_{1,\mathrm{loc}}(\mathbb{R}^d)$; in the definition of $\underline{\Phi}$ it is assumed that the matrix Φ is square and nonsingular, and $\Phi^{-1} \in L_{1,\mathrm{loc}}(\mathbb{R}^d)$. By $[\Phi^\varepsilon]$ we denote the operator of multiplication by the matrix-valued function $\Phi^\varepsilon(\mathbf{x})$.

1.2. The Steklov smoothing. The Steklov smoothing operator $S_\varepsilon^{(k)}$, $\varepsilon > 0$, acts in $L_2(\mathbb{R}^d; \mathbb{C}^k)$ (where $k \in \mathbb{N}$) and is given by

$$(S_\varepsilon^{(k)} \mathbf{u})(\mathbf{x}) = |\Omega|^{-1} \int_\Omega \mathbf{u}(\mathbf{x} - \varepsilon \mathbf{z})\, d\mathbf{z}, \quad \mathbf{u} \in L_2(\mathbb{R}^d; \mathbb{C}^k). \tag{1.1}$$

We shall omit the index k in the notation and write simply S_ε. Obviously, $S_\varepsilon \mathbf{D}^\alpha \mathbf{u} = \mathbf{D}^\alpha S_\varepsilon \mathbf{u}$ for any $\mathbf{u} \in H^\sigma(\mathbb{R}^d; \mathbb{C}^k)$ and any multiindex α such that $|\alpha| \le \sigma$. Note that

$$\| S_\varepsilon \|_{H^\sigma(\mathbb{R}^d) \to H^\sigma(\mathbb{R}^d)} \le 1, \quad \sigma \ge 0. \tag{1.2}$$

We need the following properties of the operator S_ε (see [ZhPas1, Lemmas 1.1 and 1.2] or [PSu, Propositions 3.1 and 3.2]).

Proposition 1.1. *For any function $\mathbf{u} \in H^1(\mathbb{R}^d; \mathbb{C}^k)$, we have*

$$\| S_\varepsilon \mathbf{u} - \mathbf{u} \|_{L_2(\mathbb{R}^d)} \le \varepsilon r_1 \| \mathbf{D} \mathbf{u} \|_{L_2(\mathbb{R}^d)},$$

where $2r_1 = \operatorname{diam}\Omega$.

Proposition 1.2. *Let Φ be a Γ-periodic function in $\mathbb{R}^d$ such that $\Phi \in L_2(\Omega)$. Then the operator $[\Phi^\varepsilon] S_\varepsilon$ is continuous in $L_2(\mathbb{R}^d)$ and*

$$\| [\Phi^\varepsilon] S_\varepsilon \|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \le |\Omega|^{-1/2} \| \Phi \|_{L_2(\Omega)}.$$

1.3. The operator $A_{D,\varepsilon}$. Let $\mathcal{O} \subset \mathbb{R}^d$ be a bounded domain of class $C^{1,1}$. In $L_2(\mathcal{O}; \mathbb{C}^n)$, we consider the operator $A_{D,\varepsilon}$ given formally by the differential expression

$$A_\varepsilon = b(\mathbf{D})^* g^\varepsilon(\mathbf{x}) b(\mathbf{D})$$

with the Dirichlet condition on $\partial\mathcal{O}$. Here $g(\mathbf{x})$ is a Γ-periodic Hermitian $(m \times m)$-matrix-valued function (in general, with complex entries). It is assumed that $g(\mathbf{x}) > 0$ and $g, g^{-1} \in L_\infty(\mathbb{R}^d)$. The differential operator $b(\mathbf{D})$ is given by $b(\mathbf{D}) = \sum_{j=1}^d b_j D_j$, where the b_j, $j = 1, \dots, d$, are constant matrices of size $m \times n$ (in general, with complex entries). Assume that $m \ge n$ and that the symbol $b(\boldsymbol{\xi}) = \sum_{j=1}^d b_j \xi_j$ of the operator $b(\mathbf{D})$ has maximal rank: $\operatorname{rank} b(\boldsymbol{\xi}) = n$ for $0 \ne \boldsymbol{\xi} \in \mathbb{R}^d$. This condition is equivalent to the estimates

$$\alpha_0 \mathbf{1}_n \le b(\boldsymbol{\theta})^* b(\boldsymbol{\theta}) \le \alpha_1 \mathbf{1}_n, \quad \boldsymbol{\theta} \in \mathbb{S}^{d-1}; \quad 0 < \alpha_0 \le \alpha_1 < \infty, \tag{1.3}$$

with some positive constants α_0 and α_1. From (1.3) it follows that

$$|b_j| \le \alpha_1^{1/2}, \quad j = 1, \dots, d. \tag{1.4}$$

The precise definition of the operator $A_{D,\varepsilon}$ is given in terms of the quadratic form

$$\mathfrak{a}_{D,\varepsilon}[\mathbf{u},\mathbf{u}] = \int_{\mathcal{O}} \langle g^\varepsilon(\mathbf{x}) b(\mathbf{D})\mathbf{u}, b(\mathbf{D})\mathbf{u}\rangle \, d\mathbf{x}, \quad \mathbf{u} \in H_0^1(\mathcal{O};\mathbb{C}^n). \tag{1.5}$$

Extending $\mathbf{u} \in H_0^1(\mathcal{O};\mathbb{C}^n)$ by zero to $\mathbb{R}^d \setminus \mathcal{O}$ and taking (1.3) into account, we find

$$\alpha_0 \|g^{-1}\|^{-1}_{L_\infty} \|\mathbf{D}\mathbf{u}\|^2_{L_2(\mathcal{O})} \le \mathfrak{a}_{D,\varepsilon}[\mathbf{u},\mathbf{u}] \le \alpha_1 \|g\|_{L_\infty} \|\mathbf{D}\mathbf{u}\|^2_{L_2(\mathcal{O})}, \quad \mathbf{u} \in H_0^1(\mathcal{O};\mathbb{C}^n). \tag{1.6}$$

1.4. Lower order terms. The operator $B_{D,\varepsilon}$. We study the selfadjoint operator $B_{D,\varepsilon}$ whose principal part coincides with A_ε. To define the lower order terms, we introduce Γ-periodic $(n \times n)$-matrix-valued functions (in general, with complex entries) a_j, $j = 1, \dots, d$, such that

$$a_j \in L_\rho(\Omega), \quad \rho = 2 \text{ for } d = 1, \quad \rho > d \text{ for } d \ge 2, \quad j = 1, \dots, d. \tag{1.7}$$

Next, let Q and Q_0 be Γ-periodic Hermitian $(n \times n)$-matrix-valued functions (with complex entries) such that

$$\begin{gathered} Q \in L_s(\Omega), \quad s = 1 \text{ for } d = 1, \quad s > d/2 \text{ for } d \ge 2; \\ Q_0(\mathbf{x}) > 0; \quad Q_0, Q_0^{-1} \in L_\infty(\mathbb{R}^d). \end{gathered} \tag{1.8}$$

For the convenience of further references, the following set of variables is called the “problem data”:

$$\begin{gathered} d,\ m,\ n,\ \rho,\ s;\ \alpha_0,\ \alpha_1,\ \|g\|_{L_\infty},\ \|g^{-1}\|_{L_\infty},\ \|a_j\|_{L_\rho(\Omega)},\ j = 1, \dots, d; \\ \|Q\|_{L_s(\Omega)};\ \|Q_0\|_{L_\infty},\ \|Q_0^{-1}\|_{L_\infty};\ \text{the parameters of the lattice } \Gamma; \ \text{ the domain } \mathcal{O}. \end{gathered} \tag{1.9}$$

In $L_2(\mathcal{O};\mathbb{C}^n)$, we consider the operator $B_{D,\varepsilon}$, $0 < \varepsilon \le 1$, formally given by the differential expression

$$B_\varepsilon = b(\mathbf{D})^* g^\varepsilon(\mathbf{x}) b(\mathbf{D}) + \sum_{j=1}^d \left(a_j^\varepsilon(\mathbf{x}) D_j + D_j a_j^\varepsilon(\mathbf{x})^* \right) + Q^\varepsilon(\mathbf{x}) + \lambda Q_0^\varepsilon(\mathbf{x}) \tag{1.10}$$

with the Dirichlet boundary condition. Here the constant λ is chosen so that the operator $B_{D,\varepsilon}$ is positive definite (see (1.16) below). The precise definition of the operator $B_{D,\varepsilon}$ is given in terms of the quadratic form

$$\begin{aligned} \mathfrak{b}_{D,\varepsilon}[\mathbf{u},\mathbf{u}] = {} & (g^\varepsilon b(\mathbf{D})\mathbf{u}, b(\mathbf{D})\mathbf{u})_{L_2(\mathcal{O})} + 2\operatorname{Re}\sum_{j=1}^d (a_j^\varepsilon D_j \mathbf{u}, \mathbf{u})_{L_2(\mathcal{O})} \\ & + (Q^\varepsilon \mathbf{u}, \mathbf{u})_{L_2(\mathcal{O})} + \lambda (Q_0^\varepsilon \mathbf{u}, \mathbf{u})_{L_2(\mathcal{O})}, \quad \mathbf{u} \in H_0^1(\mathcal{O};\mathbb{C}^n). \end{aligned} \tag{1.11}$$

Let us check that the form $\mathfrak{b}_{D,\varepsilon}$ is closed. Using the Hölder inequality and the Sobolev embedding theorem, it can be shown that for any $\nu > 0$ there exist constants $C_j(\nu) > 0$ such that

$$\|a_j^* \mathbf{u}\|^2_{L_2(\mathbb{R}^d)} \le \nu \|\mathbf{D}\mathbf{u}\|^2_{L_2(\mathbb{R}^d)} + C_j(\nu) \|\mathbf{u}\|^2_{L_2(\mathbb{R}^d)}, \quad \mathbf{u} \in H^1(\mathbb{R}^d;\mathbb{C}^n),$$

$j = 1, \dots, d$; see [Su4, (5.11)–(5.14)]. By the change of variables $\mathbf{y} := \varepsilon^{-1}\mathbf{x}$ and $\mathbf{u}(\mathbf{x}) =: \mathbf{v}(\mathbf{y})$, we deduce that

$$\begin{aligned} \|(a_j^\varepsilon)^* \mathbf{u}\|^2_{L_2(\mathbb{R}^d)} &= \int_{\mathbb{R}^d} |a_j(\varepsilon^{-1}\mathbf{x})^* \mathbf{u}(\mathbf{x})|^2 \, d\mathbf{x} = \varepsilon^d \int_{\mathbb{R}^d} |a_j(\mathbf{y})^* \mathbf{v}(\mathbf{y})|^2 \, d\mathbf{y} \\ &\le \varepsilon^d \nu \int_{\mathbb{R}^d} |\mathbf{D}_\mathbf{y} \mathbf{v}(\mathbf{y})|^2 \, d\mathbf{y} + \varepsilon^d C_j(\nu) \int_{\mathbb{R}^d} |\mathbf{v}(\mathbf{y})|^2 \, d\mathbf{y} \\ &\le \nu \|\mathbf{D}\mathbf{u}\|^2_{L_2(\mathbb{R}^d)} + C_j(\nu) \|\mathbf{u}\|^2_{L_2(\mathbb{R}^d)}, \quad \mathbf{u} \in H^1(\mathbb{R}^d;\mathbb{C}^n), \quad 0 < \varepsilon \le 1. \end{aligned}$$

Then, by (1.3), for any $\nu > 0$ there exists a constant $C(\nu) > 0$ such that

$$\sum_{j=1}^{d} \|(a_j^\varepsilon)^* \mathbf{u}\|^2_{L_2(\mathbb{R}^d)} \le \nu \|(g^\varepsilon)^{1/2} b(\mathbf{D})\mathbf{u}\|^2_{L_2(\mathbb{R}^d)} + C(\nu)\|\mathbf{u}\|^2_{L_2(\mathbb{R}^d)}, \quad \mathbf{u} \in H^1(\mathbb{R}^d;\mathbb{C}^n), \quad 0 < \varepsilon \le 1. \tag{1.12}$$

If ν is fixed, then $C(\nu)$ depends only on d, ρ, α_0, the norms $\|g^{-1}\|_{L_\infty}$, $\|a_j\|_{L_\rho(\Omega)}$, $j = 1, \dots, d$, and the parameters of the lattice Γ.

By (1.3), for $\mathbf{u} \in H^1(\mathbb{R}^d;\mathbb{C}^n)$ we have

$$\|\mathbf{D}\mathbf{u}\|^2_{L_2(\mathbb{R}^d)} \le c_1^2 \|(g^\varepsilon)^{1/2} b(\mathbf{D})\mathbf{u}\|^2_{L_2(\mathbb{R}^d)}, \tag{1.13}$$

where $c_1 := \alpha_0^{-1/2}\|g^{-1}\|^{1/2}_{L_\infty}$. Combining this with (1.12), we obtain

$$2\left| \operatorname{Re} \sum_{j=1}^{d} (D_j \mathbf{u}, (a_j^\varepsilon)^* \mathbf{u})_{L_2(\mathbb{R}^d)} \right| \le \frac{1}{4}\|(g^\varepsilon)^{1/2} b(\mathbf{D})\mathbf{u}\|^2_{L_2(\mathbb{R}^d)} + c_2\|\mathbf{u}\|^2_{L_2(\mathbb{R}^d)}, \quad \mathbf{u} \in H^1(\mathbb{R}^d;\mathbb{C}^n), \quad 0 < \varepsilon \le 1, \tag{1.14}$$

where $c_2 := 8c_1^2 C(\nu_0)$ with $\nu_0 := 2^{-6}\alpha_0\|g^{-1}\|^{-1}_{L_\infty}$.

Next, by condition (1.8) on Q, for any $\nu > 0$ there exists a constant $C_Q(\nu) > 0$ such that

$$|(Q^\varepsilon \mathbf{u}, \mathbf{u})_{L_2(\mathbb{R}^d)}| \le \nu \|\mathbf{D}\mathbf{u}\|^2_{L_2(\mathbb{R}^d)} + C_Q(\nu)\|\mathbf{u}\|^2_{L_2(\mathbb{R}^d)}, \quad \mathbf{u} \in H^1(\mathbb{R}^d;\mathbb{C}^n), \quad 0 < \varepsilon \le 1. \tag{1.15}$$

For fixed ν, the constant $C_Q(\nu)$ is controlled in terms of d, s, $\|Q\|_{L_s(\Omega)}$, and the parameters of the lattice Γ.

We fix a constant λ in (1.10) as in [MSu2, Subsection 2.8]:

$$\lambda := (C_Q(\nu_*) + c_2)\|Q_0^{-1}\|_{L_\infty} \quad \text{for} \quad \nu_* := 2^{-1}\alpha_0\|g^{-1}\|^{-1}_{L_\infty}. \tag{1.16}$$

We return to the form (1.11). Extending the function $\mathbf{u} \in H^1_0(\mathcal{O};\mathbb{C}^n)$ by zero to $\mathbb{R}^d \setminus \mathcal{O}$ and using (1.5), (1.13), (1.14), and (1.15) with $\nu = \nu_*$, we obtain a lower estimate for the form (1.11):

$$\mathfrak{b}_{D,\varepsilon}[\mathbf{u},\mathbf{u}] \ge \frac{1}{4}\mathfrak{a}_{D,\varepsilon}[\mathbf{u},\mathbf{u}] \ge c_*\|\mathbf{D}\mathbf{u}\|^2_{L_2(\mathcal{O})}, \quad \mathbf{u} \in H^1(\mathcal{O};\mathbb{C}^n); \tag{1.17}$$

$$c_* := \frac{1}{4}\alpha_0\|g^{-1}\|^{-1}_{L_\infty}. \tag{1.18}$$

Next, by (1.6), (1.14), and (1.15) with $\nu = 1$, we have

$$\mathfrak{b}_{D,\varepsilon}[\mathbf{u},\mathbf{u}] \le C_*\|\mathbf{u}\|^2_{H^1(\mathbb{R}^d)}, \quad \mathbf{u} \in H^1(\mathcal{O};\mathbb{C}^n),$$

where $C_* := \max\{\frac{5}{4}\alpha_1\|g\|_{L_\infty} + 1; C_Q(1) + \lambda\|Q_0\|_{L_\infty} + c_2\}$. Thus, the form $\mathfrak{b}_{D,\varepsilon}$ is closed. The corresponding selfadjoint operator in $L_2(\mathcal{O};\mathbb{C}^n)$ is denoted by $B_{D,\varepsilon}$.

By the Friedrichs inequality, (1.17) implies that

$$\mathfrak{b}_{D,\varepsilon}[\mathbf{u},\mathbf{u}] \ge c_*(\operatorname{diam}\mathcal{O})^{-2}\|\mathbf{u}\|^2_{L_2(\mathcal{O})}, \quad \mathbf{u} \in H^1_0(\mathcal{O};\mathbb{C}^n). \tag{1.19}$$

Hence, the operator $B_{D,\varepsilon}$ is positive definite. By (1.17) and (1.19),

$$\|\mathbf{u}\|_{H^1(\mathcal{O})} \le c_3\|B^{1/2}_{D,\varepsilon}\mathbf{u}\|_{L_2(\mathcal{O})}, \quad \mathbf{u} \in H^1_0(\mathcal{O};\mathbb{C}^n); \tag{1.20}$$

$$c_3 := c_*^{-1/2}\left(1 + (\operatorname{diam}\mathcal{O})^2\right)^{1/2}. \tag{1.21}$$

We also need an auxiliary operator $\widetilde{B}_{D,\varepsilon}$. We factorize the matrix $Q_0(\mathbf{x})$: there exists a Γ-periodic matrix-valued function $f(\mathbf{x})$ such that f, $f^{-1} \in L_\infty(\mathbb{R}^d)$ and

$$Q_0(\mathbf{x}) = (f(\mathbf{x})^*)^{-1} f(\mathbf{x})^{-1}. \tag{1.22}$$

(For instance, one can choose $f(\mathbf{x}) = Q_0(\mathbf{x})^{-1/2}$.) Let $\widetilde{B}_{D,\varepsilon}$ be the selfadjoint operator in $L_2(\mathcal{O};\mathbb{C}^n)$ generated by the quadratic form

$$\widetilde{\mathfrak{b}}_{D,\varepsilon}[\mathbf{u},\mathbf{u}] := \mathfrak{b}_{D,\varepsilon}[f^\varepsilon \mathbf{u}, f^\varepsilon \mathbf{u}] \tag{1.23}$$

on the domain $\operatorname{Dom}\widetilde{\mathfrak{b}}_{D,\varepsilon} := \{\mathbf{u} \in L_2(\mathcal{O};\mathbb{C}^n) : f^\varepsilon \mathbf{u} \in H^1_0(\mathcal{O};\mathbb{C}^n)\}$. In other words, $\widetilde{B}_{D,\varepsilon} = (f^\varepsilon)^* B_{D,\varepsilon} f^\varepsilon$. Let $\widetilde{B}_\varepsilon$ denote the differential expression $(f^\varepsilon)^* B_\varepsilon f^\varepsilon$. Note that

$$(B_{D,\varepsilon} - \zeta Q_0^\varepsilon)^{-1} = f^\varepsilon (\widetilde{B}_{D,\varepsilon} - \zeta I)^{-1} (f^\varepsilon)^*. \tag{1.24}$$

1.5. The effective matrix and its properties. The effective operator for $A_{D,\varepsilon}$ is given by the differential expression $A^0 = b(\mathbf{D})^* g^0 b(\mathbf{D})$ with the Dirichlet condition on $\partial\mathcal{O}$. Here g^0 is a constant *effective* matrix of size $m \times m$. The matrix g^0 is expressed in terms of the solution of an auxiliary problem on the cell. Let an $(n\times m)$-matrix-valued function $\Lambda(\mathbf{x})$ be the (weak) Γ-periodic solution of the problem

$$b(\mathbf{D})^* g(\mathbf{x})(b(\mathbf{D})\Lambda(\mathbf{x}) + \mathbf{1}_m) = 0, \quad \int_\Omega \Lambda(\mathbf{x})\, d\mathbf{x} = 0. \tag{1.25}$$

Then the effective matrix is given by

$$g^0 := |\Omega|^{-1} \int_\Omega \widetilde{g}(\mathbf{x})\, d\mathbf{x}, \tag{1.26}$$

$$\widetilde{g}(\mathbf{x}) := g(\mathbf{x})(b(\mathbf{D})\Lambda(\mathbf{x}) + \mathbf{1}_m). \tag{1.27}$$

It can be checked that the matrix g^0 is positive definite.

In accordance with [BSu3, (6.28) and Subsection 7.3], the solution of problem (1.25) satisfies

$$\|\Lambda\|_{H^1(\Omega)} \le M. \tag{1.28}$$

Here the constant M depends only on m, α_0, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, and the parameters of the lattice Γ.

The effective matrix satisfies the estimates known as the Voigt–Reuss bracketing (see, e.g., [BSu2, Chapter 3, Theorem 1.5]).

Proposition 1.3. *Let g^0 be the effective matrix* (1.26)*. Then*

$$\underline{g} \le g^0 \le \overline{g}. \tag{1.29}$$

If $m = n$, then $g^0 = \underline{g}$.

From (1.29) it follows that

$$|g^0| \le \|g\|_{L_\infty}, \quad |(g^0)^{-1}| \le \|g^{-1}\|_{L_\infty}. \tag{1.30}$$

Now we distinguish the cases where one of the inequalities in (1.29) becomes an identity, see [BSu2, Chapter 3, Propositions 1.6 and 1.7].

Proposition 1.4. *The identity $g^0 = \overline{g}$ is equivalent to the relations*

$$b(\mathbf{D})^* \mathbf{g}_k(\mathbf{x}) = 0, \quad k = 1,\dots,m, \tag{1.31}$$

where the $\mathbf{g}_k(\mathbf{x})$, $k = 1,\dots,m$, are the columns of the matrix $g(\mathbf{x})$.

Proposition 1.5. *The identity $g^0 = \underline{g}$ is equivalent to the representations*

$$\mathbf{l}_k(\mathbf{x}) = \mathbf{l}_k^0 + b(\mathbf{D})\mathbf{w}_k, \quad \mathbf{l}_k^0 \in \mathbb{C}^m, \quad \mathbf{w}_k \in \widetilde{H}^1(\Omega;\mathbb{C}^m), \quad k = 1,\dots,m, \tag{1.32}$$

where the $\mathbf{l}_k(\mathbf{x})$, $k = 1,\dots,m$, are the columns of the matrix $g(\mathbf{x})^{-1}$.

1.6. The effective operator. To describe how the lower order terms of the operator $B_{D,\varepsilon}$ are homogenized, we consider a Γ-periodic $(n\times n)$-matrix-valued function $\widetilde{\Lambda}(\mathbf{x})$ that is the (weak) solution of the problem

$$b(\mathbf{D})^* g(\mathbf{x}) b(\mathbf{D})\widetilde{\Lambda}(\mathbf{x}) + \sum_{j=1}^{d} D_j a_j(\mathbf{x})^* = 0, \quad \int_\Omega \widetilde{\Lambda}(\mathbf{x})\, d\mathbf{x} = 0. \tag{1.33}$$

By [Su4, (7.51) and (7.52)], we have

$$\|\widetilde{\Lambda}\|_{H^1(\Omega)} \le \widetilde{M}, \tag{1.34}$$

where the constant $\widetilde{M}$ depends only on n, ρ, α_0, $\|g^{-1}\|_{L_\infty}$, $\|a_j\|_{L_\rho(\Omega)}$, $j = 1, \dots, d$, and the parameters of the lattice Γ.

Next, we define constant matrices V and W as follows:

$$V := |\Omega|^{-1} \int_\Omega (b(\mathbf{D})\Lambda(\mathbf{x}))^* g(\mathbf{x}) (b(\mathbf{D})\widetilde{\Lambda}(\mathbf{x}))\, d\mathbf{x}, \tag{1.35}$$

$$W := |\Omega|^{-1} \int_\Omega (b(\mathbf{D})\widetilde{\Lambda}(\mathbf{x}))^* g(\mathbf{x}) (b(\mathbf{D})\widetilde{\Lambda}(\mathbf{x}))\, d\mathbf{x}. \tag{1.36}$$

In $L_2(\mathcal{O};\mathbb{C}^n)$, consider the quadratic form

$$\begin{aligned} \mathfrak{b}_D^0[\mathbf{u},\mathbf{u}] = (g^0 b(\mathbf{D})\mathbf{u}, b(\mathbf{D})\mathbf{u})_{L_2(\mathcal{O})} + 2\operatorname{Re}\sum_{j=1}^{d} (\overline{a_j} D_j \mathbf{u}, \mathbf{u})_{L_2(\mathcal{O})} - 2\operatorname{Re}(V\mathbf{u}, b(\mathbf{D})\mathbf{u})_{L_2(\mathcal{O})} \\ - (W\mathbf{u},\mathbf{u})_{L_2(\mathcal{O})} + (\overline{Q}\mathbf{u},\mathbf{u})_{L_2(\mathcal{O})} + \lambda(\overline{Q_0}\mathbf{u},\mathbf{u})_{L_2(\mathcal{O})}, \quad \mathbf{u} \in H_0^1(\mathcal{O};\mathbb{C}^n). \end{aligned}$$

The following estimates were proved in [MSu3, (2.22) and (2.23)]:

$$c_* \|\mathbf{D}\mathbf{u}\|^2_{L_2(\mathcal{O})} \le \mathfrak{b}_D^0[\mathbf{u},\mathbf{u}] \le c_4 \|\mathbf{u}\|^2_{H^1(\mathcal{O})}, \quad \mathbf{u} \in H_0^1(\mathcal{O};\mathbb{C}^n), \tag{1.37}$$

$$\mathfrak{b}_D^0[\mathbf{u},\mathbf{u}] \ge c_* (\operatorname{diam} \mathcal{O})^{-2} \|\mathbf{u}\|^2_{L_2(\mathcal{O})}, \quad \mathbf{u} \in H_0^1(\mathcal{O};\mathbb{C}^n). \tag{1.38}$$

Here the constant c_4 depends only on the problem data (1.9). The selfadjoint operator in $L_2(\mathcal{O};\mathbb{C}^n)$ corresponding to the form $\mathfrak{b}_D^0$ is denoted by B_D^0. By (1.37) and (1.38),

$$\|\mathbf{u}\|_{H^1(\mathcal{O})} \le c_3 \|(B_D^0)^{1/2}\mathbf{u}\|_{L_2(\mathcal{O})}, \quad \mathbf{u} \in H_0^1(\mathcal{O};\mathbb{C}^n), \tag{1.39}$$

where c_3 is given by (1.21).

Due to the condition $\partial\mathcal{O} \in C^{1,1}$, the operator B_D^0 is given by

$$B^0 = b(\mathbf{D})^* g^0 b(\mathbf{D}) - b(\mathbf{D})^* V - V^* b(\mathbf{D}) + \sum_{j=1}^{d} (\overline{a_j + a_j^*}) D_j - W + \overline{Q} + \lambda \overline{Q_0} \tag{1.40}$$

on the domain $H^2(\mathcal{O};\mathbb{C}^n) \cap H_0^1(\mathcal{O};\mathbb{C}^n)$, and we have

$$\|(B_D^0)^{-1}\|_{L_2(\mathcal{O})\to H^2(\mathcal{O})} \le \widehat{c}. \tag{1.41}$$

Here the constant $\widehat{c}$ depends only on the problem data (1.9). To justify this, we refer the reader to the theorems about regularity of solutions of the strongly elliptic systems (see [McL, Chapter 4]).

Remark 1.6. Instead of the condition $\partial\mathcal{O} \in C^{1,1}$, one could impose the following implicit condition: a bounded Lipschitz domain $\mathcal{O} \subset \mathbb{R}^d$ is such that estimate (1.41) holds true. For such domains the results of the paper remain valid. In the case of scalar elliptic operators, wide conditions on $\partial\mathcal{O}$ ensuring estimate (1.41) can be found in [KoE] and [MaSh, Chapter 7] (in particular, it suffices to assume that $\partial\mathcal{O} \in C^\alpha$, $\alpha > 3/2$).

Denote

$$f_0 := \left(\overline{Q_0}\right)^{-1/2}. \tag{1.42}$$

By (1.22),

$$|f_0| \le \|f\|_{L_\infty} = \|Q_0^{-1}\|_{L_\infty}^{1/2}, \quad |f_0^{-1}| \le \|f^{-1}\|_{L_\infty} = \|Q_0\|_{L_\infty}^{1/2}. \tag{1.43}$$

In what follows, we shall need the operator $\widetilde{B}_D^0 := f_0 B_D^0 f_0$ corresponding to the quadratic form

$$\widetilde{\mathfrak{b}}_D^0[\mathbf{u},\mathbf{u}] := \mathfrak{b}_D^0[f_0\mathbf{u}, f_0\mathbf{u}], \quad \mathbf{u} \in H_0^1(\mathcal{O};\mathbb{C}^n). \tag{1.44}$$

Note that $(B_D^0 - \zeta\overline{Q_0})^{-1} = f_0(\widetilde{B}_D^0 - \zeta I)^{-1} f_0$.

1.7. Approximation of the generalized resolvent $(B_{D,\varepsilon} - \zeta Q_0^\varepsilon)^{-1}$**.** Now we formulate the results of the paper [MSu3], where the behavior of the generalized resolvent $(B_{D,\varepsilon} - \zeta Q_0^\varepsilon)^{-1}$ was studied. Suppose that $\zeta \in \mathbb{C} \setminus \mathbb{R}_+$ and $|\zeta| \ge 1$. The principal term of approximation of the generalized resolvent $(B_{D,\varepsilon} - \zeta Q_0^\varepsilon)^{-1}$ was found in [MSu3, Theorem 2.5]; approximation of this resolvent in the $(L_2 \to H^1)$-norm with the corrector taken into account was found in [MSu3, Theorem 2.6]; an appropriate approximation of the operator $g^\varepsilon b(\mathbf{D})(B_{D,\varepsilon} - \zeta Q_0^\varepsilon)^{-1}$ (corresponding to the "flux") was obtained in [MSu3, Proposition 10.7].

We choose numbers $\varepsilon_0, \varepsilon_1 \in (0,1]$ in accordance with the following condition.

Condition 1.7. *Let $\mathcal{O} \subset \mathbb{R}^d$ be a bounded domain. Denote*

$$(\partial\mathcal{O})_\varepsilon := \left\{\mathbf{x} \in \mathbb{R}^d \,:\, \operatorname{dist}\{\mathbf{x};\partial\mathcal{O}\} < \varepsilon\right\}.$$

Suppose that there exists a number $\varepsilon_0 \in (0,1]$ such that the strip $(\partial\mathcal{O})_{\varepsilon_0}$ can be covered by finitely many open sets that admit diffeomorphisms of class $C^{0,1}$ rectifying the boundary $\partial\mathcal{O}$. Denote $\varepsilon_1 := \varepsilon_0(1+r_1)^{-1}$, where $2r_1 = \operatorname{diam}\Omega$.

Obviously, the number ε_1 depends only on the domain $\mathcal{O}$ and the lattice Γ. Note that Condition 1.7 is ensured by the assumption that $\partial\mathcal{O}$ is Lipschitz; we have imposed a more restrictive condition $\partial\mathcal{O} \in C^{1,1}$ in order to ensure estimate (1.41).

Theorem 1.8 ([MSu3])**.** *Let $\mathcal{O} \subset \mathbb{R}^d$ be a bounded domain of class $C^{1,1}$. Suppose that the assumptions of Subsections* 1.3–1.6 *are satisfied. Let*

$$\zeta = |\zeta| e^{i\phi} \in \mathbb{C} \setminus \mathbb{R}_+, \quad |\zeta| \ge 1.$$

Denote

$$c(\phi) := \begin{cases} |\sin\phi|^{-1}, & \phi \in (0,\pi/2) \cup (3\pi/2, 2\pi), \\ 1, & \phi \in [\pi/2, 3\pi/2]. \end{cases}$$

Suppose that ε_1 is subject to Condition 1.7*. Then for $0 < \varepsilon \le \varepsilon_1$ we have*

$$\left\|(B_{D,\varepsilon} - \zeta Q_0^\varepsilon)^{-1} - (B_D^0 - \zeta\overline{Q_0})^{-1}\right\|_{L_2(\mathcal{O}) \to L_2(\mathcal{O})} \le C_1 c(\phi)^5 \varepsilon |\zeta|^{-1/2}.$$

The constant C_1 depends only on the problem data (1.9)*.*

We fix a continuous linear extension operator

$$P_\mathcal{O} \,:\, H^\sigma(\mathcal{O};\mathbb{C}^n) \to H^\sigma(\mathbb{R}^d;\mathbb{C}^n), \quad \sigma \ge 0. \tag{1.45}$$

Such a "universal" extension operator exists for any bounded Lipschitz domain (see [R]). We have

$$\|P_\mathcal{O}\|_{H^\sigma(\mathcal{O}) \to H^\sigma(\mathbb{R}^d)} \le C_\mathcal{O}^{(\sigma)}, \quad \sigma \ge 0, \tag{1.46}$$

where the constant $C_{\mathcal{O}}^{(\sigma)}$ depends only on σ and the domain $\mathcal{O}$. By $R_{\mathcal{O}}$ we denote the operator of restriction of functions in $\mathbb{R}^d$ to the domain $\mathcal{O}$. We put

$$K_D(\varepsilon;\zeta) := R_{\mathcal{O}}\big([\Lambda^\varepsilon]b(\mathbf{D}) + [\widetilde{\Lambda}^\varepsilon]\big)S_\varepsilon P_{\mathcal{O}}(B_D^0 - \zeta\overline{Q_0})^{-1}. \tag{1.47}$$

The corrector (1.47) is a continuous mapping of $L_2(\mathcal{O};\mathbb{C}^n)$ to $H^1(\mathcal{O};\mathbb{C}^n)$. This can easily be checked with the help of Proposition 1.2 and the relations $\Lambda,\ \widetilde{\Lambda} \in \widetilde{H}^1(\Omega)$. Note that $\|\varepsilon K_D(\varepsilon;\zeta)\|_{L_2(\mathcal{O})\to H^1(\mathcal{O})} = O(1)$ for small ε and fixed ζ.

Theorem 1.9 ([MSu3]). *Suppose that the assumptions of Theorem* 1.8 *are satisfied. Let* $K_D(\varepsilon;\zeta)$ *be given by* (1.47). *Then for* $\zeta \in \mathbb{C}\setminus\mathbb{R}_+$, $|\zeta|\ge 1$, *and* $0<\varepsilon\le\varepsilon_1$ *we have*

$$\begin{aligned}\big\|(B_{D,\varepsilon} - \zeta Q_0^\varepsilon)^{-1} - (B_D^0 - \zeta\overline{Q_0})^{-1} - \varepsilon K_D(\varepsilon;\zeta)\big\|_{L_2(\mathcal{O})\to H^1(\mathcal{O})}& \\ \le C_2 c(\phi)^2\varepsilon^{1/2}|\zeta|^{-1/4} + C_3 c(\phi)^4\varepsilon.&\end{aligned} \tag{1.48}$$

Let $\widetilde{g}(\mathbf{x})$ *be the matrix-valued function* (1.27). *We put*

$$G_D(\varepsilon;\zeta) := \widetilde{g}^\varepsilon S_\varepsilon b(\mathbf{D})P_{\mathcal{O}}(B_D^0 - \zeta\overline{Q_0})^{-1} + g^\varepsilon\big(b(\mathbf{D})\widetilde{\Lambda}\big)^\varepsilon S_\varepsilon P_{\mathcal{O}}(B_D^0 - \zeta\overline{Q_0})^{-1}. \tag{1.49}$$

Then for $\zeta\in\mathbb{C}\setminus\mathbb{R}_+$, $|\zeta|\ge1$, *and* $0<\varepsilon\le\varepsilon_1$ *the operator* $g^\varepsilon b(\mathbf{D})(B_{D,\varepsilon} - \zeta Q_0^\varepsilon)^{-1}$ *corresponding to the "flux" satisfies*

$$\big\|g^\varepsilon b(\mathbf{D})(B_{D,\varepsilon} - \zeta Q_0^\varepsilon)^{-1} - G_D(\varepsilon;\zeta)\big\|_{L_2(\mathcal{O})\to L_2(\mathcal{O})} \le \widetilde{C}_2 c(\phi)^{5/2}\varepsilon^{1/2}|\zeta|^{-1/4}. \tag{1.50}$$

The constants C_2, C_3, *and* $\widetilde{C}_2$ *depend only on the problem data* (1.9).

In [MSu3, Theorem 9.2], estimates in a wider domain of the spectral parameter were obtained. It was assumed that $\zeta\in\mathbb{C}\setminus[c_\flat,\infty)$, where $c_\flat$ is a common lower bound of the operators $\widetilde{B}_{D,\varepsilon}$ and $\widetilde{B}_D^0$. We put

$$c_\flat := 4^{-1}\alpha_0\|g^{-1}\|_{L_\infty}^{-1}\|Q_0\|_{L_\infty}^{-1}(\operatorname{diam}\mathcal{O})^{-2}, \tag{1.51}$$

using relations (1.18), (1.19), (1.22), (1.23), (1.38), (1.43), and (1.44).

Theorem 1.10 ([MSu3]). *Let* $\mathcal{O}\subset\mathbb{R}^d$ *be a bounded domain of class* $C^{1,1}$. *Suppose that the assumptions of Subsections* 1.3–1.6 *are satisfied. Let* $K_D(\varepsilon;\zeta)$ *be the corrector* (1.47) *and let* $G_D(\varepsilon;\zeta)$ *be the operator* (1.49). *Suppose that* $\zeta\in\mathbb{C}\setminus[c_\flat,\infty)$, *where* $c_\flat$ *is given by* (1.51). *Denote* $\psi := \arg(\zeta - c_\flat)$, $0<\psi<2\pi$, *and*

$$\varrho_\flat(\zeta) := \begin{cases} c(\psi)^2|\zeta - c_\flat|^{-2}, & |\zeta - c_\flat| < 1,\\ c(\psi)^2, & |\zeta - c_\flat|\ge 1.\end{cases} \tag{1.52}$$

Suppose that the number ε_1 *is subject to Condition* 1.7. *For* $0<\varepsilon\le\varepsilon_1$ *we have*

$$\big\|(B_{D,\varepsilon} - \zeta Q_0^\varepsilon)^{-1} - (B_D^0 - \zeta\overline{Q_0})^{-1}\big\|_{L_2(\mathcal{O})\to L_2(\mathcal{O})} \le C_4\varepsilon\varrho_\flat(\zeta), \tag{1.53}$$

$$\begin{aligned}\big\|(B_{D,\varepsilon} - \zeta Q_0^\varepsilon)^{-1} - (B_D^0 - \zeta\overline{Q_0})^{-1} - \varepsilon K_D(\varepsilon;\zeta)\big\|_{L_2(\mathcal{O})\to H^1(\mathcal{O})}& \\ \le C_5\big(\varepsilon^{1/2}\varrho_\flat(\zeta)^{1/2} + \varepsilon|1+\zeta|^{1/2}\varrho_\flat(\zeta)\big),&\end{aligned} \tag{1.54}$$

$$\begin{aligned}\big\|g^\varepsilon b(\mathbf{D})(B_{D,\varepsilon} - \zeta Q_0^\varepsilon)^{-1} - G_D(\varepsilon;\zeta)\big\|_{L_2(\mathcal{O})\to L_2(\mathcal{O})}& \\ \le \widetilde{C}_5\big(\varepsilon^{1/2}\varrho_\flat(\zeta)^{1/2} + \varepsilon|1+\zeta|^{1/2}\varrho_\flat(\zeta)\big).&\end{aligned} \tag{1.55}$$

The constants C_4, C_5, *and* $\widetilde{C}_5$ *depend only on the problem data* (1.9).

Remark 1.11. 1) In (1.52), the expression $c(\psi)^2|\zeta - c_\flat|^{-2}$ is inverse to the square of the distance from ζ to $[c_\flat,\infty)$.

2) The number (1.51) in Theorem 1.10 can be replaced by any common lower bound for the operators $\widetilde{B}_{D,\varepsilon}$ and $\widetilde{B}_D^0$. Let $\kappa>0$ be an arbitrarily small number. By (1.53) (with $\zeta=0$), $B_{D,\varepsilon}$ converges to B_D^0 in the norm-resolvent sense. Therefore, for sufficiently

small ε we can take $c_\flat = \lambda_1^0 \|Q_0\|^{-1}_{L_\infty} - \kappa$, where λ_1^0 is the first eigenvalue of the operator B_D^0. Under this choice of $c_\flat$, the constants in estimates become dependent on κ.

3) It makes sense to use estimates (1.53)–(1.55) for bounded values of $|\zeta|$ and small $\varepsilon\varrho_\flat(\zeta)$. In this case, the value $\varepsilon^{1/2}\varrho_\flat(\zeta)^{1/2} + \varepsilon|1+\zeta|^{1/2}\varrho_\flat(\zeta)$ is controlled in terms of $C\varepsilon^{1/2}\varrho_\flat(\zeta)^{1/2}$. For large $|\zeta|$ and for ϕ separated away from the points 0 and 2π, it is preferable to use Theorems 1.8 and 1.9.

1.8. Removal of the smoothing operator from the corrector. It turns out that the smoothing operator in the corrector can be removed under some additional assumptions on the matrix-valued functions $\Lambda(\mathbf{x})$ and $\widetilde{\Lambda}(\mathbf{x})$.

Condition 1.12. *Suppose that the* Γ*-periodic solution* $\Lambda(\mathbf{x})$ *of problem* (1.25) *is bounded, i.e.,* $\Lambda \in L_\infty(\mathbb{R}^d)$.

Some cases where Condition 1.12 is satisfied were distinguished in [BSu4, Lemma 8.7].

Proposition 1.13 ([BSu4])**.** *Suppose that at least one of the following assumptions is satisfied*:
1°) $d \le 2$;
2°) *the dimension* $d \ge 1$ *is arbitrary, and the differential expression* A_ε *is given by* $A_\varepsilon = \mathbf{D}^* g^\varepsilon(\mathbf{x})\mathbf{D}$*, where* $g(\mathbf{x})$ *is a symmetric matrix with real entries*;
3°) *the dimension* d *is arbitrary, and* $g^0 = \underline{g}$*, i.e., relations* (1.32) *are satisfied.*
Then Condition 1.12 *is fulfilled.*

In order to remove S_ε from the term of the corrector involving $\widetilde{\Lambda}^\varepsilon$, it suffices to impose the following condition.

Condition 1.14. *Suppose that the* Γ*-periodic solution* $\widetilde{\Lambda}(\mathbf{x})$ *of problem* (1.33) *is such that*

$$\widetilde{\Lambda} \in L_p(\Omega), \quad p = 2 \text{ for } d = 1, \quad p > 2 \text{ for } d = 2, \quad p = d \text{ for } d \ge 3.$$

The following result was checked in [Su4, Proposition 8.11].

Proposition 1.15 ([Su4])**.** *Suppose that at least one of the following assumptions is satisfied*:
1°) $d \le 4$;
2°) *the dimension* d *is arbitrary, and* A_ε *is given by* $A_\varepsilon = \mathbf{D}^* g^\varepsilon(\mathbf{x})\mathbf{D}$*, where* $g(\mathbf{x})$ *is a symmetric matrix with real entries.*
Then Condition 1.14 *is satisfied.*

Remark 1.16. If $A_\varepsilon = \mathbf{D}^* g^\varepsilon(\mathbf{x})\mathbf{D}$, where $g(\mathbf{x})$ is a symmetric matrix with real entries, then from [LaU, Chapter III, Theorem 13.1] it follows that $\Lambda, \widetilde{\Lambda} \in L_\infty$ and the norm $\|\Lambda\|_{L_\infty}$ does not exceed a constant depending on d, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, and Ω, while the norm $\|\widetilde{\Lambda}\|_{L_\infty}$ is controlled in terms of d, ρ, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, $\|a_j\|_{L_\rho(\Omega)}$, $j = 1, \dots, d$, and Ω. In this case, Conditions 1.12 and 1.14 are fulfilled.

In [MSu3, Theorem 7.6], the following result was obtained.

Theorem 1.17 ([MSu3])**.** *Under the assumptions of Theorem* 1.9*, suppose that* $\Lambda(\mathbf{x})$ *is subject to Condition* 1.12*, and* $\widetilde{\Lambda}(\mathbf{x})$ *satisfies Condition* 1.14*. We put*

$$K_D^0(\varepsilon;\zeta) := (\Lambda^\varepsilon b(\mathbf{D}) + \widetilde{\Lambda}^\varepsilon)(B_D^0 - \zeta\overline{Q_0})^{-1}, \tag{1.56}$$

$$G_D^0(\varepsilon;\zeta) := \widetilde{g}^\varepsilon b(\mathbf{D})(B_D^0 - \zeta\overline{Q_0})^{-1} + g^\varepsilon\big(b(\mathbf{D})\widetilde{\Lambda}\big)^\varepsilon(B_D^0 - \zeta\overline{Q_0})^{-1}. \tag{1.57}$$

Then for $\zeta \in \mathbb{C} \setminus \mathbb{R}_+$, $|\zeta| \geq 1$, *and* $0 < \varepsilon \leq \varepsilon_1$ *we have*

$$\begin{aligned} &\left\|(B_{D,\varepsilon} - \zeta Q_0^\varepsilon)^{-1} - (B_D^0 - \zeta \overline{Q_0})^{-1} - \varepsilon K_D^0(\varepsilon;\zeta)\right\|_{L_2(\mathcal{O})\to H^1(\mathcal{O})} \\ &\qquad\qquad\qquad\qquad\qquad\qquad \leq C_2 c(\phi)^2 \varepsilon^{1/2}|\zeta|^{-1/4} + C_6 c(\phi)^4 \varepsilon, \\ &\left\|g^\varepsilon b(\mathbf{D})(B_{D,\varepsilon} - \zeta Q_0^\varepsilon)^{-1} - G_D^0(\varepsilon;\zeta)\right\|_{L_2(\mathcal{O})\to L_2(\mathcal{O})} \leq \widetilde{C}_2 c(\phi)^2 \varepsilon^{1/2}|\zeta|^{-1/4} + \widetilde{C}_6 c(\phi)^4 \varepsilon. \end{aligned}$$

Here the constants C_2, $\widetilde{C}_2$ *are as in* (1.48) *and* (1.50). *The constants* C_6 *and* $\widetilde{C}_6$ *depend only on the problem data* (1.9), p, *and the norms* $\|\Lambda\|_{L_\infty}$, $\|\widetilde{\Lambda}\|_{L_p(\Omega)}$.

Approximations in a wider domain of the spectral parameter were found in [MSu3, Theorem 9.8].

Theorem 1.18 ([MSu3]). *Under the assumptions of Theorem* 1.10 *and Conditions* 1.12, 1.14, *let* $K_D^0(\varepsilon;\zeta)$ *be the corrector* (1.56). *Let* $G_D^0(\varepsilon;\zeta)$ *be given by* (1.57). *Then for* $0 < \varepsilon \leq \varepsilon_1$ *and* $\zeta \in \mathbb{C} \setminus [c_\flat, \infty)$ *we have*

$$\begin{aligned} &\left\|(B_{D,\varepsilon} - \zeta Q_0^\varepsilon)^{-1} - (B_D^0 - \zeta \overline{Q_0})^{-1} - \varepsilon K_D^0(\varepsilon;\zeta)\right\|_{L_2(\mathcal{O})\to H^1(\mathcal{O})} \\ &\qquad\qquad\qquad\qquad\qquad\qquad \leq C_7\big(\varepsilon^{1/2}\varrho_\flat(\zeta)^{1/2} + \varepsilon|1+\zeta|^{1/2}\varrho_\flat(\zeta)\big), \\ &\left\|g^\varepsilon b(\mathbf{D})(B_{D,\varepsilon} - \zeta Q_0^\varepsilon)^{-1} - G_D^0(\varepsilon;\zeta)\right\|_{L_2(\mathcal{O})\to L_2(\mathcal{O})} \leq \widetilde{C}_7\big(\varepsilon^{1/2}\varrho_\flat(\zeta)^{1/2} + \varepsilon|1+\zeta|^{1/2}\varrho_\flat(\zeta)\big). \end{aligned}$$

Here the constants C_7 *and* $\widetilde{C}_7$ *depend only on the problem data* (1.9), p, *and the norms* $\|\Lambda\|_{L_\infty}$, $\|\widetilde{\Lambda}\|_{L_p(\Omega)}$.

Recalling [MSu3, Remarks 7.9 and 9.9], we observe the following.

Remark 1.19. If only Condition 1.12 (respectively, Condition 1.14) is satisfied, then the smoothing operator S_ε can be removed from the term of the corrector involving Λ^ε (respectively, from the term involving $\widetilde{\Lambda}^\varepsilon$).

1.9. The case where the corrector is equal to zero. Suppose that $g^0 = \overline{g}$, i.e., relations (1.31) hold true. Then the Γ-periodic solution of problem (1.25) is equal to zero: $\Lambda(\mathbf{x}) = 0$. Suppose in addition that

$$\sum_{j=1}^{d} D_j a_j(\mathbf{x})^* = 0. \tag{1.58}$$

Then the Γ-periodic solution of problem (1.33) is also equal to zero: $\widetilde{\Lambda}(\mathbf{x}) = 0$. By [MSu3, Propositions 7.10 and 9.12], in this case the following $(L_2 \to H^1)$-estimate of sharp order $O(\varepsilon)$ is valid.

Proposition 1.20 ([MSu3]). *Suppose that relations* (1.31) *and* (1.58) *are satisfied.*
1°. *Under the assumptions of Theorem* 1.8, *for* $\zeta \in \mathbb{C} \setminus \mathbb{R}_+$, $|\zeta| \geq 1$, *and* $0 < \varepsilon \leq 1$ *we have*

$$\left\|(B_{D,\varepsilon} - \zeta Q_0^\varepsilon)^{-1} - (B_D^0 - \zeta \overline{Q_0})^{-1}\right\|_{L_2(\mathcal{O})\to H^1(\mathcal{O})} \leq C_8 c(\phi)^4 \varepsilon. \tag{1.59}$$

2°. *Under the assumptions of Theorem* 1.10, *for* $\zeta \in \mathbb{C} \setminus [c_\flat, \infty)$ *and* $0 < \varepsilon \leq 1$ *we have*

$$\left\|(B_{D,\varepsilon} - \zeta Q_0^\varepsilon)^{-1} - (B_D^0 - \zeta \overline{Q_0})^{-1}\right\|_{L_2(\mathcal{O})\to H^1(\mathcal{O})} \leq (C_9 + C_{10}|1+\zeta|^{1/2})\varepsilon\varrho_\flat(\zeta). \tag{1.60}$$

The constants C_8, C_9, *and* C_{10} *depend only on the problem data* (1.9).

1.10. Estimates in a strictly interior subdomain. It is possible to improve the H^1-estimates in a strictly interior subdomain $\mathcal{O}'$ of the domain $\mathcal{O}$. In Theorems 8.1 and 9.14 of [MSu3], the following result was obtained.

Theorem 1.21 ([MSu3]). *Let $\mathcal{O}'$ be a strictly interior subdomain of the domain $\mathcal{O}$. Denote*

$$\delta := \min\{1; \operatorname{dist}\{\mathcal{O}'; \partial\mathcal{O}\}\}. \tag{1.61}$$

1°. *Under the assumptions of Theorem* 1.9, *for $\zeta \in \mathbb{C} \setminus \mathbb{R}_+$, $|\zeta| \geq 1$, and $0 < \varepsilon \leq \varepsilon_1$ we have*

$$\begin{aligned}
&\big\|(B_{D,\varepsilon} - \zeta Q_0^\varepsilon)^{-1} - (B_D^0 - \zeta\overline{Q_0})^{-1} - \varepsilon K_D(\varepsilon;\zeta)\big\|_{L_2(\mathcal{O})\to H^1(\mathcal{O}')} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \leq c(\phi)^6 \varepsilon (C_{11}'|\zeta|^{-1/2}\delta^{-1} + C_{11}''), \\
&\big\|g^\varepsilon b(\mathbf{D})(B_{D,\varepsilon} - \zeta Q_0^\varepsilon)^{-1} - G_D(\varepsilon;\zeta)\big\|_{L_2(\mathcal{O})\to L_2(\mathcal{O}')} \leq c(\phi)^6 \varepsilon \big(\widetilde{C}_{11}'|\zeta|^{-1/2}\delta^{-1} + \widetilde{C}_{11}''\big).
\end{aligned}$$

The constants C_{11}', C_{11}'', $\widetilde{C}_{11}'$, and $\widetilde{C}_{11}''$ depend only on the problem data (1.9).
2°. *Under the assumptions of Theorem* 1.10, *for $\zeta \in \mathbb{C} \setminus [c_\flat, \infty)$ and $0 < \varepsilon \leq \varepsilon_1$ we have*

$$\begin{aligned}
&\big\|(B_{D,\varepsilon} - \zeta Q_0^\varepsilon)^{-1} - (B_D^0 - \zeta\overline{Q_0})^{-1} - \varepsilon K_D(\varepsilon;\zeta)\big\|_{L_2(\mathcal{O})\to H^1(\mathcal{O}')} \\
&\qquad\qquad \leq \varepsilon\big(C_{12}'\delta^{-1}\varrho_\flat(\zeta)^{1/2} + C_{12}''|1+\zeta|^{1/2}\varrho_\flat(\zeta)\big),
\end{aligned} \tag{1.62}$$

$$\begin{aligned}
&\big\|g^\varepsilon b(\mathbf{D})(B_{D,\varepsilon} - \zeta Q_0^\varepsilon)^{-1} - G_D(\varepsilon;\zeta)\big\|_{L_2(\mathcal{O})\to L_2(\mathcal{O}')} \\
&\qquad\qquad \leq \varepsilon\big(\widetilde{C}_{12}'\delta^{-1}\varrho_\flat(\zeta)^{1/2} + \widetilde{C}_{12}''|1+\zeta|^{1/2}\varrho_\flat(\zeta)\big).
\end{aligned} \tag{1.63}$$

The constants C_{12}', C_{12}'', and $\widetilde{C}_{12}'$, $\widetilde{C}_{12}''$ depend only on the problem data (1.9).

If the matrix-valued functions $\Lambda(\mathbf{x})$ and $\widetilde{\Lambda}(\mathbf{x})$ satisfy some additional assumptions, this result remains true with a simpler corrector. Approximations for $\zeta \in \mathbb{C} \setminus \mathbb{R}_+$, $|\zeta| \geq 1$, were found in [MSu3, Theorem 8.2].

Theorem 1.22 ([MSu3]). *Suppose that the assumptions of Theorem* 1.21(1°) *are satisfied. Suppose that the matrix-valued functions $\Lambda(\mathbf{x})$ and $\widetilde{\Lambda}(\mathbf{x})$ satisfy Conditions* 1.12 *and* 1.14, *respectively. Let $K_D^0(\varepsilon;\zeta)$ and $G_D^0(\varepsilon;\zeta)$ be the operators defined by* (1.56) *and* (1.57). *Then for $0 < \varepsilon \leq \varepsilon_1$ and $\zeta \in \mathbb{C} \setminus \mathbb{R}_+$, $|\zeta| \geq 1$, we have*

$$\begin{aligned}
&\big\|(B_{D,\varepsilon} - \zeta Q_0^\varepsilon)^{-1} - (B_D^0 - \zeta\overline{Q_0})^{-1} - \varepsilon K_D^0(\varepsilon;\zeta)\big\|_{L_2(\mathcal{O})\to H^1(\mathcal{O}')} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \leq c(\phi)^6 \varepsilon (C_{11}'|\zeta|^{-1/2}\delta^{-1} + C_{13}), \\
&\big\|g^\varepsilon b(\mathbf{D})(B_{D,\varepsilon} - \zeta Q_0^\varepsilon)^{-1} - G_D^0(\varepsilon;\zeta)\big\|_{L_2(\mathcal{O})\to L_2(\mathcal{O}')} \leq c(\phi)^6 \varepsilon (\widetilde{C}_{11}'|\zeta|^{-1/2}\delta^{-1} + \widetilde{C}_{13}).
\end{aligned}$$

The constants C_{11}' and $\widetilde{C}_{11}'$ are as in Theorem 1.21. *The constants C_{13} and $\widetilde{C}_{13}$ depend on the problem data* (1.9), *p, and the norms $\|\Lambda\|_{L_\infty}$, $\|\widetilde{\Lambda}\|_{L_p(\Omega)}$.*

Approximations in a wider domain of the parameter ζ were obtained in [MSu3, Theorem 9.15].

Theorem 1.23 ([MSu3]). *Suppose that the assumptions of Theorem* 1.21(2°) *are satisfied. Suppose that the matrix-valued functions $\Lambda(\mathbf{x})$ and $\widetilde{\Lambda}(\mathbf{x})$ are subject to Conditions* 1.12 *and* 1.14, *respectively. Let $K_D^0(\varepsilon;\zeta)$ be the corrector* (1.56), *and let $G_D^0(\varepsilon;\zeta)$*

be the operator (1.57). *Then for* $\zeta \in \mathbb{C} \setminus [c_\flat, \infty)$ *and* $0 < \varepsilon \le \varepsilon_1$ *we have*

$$\begin{aligned} &\big\|(B_{D,\varepsilon} - \zeta Q_0^\varepsilon)^{-1} - (B_D^0 - \zeta \overline{Q_0})^{-1} - \varepsilon K_D^0(\varepsilon;\zeta)\big\|_{L_2(\mathcal{O}) \to H^1(\mathcal{O}')} \\ &\qquad\qquad \le \varepsilon\big(C_{12}'\delta^{-1}\varrho_\flat(\zeta)^{1/2} + C_{14}|1+\zeta|^{1/2}\varrho_\flat(\zeta)\big), \\ &\big\|g^\varepsilon b(\mathbf{D})(B_{D,\varepsilon} - \zeta Q_0^\varepsilon)^{-1} - G_D^0(\varepsilon;\zeta)\big\|_{L_2(\mathcal{O}) \to L_2(\mathcal{O}')} \\ &\qquad\qquad \le \varepsilon\big(\widetilde{C}_{12}'\delta^{-1}\varrho_\flat(\zeta)^{1/2} + \widetilde{C}_{14}|1+\zeta|^{1/2}\varrho_\flat(\zeta)\big). \end{aligned}$$

Here the constants C_{12}' *and* $\widetilde{C}_{12}'$ *are as in* (1.62) *and* (1.63). *The constants* C_{14} *and* $\widetilde{C}_{14}$ *depend on the problem data* (1.9), p, *and the norms* $\|\Lambda\|_{L_\infty}$, $\|\widetilde{\Lambda}\|_{L_p(\Omega)}$.

§2. Statement of the problem. Main results

2.1. Statement of the problem. We study the behavior of the solution of the first initial boundary-value problem

$$\begin{cases} Q_0^\varepsilon(\mathbf{x})\frac{\partial \mathbf{u}_\varepsilon}{\partial t}(\mathbf{x},t) = -B_\varepsilon \mathbf{u}_\varepsilon(\mathbf{x},t), & \mathbf{x} \in \mathcal{O}, \quad t > 0; \\ \mathbf{u}_\varepsilon(\,\cdot\,,t)|_{\partial\mathcal{O}} = 0, & t > 0; \\ Q_0^\varepsilon(\mathbf{x})\mathbf{u}_\varepsilon(\mathbf{x},0) = \boldsymbol{\varphi}(\mathbf{x}), & \mathbf{x} \in \mathcal{O}. \end{cases} \tag{2.1}$$

Here $\boldsymbol{\varphi} \in L_2(\mathcal{O};\mathbb{C}^n)$. (The solution is understood in the weak sense.) Let us find a relationship between $\mathbf{u}_\varepsilon(\,\cdot\,,t)$ and $\boldsymbol{\varphi}$. By (1.22), the function $\mathbf{s}_\varepsilon(\mathbf{x},t) := \big(f^\varepsilon(\mathbf{x})\big)^{-1}\mathbf{u}_\varepsilon(\mathbf{x},t)$ is the solution of the problem

$$\begin{cases} \frac{\partial \mathbf{s}_\varepsilon}{\partial t}(\mathbf{x},t) = -\widetilde{B}_\varepsilon \mathbf{s}_\varepsilon(\mathbf{x},t), & \mathbf{x} \in \mathcal{O}, \quad t > 0; \\ \mathbf{s}_\varepsilon(\,\cdot\,,t)|_{\partial\mathcal{O}} = 0, & t > 0; \\ \mathbf{s}_\varepsilon(\mathbf{x},0) = (f^\varepsilon(\mathbf{x}))^*\boldsymbol{\varphi}(\mathbf{x}), & \mathbf{x} \in \mathcal{O}. \end{cases}$$

Then $\mathbf{s}_\varepsilon(\,\cdot\,,t) = e^{-\widetilde{B}_{D,\varepsilon}t}(f^\varepsilon)^*\boldsymbol{\varphi}$ and $\mathbf{u}_\varepsilon(\,\cdot\,,t) = f^\varepsilon \mathbf{s}_\varepsilon(\,\cdot\,,t) = f^\varepsilon e^{-\widetilde{B}_{D,\varepsilon}t}(f^\varepsilon)^*\boldsymbol{\varphi}$.

Our goal is to study the behavior of the generalized solution $\mathbf{u}_\varepsilon$ of the first initial boundary-value problem (2.1) in the small period limit. In other words, we are interested in approximations of the sandwiched operator exponential $f^\varepsilon e^{-\widetilde{B}_{D,\varepsilon}t}(f^\varepsilon)^*$ for small ε.

The corresponding effective problem is given by

$$\begin{cases} \overline{Q_0}\frac{\partial \mathbf{u}_0}{\partial t}(\mathbf{x},t) = -B^0 \mathbf{u}_0(\mathbf{x},t), & \mathbf{x} \in \mathcal{O}, \quad t > 0; \\ \mathbf{u}_0(\,\cdot\,,t)|_{\partial\mathcal{O}} = 0, & t > 0; \\ \overline{Q_0}\mathbf{u}_0(\mathbf{x},0) = \boldsymbol{\varphi}(\mathbf{x}), & \mathbf{x} \in \mathcal{O}. \end{cases} \tag{2.2}$$

By (1.42), the solution of the effective problem is given by

$$\mathbf{u}_0(\,\cdot\,,t) = f_0 e^{-\widetilde{B}_D^0 t} f_0 \boldsymbol{\varphi}(\,\cdot\,). \tag{2.3}$$

2.2. The properties of the operator exponential. We prove the following simple statement about estimates for the operator exponentials $e^{-\widetilde{B}_{D,\varepsilon}t}$ and $e^{-\widetilde{B}_D^0 t}$.

Lemma 2.1. *For* $0 < \varepsilon \le 1$ *we have*:

$$\big\|e^{-\widetilde{B}_{D,\varepsilon}t}\big\|_{L_2(\mathcal{O}) \to L_2(\mathcal{O})} \le e^{-c_\flat t}, \quad t \ge 0, \tag{2.4}$$

$$\big\|f^\varepsilon e^{-\widetilde{B}_{D,\varepsilon}t}\big\|_{L_2(\mathcal{O}) \to H^1(\mathcal{O})} \le c_3 t^{-1/2} e^{-c_\flat t/2}, \quad t > 0, \tag{2.5}$$

$$\big\|e^{-\widetilde{B}_D^0 t}\big\|_{L_2(\mathcal{O}) \to L_2(\mathcal{O})} \le e^{-c_\flat t}, \quad t \ge 0, \tag{2.6}$$

$$\big\|f_0 e^{-\widetilde{B}_D^0 t}\big\|_{L_2(\mathcal{O}) \to H^1(\mathcal{O})} \le c_3 t^{-1/2} e^{-c_\flat t/2}, \quad t > 0, \tag{2.7}$$

$$\big\|f_0 e^{-\widetilde{B}_D^0 t}\big\|_{L_2(\mathcal{O}) \to H^2(\mathcal{O})} \le \widetilde{c}\, t^{-1} e^{-c_\flat t/2}, \quad t > 0. \tag{2.8}$$

Here the constants c_3 *and* $c_\flat$ *are given by* (1.21) *and* (1.51)*. The constant* $\widetilde{c}$ *depends only on the problem data* (1.9)*.*

Proof. Since the number $c_\flat$ defined by (1.51) is a common lower bound for the operators $\widetilde{B}_{D,\varepsilon}$ and $\widetilde{B}_D^0$, estimates (2.4) and (2.6) are obvious.

By (1.20) and (1.23),

$$
\begin{aligned}
\big\| f^\varepsilon e^{-\widetilde{B}_{D,\varepsilon} t} \big\|_{L_2(\mathcal{O})\to H^1(\mathcal{O})} &\le c_3 \big\| B_{D,\varepsilon}^{1/2} f^\varepsilon e^{-\widetilde{B}_{D,\varepsilon} t} \big\|_{L_2(\mathcal{O})\to L_2(\mathcal{O})} \\
&= c_3 \big\| \widetilde{B}_{D,\varepsilon}^{1/2} e^{-\widetilde{B}_{D,\varepsilon} t} \big\|_{L_2(\mathcal{O})\to L_2(\mathcal{O})}.
\end{aligned}
\tag{2.9}
$$

Since $\widetilde{B}_{D,\varepsilon} \ge c_\flat I$, we have

$$
\begin{aligned}
\big\| \widetilde{B}_{D,\varepsilon}^{1/2} e^{-\widetilde{B}_{D,\varepsilon} t} \big\|_{L_2(\mathcal{O})\to L_2(\mathcal{O})} &\le \sup_{x \ge c_\flat} x^{1/2} e^{-xt} \\
&\le e^{-c_\flat t/2} \sup_{x\ge c_\flat} x^{1/2} e^{-xt/2} \le t^{-1/2} e^{-c_\flat t/2}.
\end{aligned}
\tag{2.10}
$$

Combining this with (2.9), we obtain inequality (2.5). Similarly, (1.39) and (1.44) imply estimate (2.7).

From (1.41), (1.43), and the identity $\widetilde{B}_D^0 = f_0 B_D^0 f_0$ it follows that

$$
\begin{aligned}
\big\| f_0 e^{-\widetilde{B}_D^0 t} \big\|_{L_2(\mathcal{O})\to H^2(\mathcal{O})} &\le \widehat{c} \big\| B_D^0 f_0 e^{-\widetilde{B}_D^0 t} \big\|_{L_2(\mathcal{O})\to L_2(\mathcal{O})} \\
&\le \widehat{c} \| f^{-1} \|_{L_\infty} \big\| \widetilde{B}_D^0 e^{-\widetilde{B}_D^0 t} \big\|_{L_2(\mathcal{O})\to L_2(\mathcal{O})}.
\end{aligned}
$$

Hence,

$$
\big\| f_0 e^{-\widetilde{B}_D^0 t} \big\|_{L_2(\mathcal{O})\to H^2(\mathcal{O})} \le \widehat{c} \| f^{-1} \|_{L_\infty} \sup_{x\ge c_\flat} x e^{-xt} \le \widehat{c} \| f^{-1} \|_{L_\infty} t^{-1} e^{-c_\flat t/2}.
$$

This proves estimate (2.8) with the constant $\widetilde{c} = \widehat{c} \| f^{-1} \|_{L_\infty}$. ☐

2.3. Approximation of the solution in $L_2(\mathcal{O};\mathbb{C}^n)$.

Theorem 2.2. *Let* $\mathcal{O} \subset \mathbb{R}^d$ *be a bounded domain of class* $C^{1,1}$*. Suppose that the assumptions of Subsections* 1.3–1.6 *are satisfied. Let* $B_{D,\varepsilon}$ *be the operator in* $L_2(\mathcal{O};\mathbb{C}^n)$ *corresponding to the quadratic form* (1.11)*. Let* B_D^0 *be the operator in* $L_2(\mathcal{O};\mathbb{C}^n)$ *given by the expression* (1.40) *on* $H^2(\mathcal{O};\mathbb{C}^n) \cap H_0^1(\mathcal{O};\mathbb{C}^n)$*. We put* $\widetilde{B}_{D,\varepsilon} = (f^\varepsilon)^* B_{D,\varepsilon} f^\varepsilon$ *and* $\widetilde{B}_D^0 = f_0 B_D^0 f_0$*, where the matrix-valued function* f *is defined by* (1.22)*, and the matrix* f_0 *is given by* (1.42)*. Let* $\mathbf{u}_\varepsilon$ *be the solution of problem* (2.1)*, and let* $\mathbf{u}_0$ *be the solution of the corresponding effective problem* (2.2)*. Suppose that the number* ε_1 *is subject to Condition* 1.7*. Then for* $0 < \varepsilon \le \varepsilon_1$ *we have*

$$
\| \mathbf{u}_\varepsilon(\,\cdot\,,t) - \mathbf{u}_0(\,\cdot\,,t) \|_{L_2(\mathcal{O})} \le C_{15} \varepsilon (t+\varepsilon^2)^{-1/2} e^{-c_\flat t/2} \| \boldsymbol{\varphi} \|_{L_2(\mathcal{O})}, \quad t \ge 0.
$$

In operator terms,

$$
\begin{aligned}
\big\| f^\varepsilon e^{-\widetilde{B}_{D,\varepsilon} t} (f^\varepsilon)^* - f_0 e^{-\widetilde{B}_D^0 t} f_0 \big\|_{L_2(\mathcal{O})\to L_2(\mathcal{O})} \le C_{15} \varepsilon (t+\varepsilon^2)^{-1/2} e^{-c_\flat t/2}&, \\
t \ge 0&.
\end{aligned}
\tag{2.11}
$$

Here the constant $c_\flat$ *is given by* (1.51)*. The constant* C_{15} *depends only on the problem data* (1.9)*.*

Proof. The proof is based on the results of Theorems 1.8, 1.10, and representations for the exponentials of the operators $\widetilde{B}_{D,\varepsilon}$, $\widetilde{B}_D^0$ in terms of the contour integrals of the corresponding resolvents.

We have (see, e.g., [Ka, Chapter IX, Section 1.6])

$$e^{-\widetilde{B}_{D,\varepsilon}t} = -\frac{1}{2\pi i}\int_\gamma e^{-\zeta t}(\widetilde{B}_{D,\varepsilon} - \zeta I)^{-1}\, d\zeta, \quad t > 0. \tag{2.12}$$

Here $\gamma \subset \mathbb{C}$ is a contour enclosing the spectrum of the operator $\widetilde{B}_{D,\varepsilon}$ in positive direction. The exponential of the operator $\widetilde{B}_D^0$ satisfies a similar representation. Since the constant (1.51) is a common lower bound of the operators $\widetilde{B}_{D,\varepsilon}$ and $\widetilde{B}_D^0$, it is convenient to choose the contour of integration as follows:

$$\gamma = \{\zeta \in \mathbb{C} : \operatorname{Im}\zeta \geq 0, \operatorname{Re}\zeta = \operatorname{Im}\zeta + c_\flat/2\} \cup \{\zeta \in \mathbb{C} : \operatorname{Im}\zeta \leq 0, \operatorname{Re}\zeta = -\operatorname{Im}\zeta + c_\flat/2\}.$$

Multiplying (2.12) by f^ε from the left and by $(f^\varepsilon)^*$ from the right and using identity (1.24), we obtain

$$f^\varepsilon e^{-\widetilde{B}_{D,\varepsilon}t}(f^\varepsilon)^* = -\frac{1}{2\pi i}\int_\gamma e^{-\zeta t}(B_{D,\varepsilon} - \zeta Q_0^\varepsilon)^{-1}\, d\zeta, \quad t > 0.$$

Similarly,

$$f_0 e^{-\widetilde{B}_D^0 t} f_0 = -\frac{1}{2\pi i}\int_\gamma e^{-\zeta t}(B_D^0 - \zeta\overline{Q_0})^{-1}\, d\zeta, \quad t > 0.$$

Hence,

$$\begin{aligned} &f^\varepsilon e^{-\widetilde{B}_{D,\varepsilon}t}(f^\varepsilon)^* - f_0 e^{-\widetilde{B}_D^0 t} f_0 \\ &= -\frac{1}{2\pi i}\int_\gamma e^{-\zeta t}\left((B_{D,\varepsilon} - \zeta Q_0^\varepsilon)^{-1} - (B_D^0 - \zeta\overline{Q_0})^{-1}\right) d\zeta. \end{aligned} \tag{2.13}$$

By Theorems 1.8 and 1.10, we estimate the difference of the generalized resolvents for $\zeta \in \gamma$ uniformly in $\arg\zeta$. Recall the notation $\psi = \arg(\zeta - c_\flat)$. Note that for $\zeta \in \gamma$ and $\psi = \pi/2$ or $\psi = 3\pi/2$ we have $|\zeta| = \sqrt{5}c_\flat/2$. We apply Theorem 1.10 for $\zeta \in \gamma$ with $|\zeta| \leq \check{c}$, where

$$\check{c} := \max\{1; \sqrt{5}c_\flat/2\}. \tag{2.14}$$

Obviously, $\psi \in (\pi/4, 7\pi/4)$ on the contour γ, and

$$\rho_\flat(\zeta) \leq 2\max\{1; 8c_\flat^{-2}\} =: \mathfrak{C}, \quad \zeta \in \gamma. \tag{2.15}$$

Therefore, (1.53) implies

$$\begin{aligned} &\left\|(B_{D,\varepsilon} - \zeta Q_0^\varepsilon)^{-1} - (B_D^0 - \zeta\overline{Q_0})^{-1}\right\|_{L_2(\mathcal{O})\to L_2(\mathcal{O})} \leq C_4\mathfrak{C}\varepsilon \leq C'_{15}|\zeta|^{-1/2}\varepsilon, \\ &\qquad \zeta \in \gamma, \quad |\zeta| \leq \check{c}, \quad 0 < \varepsilon \leq \varepsilon_1; \quad C'_{15} := C_4\mathfrak{C}\check{c}^{1/2}. \end{aligned} \tag{2.16}$$

For the other $\zeta \in \gamma$, we have

$$|\sin\phi| \geq 5^{-1/2}, \quad \zeta \in \gamma, \quad |\zeta| > \check{c}, \tag{2.17}$$

and, by Theorem 1.8,

$$\begin{aligned} &\left\|(B_{D,\varepsilon} - \zeta Q_0^\varepsilon)^{-1} - (B_D^0 - \zeta\overline{Q_0})^{-1}\right\|_{L_2(\mathcal{O})\to L_2(\mathcal{O})} \leq C''_{15}|\zeta|^{-1/2}\varepsilon, \\ &\qquad \zeta \in \gamma, \quad |\zeta| > \check{c}, \quad 0 < \varepsilon \leq \varepsilon_1, \end{aligned} \tag{2.18}$$

where $C''_{15} := 5^{5/2}C_1$. As a result, combining (2.16) and (2.18), for $0 < \varepsilon \leq \varepsilon_1$ we have

$$\left\|(B_{D,\varepsilon} - \zeta Q_0^\varepsilon)^{-1} - (B_D^0 - \zeta\overline{Q_0})^{-1}\right\|_{L_2(\mathcal{O})\to L_2(\mathcal{O})} \leq \widehat{C}_{15}|\zeta|^{-1/2}\varepsilon, \quad \zeta \in \gamma, \tag{2.19}$$

where $\widehat{C}_{15} := \max\{C'_{15}; C''_{15}\}$.

From (2.13) and (2.19) it follows that

$$\left\|f^\varepsilon e^{-\widetilde{B}_{D,\varepsilon}t}(f^\varepsilon)^* - f_0 e^{-\widetilde{B}_D^0 t} f_0\right\|_{L_2(\mathcal{O})\to L_2(\mathcal{O})} \leq 2\pi^{-1}\widehat{C}_{15}\varepsilon t^{-1/2}\Gamma(1/2)e^{-c_\flat t/2}.$$

Taking into account that $\Gamma(1/2) = \pi^{1/2}$, we find

$$\begin{aligned}\big\|f^\varepsilon e^{-\widetilde{B}_{D,\varepsilon} t}(f^\varepsilon)^* - f_0 e^{-\widetilde{B}^0_D t} f_0\big\|_{L_2(\mathcal{O})\to L_2(\mathcal{O})} &\le 2\pi^{-1/2}\widehat{C}_{15}\varepsilon t^{-1/2} e^{-c_\flat t/2} \\ &\le \check{C}_{15}\varepsilon(t+\varepsilon^2)^{-1/2} e^{-c_\flat t/2}, \quad t \ge \varepsilon^2,\end{aligned} \tag{2.20}$$

where $\check{C}_{15} := 2\sqrt{2}\pi^{-1/2}\widehat{C}_{15}$. For $t \le \varepsilon^2$ we use the rough estimate

$$\begin{aligned}\big\|f^\varepsilon e^{-\widetilde{B}_{D,\varepsilon} t}(f^\varepsilon)^* - f_0 e^{-\widetilde{B}^0_D t} f_0\big\|_{L_2(\mathcal{O})\to L_2(\mathcal{O})} &\le 2\|f\|^2_{L_\infty} e^{-c_\flat t} \\ &\le 2\sqrt{2}\|f\|^2_{L_\infty}\varepsilon(t+\varepsilon^2)^{-1/2} e^{-c_\flat t/2}, \quad t \le \varepsilon^2.\end{aligned} \tag{2.21}$$

Relations (2.20) and (2.21) imply the required inequality (2.11) with the constant $C_{15} := \max\{\check{C}_{15}; 2\sqrt{2}\|f\|^2_{L_\infty}\}$. $\square$

2.4. Approximation of the solution in $H^1(\mathcal{O};\mathbb{C}^n)$. We introduce a *corrector*

$$\mathcal{K}_D(t;\varepsilon) := R_{\mathcal{O}}\big([\Lambda^\varepsilon]S_\varepsilon b(\mathbf{D}) + [\widetilde{\Lambda}^\varepsilon]S_\varepsilon\big)P_{\mathcal{O}} f_0 e^{-\widetilde{B}^0_D t} f_0. \tag{2.22}$$

For $t > 0$ the operator (2.22) is a continuous mapping of $L_2(\mathcal{O};\mathbb{C}^n)$ to $H^1(\mathcal{O};\mathbb{C}^n)$. Indeed, by (2.8), for $t > 0$ the operator $f_0 e^{-\widetilde{B}^0_D t} f_0$ acts continuously from $L_2(\mathcal{O};\mathbb{C}^n)$ to $H^2(\mathcal{O};\mathbb{C}^n)$. Hence, the operator $b(\mathbf{D})P_{\mathcal{O}} f_0 e^{-\widetilde{B}^0_D t} f_0$ is continuous from $L_2(\mathcal{O};\mathbb{C}^n)$ to $H^1(\mathbb{R}^d;\mathbb{C}^m)$. Obviously, the operator $P_{\mathcal{O}} f_0 e^{-\widetilde{B}^0_D t} f_0$ is also continuous from $L_2(\mathcal{O};\mathbb{C}^n)$ to $H^1(\mathbb{R}^d;\mathbb{C}^n)$. It remains to use the continuity of the operators $[\Lambda^\varepsilon]S_\varepsilon\colon H^1(\mathbb{R}^d;\mathbb{C}^m) \to H^1(\mathbb{R}^d;\mathbb{C}^n)$ and $[\widetilde{\Lambda}^\varepsilon]S_\varepsilon\colon H^1(\mathbb{R}^d;\mathbb{C}^n) \to H^1(\mathbb{R}^d;\mathbb{C}^n)$, which follows from Proposition 1.2 and the relations $\Lambda, \widetilde{\Lambda} \in \widetilde{H}^1(\Omega)$.

We put $\widetilde{\mathbf{u}}_0(\,\cdot\,,t) := P_{\mathcal{O}}\mathbf{u}_0(\,\cdot\,,t)$. By $\mathbf{v}_\varepsilon$ we denote the first order approximation of the solution $\mathbf{u}_\varepsilon$ of problem (2.1):

$$\begin{aligned}\widetilde{\mathbf{v}}_\varepsilon(\,\cdot\,,t) = \widetilde{\mathbf{u}}_0(\,\cdot\,,t) + \varepsilon\Lambda^\varepsilon S_\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_0(\,\cdot\,,t) + \varepsilon\widetilde{\Lambda}^\varepsilon S_\varepsilon\widetilde{\mathbf{u}}_0(\,\cdot\,,t),\\ \mathbf{v}_\varepsilon(\,\cdot\,,t) := \widetilde{\mathbf{v}}_\varepsilon(\,\cdot\,,t)|_{\mathcal{O}}.\end{aligned} \tag{2.23}$$

So, $\mathbf{v}_\varepsilon(\,\cdot\,,t) = f_0 e^{-\widetilde{B}^0_D t} f_0\boldsymbol{\varphi}(\,\cdot\,) + \varepsilon\mathcal{K}_D(t;\varepsilon)\boldsymbol{\varphi}(\,\cdot\,)$.

Theorem 2.3. *Under the assumptions of Theorem* 2.2, *suppose that the matrix-valued functions* $\Lambda(\mathbf{x})$ *and* $\widetilde{\Lambda}(\mathbf{x})$ *are* Γ*-periodic solutions of the problems* (1.25) *and* (1.33), *respectively. Let* S_ε *be the Steklov smoothing operator* (1.1), *and let* $P_{\mathcal{O}}$ *be the extension operator* (1.45). *We put* $\widetilde{\mathbf{u}}_0(\,\cdot\,,t) = P_{\mathcal{O}}\mathbf{u}_0(\,\cdot\,,t)$. *Let* $\mathbf{v}_\varepsilon$ *be defined by* (2.23). *Then for* $0 < \varepsilon \le \varepsilon_1$ *and* $t > 0$ *we have*

$$\|\mathbf{u}_\varepsilon(\,\cdot\,,t) - \mathbf{v}_\varepsilon(\,\cdot\,,t)\|_{H^1(\mathcal{O})} \le C_{16}(\varepsilon^{1/2}t^{-3/4} + \varepsilon t^{-1})e^{-c_\flat t/2}\|\boldsymbol{\varphi}\|_{L_2(\mathcal{O})}.$$

In the operator terms,

$$\begin{aligned}\big\|f^\varepsilon e^{-\widetilde{B}_{D,\varepsilon} t}(f^\varepsilon)^* - f_0 e^{-\widetilde{B}^0_D t} f_0 - \varepsilon\mathcal{K}_D(t;\varepsilon)\big\|_{L_2(\mathcal{O})\to H^1(\mathcal{O})}\\ \le C_{16}(\varepsilon^{1/2}t^{-3/4} + \varepsilon t^{-1})e^{-c_\flat t/2},\end{aligned} \tag{2.24}$$

where $\mathcal{K}_D(t;\varepsilon)$ *is the corrector* (2.22). *Suppose that the matrix-valued function* $\widetilde{g}(\mathbf{x})$ *is defined by* (1.27). *For* $0 < \varepsilon \le \varepsilon_1$ *and* $t > 0$, *the flux* $\mathbf{p}_\varepsilon := g^\varepsilon b(\mathbf{D})\mathbf{u}_\varepsilon$ *satisfies*

$$\begin{aligned}\big\|\mathbf{p}_\varepsilon(\,\cdot\,,t) - \widetilde{g}^\varepsilon S_\varepsilon b(\mathbf{D})\widetilde{\mathbf{u}}_0(\,\cdot\,,t) - g^\varepsilon\big(b(\mathbf{D})\widetilde{\Lambda}\big)^\varepsilon S_\varepsilon\widetilde{\mathbf{u}}_0(\,\cdot\,,t)\big\|_{L_2(\mathcal{O})}\\ \le \widetilde{C}_{16}\varepsilon^{1/2}t^{-3/4}e^{-c_\flat t/2}\|\boldsymbol{\varphi}\|_{L_2(\mathcal{O})}.\end{aligned}$$

In the operator terms,

$$\big\|g^\varepsilon b(\mathbf{D}) f^\varepsilon e^{-\widetilde{B}_{D,\varepsilon} t}(f^\varepsilon)^* - \mathcal{G}_D(t;\varepsilon)\big\|_{L_2(\mathcal{O})\to L_2(\mathcal{O})} \le \widetilde{C}_{16}\varepsilon^{1/2}t^{-3/4}e^{-c_\flat t/2}. \tag{2.25}$$

Here
$$\mathcal{G}_D(t;\varepsilon) := \widetilde{g}^\varepsilon S_\varepsilon b(\mathbf{D}) P_{\mathcal{O}} f_0 e^{-\widetilde{B}_D^0 t} f_0 + g^\varepsilon \bigl(b(\mathbf{D})\widetilde{\Lambda}\bigr)^\varepsilon S_\varepsilon P_{\mathcal{O}} f_0 e^{-\widetilde{B}_D^0 t} f_0.$$
The constants C_{16} and $\widetilde{C}_{16}$ depend only on the problem data (1.9).

Proof. As in the proof of Theorem 2.2, we use representations for the sandwiched operator exponentials in terms of the contour integrals of the corresponding generalized resolvents. We have

$$\begin{aligned} &f^\varepsilon e^{-\widetilde{B}_{D,\varepsilon} t}(f^\varepsilon)^* - f_0 e^{-\widetilde{B}_D^0 t} f_0 - \varepsilon \mathcal{K}_D(t;\varepsilon) \\ &\quad= -\frac{1}{2\pi i}\int_\gamma e^{-\zeta t}\bigl((B_{D,\varepsilon} - \zeta Q_0^\varepsilon)^{-1} - (B_D^0 - \zeta\overline{Q_0})^{-1} - \varepsilon K_D(\varepsilon;\zeta)\bigr)\,d\zeta. \end{aligned} \tag{2.26}$$

Here $K_D(\varepsilon;\zeta)$ is the operator (1.47).

As in (2.16)–(2.19), using Theorems 1.9 and 1.10, we get

$$\begin{aligned} &\bigl\|(B_{D,\varepsilon} - \zeta Q_0^\varepsilon)^{-1} - (B_D^0 - \zeta\overline{Q_0})^{-1} - \varepsilon K_D(\varepsilon;\zeta)\bigr\|_{L_2(\mathcal{O})\to H^1(\mathcal{O})} \\ &\qquad\le \widehat{C}_{16}\bigl(\varepsilon^{1/2}|\zeta|^{-1/4} + \varepsilon\bigr), \quad \zeta\in\gamma, \quad 0<\varepsilon\le\varepsilon_1, \end{aligned} \tag{2.27}$$

with the constant $\widehat{C}_{16} := \max\{C'_{16}; C''_{16}\}$, where $C'_{16} := (1+\check{c})^{1/2} C_5 \mathfrak{C}$ and $C''_{16} := \max\{5C_2; 25C_3\}$. Relations (2.26) and (2.27) imply the required estimate (2.24) with the constant $C_{16} := 2\pi^{-1}\Gamma(3/4)\widehat{C}_{16}$.

Similarly, the identity

$$\begin{aligned} &g^\varepsilon b(\mathbf{D}) f^\varepsilon e^{-\widetilde{B}_{D,\varepsilon} t}(f^\varepsilon)^* - \mathcal{G}_D(t;\varepsilon) \\ &\quad= -\frac{1}{2\pi i}\int_\gamma e^{-\zeta t}\bigl(g^\varepsilon b(\mathbf{D})(B_{D,\varepsilon} - \zeta Q_0^\varepsilon)^{-1} - G_D(\varepsilon;\zeta)\bigr)\,d\zeta \end{aligned} \tag{2.28}$$

and estimates (1.50), (1.55) yield inequality (2.25) with the constant

$$\widetilde{C}_{16} := 2\pi^{-1}\Gamma(3/4)\max\bigl\{5^{5/4}\widetilde{C}_2; 2\check{c}^{1/4}(1+\check{c})^{1/2}\widetilde{C}_5\mathfrak{C}\bigr\}. \qquad \square$$

Remark 1.11(2) leads to the following statement.

Remark 2.4. Let λ_1^0 be the first eigenvalue of the operator B_D^0, and let $\varkappa > 0$ be an arbitrarily small number. Due to the norm-resolvent convergence, for sufficiently small $\varepsilon_\circ$ the number $\lambda_1^0\|Q_0\|_{L_\infty}^{-1} - \varkappa/2$ is a common lower bound for the operators $\widetilde{B}_{D,\varepsilon}$ for all $0<\varepsilon\le\varepsilon_\circ$. Therefore, we can shift the integration contour so that it will intersect the real axis at the point $\mathfrak{c} := \lambda_1^0\|Q_0\|_{L_\infty}^{-1} - \varkappa$ instead of $c_\flat/2$. In this way, we obtain estimates (2.11), (2.24), and (2.25) with $e^{-c_\flat t/2}$ replaced by $e^{-\mathfrak{c}t}$ on the right-hand sides. The constants in estimates become dependent on $\varkappa$.

2.5. Estimates for small time. Note that for $0<t<\varepsilon^2$ it makes no sense to apply estimates (2.24) and (2.25), because it is better to use the following simple statement (which is valid, however, for all $t>0$).

Proposition 2.5. *Under the assumptions of Theorem* 2.2, *for $t>0$ and $0<\varepsilon\le 1$ we have*

$$\bigl\|f^\varepsilon e^{-\widetilde{B}_{D,\varepsilon} t}(f^\varepsilon)^* - f_0 e^{-\widetilde{B}_D^0 t} f_0\bigr\|_{L_2(\mathcal{O})\to H^1(\mathcal{O})} \le C_{17} t^{-1/2} e^{-c_\flat t/2}, \tag{2.29}$$

$$\bigl\|g^\varepsilon b(\mathbf{D}) f^\varepsilon e^{-\widetilde{B}_{D,\varepsilon} t}(f^\varepsilon)^*\bigr\|_{L_2(\mathcal{O})\to L_2(\mathcal{O})} \le \widetilde{C}_{17} t^{-1/2} e^{-c_\flat t/2}, \tag{2.30}$$

$$\bigl\|g^0 b(\mathbf{D}) f_0 e^{-\widetilde{B}_D^0 t} f_0\bigr\|_{L_2(\mathcal{O})\to L_2(\mathcal{O})} \le \widetilde{C}_{17} t^{-1/2} e^{-c_\flat t/2}, \tag{2.31}$$

where the constants $C_{17} := 2c_3\|f\|_{L_\infty}$ and $\widetilde{C}_{17} := \|g\|_{L_\infty}^{1/2}\|f\|_{L_\infty}$ depend only on the problem data (1.9).

Proof. Inequality (2.29) follows from (1.43), (2.5), and (2.7).

Next, by (1.23),

$$\big\|g^\varepsilon b(\mathbf{D}) f^\varepsilon e^{-\widetilde{B}_{D,\varepsilon} t}(f^\varepsilon)^*\big\|_{L_2(\mathcal{O})\to L_2(\mathcal{O})} \le \|g\|_{L_\infty}^{1/2}\|f\|_{L_\infty}\big\|\widetilde{B}_{D,\varepsilon}^{1/2} e^{-\widetilde{B}_{D,\varepsilon} t}\big\|_{L_2(\mathcal{O})\to L_2(\mathcal{O})}.$$

Together with (2.10), this yields (2.30). By (1.43) and (1.44), estimate (2.31) is checked similarly. □

2.6. Removal of the smoothing operator S_ε from the corrector. It is possible to remove the smoothing operator from the corrector if the matrix-valued functions $\Lambda(\mathbf{x})$ and $\widetilde{\Lambda}(\mathbf{x})$ satisfy Conditions 1.12 and 1.14, respectively. The following result is checked like in Theorem 2.3 by using Theorems 1.17 and 1.18.

Theorem 2.6. *Under the assumptions of Theorem* 2.3, *suppose that the matrix-valued functions* $\Lambda(\mathbf{x})$ *and* $\widetilde{\Lambda}(\mathbf{x})$ *satisfy Conditions* 1.12 *and* 1.14, *respectively. Put*

$$\mathcal{K}_D^0(t;\varepsilon) := (\Lambda^\varepsilon b(\mathbf{D}) + \widetilde{\Lambda}^\varepsilon) f_0 e^{-\widetilde{B}_D^0 t} f_0, \tag{2.32}$$

$$\mathcal{G}_D^0(t;\varepsilon) := \widetilde{g}^\varepsilon b(\mathbf{D}) f_0 e^{-\widetilde{B}_D^0 t} f_0 + g^\varepsilon \big(b(\mathbf{D})\widetilde{\Lambda}\big)^\varepsilon f_0 e^{-\widetilde{B}_D^0 t} f_0. \tag{2.33}$$

Then for $t>0$ *and* $0<\varepsilon\le\varepsilon_1$ *we have*

$$\begin{aligned}\big\|f^\varepsilon e^{-\widetilde{B}_{D,\varepsilon} t}(f^\varepsilon)^* - f_0 e^{-\widetilde{B}_D^0 t} f_0 - \varepsilon \mathcal{K}_D^0(t;\varepsilon)\big\|_{L_2(\mathcal{O})\to H^1(\mathcal{O})}&\\ \le C_{18}\big(\varepsilon^{1/2} t^{-3/4} + \varepsilon t^{-1}\big) e^{-c_\flat t/2}&,\\ \big\|g^\varepsilon b(\mathbf{D}) f^\varepsilon e^{-\widetilde{B}_{D,\varepsilon} t}(f^\varepsilon)^* - \mathcal{G}_D^0(t;\varepsilon)\big\|_{L_2(\mathcal{O})\to L_2(\mathcal{O})}&\\ \le \widetilde{C}_{18}\big(\varepsilon^{1/2} t^{-3/4} + \varepsilon t^{-1}\big) e^{-c_\flat t/2}&.\end{aligned}$$

The constants C_{18} *and* $\widetilde{C}_{18}$ *depend on the problem data* (1.9), p, *and the norms* $\|\Lambda\|_{L_\infty}$ *and* $\|\widetilde{\Lambda}\|_{L_p(\Omega)}$.

Using Remark 1.19, we observe the following.

Remark 2.7. If only Condition 1.12 (respectively, Condition 1.14) is satisfied, then the smoothing operator S_ε can be removed from the term of the corrector involving Λ^ε (respectively, $\widetilde{\Lambda}^\varepsilon$).

2.7. The case of smooth boundary. It is also possible to remove the smoothing operator S_ε from the corrector by increasing smoothness of the boundary. In this subsection, we consider the case where $d\ge 3$, because for $d\le 2$ we can apply Theorem 2.6 (see Propositions 1.13 and 1.15).

Lemma 2.8. *Let* $k\ge 2$ *be an integer. Let* $\mathcal{O}\subset\mathbb{R}^d$ *be a bounded domain with boundary* $\partial\mathcal{O}$ *of class* $C^{k-1,1}$. *Then for* $t>0$ *the operator* $e^{-\widetilde{B}_D^0 t}$ *is a continuous mapping from* $L_2(\mathcal{O};\mathbb{C}^n)$ *to* $H^q(\mathcal{O};\mathbb{C}^n)$, $0\le q\le k$, *and*

$$\big\|e^{-\widetilde{B}_D^0 t}\big\|_{L_2(\mathcal{O})\to H^q(\mathcal{O})} \le \widehat{\mathrm{C}}_q t^{-q/2} e^{-c_\flat t/2}, \quad t>0. \tag{2.34}$$

The constant $\widehat{\mathrm{C}}_q$ *depends only on* q *and the problem data* (1.9).

Proof. It suffices to check estimate (2.34) in the case where $q\in[0,k]$ is an integer; then the result for nonintegral q follows by interpolation. For $q=0,1,2$ estimate (2.34) was already proved (see Lemma 2.1).

So, let q be an integer such that $2\le q\le k$. By theorems about regularity of solutions of strongly elliptic systems (see, e.g., [McL, Chapter 4]), the operator $(\widetilde{B}_D^0)^{-1}$ acts continuously from $H^\sigma(\mathcal{O};\mathbb{C}^n)$ to $H^{\sigma+2}(\mathcal{O};\mathbb{C}^n)$ under the assumption $\partial\mathcal{O}\in C^{\sigma+1,1}$, where $\sigma\in\mathbb{Z}_+$. We also take into account that the operator $(\widetilde{B}_D^0)^{-1/2}$ is continuous from $L_2(\mathcal{O};\mathbb{C}^n)$ to $H^1(\mathcal{O};\mathbb{C}^n)$. It follows that, under the assumptions of the lemma, for an integer $q\in[2,k]$ the operator $(\widetilde{B}_D^0)^{-q/2}$ is a continuous mapping from $L_2(\mathcal{O};\mathbb{C}^n)$ to $H^q(\mathcal{O};\mathbb{C}^n)$. We have

$$\|(\widetilde{B}_D^0)^{-q/2}\|_{L_2(\mathcal{O})\to H^q(\mathcal{O})}\le \check{\mathrm{C}}_q, \tag{2.35}$$

where the constant $\check{\mathrm{C}}_q$ depends on q and the problem data (1.9). From (2.35) it follows that

$$\begin{aligned}\|e^{-\widetilde{B}_D^0 t}\|_{L_2(\mathcal{O})\to H^q(\mathcal{O})}&\le \check{\mathrm{C}}_q\|(\widetilde{B}_D^0)^{q/2}e^{-\widetilde{B}_D^0 t}\|_{L_2(\mathcal{O})\to L_2(\mathcal{O})}\le \check{\mathrm{C}}_q\sup_{x\ge c_\flat}x^{q/2}e^{-xt}\\&\le \check{\mathrm{C}}_q t^{-q/2}e^{-c_\flat t/2}\sup_{x\ge 0}x^{q/2}e^{-x/2}\le \widehat{\mathrm{C}}_q t^{-q/2}e^{-c_\flat t/2},\end{aligned}$$

where $\widehat{\mathrm{C}}_q:=\check{\mathrm{C}}_q(q/e)^{q/2}$. □

Using Lemma 2.8, the properties of the matrix-valued functions $\Lambda(\mathbf{x})$ and $\widetilde{\Lambda}(\mathbf{x})$, and the properties of the operator S_ε, we can estimate the difference of the correctors (2.22) and (2.32).

Lemma 2.9. *Let $d\ge 3$. Let $\mathcal{O}\subset\mathbb{R}^d$ be a bounded domain of class $C^{d/2,1}$ if d is even and of class $C^{(d+1)/2,1}$ if d is odd. Let $\mathcal{K}_D(t;\varepsilon)$ be the operator* (2.22)*, and let $\mathcal{K}_D^0(t;\varepsilon)$ be the operator* (2.32)*. Then for $0<\varepsilon\le 1$ and $t>0$ we have*

$$\|\mathcal{K}_D(t;\varepsilon)-\mathcal{K}_D^0(t;\varepsilon)\|_{L_2(\mathcal{O})\to H^1(\mathcal{O})}\le \widehat{\mathcal{C}}_d(t^{-1}+t^{-d/4-1/2})e^{-c_\flat t/2}. \tag{2.36}$$

The constant $\widehat{\mathcal{C}}_d$ depends only on the problem data (1.9)*.*

Lemma 2.9 and Theorem 2.3 imply the following result.

Theorem 2.10. *Under the assumptions of Theorem* 2.2*, suppose that $d\ge 3$ and that the domain $\mathcal{O}$ is as in Lemma* 2.9*. Let $\mathcal{K}_D^0(t;\varepsilon)$ be the corrector* (2.32)*. Let $\mathcal{G}_D^0(t;\varepsilon)$ be the operator* (2.33)*. Then for $t>0$ and $0<\varepsilon\le\varepsilon_1$ we have*

$$\begin{aligned}&\|f^\varepsilon e^{-\widetilde{B}_{D,\varepsilon}t}(f^\varepsilon)^*-f_0e^{-\widetilde{B}_D^0 t}f_0-\varepsilon\mathcal{K}_D^0(t;\varepsilon)\|_{L_2(\mathcal{O})\to H^1(\mathcal{O})}\\&\qquad\le \mathcal{C}_d(\varepsilon^{1/2}t^{-3/4}+\varepsilon t^{-d/4-1/2})e^{-c_\flat t/2},\end{aligned} \tag{2.37}$$

$$\begin{aligned}&\|g^\varepsilon b(\mathbf{D})f^\varepsilon e^{-\widetilde{B}_{D,\varepsilon}t}(f^\varepsilon)^*-\mathcal{G}_D^0(t;\varepsilon)\|_{L_2(\mathcal{O})\to L_2(\mathcal{O})}\\&\qquad\le \widetilde{\mathcal{C}}_d(\varepsilon^{1/2}t^{-3/4}+\varepsilon t^{-d/4-1/2})e^{-c_\flat t/2}.\end{aligned} \tag{2.38}$$

The constants $\mathcal{C}_d$ and $\widetilde{\mathcal{C}}_d$ depend only on the problem data (1.9)*.*

The proofs of Lemma 2.9 and Theorem 2.10 are presented in the Appendix (see §7) in order not to overload the main presentation. Clearly, it is convenient to apply Theorem 2.10 if t is separated away from zero. For small t the order of the factor $(\varepsilon^{1/2}t^{-3/4}+\varepsilon t^{-d/4-1/2})$ grows with dimension. This is the "price" for the removal of the smoothing operator.

Remark 2.11. Instead of the smoothness assumption on $\partial\mathcal{O}$ as in Lemma 2.9, we could impose the following implicit condition: a bounded domain $\mathcal{O}$ with Lipschitz boundary is such that estimate (2.34) is fulfilled for $q=d/2+1$. In such domains the statements of Lemma 2.9 and Theorem 2.10 remain valid.

2.8. The case of zero corrector. Suppose that $g^0 = \overline{g}$, i.e., relations (1.31) are true. Suppose also that condition (1.58) is satisfied. Then the Γ-periodic solutions of problems (1.25) and (1.33) are equal to zero: $\Lambda(\mathbf{x}) = 0$ and $\widetilde{\Lambda}(\mathbf{x}) = 0$. Using Proposition 1.20, we obtain the following result.

Proposition 2.12. *Suppose that relations* (1.31) *and* (1.58) *are satisfied. Then, under the assumptions of Theorem* 2.2, *for* $0 < \varepsilon \le 1$ *we have*

$$\big\| f^\varepsilon e^{-\widetilde{B}_{D,\varepsilon} t}(f^\varepsilon)^* - f_0 e^{-\widetilde{B}_D^0 t} f_0 \big\|_{L_2(\mathcal{O}) \to H^1(\mathcal{O})} \le C_{19} \varepsilon t^{-1} e^{-c_\flat t/2}, \quad t > 0, \tag{2.39}$$

where the constant C_{19} *depends only on the problem data* (1.9).

Proof. We use identity (2.13). For $|\zeta| \le \check{c}$, where $\check{c}$ is the constant (2.14), we employ (1.60) and (2.15). For $|\zeta| > \check{c}$ we apply (1.59) and (2.17). As a result, we see that, for $0 < \varepsilon \le 1$,

$$\begin{aligned} &\big\| (B_{D,\varepsilon} - \zeta Q_0^\varepsilon)^{-1} - (B_D^0 - \zeta \overline{Q_0})^{-1} \big\|_{L_2(\mathcal{O}) \to H^1(\mathcal{O})} \le \widehat{C}_{19} \varepsilon, \quad \zeta \in \gamma; \\ &\widehat{C}_{19} := \max\big\{ \big(C_9 + C_{10}(1 + \check{c})^{1/2}\big)\mathfrak{C}; 25 C_8 \big\}. \end{aligned}$$

Together with (2.13), this yields (2.39) with the constant $C_{19} := 2\pi^{-1} \widehat{C}_{19}$. $\square$

2.9. Special case. Now, we assume that $g^0 = \underline{g}$, i.e., relations (1.32) are satisfied. Then, by Proposition 1.13(3°), Condition 1.12 is fulfilled. By [BSu3, Remark 3.5], the matrix-valued function (1.27) is constant and coincides with g^0, i.e., $\widetilde{g}(\mathbf{x}) = g^0 = \underline{g}$. Thus, $\widetilde{g}^\varepsilon b(\mathbf{D}) f_0 e^{-\widetilde{B}_D^0 t} f_0 = g^0 b(\mathbf{D}) f_0 e^{-\widetilde{B}_D^0 t} f_0$.

Suppose moreover that (1.58) is true. Then $\widetilde{\Lambda}(\mathbf{x}) = 0$. The following result can be deduced from Theorem 2.3 and Proposition 1.1.

Proposition 2.13. *Suppose that relations* (1.32) *and* (1.58) *are satisfied. Then, under the assumptions of Theorem* 2.2, *for* $0 < \varepsilon \le \varepsilon_1$ *and* $t > 0$ *we have*

$$\big\| g^\varepsilon b(\mathbf{D}) f^\varepsilon e^{-\widetilde{B}_{D,\varepsilon} t}(f^\varepsilon)^* - g^0 b(\mathbf{D}) f_0 e^{-\widetilde{B}_D^0 t} f_0 \big\|_{L_2(\mathcal{O}) \to L_2(\mathcal{O})} \le \widetilde{C}'_{16} \varepsilon^{1/2} t^{-3/4} e^{-c_\flat t/2}. \tag{2.40}$$

The constant $\widetilde{C}'_{16}$ *depends only on the problem data* (1.9).

Proof. From Theorem 2.3 it follows that

$$\begin{aligned} &\big\| g^\varepsilon b(\mathbf{D}) f^\varepsilon e^{-\widetilde{B}_{D,\varepsilon} t}(f^\varepsilon)^* - g^0 S_\varepsilon b(\mathbf{D}) P_{\mathcal{O}} f_0 e^{-\widetilde{B}_D^0 t} f_0 \big\|_{L_2(\mathcal{O}) \to L_2(\mathcal{O})} \\ &\qquad \le \widetilde{C}_{16} \varepsilon^{1/2} t^{-3/4} e^{-c_\flat t/2}. \end{aligned} \tag{2.41}$$

On the one hand, Proposition 1.1 and relations (1.3), (1.30), (1.43), (1.46), (2.8) imply that

$$\begin{aligned} &\big\| g^0 (S_\varepsilon - I) b(\mathbf{D}) P_{\mathcal{O}} f_0 e^{-\widetilde{B}_D^0 t} f_0 \big\|_{L_2(\mathcal{O}) \to L_2(\mathbb{R}^d)} \\ &\le \varepsilon \|g\|_{L_\infty} r_1 \alpha_1^{1/2} \| P_{\mathcal{O}} f_0 e^{-\widetilde{B}_D^0 t} f_0 \|_{L_2(\mathcal{O}) \to H^2(\mathbb{R}^d)} \\ &\le \varepsilon \|g\|_{L_\infty} \|f\|_{L_\infty} r_1 \alpha_1^{1/2} C_{\mathcal{O}}^{(2)} \widetilde{c} t^{-1} e^{-c_\flat t/2}. \end{aligned} \tag{2.42}$$

On the other hand, from (1.2), (1.3), (1.30), (1.43), (1.46), and (2.7) it follows that

$$\begin{aligned} &\big\| g^0 (S_\varepsilon - I) b(\mathbf{D}) P_{\mathcal{O}} f_0 e^{-\widetilde{B}_D^0 t} f_0 \big\|_{L_2(\mathcal{O}) \to L_2(\mathbb{R}^d)} \\ &\le 2 \|g\|_{L_\infty} \alpha_1^{1/2} \| P_{\mathcal{O}} f_0 e^{-\widetilde{B}_D^0 t} f_0 \|_{L_2(\mathcal{O}) \to H^1(\mathbb{R}^d)} \\ &\le 2 \|g\|_{L_\infty} \|f\|_{L_\infty} \alpha_1^{1/2} C_{\mathcal{O}}^{(1)} c_3 t^{-1/2} e^{-c_\flat t/2}. \end{aligned} \tag{2.43}$$

By (2.42) and (2.43),

$$\big\| g^0 (S_\varepsilon - I) b(\mathbf{D}) P_{\mathcal{O}} f_0 e^{-\widetilde{B}_D^0 t} f_0 \big\|_{L_2(\mathcal{O}) \to L_2(\mathbb{R}^d)} \leq \check{C}_{16} \varepsilon^{1/2} t^{-3/4} e^{-c_\flat t/2},$$

where $\check{C}_{16} := \|g\|_{L_\infty} \|f\|_{L_\infty} \alpha_1^{1/2} \big(2 r_1 C_{\mathcal{O}}^{(1)} C_{\mathcal{O}}^{(2)} \widetilde{c} c_3 \big)^{1/2}$. Combining this with (2.41), we obtain estimate (2.40) with the constant $\widetilde{C}'_{16} := \widetilde{C}_{16} + \check{C}_{16}$. $\square$

2.10. Estimates in a strictly interior subdomain. Using Theorem 1.21, we improve error estimates in a strictly interior subdomain.

Theorem 2.14. *Under the assumptions of Theorem* 2.3*, let $\mathcal{O}'$ be a strictly interior subdomain of the domain $\mathcal{O}$, and let δ be defined as in* (1.61)*. Then for $0 < \varepsilon \leq \varepsilon_1$ and $t > 0$ we have*

$$\begin{aligned} &\big\| f^\varepsilon e^{-\widetilde{B}_{D,\varepsilon} t} (f^\varepsilon)^* - f_0 e^{-\widetilde{B}_D^0 t} f_0 - \varepsilon \mathcal{K}_D(t;\varepsilon) \big\|_{L_2(\mathcal{O}) \to H^1(\mathcal{O}')} \\ &\qquad\qquad \leq \varepsilon (C_{20} t^{-1/2} \delta^{-1} + C_{21} t^{-1}) e^{-c_\flat t/2}, \\ &\big\| g^\varepsilon b(\mathbf{D}) f^\varepsilon e^{-\widetilde{B}_{D,\varepsilon} t} (f^\varepsilon)^* - \mathcal{G}_D(t;\varepsilon) \big\|_{L_2(\mathcal{O}) \to L_2(\mathcal{O}')} \\ &\qquad\qquad \leq \varepsilon (\widetilde{C}_{20} t^{-1/2} \delta^{-1} + \widetilde{C}_{21} t^{-1}) e^{-c_\flat t/2}. \end{aligned} \tag{2.44}$$

The constants $C_{20}, C_{21}, \widetilde{C}_{20}$, and $\widetilde{C}_{21}$ depend only on the problem data (1.9)*.*

Proof. The proof is based on application of Theorem 1.21 and relations (2.26), (2.28). Also, estimates (2.15) and (2.17) are used. We omit the details. $\square$

The following result is checked similarly with the help of Theorems 1.22 and 1.23.

Theorem 2.15. *Under the assumptions of Theorem* 2.14*, suppose that the matrix-valued functions $\Lambda(\mathbf{x})$ and $\widetilde{\Lambda}(\mathbf{x})$ satisfy Conditions* 1.12 *and* 1.14*, respectively. Let $\mathcal{K}_D^0(t;\varepsilon)$ be the corrector* (2.32)*, and let $\mathcal{G}_D^0(t;\varepsilon)$ be the operator* (2.33)*. Then for $t > 0$ and $0 < \varepsilon \leq \varepsilon_1$ we have*

$$\begin{aligned} &\big\| f^\varepsilon e^{-\widetilde{B}_{D,\varepsilon} t} (f^\varepsilon)^* - f_0 e^{-\widetilde{B}_D^0 t} f_0 - \varepsilon \mathcal{K}_D^0(t;\varepsilon) \big\|_{L_2(\mathcal{O}) \to H^1(\mathcal{O}')} \\ &\qquad\qquad\qquad \leq \varepsilon (C_{20} t^{-1/2} \delta^{-1} + C_{22} t^{-1}) e^{-c_\flat t/2}, \\ &\big\| g^\varepsilon b(\mathbf{D}) f^\varepsilon e^{-\widetilde{B}_{D,\varepsilon} t} (f^\varepsilon)^* - \mathcal{G}_D^0(t;\varepsilon) \big\|_{L_2(\mathcal{O}) \to L_2(\mathcal{O}')} \leq \varepsilon (\widetilde{C}_{20} t^{-1/2} \delta^{-1} + \widetilde{C}_{22} t^{-1}) e^{-c_\flat t/2}. \end{aligned}$$

The constants C_{20} and $\widetilde{C}_{20}$ are the same as in Theorem 2.14*. The constants C_{22} and $\widetilde{C}_{22}$ depend on the problem data* (1.9)*, p, and the norms $\|\Lambda\|_{L_\infty}$, $\|\widetilde{\Lambda}\|_{L_p(\Omega)}$.*

Note that it is possible to remove the smoothing operator S_ε from the corrector in estimates of Theorem 2.14 without any additional assumptions on the matrix-valued functions $\Lambda(\mathbf{x})$ and $\widetilde{\Lambda}(\mathbf{x})$. For this, the additional smoothness of the boundary is not required. We consider the case where $d \geq 3$ (otherwise, by Propositions 1.13 and 1.15, we can apply Theorem 2.15). We know that for $t > 0$ the operator $e^{-\widetilde{B}_D^0 t}$ acts continuously from $L_2(\mathcal{O};\mathbb{C}^n)$ to $H^2(\mathcal{O};\mathbb{C}^n)$ and estimate (2.8) is fulfilled. Moreover, the following property of "regularity improvement" inside the domain is valid: for $t > 0$ the operator $e^{-\widetilde{B}_D^0 t}$ acts continuously from $L_2(\mathcal{O};\mathbb{C}^n)$ to $H^\sigma(\mathcal{O}';\mathbb{C}^n)$ for any integer $\sigma \geq 3$. We have

$$\begin{aligned} &\big\| e^{-\widetilde{B}_D^0 t} \big\|_{L_2(\mathcal{O}) \to H^\sigma(\mathcal{O}')} \leq \mathrm{C}'_\sigma t^{-1/2} (\delta^{-2} + t^{-1})^{(\sigma-1)/2} e^{-c_\flat t/2}, \\ &\qquad\qquad\qquad\qquad t > 0, \quad \sigma \in \mathbb{N}, \quad \sigma \geq 3. \end{aligned} \tag{2.45}$$

The constant C'_σ depends on σ and the problem data (1.9). For the scalar parabolic equations, the "regularity improvement" property inside the domain was obtained in [LaSoU, Chapter 3, § 12].

In a similar way, the "regularity improvement" can be checked for the operator $\widetilde{B}^0_D$. It is easy to deduce the qualified estimates (2.45), observing that the derivatives $\mathbf{D}^\alpha \mathbf{u}_0$ (where $\mathbf{u}_0$ is the function (2.3) with $\boldsymbol{\varphi} \in L_2(\mathcal{O};\mathbb{C}^n)$) are solutions of the parabolic equation $\overline{Q_0}\partial_t \mathbf{D}^\alpha \mathbf{u}_0 = -B^0 \mathbf{D}^\alpha \mathbf{u}_0$. We multiply this equation by $\chi^2 \mathbf{D}^\alpha \mathbf{u}_0$ and integrate over the cylinder $\mathcal{O} \times (0,t)$. Here χ is a smooth cut-off function equal to zero near the lateral surface and the bottom of the cylinder. The standard analysis of the corresponding integral identity together with the already known inequalities of Lemma 2.1 leads to estimates (2.45).

Using the properties of $\Lambda(\mathbf{x})$ and $\widetilde{\Lambda}(\mathbf{x})$, and also the properties of S_ε, we can deduce the following statement from (2.45).

Lemma 2.16. *Under the assumptions of Theorem* 2.14, *let $d \geq 3$, and let $\mathcal{K}^0_D(t;\varepsilon)$ be the operator* (2.32). *Denote*

$$h_d(\delta;t) := t^{-1} + t^{-1/2}(\delta^{-2} + t^{-1})^{d/4}. \tag{2.46}$$

Let $2r_1 = \operatorname{diam}\Omega$. Then for $0 < \varepsilon \leq (4r_1)^{-1}\delta$ and $t > 0$ we have

$$\|\mathcal{K}_D(t;\varepsilon) - \mathcal{K}^0_D(t;\varepsilon)\|_{L_2(\mathcal{O})\to H^1(\mathcal{O}')} \leq \mathrm{C}''_d h_d(\delta;t) e^{-c_\flat t/2}. \tag{2.47}$$

The constant C''_d depends only on the problem data (1.9).

From Lemma 2.16 and Theorem 2.14 we deduce the following result.

Theorem 2.17. *Under the assumptions of Theorem* 2.14, *let $d \geq 3$, let $\mathcal{K}^0_D(t;\varepsilon)$ be the corrector* (2.32), *and let $\mathcal{G}^0_D(t;\varepsilon)$ be the operator* (2.33). *Let $2r_1 = \operatorname{diam}\Omega$. Then for $0 < \varepsilon \leq \min\{\varepsilon_1; (4r_1)^{-1}\delta\}$ and $t > 0$ we have*

$$\big\|f^\varepsilon e^{-\widetilde{B}_{D,\varepsilon}t}(f^\varepsilon)^* - f_0 e^{-\widetilde{B}^0_D t} f_0 - \varepsilon \mathcal{K}^0_D(t;\varepsilon)\big\|_{L_2(\mathcal{O})\to H^1(\mathcal{O}')} \leq \varepsilon \mathrm{C}_d h_d(\delta;t) e^{-c_\flat t/2}, \tag{2.48}$$

$$\big\|g^\varepsilon b(\mathbf{D}) f^\varepsilon e^{-\widetilde{B}_{D,\varepsilon}t}(f^\varepsilon)^* - \mathcal{G}^0_D(t;\varepsilon)\big\|_{L_2(\mathcal{O})\to L_2(\mathcal{O}')} \leq \varepsilon \widetilde{\mathrm{C}}_d h_d(\delta;t) e^{-c_\flat t/2}. \tag{2.49}$$

Here $h_d(\delta;t)$ is given by (2.46), *the constants C_d and $\widetilde{\mathrm{C}}_d$ depend only on the problem data* (1.9).

The proofs of Lemma 2.16 and Theorem 2.17 are presented in the Appendix (see §8) in order not to overload the main presentation. Clearly, it is convenient to apply Theorem 2.17 if t is separated away from zero. For small t the order of the factor $h_d(\delta;t)$ grows with dimension. This is the "price" for removal of the smoothing operator.

§3. Homogenization of the first initial boundary-value problem for a nonhomogeneous equation

3.1. The principal term of approximation. In this section, we study the behavior of the solution of the first initial boundary-value problem for a nonhomogeneous parabolic equation:

$$\begin{cases} Q^\varepsilon_0(\mathbf{x})\frac{\partial \mathbf{u}_\varepsilon}{\partial t}(\mathbf{x},t) = -B_\varepsilon \mathbf{u}_\varepsilon(\mathbf{x},t) + \mathbf{F}(\mathbf{x},t), & \mathbf{x}\in\mathcal{O}, \quad t>0; \\ \mathbf{u}_\varepsilon(\,\cdot\,,t)|_{\partial\mathcal{O}} = 0, & t>0; \\ Q^\varepsilon_0(\mathbf{x})\mathbf{u}_\varepsilon(\mathbf{x},0) = \boldsymbol{\varphi}(\mathbf{x}), & \mathbf{x}\in\mathcal{O}. \end{cases} \tag{3.1}$$

Here $\mathbf{F} \in \mathfrak{H}_r(T) := L_r((0,T);L_2(\mathcal{O};\mathbb{C}^n))$, $0 < T \leq \infty$, with some $1 \leq r \leq \infty$. Then

$$\mathbf{u}_\varepsilon(\,\cdot\,,t) = f^\varepsilon e^{-\widetilde{B}_{D,\varepsilon}t}(f^\varepsilon)^*\boldsymbol{\varphi}(\,\cdot\,) + \int_0^t f^\varepsilon e^{-\widetilde{B}_{D,\varepsilon}(t-\tilde{t})}(f^\varepsilon)^*\mathbf{F}(\,\cdot\,,\tilde{t})\,d\tilde{t}. \tag{3.2}$$

The corresponding effective problem takes the form

$$(3.3)\qquad \begin{cases} \overline{Q_0}\frac{\partial \mathbf{u}_0}{\partial t}(\mathbf{x},t) = -B^0\mathbf{u}_0(\mathbf{x},t) + \mathbf{F}(\mathbf{x},t), & \mathbf{x}\in\mathcal{O}, \quad t>0;\\ \mathbf{u}_0(\,\cdot\,,t)|_{\partial\mathcal{O}} = 0, & t>0;\\ \overline{Q_0}\mathbf{u}_0(\mathbf{x},0) = \boldsymbol{\varphi}(\mathbf{x}), & \mathbf{x}\in\mathcal{O}. \end{cases}$$

The solution of this problem is given by

$$(3.4)\qquad \mathbf{u}_0(\,\cdot\,,t) = f_0 e^{-\widetilde{B}^0_D t} f_0\boldsymbol{\varphi}(\,\cdot\,) + \int_0^t f_0 e^{-\widetilde{B}^0_D (t-\widetilde{t})} f_0\mathbf{F}(\,\cdot\,,\widetilde{t})\,d\widetilde{t}.$$

Subtracting (3.4) from (3.2) and using Theorem 2.2 (see (2.11)), we conclude that, for $0<\varepsilon\le\varepsilon_1$ and $t>0$,

$$\|\mathbf{u}_\varepsilon(\,\cdot\,,t) - \mathbf{u}_0(\,\cdot\,,t)\|_{L_2(\mathcal{O})} \le C_{15}\varepsilon(t+\varepsilon^2)^{-1/2}e^{-c_\flat t/2}\|\boldsymbol{\varphi}\|_{L_2(\mathcal{O})} + C_{15}\varepsilon\mathcal{L}(\varepsilon;t;\mathbf{F}),$$

where

$$\mathcal{L}(\varepsilon;t;\mathbf{F}) := \int_0^t e^{-c_\flat(t-\widetilde{t})/2}(\varepsilon^2+t-\widetilde{t})^{-1/2}\|\mathbf{F}(\,\cdot\,,\widetilde{t})\|_{L_2(\mathcal{O})}\,d\widetilde{t}.$$

Estimating the term $\mathcal{L}(\varepsilon;t;\mathbf{F})$ for the case where $1<r\le\infty$, we obtain the following result. Its proof is completely similar to that of Theorem 5.1 in [MSu1].

Theorem 3.1. *Suppose that $\mathcal{O}\subset\mathbb{R}^d$ is a bounded domain of class $C^{1,1}$. Under the assumptions of Subsections* 1.3–1.6, *let $\mathbf{u}_\varepsilon$ be the solution of problem* (3.1), *and let $\mathbf{u}_0$ be the solution of the effective problem* (3.3) *with $\boldsymbol{\varphi}\in L_2(\mathcal{O};\mathbb{C}^n)$ and $\mathbf{F}\in\mathfrak{H}_r(T)$, where $0<T\le\infty$, with some $1<r\le\infty$. Then for $0<\varepsilon\le\varepsilon_1$ and $0<t<T$ we have*

$$\|\mathbf{u}_\varepsilon(\,\cdot\,,t) - \mathbf{u}_0(\,\cdot\,,t)\|_{L_2(\mathcal{O})} \le C_{15}\varepsilon(t+\varepsilon^2)^{-1/2}e^{-c_\flat t/2}\|\boldsymbol{\varphi}\|_{L_2(\mathcal{O})} + c_r\theta(\varepsilon,r)\|\mathbf{F}\|_{\mathfrak{H}_r(T)}.$$

Here $\theta(\varepsilon,r)$ is given by

$$(3.5)\qquad \theta(\varepsilon,r) = \begin{cases} \varepsilon^{2-2/r}, & 1<r<2,\\ \varepsilon(|\ln\varepsilon|+1)^{1/2}, & r=2,\\ \varepsilon, & 2<r\le\infty. \end{cases}$$

The constant c_r depends only on r and the problem data (1.9).

By analogy with the proof of Theorem 5.2 in [MSu1], we can deduce approximation of the solution of problem (3.1) in $\mathfrak{H}_r(T)$ from Theorem 2.2.

Theorem 3.2. *Under the assumptions of Theorem* 3.1, *let $\mathbf{u}_\varepsilon$ and $\mathbf{u}_0$ be the solutions of problems* (3.1) *and* (3.3), *respectively, with $\boldsymbol{\varphi}\in L_2(\mathcal{O};\mathbb{C}^n)$ and $\mathbf{F}\in\mathfrak{H}_r(T)$, $0<T\le\infty$, for some $1\le r<\infty$. Then for $0<\varepsilon\le\varepsilon_1$ we have*

$$\|\mathbf{u}_\varepsilon - \mathbf{u}_0\|_{\mathfrak{H}_r(T)} \le c_{r'}\theta(\varepsilon,r')\|\boldsymbol{\varphi}\|_{L_2(\mathcal{O})} + C_{23}\varepsilon\|\mathbf{F}\|_{\mathfrak{H}_r(T)}.$$

Here $\theta(\varepsilon,\,\cdot\,)$ is given by (3.5), *$r^{-1}+(r')^{-1}=1$. The constant C_{23} depends only on the problem data* (1.9), *and the constant $c_{r'}$ depends on the same parameters and r.*

Remark 3.3. For the case where $\boldsymbol{\varphi}=0$ and $\mathbf{F}\in\mathfrak{H}_\infty(T)$, Theorem 3.1 implies that

$$\|\mathbf{u}_\varepsilon - \mathbf{u}_0\|_{\mathfrak{H}_\infty(T)} \le c_\infty\varepsilon\|\mathbf{F}\|_{\mathfrak{H}_\infty(T)}, \quad 0<\varepsilon\le\varepsilon_1.$$

3.2. Approximation of the solution in $H^1(\mathcal{O};\mathbb{C}^n)$. Now, we obtain approximation of the solution of problem (3.1) in the $H^1(\mathcal{O};\mathbb{C}^n)$-norm with the help of Theorem 2.3. Some difficulties arise when we treat the integral term in (3.2), because estimate (2.24) "deteriorates" for small t. *Assuming that* $t \geq \varepsilon^2$, we divide the integration interval in (3.2) into two parts: $(0, t-\varepsilon^2)$ and $(t-\varepsilon^2, t)$. On the interval $(0, t-\varepsilon^2)$ we apply (2.24), and on $(t-\varepsilon^2, t)$ we use (2.29).

Denote

$$\mathbf{w}_\varepsilon(\,\cdot\,,t) := f_0 e^{-\widetilde{B}^0_D \varepsilon^2} f_0^{-1} \mathbf{u}_0(\,\cdot\,, t-\varepsilon^2), \tag{3.6}$$

where $\mathbf{u}_0$ is the solution of problem (3.3). By (3.4),

$$\mathbf{w}_\varepsilon(\,\cdot\,,t) = f_0 e^{-\widetilde{B}^0_D t} f_0 \boldsymbol{\varphi}(\,\cdot\,) + \int_0^{t-\varepsilon^2} f_0 e^{-\widetilde{B}^0_D (t-\widetilde{t})} f_0 \mathbf{F}(\,\cdot\,,\widetilde{t})\, d\widetilde{t}.$$

The following statement can be checked much in the same way as Theorem 5.4 in [MSu1].

Theorem 3.4. *Under the assumptions of Theorem* 3.1, *suppose that* $\mathbf{u}_\varepsilon$ *and* $\mathbf{u}_0$ *are the solutions of problems* (3.1) *and* (3.3), *respectively, with* $\boldsymbol{\varphi} \in L_2(\mathcal{O};\mathbb{C}^n)$ *and* $\mathbf{F} \in \mathfrak{H}_r(T)$, $0 < T \leq \infty$, *for some* $2 < r \leq \infty$. *Let* $\mathbf{w}_\varepsilon(\,\cdot\,,t)$ *be given by* (3.6). *Let* $\Lambda(\mathbf{x})$ *and* $\widetilde{\Lambda}(\mathbf{x})$ *be the* Γ*-periodic matrix solutions of problems* (1.25) *and* (1.33), *respectively. Let* $P_{\mathcal{O}}$ *be the continuous linear extension operator* (1.45) *and* S_ε *the Steklov smoothing operator* (1.1). *We put* $\widetilde{\mathbf{w}}_\varepsilon(\,\cdot\,,t) := P_{\mathcal{O}} \mathbf{w}_\varepsilon(\,\cdot\,,t)$ *and denote*

$$\mathbf{v}_\varepsilon(\,\cdot\,,t) := \mathbf{u}_0(\,\cdot\,,t) + \varepsilon \Lambda^\varepsilon S_\varepsilon b(\mathbf{D}) \widetilde{\mathbf{w}}_\varepsilon(\,\cdot\,,t) + \varepsilon \widetilde{\Lambda}^\varepsilon S_\varepsilon \widetilde{\mathbf{w}}_\varepsilon(\,\cdot\,,t).$$

Let $\mathbf{p}_\varepsilon(\,\cdot\,,t) := g^\varepsilon b(\mathbf{D}) \mathbf{u}_\varepsilon(\,\cdot\,,t)$, *and let* $\widetilde{g}(\mathbf{x})$ *be the matrix-valued function* (1.27). *We put*

$$\mathbf{q}_\varepsilon(\,\cdot\,,t) := \widetilde{g}^\varepsilon S_\varepsilon b(\mathbf{D}) \widetilde{\mathbf{w}}_\varepsilon(\,\cdot\,,t) + g^\varepsilon \big(b(\mathbf{D})\widetilde{\Lambda}\big)^\varepsilon S_\varepsilon \widetilde{\mathbf{w}}_\varepsilon(\,\cdot\,,t).$$

Then for $0 < \varepsilon \leq \varepsilon_1$ *and* $\varepsilon^2 \leq t < T$ *we have*

$$\|\mathbf{u}_\varepsilon(\,\cdot\,,t) - \mathbf{v}_\varepsilon(\,\cdot\,,t)\|_{H^1(\mathcal{O})} \leq 2C_{16} \varepsilon^{1/2} t^{-3/4} e^{-c_\flat t/2} \|\boldsymbol{\varphi}\|_{L_2(\mathcal{O})} + \check{c}_r \omega(\varepsilon,r) \|\mathbf{F}\|_{\mathfrak{H}_r(T)},$$

$$\|\mathbf{p}_\varepsilon(\,\cdot\,,t) - \mathbf{q}_\varepsilon(\,\cdot\,,t)\|_{L_2(\mathcal{O})} \leq \widetilde{C}_{16} \varepsilon^{1/2} t^{-3/4} e^{-c_\flat t/2} \|\boldsymbol{\varphi}\|_{L_2(\mathcal{O})} + \widetilde{c}_r \omega(\varepsilon,r) \|\mathbf{F}\|_{\mathfrak{H}_r(T)}.$$

Here

$$\omega(\varepsilon,r) := \begin{cases} \varepsilon^{1-2/r}, & 2 < r < 4, \\ \varepsilon^{1/2}(|\ln\varepsilon| + 1)^{3/4}, & r = 4, \\ \varepsilon^{1/2}, & 4 < r \leq \infty, \end{cases} \tag{3.7}$$

where the constants $\check{c}_r$ *and* $\widetilde{c}_r$ *depend only on the problem data* (1.9) *and* r.

Since the right-hand side in (2.25) grows slower than the right-hand side in (2.24) as $t \to 0$, for $r > 4$ we can approximate the flux $\mathbf{p}_\varepsilon$ in terms of

$$\mathbf{h}_\varepsilon(\,\cdot\,,t) := \widetilde{g}^\varepsilon S_\varepsilon b(\mathbf{D}) \widetilde{\mathbf{u}}_0(\,\cdot\,,t) + g^\varepsilon \big(b(\mathbf{D})\widetilde{\Lambda}\big)^\varepsilon S_\varepsilon \widetilde{\mathbf{u}}_0(\,\cdot\,,t). \tag{3.8}$$

Proposition 3.5. *Under the assumptions of Theorem* 3.1, *suppose that* $\mathbf{u}_\varepsilon$ *and* $\mathbf{u}_0$ *are the solutions of problems* (3.1) *and* (3.3), *respectively, with* $\boldsymbol{\varphi} \in L_2(\mathcal{O};\mathbb{C}^n)$ *and* $\mathbf{F} \in \mathfrak{H}_r(T)$, $0 < T \leq \infty$, *for some* r *with* $4 < r \leq \infty$. *Let* $\mathbf{p}_\varepsilon(\,\cdot\,,t) = g^\varepsilon b(\mathbf{D}) \mathbf{u}_\varepsilon(\,\cdot\,,t)$ *and let* $\mathbf{h}_\varepsilon(\,\cdot\,,t)$ *be given by* (3.8). *Then for* $0 < t < T$ *and* $0 < \varepsilon \leq \varepsilon_1$ *we have*

$$\|\mathbf{p}_\varepsilon(\,\cdot\,,t) - \mathbf{h}_\varepsilon(\,\cdot\,,t)\|_{L_2(\mathcal{O})} \leq \widetilde{C}_{16} \varepsilon^{1/2} t^{-3/4} e^{-c_\flat t/2} \|\boldsymbol{\varphi}\|_{L_2(\mathcal{O})} + C_{24}^{(r)} \varepsilon^{1/2} \|\mathbf{F}\|_{\mathfrak{H}_p(t)}. \tag{3.9}$$

The constant $C_{24}^{(r)}$ *depends only on the problem data* (1.9) *and* r.

Proof. To check (3.9), we use inequality (2.25) and identities (3.2), (3.4). If $r = \infty$, we deduce (3.9) with $C_{24}^{(\infty)} := (2/c_\flat)^{1/4}\Gamma(1/4)\widetilde{C}_{16}$. If $4 < r < \infty$, we apply the Hölder inequality:

$$\|\mathbf{p}_\varepsilon(\,\cdot\,,t) - \mathbf{h}_\varepsilon(\,\cdot\,,t)\|_{L_2(\mathcal{O})} \leq \widetilde{C}_{16}\varepsilon^{1/2}t^{-3/4}e^{-c_\flat t/2}\|\boldsymbol{\varphi}\|_{L_2(\mathcal{O})} + \widetilde{C}_{16}\varepsilon^{1/2}\|\mathbf{F}\|_{\mathfrak{H}_r(t)}\mathfrak{I}_r(\varepsilon,t)^{1/r'}, \quad r^{-1} + (r')^{-1} = 1.$$

Here

$$\mathfrak{I}_r(\varepsilon,t) := \int_0^t \tau^{-3r'/4}e^{-c_\flat r'\tau/2}\,d\tau \leq (c_\flat r'/2)^{3r'/4-1}\Gamma(1-3r'/4).$$

This implies (3.9) with the constant $C_{24}^{(r)} := (c_\flat r'/2)^{3/4-1/r'}\Gamma(1-3r'/4)^{1/r'}\widetilde{C}_{16}$. $\square$

Combining Proposition 2.5 and Theorem 2.6, we deduce the following result.

Theorem 3.6. *Under the assumptions of Theorem* 3.4*, suppose that the matrix-valued functions* $\Lambda(\mathbf{x})$ *and* $\widetilde{\Lambda}(\mathbf{x})$ *satisfy Conditions* 1.12 *and* 1.14*, respectively. Denote*

$$\check{\mathbf{v}}_\varepsilon(\,\cdot\,,t) := \mathbf{u}_0(\,\cdot\,,t) + \varepsilon\Lambda^\varepsilon b(\mathbf{D})\mathbf{w}_\varepsilon(\,\cdot\,,t) + \varepsilon\widetilde{\Lambda}^\varepsilon\mathbf{w}_\varepsilon(\,\cdot\,,t), \tag{3.10}$$

$$\check{\mathbf{q}}_\varepsilon(\,\cdot\,,t) := \widetilde{g}^\varepsilon b(\mathbf{D})\mathbf{w}_\varepsilon(\,\cdot\,,t) + g^\varepsilon\big(b(\mathbf{D})\widetilde{\Lambda}\big)^\varepsilon\mathbf{w}_\varepsilon(\,\cdot\,,t). \tag{3.11}$$

Then for $0 < \varepsilon \leq \varepsilon_1$ *and* $\varepsilon^2 \leq t < T$ *we have*

$$\|\mathbf{u}_\varepsilon(\,\cdot\,,t) - \check{\mathbf{v}}_\varepsilon(\,\cdot\,,t)\|_{H^1(\mathcal{O})} \leq 2C_{18}\varepsilon^{1/2}t^{-3/4}e^{-c_\flat t/2}\|\boldsymbol{\varphi}\|_{L_2(\mathcal{O})} + c_r'\omega(\varepsilon,r)\|\mathbf{F}\|_{\mathfrak{H}_r(t)},$$

$$\|\mathbf{p}_\varepsilon(\,\cdot\,,t) - \check{\mathbf{q}}_\varepsilon(\,\cdot\,,t)\|_{L_2(\mathcal{O})} \leq 2\widetilde{C}_{18}\varepsilon^{1/2}t^{-3/4}e^{-c_\flat t/2}\|\boldsymbol{\varphi}\|_{L_2(\mathcal{O})} + c_r''\omega(\varepsilon,r)\|\mathbf{F}\|_{\mathfrak{H}_r(t)}.$$

The constants c_r' *and* c_r'' *depend only on the initial data* (1.9)*,* r*,* p*, and the norms* $\|\Lambda\|_{L_\infty}$*,* $\|\widetilde{\Lambda}\|_{L_p(\Omega)}$*.*

For the case of sufficiently smooth boundary, we could apply Theorem 2.10. However, because of the strong growth of the right-hand side in estimates (2.37), (2.38) for small t, we obtain a substantial result only in the three-dimensional case and only for $r > 4$.

Proposition 3.7. *Suppose that the assumptions of Theorem* 3.4 *are satisfied with* $d = 3$ *and* $r > 4$*. Suppose that* $\partial\mathcal{O} \in C^{2,1}$*. Let* $\check{\mathbf{v}}_\varepsilon$ *and* $\check{\mathbf{q}}_\varepsilon$ *be given by* (3.10) *and* (3.11)*. Then for* $0 < \varepsilon \leq \varepsilon_1$ *and* $\varepsilon^2 \leq t < T$ *we have*

$$\|\mathbf{u}_\varepsilon(\,\cdot\,,t) - \check{\mathbf{v}}_\varepsilon(\,\cdot\,,t)\|_{H^1(\mathcal{O})} \leq \mathcal{C}_3(\varepsilon^{1/2}t^{-3/4} + \varepsilon t^{-5/4})e^{-c_\flat t/2}\|\boldsymbol{\varphi}\|_{L_2(\mathcal{O})} + \widetilde{c}_r'\varepsilon^{1/2-2/r}\|\mathbf{F}\|_{\mathfrak{H}_r(t)},$$

$$\|\mathbf{p}_\varepsilon(\,\cdot\,,t) - \check{\mathbf{q}}_\varepsilon(\,\cdot\,,t)\|_{L_2(\mathcal{O})} \leq \widetilde{\mathcal{C}}_3(\varepsilon^{1/2}t^{-3/4} + \varepsilon t^{-5/4})e^{-c_\flat t/2}\|\boldsymbol{\varphi}\|_{L_2(\mathcal{O})} + \widetilde{c}_r''\varepsilon^{1/2-2/r}\|\mathbf{F}\|_{\mathfrak{H}_r(t)}.$$

The constants $\widetilde{c}_r'$ *and* $\widetilde{c}_r''$ *depend only on the problem data* (1.9) *and* r*.*

3.3. Approximation of the solution in a strictly interior subdomain. From Theorem 2.14 and Proposition 2.5 we deduce the following result.

Theorem 3.8. *Under the assumptions of Theorem* 3.4*, let* $\mathcal{O}'$ *be a strictly interior subdomain of* $\mathcal{O}$*. Let* δ *be given by* (1.61)*. Then for* $0 < \varepsilon \leq \varepsilon_1$ *and* $\varepsilon^2 \leq t < T$ *we have*

$$\begin{aligned}&\|\mathbf{u}_\varepsilon(\,\cdot\,,t) - \mathbf{v}_\varepsilon(\,\cdot\,,t)\|_{H^1(\mathcal{O}')}\\ &\quad\leq \varepsilon(C_{20}t^{-1/2}\delta^{-1} + C_{21}t^{-1})e^{-c_\flat t/2}\|\boldsymbol{\varphi}\|_{L_2(\mathcal{O})} + k_r\vartheta(\varepsilon,\delta,r)\|\mathbf{F}\|_{\mathfrak{H}_r(t)},\\ &\|\mathbf{p}_\varepsilon(\,\cdot\,,t) - \mathbf{q}_\varepsilon(\,\cdot\,,t)\|_{L_2(\mathcal{O}')}\\ &\quad\leq \varepsilon(\widetilde{C}_{20}t^{-1/2}\delta^{-1} + \widetilde{C}_{21}t^{-1})e^{-c_\flat t/2}\|\boldsymbol{\varphi}\|_{L_2(\mathcal{O})} + \widetilde{k}_r\vartheta(\varepsilon,\delta,r)\|\mathbf{F}\|_{\mathfrak{H}_r(t)}.\end{aligned}$$

Here the constants k_r *and* $\widetilde{k}_r$ *depend only on the problem data* (1.9) *and* r*, and*

$$\vartheta(\varepsilon,\delta,r) := \begin{cases}\varepsilon\delta^{-1} + \varepsilon^{1-2/r}, & 2 < r < \infty,\\ \varepsilon\delta^{-1} + \varepsilon(|\ln\varepsilon| + 1), & r = \infty.\end{cases}$$

Finally, if Conditions 1.12 and 1.14 are fulfilled, then Theorem 2.15 implies the following result.

Theorem 3.9. *Under the assumtions of Theorem* 3.8*, suppose that the matrix-valued functions* $\Lambda(\mathbf{x})$ *and* $\widetilde{\Lambda}(\mathbf{x})$ *satisfy Conditions* 1.12 *and* 1.14*, respectively. Suppose that* $\check{\mathbf{v}}_\varepsilon$ *and* $\check{\mathbf{q}}_\varepsilon$ *are given by* (3.10) *and* (3.11)*. Then for* $0 < \varepsilon \le \varepsilon_1$ *and* $\varepsilon^2 \le t < T$ *we have*

$$\begin{aligned}
&\|\mathbf{u}_\varepsilon(\,\cdot\,,t) - \check{\mathbf{v}}_\varepsilon(\,\cdot\,,t)\|_{H^1(\mathcal{O}')} \\
&\qquad\le \varepsilon(C_{20}t^{-1/2}\delta^{-1} + C_{22}t^{-1})e^{-c_\flat t/2}\|\boldsymbol{\varphi}\|_{L_2(\mathcal{O})} + \check{k}_r\vartheta(\varepsilon,\delta,r)\|\mathbf{F}\|_{\mathfrak{H}_r(t)},\\
&\|\mathbf{p}_\varepsilon(\,\cdot\,,t) - \check{\mathbf{q}}_\varepsilon(\,\cdot\,,t)\|_{L_2(\mathcal{O}')} \\
&\qquad\le \varepsilon(\widetilde{C}_{20}t^{-1/2}\delta^{-1} + \widetilde{C}_{22}t^{-1})e^{-c_\flat t/2}\|\boldsymbol{\varphi}\|_{L_2(\mathcal{O})} + \widehat{k}_r\vartheta(\varepsilon,\delta,r)\|\mathbf{F}\|_{\mathfrak{H}_r(t)}.
\end{aligned}$$

The constants $\check{k}_r$ *and* $\widehat{k}_r$ *depend only on the problem data* (1.9) *and also on* r*,* p*, and the norms* $\|\Lambda\|_{L_\infty}$*,* $\|\widetilde{\Lambda}\|_{L_p(\Omega)}$*.*

Applications

For elliptic systems in the entire space $\mathbb{R}^d$, the examples considered below were studied in [Su4, MSu2]. For elliptic systems in a bounded domain, these examples were considered in [MSu3].

§4. Scalar elliptic operator with a singular potential

4.1. Description of the operator. We consider the case where $n = 1$, $m = d$, $b(\mathbf{D}) = \mathbf{D}$, and $g(\mathbf{x})$ is a Γ-periodic symmetric $(d \times d)$-matrix-valued function *with real entries* such that $g, g^{-1} \in L_\infty$ and $g(\mathbf{x}) > 0$. Obviously (see (1.3)), $\alpha_0 = \alpha_1 = 1$ and $b(\mathbf{D})^* g^\varepsilon(\mathbf{x}) b(\mathbf{D}) = -\operatorname{div} g^\varepsilon(\mathbf{x})\nabla$.

Next, let $\mathbf{A}(\mathbf{x}) = \operatorname{col}\{A_1(\mathbf{x}), \dots, A_d(\mathbf{x})\}$, where the $A_j(\mathbf{x})$, $j = 1, \dots, d$, are Γ-periodic real-valued functions such that

$$A_j \in L_\rho(\Omega), \quad \rho = 2 \text{ for } d = 1, \quad \rho > d \text{ for } d \ge 2; \quad j = 1, \dots, d. \tag{4.1}$$

Let $v(\mathbf{x})$ and $\mathcal{V}(\mathbf{x})$ be real-valued Γ-periodic functions such that

$$v, \mathcal{V} \in L_s(\Omega), \quad s = 1 \ \text{ for } d = 1, \quad s > d/2 \ \text{ for } \ d \ge 2; \quad \int_\Omega v(\mathbf{x})\, d\mathbf{x} = 0. \tag{4.2}$$

In $L_2(\mathcal{O})$, we consider the operator $\mathfrak{B}_{D,\varepsilon}$ given formally by the differential expression

$$\mathfrak{B}_\varepsilon = (\mathbf{D} - \mathbf{A}^\varepsilon(\mathbf{x}))^* g^\varepsilon(\mathbf{x})(\mathbf{D} - \mathbf{A}^\varepsilon(\mathbf{x})) + \varepsilon^{-1}v^\varepsilon(\mathbf{x}) + \mathcal{V}^\varepsilon(\mathbf{x}) \tag{4.3}$$

with the Dirichlet condition on $\partial\mathcal{O}$. The precise definition of the operator $\mathfrak{B}_{D,\varepsilon}$ is given in terms of the quadratic form

$$\mathfrak{b}_{D,\varepsilon}[u,u] = \int_\mathcal{O} \left(\langle g^\varepsilon(\mathbf{D} - \mathbf{A}^\varepsilon)u, (\mathbf{D} - \mathbf{A}^\varepsilon)u\rangle + (\varepsilon^{-1}v^\varepsilon + \mathcal{V}^\varepsilon)|u|^2\right) d\mathbf{x}, \quad u \in H^1_0(\mathcal{O}).$$

It is easily seen (cf. [Su4, Subsection 13.1]) that (4.3) can be written as

$$\mathfrak{B}_\varepsilon = \mathbf{D}^* g^\varepsilon(\mathbf{x})\mathbf{D} + \sum_{j=1}^d \left(a_j^\varepsilon(\mathbf{x})D_j + D_j(a_j^\varepsilon(\mathbf{x}))^*\right) + Q^\varepsilon(\mathbf{x}). \tag{4.4}$$

Here $Q(\mathbf{x})$ is a real-valued function defined by

$$Q(\mathbf{x}) = \mathcal{V}(\mathbf{x}) + \langle g(\mathbf{x})\mathbf{A}(\mathbf{x}), \mathbf{A}(\mathbf{x})\rangle. \tag{4.5}$$

The complex-valued functions $a_j(\mathbf{x})$ are given by

$$a_j(\mathbf{x}) = -\eta_j(\mathbf{x}) + i\xi_j(\mathbf{x}), \quad j = 1, \dots, d. \tag{4.6}$$

Here the $\eta_j(\mathbf{x})$ are the components of the vector-valued function $\boldsymbol{\eta}(\mathbf{x}) = g(\mathbf{x})\mathbf{A}(\mathbf{x})$, and the functions $\xi_j(\mathbf{x})$ are defined by $\xi_j(\mathbf{x}) = -\partial_j \Phi(\mathbf{x})$, where $\Phi(\mathbf{x})$ is the Γ-periodic solution of the problem $\Delta\Phi(\mathbf{x}) = v(\mathbf{x})$, $\int_\Omega \Phi(\mathbf{x})\, d\mathbf{x} = 0$. We have

$$v(\mathbf{x}) = -\sum_{j=1}^{d} \partial_j \xi_j(\mathbf{x}). \tag{4.7}$$

It is easy to check that the functions (4.6) satisfy condition (1.7) with a suitable ρ' depending on ρ and s, and that the norms $\|a_j\|_{L_{\rho'}(\Omega)}$ are controlled in terms of $\|g\|_{L_\infty}$, $\|\mathbf{A}\|_{L_\rho(\Omega)}$, $\|v\|_{L_s(\Omega)}$, and the parameters of the lattice Γ. (See [Su4, Subsection 13.1].) The function (4.5) satisfies condition (1.8) with a suitable $s' = \min\{s; \rho/2\}$.

Let $Q_0(\mathbf{x})$ be a positive definite and bounded Γ-periodic function. As in (1.10), we introduce a positive definite operator $\mathcal{B}_{D,\varepsilon} := \mathfrak{B}_{D,\varepsilon} + \lambda Q_0^\varepsilon$. Here the constant λ is chosen in accordance with condition (1.16) for the operator $\mathcal{B}_{D,\varepsilon}$ with the coefficients g, a_j, $j = 1, \dots, d$, Q, and Q_0 defined above. The operator $\mathcal{B}_{D,\varepsilon}$ is given by

$$\mathcal{B}_\varepsilon = (\mathbf{D} - \mathbf{A}^\varepsilon(\mathbf{x}))^* g^\varepsilon(\mathbf{x})(\mathbf{D} - \mathbf{A}^\varepsilon(\mathbf{x})) + \varepsilon^{-1} v^\varepsilon(\mathbf{x}) + \mathcal{V}^\varepsilon(\mathbf{x}) + \lambda Q_0^\varepsilon(\mathbf{x}). \tag{4.8}$$

We are interested in the behavior of the exponential of the operator $\widetilde{\mathcal{B}}_{D,\varepsilon} := f^\varepsilon \mathcal{B}_{D,\varepsilon} f^\varepsilon$, where $f(\mathbf{x}) := Q_0(\mathbf{x})^{-1/2}$.

For the scalar elliptic operator (4.8), the problem data (1.9) reduce to the following set of parameters:

$$\begin{aligned} &d, \rho, s; \|g\|_{L_\infty}, \|g^{-1}\|_{L_\infty}, \|\mathbf{A}\|_{L_\rho(\Omega)}, \|v\|_{L_s(\Omega)}, \|\mathcal{V}\|_{L_s(\Omega)}, \\ &\|Q_0\|_{L_\infty}, \|Q_0^{-1}\|_{L_\infty}; \text{ the parameters of the lattice } \Gamma; \text{ the domain } \mathcal{O}. \end{aligned} \tag{4.9}$$

4.2. The effective operator. Let us write out the effective operator. In the case under consideration, the Γ-periodic solution of problem (1.25) is a row: $\Lambda(\mathbf{x}) = i\Psi(\mathbf{x})$, $\Psi(\mathbf{x}) = \big(\psi_1(\mathbf{x}), \dots, \psi_d(\mathbf{x})\big)$, where $\psi_j \in \widetilde{H}^1(\Omega)$ is the solution of the problem

$$\operatorname{div} g(\mathbf{x})(\nabla \psi_j(\mathbf{x}) + \mathbf{e}_j) = 0, \quad \int_\Omega \psi_j(\mathbf{x})\, d\mathbf{x} = 0.$$

Here the $\mathbf{e}_j$, $j = 1, \dots, d$, form the standard orthonormal basis in $\mathbb{R}^d$. Clearly, the functions $\psi_j(\mathbf{x})$ are real-valued, and the entries of $\Lambda(\mathbf{x})$ are purely imaginary. By (1.27), the columns of the $(d \times d)$-matrix-valued function $\widetilde{g}(\mathbf{x})$ are the vector-valued functions $g(\mathbf{x})(\nabla\psi_j(\mathbf{x}) + \mathbf{e}_j)$, $j = 1, \dots, d$. The effective matrix is defined as in (1.26): $g^0 = |\Omega|^{-1} \int_\Omega \widetilde{g}(\mathbf{x})\, d\mathbf{x}$. Clearly, $\widetilde{g}(\mathbf{x})$ and g^0 have real entries.

By (4.6) and (4.7), the periodic solution of problem (1.33) is represented as $\widetilde{\Lambda}(\mathbf{x}) = \widetilde{\Lambda}_1(\mathbf{x}) + i\widetilde{\Lambda}_2(\mathbf{x})$, where the real-valued Γ-periodic functions $\widetilde{\Lambda}_1(\mathbf{x})$ and $\widetilde{\Lambda}_2(\mathbf{x})$ are the solutions of the problems

$$\begin{aligned} &-\operatorname{div} g(\mathbf{x})\nabla\widetilde{\Lambda}_1(\mathbf{x}) + v(\mathbf{x}) = 0, && \int_\Omega \widetilde{\Lambda}_1(\mathbf{x})\, d\mathbf{x} = 0; \\ &-\operatorname{div} g(\mathbf{x})\nabla\widetilde{\Lambda}_2(\mathbf{x}) + \operatorname{div} g(\mathbf{x})\mathbf{A}(\mathbf{x}) = 0, && \int_\Omega \widetilde{\Lambda}_2(\mathbf{x})\, d\mathbf{x} = 0. \end{aligned}$$

The column V (see (1.35)) has the form $V = V_1 + iV_2$, where V_1, V_2 are the columns with real entries defined by

$$V_1 = |\Omega|^{-1} \int_\Omega (\nabla\Psi(\mathbf{x}))^t g(\mathbf{x}) \nabla\widetilde{\Lambda}_2(\mathbf{x})\, d\mathbf{x}, \quad V_2 = -|\Omega|^{-1} \int_\Omega (\nabla\Psi(\mathbf{x}))^t g(\mathbf{x}) \nabla\widetilde{\Lambda}_1(\mathbf{x})\, d\mathbf{x}.$$

By (1.36), the constant W is given by

$$W = |\Omega|^{-1} \int_\Omega \big(\langle g(\mathbf{x})\nabla\widetilde{\Lambda}_1(\mathbf{x}), \nabla\widetilde{\Lambda}_1(\mathbf{x})\rangle + \langle g(\mathbf{x})\nabla\widetilde{\Lambda}_2(\mathbf{x}), \nabla\widetilde{\Lambda}_2(\mathbf{x})\rangle\big)\, d\mathbf{x}.$$

The effective operator for $\mathcal{B}_{D,\varepsilon}$ acts as follows:

$$\mathcal{B}_D^0 u = -\operatorname{div} g^0 \nabla u + 2i\langle \nabla u, V_1 + \overline{\boldsymbol{\eta}}\rangle + (-W + \overline{Q} + \lambda \overline{Q_0})u, \quad u \in H^2(\mathcal{O}) \cap H_0^1(\mathcal{O}).$$

The corresponding differential expression can be written as

$$\mathcal{B}^0 = (\mathbf{D} - \mathbf{A}^0)^* g^0 (\mathbf{D} - \mathbf{A}^0) + \mathcal{V}^0 + \lambda \overline{Q_0}, \tag{4.10}$$

where

$$\mathbf{A}^0 = (g^0)^{-1}(V_1 + \overline{g\mathbf{A}}), \quad \mathcal{V}^0 = \overline{\mathcal{V}} + \overline{\langle g\mathbf{A}, \mathbf{A}\rangle} - \langle g^0 \mathbf{A}^0, \mathbf{A}^0\rangle - W.$$

Let $f_0 := (\overline{Q_0})^{-1/2}$. Denote $\widetilde{\mathcal{B}}_D^0 := f_0 \mathcal{B}_D^0 f_0$.

4.3. Approximation of the sandwiched operator exponential. In accordance with Remark 1.16, Conditions 1.12 and 1.14 are satisfied in the case under consideration, and the norms $\|\Lambda\|_{L_\infty}$ and $\|\widetilde{\Lambda}\|_{L_\infty}$ are estimated in terms of the problem data (4.9). Therefore, we can use a corrector that involves no smoothing operator:

$$\mathcal{K}_D^0(t;\varepsilon) := \big([\Lambda^\varepsilon]\mathbf{D} + [\widetilde{\Lambda}^\varepsilon]\big) f_0 e^{-\widetilde{\mathcal{B}}_D^0 t} f_0 = \big([\Psi^\varepsilon]\nabla + [\widetilde{\Lambda}^\varepsilon]\big) f_0 e^{-\widetilde{\mathcal{B}}_D^0 t} f_0. \tag{4.11}$$

The operator (2.33) takes the form $\mathcal{G}_D^0(t;\varepsilon) = -i\mathfrak{G}_D^0(t;\varepsilon)$, where

$$\mathfrak{G}_D^0(t;\varepsilon) = \widetilde{g}^\varepsilon \nabla f_0 e^{-\widetilde{\mathcal{B}}_D^0 t} f_0 + g^\varepsilon (\nabla \widetilde{\Lambda})^\varepsilon f_0 e^{-\widetilde{\mathcal{B}}_D^0 t} f_0. \tag{4.12}$$

Theorems 2.2 and 2.6 imply the following result.

Proposition 4.1. *Under the assumptions of Subsections* 4.1 *and* 4.2, *suppose that the operators* $\mathcal{K}_D^0(t;\varepsilon)$ *and* $\mathfrak{G}_D^0(t;\varepsilon)$ *are given by* (4.11) *and* (4.12), *respectively. Suppose that the number* ε_1 *is subject to Condition* 1.7. *Then for* $0 < \varepsilon \le \varepsilon_1$ *we have*

$$\big\| f^\varepsilon e^{-\widetilde{\mathcal{B}}_{D,\varepsilon} t} f^\varepsilon - f_0 e^{-\widetilde{\mathcal{B}}_D^0 t} f_0 \big\|_{L_2(\mathcal{O}) \to L_2(\mathcal{O})} \le C_{15} \varepsilon (t + \varepsilon^2)^{-1/2} e^{-c_\flat t/2}, \quad t \ge 0;$$

$$\big\| f^\varepsilon e^{-\widetilde{\mathcal{B}}_{D,\varepsilon} t} f^\varepsilon - f_0 e^{-\widetilde{\mathcal{B}}_D^0 t} f_0 - \varepsilon \mathcal{K}_D^0(t;\varepsilon) \big\|_{L_2(\mathcal{O}) \to H^1(\mathcal{O})} \le C_{18} (\varepsilon^{1/2} t^{-3/4} + \varepsilon t^{-1}) e^{-c_\flat t/2}, \quad t > 0;$$

$$\big\| g^\varepsilon \nabla f^\varepsilon e^{-\widetilde{\mathcal{B}}_{D,\varepsilon} t} f^\varepsilon - \mathfrak{G}_D^0(t;\varepsilon) \big\|_{L_2(\mathcal{O}) \to L_2(\mathcal{O})} \le \widetilde{C}_{18} (\varepsilon^{1/2} t^{-3/4} + \varepsilon t^{-1}) e^{-c_\flat t/2}, \quad t > 0.$$

The constants C_{15}, C_{18}, *and* $\widetilde{C}_{18}$ *depend only on the problem data* (4.9).

4.4. Homogenization of the first initial boundary-value problem for a parabolic equation with singular potential. Consider the first initial boundary-value problem for a nonhomogeneous parabolic equation with singular potential:

$$\begin{cases} Q_0^\varepsilon(\mathbf{x}) \frac{\partial u_\varepsilon}{\partial t}(\mathbf{x}, t) = -(\mathbf{D} - \mathbf{A}^\varepsilon(\mathbf{x}))^* g^\varepsilon(\mathbf{x}) (\mathbf{D} - \mathbf{A}^\varepsilon(\mathbf{x})) u_\varepsilon(\mathbf{x}, t) \\ \qquad - \big(\varepsilon^{-1} v^\varepsilon(\mathbf{x}) + \mathcal{V}^\varepsilon(\mathbf{x}) + \lambda Q_0^\varepsilon(\mathbf{x})\big) u_\varepsilon(\mathbf{x}, t) + F(\mathbf{x}, t), & \mathbf{x} \in \mathcal{O}, \quad t > 0; \\ u_\varepsilon(\,\cdot\,, t)|_{\partial\mathcal{O}} = 0, & t > 0; \\ Q_0^\varepsilon(\mathbf{x}) u_\varepsilon(\mathbf{x}, 0) = \varphi(\mathbf{x}), & \mathbf{x} \in \mathcal{O}. \end{cases}$$

Here $\varphi \in L_2(\mathcal{O})$ and $F \in \mathfrak{H}_r(T) := L_r((0,T); L_2(\mathcal{O}))$, $0 < T \le \infty$, for some $1 \le r \le \infty$.

By (3.3) and (4.10), the effective problem takes the form

$$\begin{cases} \overline{Q_0} \frac{\partial u_0}{\partial t}(\mathbf{x}, t) = -(\mathbf{D} - \mathbf{A}^0)^* g^0 (\mathbf{D} - \mathbf{A}^0) u_0(\mathbf{x}, t) \\ \qquad - \big(\mathcal{V}^0 + \lambda \overline{Q_0}\big) u_0(\mathbf{x}, t) + F(\mathbf{x}, t), & \mathbf{x} \in \mathcal{O}, \quad t > 0; \\ u_0(\,\cdot\,, t)|_{\partial\mathcal{O}} = 0, & t > 0; \\ \overline{Q_0} u_0(\mathbf{x}, 0) = \varphi(\mathbf{x}), & \mathbf{x} \in \mathcal{O}. \end{cases}$$

Applying Theorems 3.1 and 3.6, we arrive at the following result.

Proposition 4.2. *Suppose that the number ε_1 is subject to Condition* 1.7. *Under the assumptions of Subsection* 4.4, *let $1 < r \leq \infty$. Then for $0 < \varepsilon \leq \varepsilon_1$ and $0 < t < T$ we have*

$$\|u_\varepsilon(\,\cdot\,,t) - u_0(\,\cdot\,,t)\|_{L_2(\mathcal{O})} \leq C_{15}\varepsilon(t+\varepsilon^2)^{-1/2}e^{-c_\flat t/2}\|\varphi\|_{L_2(\mathcal{O})} + c_r\theta(\varepsilon,r)\|F\|_{\mathfrak{H}_r(T)}.$$

Here $\theta(\varepsilon,r)$ is given by (3.5).

Assuming that $t \geq \varepsilon^2$, we put $w_\varepsilon(\,\cdot\,,t) := f_0 e^{-\widetilde{\mathcal{B}}^0_D\varepsilon^2} f_0^{-1} u_0(\,\cdot\,,t-\varepsilon^2)$. Denote $\breve{v}_\varepsilon(\,\cdot\,,t) := u_0(\,\cdot\,,t) + \varepsilon\Psi^\varepsilon\nabla w_\varepsilon(\,\cdot\,,t) + \varepsilon\widetilde{\Lambda}^\varepsilon w_\varepsilon(\,\cdot\,,t)$ and $\breve{q}_\varepsilon(\,\cdot\,,t) := \widetilde{g}^\varepsilon\nabla w_\varepsilon(\,\cdot\,,t) + g^\varepsilon\big(\nabla\widetilde{\Lambda}\big)^\varepsilon w_\varepsilon(\,\cdot\,,t)$. Moreover, assume that $2 < r \leq \infty$. Then for $0 < \varepsilon \leq \varepsilon_1$ and $\varepsilon^2 \leq t < T$ we have

$$\|u_\varepsilon(\,\cdot\,,t) - \breve{v}_\varepsilon(\,\cdot\,,t)\|_{H^1(\mathcal{O})} \leq 2C_{18}\varepsilon^{1/2}t^{-3/4}e^{-c_\flat t/2}\|\varphi\|_{L_2(\mathcal{O})} + c_r'\omega(\varepsilon,r)\|F\|_{\mathfrak{H}_r(t)},$$

$$\|g^\varepsilon\nabla u_\varepsilon(\,\cdot\,,t) - \breve{q}_\varepsilon(\,\cdot\,,t)\|_{L_2(\mathcal{O})} \leq 2\widetilde{C}_{18}\varepsilon^{1/2}t^{-3/4}e^{-c_\flat t/2}\|\varphi\|_{L_2(\mathcal{O})} + c_r''\omega(\varepsilon,r)\|F\|_{\mathfrak{H}_r(t)}.$$

Here $\omega(\varepsilon,r)$ is given by (3.7). *The constants C_{15}, C_{18}, and $\widetilde{C}_{18}$ depend only on the problem data* (4.9). *The constants c_r, c_r', and c_r'' depend on the same parameters and also on r.*

§5. The scalar operator with a strongly singular potential of order ε^{-2}

Homogenization of the first initial boundary-value problem for parabolic equation with a strongly singular potential was studied in [AlCPiSiVa]. Some motivations can be found in [AlCPiSiVa, §1]). However, the results of [AlCPiSiVa] cannot be formulated in the uniform operator topology.

5.1. Description of the operator. Let $\breve{g}(\mathbf{x})$ be a Γ-periodic symmetric $(d\times d)$-matrix-valued function in $\mathbb{R}^d$ with real entries such that $\breve{g}, \breve{g}^{-1} \in L_\infty$ and $\breve{g}(\mathbf{x}) > 0$. Let $\breve{v}(\mathbf{x})$ be a real-valued Γ-periodic function such that

$$\breve{v} \in L_s(\Omega), \quad s = 1 \ \text{ for } \ d = 1, \quad s > d/2 \ \text{ for } \ d \geq 2.$$

Let $\breve{\mathcal{A}}$ denote the operator in $L_2(\mathbb{R}^d)$ that corresponds to the quadratic form

$$\int_{\mathbb{R}^d} \big(\langle \breve{g}(\mathbf{x})\mathbf{D}u, \mathbf{D}u\rangle + \breve{v}(\mathbf{x})|u|^2\big)\, d\mathbf{x}, \quad u \in H^1(\mathbb{R}^d).$$

Adding a constant to the potential $\breve{v}(\mathbf{x})$, we assume that the bottom of the spectrum of $\breve{\mathcal{A}}$ is the point zero. Then the operator $\breve{\mathcal{A}}$ admits a factorization with the help of the eigenfunction of the operator $\mathbf{D}^*\breve{g}(\mathbf{x})\mathbf{D} + \breve{v}(\mathbf{x})$ on the cell Ω (with periodic boundary conditions) corresponding to the eigenvalue $\lambda = 0$ (see [BSu2, Chapter 6, Subsection 1.1]). Apparently, this factorization trick was used in homogenization problems for the first time in [Zh1, K].

In $L_2(\mathcal{O})$, we consider the operator $\breve{\mathcal{A}}_D$ given by the expression $\mathbf{D}^*\breve{g}(\mathbf{x})\mathbf{D} + \breve{v}(\mathbf{x})$ with the Dirichlet condition on $\partial\mathcal{O}$. The precise definition of $\breve{\mathcal{A}}_D$ is given in terms of the quadratic form

$$\breve{\mathfrak{a}}_D[u,u] = \int_{\mathcal{O}} \big(\langle \breve{g}(\mathbf{x})\mathbf{D}u, \mathbf{D}u\rangle + \breve{v}(\mathbf{x})|u|^2\big)\, d\mathbf{x}, \quad u \in H^1_0(\mathcal{O}). \tag{5.1}$$

The operator $\breve{\mathcal{A}}_D$ inherits factorization of the operator $\breve{\mathcal{A}}$. To describe this factorization, we consider the equation

$$\mathbf{D}^*\breve{g}(\mathbf{x})\mathbf{D}\omega(\mathbf{x}) + \breve{v}(\mathbf{x})\omega(\mathbf{x}) = 0. \tag{5.2}$$

There exists a Γ-periodic solution $\omega \in \widetilde{H}^1(\Omega)$ of this equation defined up to a constant factor. We can fix this factor so that $\omega(\mathbf{x}) > 0$ and

$$\int_\Omega \omega^2(\mathbf{x})\, d\mathbf{x} = |\Omega|. \tag{5.3}$$

Moreover, the solution is positive definite and bounded: $0 < \omega_0 \le \omega(\mathbf{x}) \le \omega_1 < \infty$. The norms $\|\omega\|_{L_\infty}$ and $\|\omega^{-1}\|_{L_\infty}$ are controlled in terms of $\|\breve{g}\|_{L_\infty}$, $\|\breve{g}^{-1}\|_{L_\infty}$, and $\|\breve{v}\|_{L_s(\Omega)}$. Note that ω and ω^{-1} are multipliers in $H_0^1(\mathcal{O})$.

Substituting $u = \omega z$ and taking (5.2) into account, we represent the form (5.1) as

$$\breve{\mathfrak{a}}_D[u,u] = \int_{\mathcal{O}} \omega(\mathbf{x})^2 \langle \breve{g}(\mathbf{x})\mathbf{D}z, \mathbf{D}z\rangle\, d\mathbf{x}, \quad u = \omega z, \quad z \in H_0^1(\mathcal{O}).$$

Hence, the differential expression for the operator $\breve{\mathcal{A}}_D$ admits the factorization

$$\breve{\mathcal{A}} = \omega^{-1}\mathbf{D}^* g \mathbf{D}\omega^{-1}, \quad g = \omega^2 \breve{g}. \tag{5.4}$$

Now, we consider the operator $\breve{\mathcal{A}}_{D,\varepsilon}$ with rapidly oscillating coefficients acting in $L_2(\mathcal{O})$ and given by

$$\breve{\mathcal{A}}_\varepsilon = (\omega^\varepsilon)^{-1}\mathbf{D}^* g^\varepsilon \mathbf{D}(\omega^\varepsilon)^{-1}, \quad g = \omega^2 \breve{g}, \tag{5.5}$$

with the Dirichlet boundary condition. In the initial terms, (5.5) takes the form

$$\breve{\mathcal{A}}_\varepsilon = \mathbf{D}^* \breve{g}^\varepsilon \mathbf{D} + \varepsilon^{-2}\breve{v}^\varepsilon. \tag{5.6}$$

Next, let $\mathbf{A}(\mathbf{x}) = \operatorname{col}\{A_1(\mathbf{x}), \dots, A_d(\mathbf{x})\}$, where the $A_j(\mathbf{x})$ are Γ-periodic real-valued functions satisfying (4.1). Let $\widehat{v}(\mathbf{x})$ and $\breve{\mathcal{V}}(\mathbf{x})$ be Γ-periodic real-valued functions such that

$$\widehat{v}, \breve{\mathcal{V}} \in L_s(\Omega), \quad s = 1 \text{ for } d = 1, \quad s > d/2 \text{ for } d \ge 2; \int_\Omega \widehat{v}(\mathbf{x})\omega^2(\mathbf{x})\, d\mathbf{x} = 0. \tag{5.7}$$

In $L_2(\mathcal{O})$, we consider the operator $\widetilde{\mathfrak{B}}_{D,\varepsilon}$ given formally by the differential expression

$$\widetilde{\mathfrak{B}}_\varepsilon = (\mathbf{D} - \mathbf{A}^\varepsilon)^* \breve{g}^\varepsilon (\mathbf{D} - \mathbf{A}^\varepsilon) + \varepsilon^{-2}\breve{v}^\varepsilon + \varepsilon^{-1}\widehat{v}^\varepsilon + \breve{\mathcal{V}}^\varepsilon$$

with the Dirichlet condition on $\partial\mathcal{O}$. The precise definition is given in terms of a quadratic form.

We put

$$v(\mathbf{x}) := \widehat{v}(\mathbf{x})\omega^2(\mathbf{x}), \quad \mathcal{V}(\mathbf{x}) := \breve{\mathcal{V}}(\mathbf{x})\omega^2(\mathbf{x}). \tag{5.8}$$

By (5.5) and (5.6), we have $\widetilde{\mathfrak{B}}_{D,\varepsilon} = (\omega^\varepsilon)^{-1}\mathfrak{B}_{D,\varepsilon}(\omega^\varepsilon)^{-1}$, where the operator $\mathfrak{B}_{D,\varepsilon}$ is given by (4.3) with the Dirichlet condition on $\partial\mathcal{O}$; g is defined by (5.4), and v, $\mathcal{V}$ are given by (5.8). By (5.7) and the properties of ω, the coefficients v and $\mathcal{V}$ satisfy (4.2). Then the operator $\mathfrak{B}_{D,\varepsilon}$ can be represented as in (4.4), where the a_j, $j = 1, \dots, d$, and Q are constructed in terms of g, $\mathbf{A}$, v, and $\mathcal{V}$ in accordance with (4.5), (4.6).

The constant λ is chosen as in (1.16) for the operator with the same coefficients g, a_j, $j = 1, \dots, d$, and Q as the coefficients of $\mathfrak{B}_{D,\varepsilon}$, and with $Q_0(\mathbf{x}) := \omega^2(\mathbf{x})$. Then the operators $\widetilde{\mathcal{B}}_{D,\varepsilon} := \widetilde{\mathfrak{B}}_{D,\varepsilon} + \lambda I$ and $\mathcal{B}_{D,\varepsilon} := \mathfrak{B}_{D,\varepsilon} + \lambda Q_0^\varepsilon$ are related by $\widetilde{\mathcal{B}}_{D,\varepsilon} = (\omega^\varepsilon)^{-1}\mathcal{B}_{D,\varepsilon}(\omega^\varepsilon)^{-1}$.

The following set of parameters is called the "problem data":

$$\begin{aligned} &d, \rho, s; \quad \|\breve{g}\|_{L_\infty}, \ \|\breve{g}^{-1}\|_{L_\infty}, \ \|\mathbf{A}\|_{L_\rho(\Omega)}, \ \|\breve{v}\|_{L_s(\Omega)}, \ \|\widehat{v}\|_{L_s(\Omega)}, \ \|\breve{\mathcal{V}}\|_{L_s(\Omega)}; \\ &\text{the parameters of the lattice } \Gamma; \quad \text{the domain } \mathcal{O}. \end{aligned} \tag{5.9}$$

5.2. Homogenization of the first initial boundary-value problem for the parabolic equation with strongly singular potential. We apply Proposition 4.1 to the operator $\widetilde{\mathcal{B}}_{D,\varepsilon}$ described in Subsection 5.1. We have $f(\mathbf{x}) = \omega(\mathbf{x})^{-1}$, whence, by (5.3), $f_0 = 1$ and $\widetilde{\mathcal{B}}_D^0 = \mathcal{B}_D^0$. The coefficients g^0, $\mathbf{A}^0$, and $\mathcal{V}^0$ of the effective operator are constructed in terms of g, $\mathbf{A}$, v, and $\mathcal{V}$ (see (5.5) and (5.8)), as described in Subsection 4.2. We apply the results to homogenization of the solution of the first initial boundary-value problem

$$\begin{cases} \frac{\partial u_\varepsilon}{\partial t}(\mathbf{x},t) = -(\mathbf{D}-\mathbf{A}^\varepsilon(\mathbf{x}))^* \breve{g}^\varepsilon(\mathbf{x})(\mathbf{D}-\mathbf{A}^\varepsilon(\mathbf{x}))u_\varepsilon(\mathbf{x},t) & \\ \qquad\qquad - \left(\varepsilon^{-2}\breve{v}^\varepsilon + \varepsilon^{-1}\widehat{v}^\varepsilon(\mathbf{x}) + \breve{\mathcal{V}}^\varepsilon(\mathbf{x}) + \lambda I\right)u_\varepsilon(\mathbf{x},t), & \mathbf{x}\in\mathcal{O}, \quad t>0; \\ u_\varepsilon(\,\cdot\,,t)|_{\partial\mathcal{O}} = 0, & t>0; \\ u_\varepsilon(\mathbf{x},0) = \omega^\varepsilon(\mathbf{x})^{-1}\varphi(\mathbf{x}), & \mathbf{x}\in\mathcal{O}. \end{cases}$$

Here $\varphi \in L_2(\mathcal{O})$. (For simplicity, we consider a homogeneous equation.) Then $u_\varepsilon(\,\cdot\,,t) = e^{-\widetilde{\mathcal{B}}_{D,\varepsilon}t}(\omega^\varepsilon)^{-1}\varphi$.

Let u_0 be the solution of the homogenized problem

$$\begin{cases} \frac{\partial u_0}{\partial t}(\mathbf{x},t) = -(\mathbf{D}-\mathbf{A}^0)^* g^0(\mathbf{D}-\mathbf{A}^0)u_0(\mathbf{x},t) - \left(\mathcal{V}^0+\lambda\right)u_0(\mathbf{x},t), & \mathbf{x}\in\mathcal{O}, \ t>0; \\ u_0(\,\cdot\,,t)|_{\partial\mathcal{O}} = 0, & t>0; \\ u_0(\mathbf{x},0) = \varphi(\mathbf{x}), & \mathbf{x}\in\mathcal{O}. \end{cases}$$

Proposition 4.1 implies the following result.

Proposition 5.1. *Under the assumptions of Subsection* 5.2, *denote*

$$\breve{v}_\varepsilon(\,\cdot\,,t) := u_0(\,\cdot\,,t) + \varepsilon\Psi^\varepsilon\nabla u_0(\,\cdot\,,t) + \varepsilon\widetilde{\Lambda}^\varepsilon u_0(\,\cdot\,,t),$$
$$\breve{q}_\varepsilon(\,\cdot\,,t) := \widetilde{g}^\varepsilon\nabla u_0(\,\cdot\,,t) + g^\varepsilon(\nabla\widetilde{\Lambda})^\varepsilon u_0(\,\cdot\,,t).$$

Then for $0<\varepsilon\le\varepsilon_1$ *we have*

$$\|(\omega^\varepsilon)^{-1}u_\varepsilon(\,\cdot\,,t) - u_0(\,\cdot\,,t)\|_{L_2(\mathcal{O})} \le C_{15}\varepsilon(t+\varepsilon^2)^{-1/2}e^{-c_\flat t/2}\|\varphi\|_{L_2(\mathcal{O})}, \quad t\ge 0;$$
$$\|(\omega^\varepsilon)^{-1}u_\varepsilon(\,\cdot\,,t) - \breve{v}_\varepsilon(\,\cdot\,,t)\|_{H^1(\mathcal{O})} \le C_{18}(\varepsilon^{1/2}t^{-3/4}+\varepsilon t^{-1})e^{-c_\flat t/2}\|\varphi\|_{L_2(\mathcal{O})},$$
$$\|g^\varepsilon\nabla(\omega^\varepsilon)^{-1}u_\varepsilon(\,\cdot\,,t) - \breve{q}_\varepsilon(\,\cdot\,,t)\|_{L_2(\mathcal{O})} \le \widetilde{C}_{18}(\varepsilon^{1/2}t^{-3/4}+\varepsilon t^{-1})e^{-c_\flat t/2}\|\varphi\|_{L_2(\mathcal{O})},$$

$t>0$. *The constants* C_{15}, C_{18}, *and* $\widetilde{C}_{18}$ *depend on the problem data* (5.9).

Note that, in the presence of a strongly singular potential in the equation, not the solution u_ε itself, but rather the product $(\omega^\varepsilon)^{-1}u_\varepsilon$ admits a "good approximation". This shows that the nature of the results of §5 differs from that of §4.

Appendix

In the Appendix, we consider the case where $d\ge 3$ and justify the removal of the smoothing operator S_ε in the case of sufficiently smooth boundary (Lemma 2.9 and Theorem 2.10) and in the case of a strictly interior subdomain (Lemma 2.16 and Theorem 2.17).

§6. The properties of the matrix-valued functions Λ and $\widetilde{\Lambda}$

We need the following results; see [PSu, Lemma 2.3] and [MSu2, Lemma 3.4].

Lemma 6.1. *Let* Λ *be the* Γ*-periodic solution of problem* (1.25). *Then for any function* $u\in C_0^\infty(\mathbb{R}^d)$ *and* $\varepsilon>0$ *we have*

$$\int_{\mathbb{R}^d} |(\mathbf{D}\Lambda)^\varepsilon(\mathbf{x})|^2|u(\mathbf{x})|^2\,d\mathbf{x} \le \beta_1\|u\|^2_{L_2(\mathbb{R}^d)} + \beta_2\varepsilon^2\int_{\mathbb{R}^d}|\Lambda^\varepsilon(\mathbf{x})|^2|\mathbf{D}u(\mathbf{x})|^2\,d\mathbf{x}.$$

The constants β_1 *and* β_2 *depend on* m, d, α_0, α_1, $\|g\|_{L_\infty}$, *and* $\|g^{-1}\|_{L_\infty}$.

Lemma 6.2. *Let* $\widetilde{\Lambda}$ *be the* Γ*-periodic solution of problem* (1.33). *Then for any function* $u \in C_0^\infty(\mathbb{R}^d)$ *and* $0 < \varepsilon \le 1$ *we have*

$$\int_{\mathbb{R}^d} |(\mathbf{D}\widetilde{\Lambda})^\varepsilon(\mathbf{x})|^2 |u(\mathbf{x})|^2 \, d\mathbf{x} \le \widetilde{\beta}_1 \|u\|^2_{H^1(\mathbb{R}^d)} + \widetilde{\beta}_2 \varepsilon^2 \int_{\mathbb{R}^d} |\widetilde{\Lambda}^\varepsilon(\mathbf{x})|^2 |\mathbf{D}u(\mathbf{x})|^2 \, d\mathbf{x}.$$

The constants $\widetilde{\beta}_1$ *and* $\widetilde{\beta}_2$ *depend only on* n, d, α_0, α_1, ρ, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, *the norms* $\|a_j\|_{L_\rho(\Omega)}$, $j = 1, \dots, d$, *and the parameters of the lattice* Γ.

Below in §7 we shall need the following multiplier properties of the matrix-valued functions $\Lambda(\mathbf{x})$ and $\widetilde{\Lambda}(\mathbf{x})$.

Lemma 6.3. *Suppose that a matrix-valued function* $\Lambda(\mathbf{x})$ *is the* Γ*-periodic solution of problem* (1.25). *Let* $d \ge 3$ *and put* $l = d/2$.
1°. *For* $0 < \varepsilon \le 1$ *and* $\mathbf{u} \in H^{l-1}(\mathbb{R}^d; \mathbb{C}^m)$ *we have* $\Lambda^\varepsilon \mathbf{u} \in L_2(\mathbb{R}^d; \mathbb{C}^n)$ *and*

$$\|\Lambda^\varepsilon \mathbf{u}\|_{L_2(\mathbb{R}^d)} \le C^{(0)} \|\mathbf{u}\|_{H^{l-1}(\mathbb{R}^d)}. \tag{6.1}$$

2°. *For* $0 < \varepsilon \le 1$ *and* $\mathbf{u} \in H^l(\mathbb{R}^d; \mathbb{C}^m)$ *we have* $\Lambda^\varepsilon \mathbf{u} \in H^1(\mathbb{R}^d; \mathbb{C}^n)$ *and*

$$\|\Lambda^\varepsilon \mathbf{u}\|_{H^1(\mathbb{R}^d)} \le C^{(1)} \varepsilon^{-1} \|\mathbf{u}\|_{L_2(\mathbb{R}^d)} + C^{(2)} \|\mathbf{u}\|_{H^l(\mathbb{R}^d)}. \tag{6.2}$$

The constants $C^{(0)}$, $C^{(1)}$, *and* $C^{(2)}$ *depend on* m, d, α_0, α_1, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, *and the parameters of the lattice* Γ.

Proof. It suffices to check (6.1) and (6.2) for $\mathbf{u} \in C_0^\infty(\mathbb{R}^d; \mathbb{C}^m)$. Substituting $\mathbf{x} = \varepsilon \mathbf{y}$, $\varepsilon^{d/2} \mathbf{u}(\mathbf{x}) = \mathbf{U}(\mathbf{y})$, we obtain

$$\begin{aligned} \|\Lambda^\varepsilon \mathbf{u}\|^2_{L_2(\mathbb{R}^d)} &\le \int_{\mathbb{R}^d} |\Lambda(\varepsilon^{-1}\mathbf{x})|^2 |\mathbf{u}(\mathbf{x})|^2 \, d\mathbf{x} = \int_{\mathbb{R}^d} |\Lambda(\mathbf{y})|^2 |\mathbf{U}(\mathbf{y})|^2 \, d\mathbf{y} \\ &= \sum_{\mathbf{a} \in \Gamma} \int_{\Omega + \mathbf{a}} |\Lambda(\mathbf{y})|^2 |\mathbf{U}(\mathbf{y})|^2 \, d\mathbf{y} \le \sum_{\mathbf{a} \in \Gamma} \|\Lambda\|^2_{L_{2\nu}(\Omega)} \|\mathbf{U}\|^2_{L_{2\nu'}(\Omega + \mathbf{a})}, \end{aligned} \tag{6.3}$$

where $\nu^{-1} + (\nu')^{-1} = 1$. We choose ν so that the embedding $H^1(\Omega) \hookrightarrow L_{2\nu}(\Omega)$ is continuous, i.e., $\nu = d(d-2)^{-1}$. Then

$$\|\Lambda\|^2_{L_{2\nu}(\Omega)} \le c_\Omega \|\Lambda\|^2_{H^1(\Omega)}, \tag{6.4}$$

where the constant c_Ω depends only on the dimension d and the lattice Γ. We have $2\nu' = d$. Since the embedding $H^{l-1}(\Omega) \hookrightarrow L_d(\Omega)$ is continuous, we have

$$\|\mathbf{U}\|^2_{L_d(\Omega + \mathbf{a})} \le c'_\Omega \|\mathbf{U}\|^2_{H^{l-1}(\Omega + \mathbf{a})}, \tag{6.5}$$

where the constant c'_Ω depends only on the dimension d and the lattice Γ. Now, from (6.3)–(6.5) it follows that

$$\int_{\mathbb{R}^d} |\Lambda^\varepsilon(\mathbf{x})|^2 |\mathbf{u}(\mathbf{x})|^2 \, d\mathbf{x} \le c_\Omega c'_\Omega \|\Lambda\|^2_{H^1(\Omega)} \|\mathbf{U}\|^2_{H^{l-1}(\mathbb{R}^d)}. \tag{6.6}$$

Obviously, for $0 < \varepsilon \le 1$ we have $\|\mathbf{U}\|_{H^{l-1}(\mathbb{R}^d)} \le \|\mathbf{u}\|_{H^{l-1}(\mathbb{R}^d)}$. Combining this with (1.28) and (6.6), we see that

$$\int_{\mathbb{R}^d} |\Lambda^\varepsilon(\mathbf{x})|^2 |\mathbf{u}(\mathbf{x})|^2 \, d\mathbf{x} \le c_\Omega c'_\Omega M^2 \|\mathbf{u}\|^2_{H^{l-1}(\mathbb{R}^d)}, \quad \mathbf{u} \in C_0^\infty(\mathbb{R}^d; \mathbb{C}^m), \tag{6.7}$$

which proves estimate (6.1) with the constant $C^{(0)} := (c_\Omega c'_\Omega)^{1/2} M$.

Next, by Lemma 6.1,

$$\begin{aligned} \|\mathbf{D}(\Lambda^\varepsilon \mathbf{u})\|^2_{L_2(\mathbb{R}^d)} &\le 2\varepsilon^{-2} \int_{\mathbb{R}^d} |(\mathbf{D}\Lambda)^\varepsilon(\mathbf{x})\mathbf{u}(\mathbf{x})|^2\,d\mathbf{x} + 2\int_{\mathbb{R}^d} |\Lambda^\varepsilon(\mathbf{x})|^2 |\mathbf{D}\mathbf{u}(\mathbf{x})|^2\,d\mathbf{x} \\ &\le 2\beta_1\varepsilon^{-2}\int_{\mathbb{R}^d} |\mathbf{u}(\mathbf{x})|^2\,d\mathbf{x} + 2(1+\beta_2)\int_{\mathbb{R}^d} |\Lambda^\varepsilon(\mathbf{x})|^2 |\mathbf{D}\mathbf{u}(\mathbf{x})|^2\,d\mathbf{x}. \end{aligned} \tag{6.8}$$

From (6.7) (with $\mathbf{u}$ replaced by the derivatives $\partial_j \mathbf{u}$) it follows that

$$\int_{\mathbb{R}^d} |\Lambda^\varepsilon(\mathbf{x})|^2 |\mathbf{D}\mathbf{u}(\mathbf{x})|^2\,d\mathbf{x} \le c_\Omega c'_\Omega M^2 \|\mathbf{u}\|^2_{H^l(\mathbb{R}^d)}, \quad \mathbf{u} \in C_0^\infty(\mathbb{R}^d;\mathbb{C}^m). \tag{6.9}$$

As a result, relations (6.7)–(6.9) imply inequality (6.2) with the constants $C^{(1)} := (2\beta_1)^{1/2}$ and $C^{(2)} := M(3+2\beta_2)^{1/2}(c_\Omega c'_\Omega)^{1/2}$. $\square$

Using the extension operator $P_{\mathcal{O}}$ satisfying estimates (1.46), we deduce the following statement from Lemma 6.3(1°).

Corollary 6.4. *Under the assumptions of Lemma* 6.3, *the operator* $[\Lambda^\varepsilon]$ *acts continuously from* $H^{l-1}(\mathcal{O};\mathbb{C}^m)$ *to* $L_2(\mathcal{O};\mathbb{C}^n)$, *and*

$$\|[\Lambda^\varepsilon]\|_{H^{l-1}(\mathcal{O})\to L_2(\mathcal{O})} \le C^{(0)} C_{\mathcal{O}}^{(l-1)}.$$

The following statement can be checked much as Lemma 6.3, by using Lemma 6.2 and estimate (1.34).

Lemma 6.5. *Suppose that a matrix-valued function* $\widetilde{\Lambda}(\mathbf{x})$ *is the* Γ*-periodic solution of problem* (1.33). *Let* $d \ge 3$ *and* $l = d/2$.
1°. *For* $0 < \varepsilon \le 1$ *and* $\mathbf{u} \in H^{l-1}(\mathbb{R}^d;\mathbb{C}^n)$, *we have* $\widetilde{\Lambda}^\varepsilon \mathbf{u} \in L_2(\mathbb{R}^d;\mathbb{C}^n)$ *and*

$$\|\widetilde{\Lambda}^\varepsilon \mathbf{u}\|_{L_2(\mathbb{R}^d)} \le \widetilde{C}^{(0)} \|\mathbf{u}\|_{H^{l-1}(\mathbb{R}^d)}.$$

2°. *For* $0 < \varepsilon \le 1$ *and* $\mathbf{u} \in H^{l}(\mathbb{R}^d;\mathbb{C}^n)$, *we have* $\widetilde{\Lambda}^\varepsilon \mathbf{u} \in H^1(\mathbb{R}^d;\mathbb{C}^n)$ *and*

$$\|\widetilde{\Lambda}^\varepsilon \mathbf{u}\|_{H^1(\mathbb{R}^d)} \le \widetilde{C}^{(1)}\varepsilon^{-1} \|\mathbf{u}\|_{H^{1}(\mathbb{R}^d)} + \widetilde{C}^{(2)} \|\mathbf{u}\|_{H^{l}(\mathbb{R}^d)}.$$

The constants

$$\widetilde{C}^{(0)} := (c_\Omega c'_\Omega)^{1/2}\widetilde{M}, \quad \widetilde{C}^{(1)} := (2\widetilde{\beta}_1)^{1/2}, \quad \widetilde{C}^{(2)} := \sqrt{2}(\widetilde{\beta}_2+1)^{1/2}(c_\Omega c'_\Omega)^{1/2}\widetilde{M}$$

depend only on the problem data (1.9).

The extension operator $P_{\mathcal{O}}$ allows us to deduce the following corollary from Lemma 6.5(1°).

Corollary 6.6. *Under the assumptions of Lemma* 6.5, *the operator* $[\widetilde{\Lambda}^\varepsilon]$ *acts continuously from* $H^{l-1}(\mathcal{O};\mathbb{C}^n)$ *to* $L_2(\mathcal{O};\mathbb{C}^n)$, *and*

$$\|[\widetilde{\Lambda}^\varepsilon]\|_{H^{l-1}(\mathcal{O})\to L_2(\mathcal{O})} \le \widetilde{C}^{(0)} C_{\mathcal{O}}^{(l-1)}.$$

§7. Removal of the smoothing operator from the corrector in the case of sufficiently smooth boundary

7.1. *Proof of Theorem* 2.10. Suppose that the assumptions of Lemma 2.9 are satisfied. Let $\mathbf{u}_0$ be given by (2.3), where $\boldsymbol{\varphi} \in L_2(\mathcal{O};\mathbb{C}^n)$. We put

$$\widetilde{\mathbf{u}}_0(\,\cdot\,,t) = P_{\mathcal{O}}\mathbf{u}_0(\,\cdot\,,t).$$

By (2.22) and (2.32), we have

$$\mathcal{K}_D(t;\varepsilon)\boldsymbol{\varphi} = \big(\Lambda^\varepsilon S_\varepsilon b(\mathbf{D}) + \widetilde{\Lambda}^\varepsilon S_\varepsilon\big)\widetilde{\mathbf{u}}_0(\,\cdot\,,t), \tag{7.1}$$

$$\mathcal{K}_D^0(t;\varepsilon)\boldsymbol{\varphi} = \big(\Lambda^\varepsilon b(\mathbf{D}) + \widetilde{\Lambda}^\varepsilon\big)\mathbf{u}_0(\,\cdot\,,t). \tag{7.2}$$

We need to estimate the following quantity:

$$\begin{aligned}(7.3)\qquad &\|\mathcal{K}_D(t;\varepsilon)\boldsymbol{\varphi} - \mathcal{K}_D^0(t;\varepsilon)\boldsymbol{\varphi}\|_{H^1(\mathcal{O})}\\ &\leq \|\Lambda^\varepsilon\big((S_\varepsilon - I)b(\mathbf{D})\widetilde{\mathbf{u}}_0\big)(\,\cdot\,,t)\|_{H^1(\mathbb{R}^d)} + \|\widetilde{\Lambda}^\varepsilon\big((S_\varepsilon - I)\widetilde{\mathbf{u}}_0\big)(\,\cdot\,,t)\|_{H^1(\mathbb{R}^d)}.\end{aligned}$$

Under the above assumptions, Lemma 2.8 shows that $\mathbf{u}_0 \in H^{l+1}(\mathcal{O};\mathbb{C}^n)$, whence $\widetilde{\mathbf{u}}_0 \in H^{l+1}(\mathbb{R}^d;\mathbb{C}^n)$. This makes it possible to apply Lemma 6.3(2°) to estimate the first summand on the right-hand side of (7.3):

$$\begin{aligned}(7.4)\qquad &\|\Lambda^\varepsilon\big((S_\varepsilon - I)b(\mathbf{D})\widetilde{\mathbf{u}}_0\big)(\,\cdot\,,t)\|_{H^1(\mathbb{R}^d)}\\ &\leq C^{(1)}\varepsilon^{-1}\|\big((S_\varepsilon - I)b(\mathbf{D})\widetilde{\mathbf{u}}_0\big)(\,\cdot\,,t)\|_{L_2(\mathbb{R}^d)} + C^{(2)}\|\big((S_\varepsilon - I)b(\mathbf{D})\widetilde{\mathbf{u}}_0\big)(\,\cdot\,,t)\|_{H^l(\mathbb{R}^d)},\end{aligned}$$

where $l = d/2$. The first term on the right-hand side of (7.4) is estimated with the help of Proposition 1.1 and formulas (1.3), (1.43), (1.46), (2.3), and (2.8):

$$\begin{aligned}(7.5)\qquad \varepsilon^{-1}\|\big((S_\varepsilon - I)b(\mathbf{D})\widetilde{\mathbf{u}}_0\big)(\,\cdot\,,t)\|_{L_2(\mathbb{R}^d)} &\leq r_1\|\mathbf{D}b(\mathbf{D})\widetilde{\mathbf{u}}_0(\,\cdot\,,t)\|_{L_2(\mathbb{R}^d)}\\ &\leq r_1\alpha_1^{1/2}C_{\mathcal{O}}^{(2)}\|\mathbf{u}_0(\,\cdot\,,t)\|_{H^2(\mathcal{O})} \leq C^{(3)}t^{-1}e^{-c_\flat t/2}\|\boldsymbol{\varphi}\|_{L_2(\mathcal{O})},\end{aligned}$$

where $C^{(3)} := r_1\alpha_1^{1/2}C_{\mathcal{O}}^{(2)}\widetilde{c}\|f\|_{L_\infty}$. To estimate the second term on the right-hand side of (7.4), we apply (1.2) and (1.3):

$$(7.6)\qquad \big\|\big((S_\varepsilon - I)b(\mathbf{D})\widetilde{\mathbf{u}}_0\big)(\,\cdot\,,t)\big\|_{H^l(\mathbb{R}^d)} \leq 2\alpha_1^{1/2}\|\widetilde{\mathbf{u}}_0(\,\cdot\,,t)\|_{H^{l+1}(\mathbb{R}^d)}.$$

By (1.43), (1.46), (2.3), and Lemma 2.8, we have

$$(7.7)\qquad \|\widetilde{\mathbf{u}}_0(\,\cdot\,,t)\|_{H^{l+1}(\mathbb{R}^d)} \leq C_{\mathcal{O}}^{(l+1)}\widehat{\mathrm{C}}_{l+1}\|f\|_{L_\infty}^2 t^{-(l+1)/2}e^{-c_\flat t/2}\|\boldsymbol{\varphi}\|_{L_2(\mathcal{O})}.$$

From (7.6) and (7.7) it follows that

$$(7.8)\qquad \big\|\big((S_\varepsilon - I)b(\mathbf{D})\widetilde{\mathbf{u}}_0\big)(\,\cdot\,,t)\big\|_{H^l(\mathbb{R}^d)} \leq C^{(4)}t^{-(l+1)/2}e^{-c_\flat t/2}\|\boldsymbol{\varphi}\|_{L_2(\mathcal{O})},$$

where $C^{(4)} := 2\alpha_1^{1/2}C_{\mathcal{O}}^{(l+1)}\widehat{\mathrm{C}}_{l+1}\|f\|_{L_\infty}^2$.

Now we estimate the second term on the right-hand side of (7.3). By Lemma 6.5(2°),

$$\begin{aligned}(7.9)\qquad &\big\|\widetilde{\Lambda}^\varepsilon\big((S_\varepsilon - I)\widetilde{\mathbf{u}}_0\big)(\,\cdot\,,t)\big\|_{H^1(\mathbb{R}^d)}\\ &\leq \widetilde{C}^{(1)}\varepsilon^{-1}\|(S_\varepsilon - I)\widetilde{\mathbf{u}}_0(\,\cdot\,,t)\|_{H^1(\mathbb{R}^d)} + \widetilde{C}^{(2)}\|(S_\varepsilon - I)\widetilde{\mathbf{u}}_0(\,\cdot\,,t)\|_{H^l(\mathbb{R}^d)}, \quad l = d/2.\end{aligned}$$

The first summand on the right-hand side of (7.9) is estimated by using Proposition 1.1 and relations (1.43), (1.46), (2.3), (2.8):

$$\begin{aligned}(7.10)\qquad \varepsilon^{-1}\|(S_\varepsilon - I)\widetilde{\mathbf{u}}_0(\,\cdot\,,t)\|_{H^1(\mathbb{R}^d)} \leq r_1C_{\mathcal{O}}^{(2)}\|\mathbf{u}_0(\,\cdot\,,t)\|_{H^2(\mathcal{O})} &\leq C^{(5)}t^{-1}e^{-c_\flat t/2}\|\boldsymbol{\varphi}\|_{L_2(\mathcal{O})};\\ C^{(5)} &:= r_1C_{\mathcal{O}}^{(2)}\widetilde{c}\|f\|_{L_\infty}.\end{aligned}$$

The second summand in (7.9) is estimated with the help of (1.2) and (7.7):

$$\begin{aligned}(7.11)\qquad \|(S_\varepsilon - I)\widetilde{\mathbf{u}}_0(\,\cdot\,,t)\|_{H^l(\mathbb{R}^d)} &\leq 2\|\widetilde{\mathbf{u}}_0(\,\cdot\,,t)\|_{H^l(\mathbb{R}^d)} \leq 2\|\widetilde{\mathbf{u}}_0(\,\cdot\,,t)\|_{H^{l+1}(\mathbb{R}^d)}\\ &\leq C^{(6)}t^{-(l+1)/2}e^{-c_\flat t/2}\|\boldsymbol{\varphi}\|_{L_2(\mathcal{O})};\\ C^{(6)} &:= 2C_{\mathcal{O}}^{(l+1)}\widehat{\mathrm{C}}_{l+1}\|f\|_{L_\infty}^2.\end{aligned}$$

As a result, relations (7.3)–(7.5) and (7.8)–(7.11) imply the inequality

$$\|\mathcal{K}_D(t;\varepsilon)\boldsymbol{\varphi} - \mathcal{K}_D^0(t;\varepsilon)\boldsymbol{\varphi}\|_{H^1(\mathcal{O})} \leq (C^{(7)}t^{-1} + C^{(8)}t^{-(l+1)/2})e^{-c_\flat t/2}\|\boldsymbol{\varphi}\|_{L_2(\mathcal{O})},$$

where $l = d/2$, $C^{(7)} := C^{(1)}C^{(3)} + \widetilde{C}^{(1)}C^{(5)}$, and $C^{(8)} := C^{(2)}C^{(4)} + \widetilde{C}^{(2)}C^{(6)}$. This proves (2.36) with the constant $\widehat{\mathcal{C}}_d := \max\{C^{(7)}; C^{(8)}\}$. □

7.2. *Proof of Theorem* 2.10. Inequality (2.37) follows directly from (2.24) and (2.36). Here, $\mathcal{C}_d := 2(\widehat{\mathcal{C}}_d + C_{16})$. We have taken into account the fact that for $t > 1$ the term εt^{-1} does not exceed $\varepsilon^{1/2} t^{-3/4}$, and for $t \leq 1$ it does not exceed $\varepsilon t^{-d/4-1/2}$ because $d \geq 3$.

Now we check (2.38). By (1.4) and (2.37),

$$\begin{aligned}\big\| g^\varepsilon b(\mathbf{D}) \big(f^\varepsilon e^{-\widetilde{B}_{D,\varepsilon} t} (f^\varepsilon)^* - f_0 e^{-\widetilde{B}_D^0 t} f_0 - \varepsilon \big(\Lambda^\varepsilon b(\mathbf{D}) + \widetilde{\Lambda}^\varepsilon \big) f_0 e^{-\widetilde{B}_D^0 t} f_0 \big) \big\|_{L_2 \to L_2} \\ \leq \|g\|_{L_\infty} (d\alpha_1)^{1/2} \mathcal{C}_d (\varepsilon^{1/2} t^{-3/4} + \varepsilon t^{-d/4-1/2}) e^{-c_\flat t/2}.\end{aligned} \tag{7.12}$$

We have

$$\begin{aligned}\varepsilon g^\varepsilon b(\mathbf{D}) \big(\Lambda^\varepsilon b(\mathbf{D}) + \widetilde{\Lambda}^\varepsilon \big) f_0 e^{-\widetilde{B}_D^0 t} f_0 = g^\varepsilon \big((b(\mathbf{D}) \Lambda)^\varepsilon + \big(b(\mathbf{D}) \widetilde{\Lambda} \big)^\varepsilon \big) f_0 e^{-\widetilde{B}_D^0 t} f_0 \\ + \varepsilon \sum_{k,j=1}^d g^\varepsilon b_k \Lambda^\varepsilon b_j D_k D_j f_0 e^{-\widetilde{B}_D^0 t} f_0 + \varepsilon \sum_{j=1}^d g^\varepsilon b_j \widetilde{\Lambda}^\varepsilon D_j f_0 e^{-\widetilde{B}_D^0 t} f_0.\end{aligned} \tag{7.13}$$

The norm of the second summand on the right-hand side of (7.13) is estimated with the help of (1.4), (1.43), Lemma 2.8, and Corollary 6.4:

$$\varepsilon \Big\| \sum_{k,j=1}^d g^\varepsilon b_k \Lambda^\varepsilon b_j D_k D_j f_0 e^{-\widetilde{B}_D^0 t} f_0 \Big\|_{L_2(\mathcal{O}) \to L_2(\mathcal{O})} \leq C^{(9)} \varepsilon t^{-(l+1)/2} e^{-c_\flat t/2}, \tag{7.14}$$

where $l = d/2$, $C^{(9)} := \alpha_1 d C^{(0)} C_{\mathcal{O}}^{(l-1)} \widehat{\mathrm{C}}_{l+1} \|g\|_{L_\infty} \|f\|^2_{L_\infty}$. The third summand on the right-hand side of (7.13) is estimated by using (1.4), (1.43), Lemma 2.8, and Corollary 6.6:

$$\varepsilon \Big\| \sum_{j=1}^d g^\varepsilon b_j \widetilde{\Lambda}^\varepsilon D_j f_0 e^{-\widetilde{B}_D^0 t} f_0 \Big\|_{L_2(\mathcal{O}) \to L_2(\mathcal{O})} \leq C^{(10)} \varepsilon t^{-(l+1)/2} e^{-c_\flat t/2}, \tag{7.15}$$

where $l = d/2$ and

$$C^{(10)} := (d\alpha_1)^{1/2} \widetilde{C}^{(0)} C_{\mathcal{O}}^{(l-1)} \widehat{\mathrm{C}}_{l+1} \|g\|_{L_\infty} \|f\|^2_{L_\infty}.$$

As a result, relations (7.12)–(7.15) imply inequality (2.38) with the constant

$$\widetilde{\mathcal{C}}_d := \|g\|_{L_\infty} (d\alpha_1)^{1/2} \mathcal{C}_d + C^{(9)} + C^{(10)}. \qquad \square$$

§8. Removal of the smoothing operator from the corrector in a strictly interior subdomain

8.1. A property of the operator S_ε. Now we proceed to estimates in a strictly interior subdomain. We start with a simple property of the operator S_ε.

Let $\mathcal{O}'$ be a strictly interior subdomain of the domain $\mathcal{O}$, and let δ be given by (1.61). Denote

$$\mathcal{O}'' := \{\mathbf{x} \in \mathcal{O} : \operatorname{dist}\{\mathbf{x}; \partial\mathcal{O}\} > \delta/2\}, \quad \mathcal{O}''' := \{\mathbf{x} \in \mathcal{O} : \operatorname{dist}\{\mathbf{x}; \partial\mathcal{O}\} > \delta/4\}.$$

Lemma 8.1. *Let S_ε be the operator* (1.1). *Put $2r_1 = \operatorname{diam} \Omega$. Suppose that $\mathbf{v} \in L_2(\mathbb{R}^d; \mathbb{C}^m)$ and $\mathbf{v} \in H^\sigma(\mathcal{O}'''; \mathbb{C}^m)$ with some $\sigma \in \mathbb{Z}_+$. Then for $0 < \varepsilon \leq (4r_1)^{-1}\delta$ we have $S_\varepsilon \mathbf{v} \in H^\sigma(\mathcal{O}''; \mathbb{C}^m)$, and*

$$\|S_\varepsilon \mathbf{v}\|_{H^\sigma(\mathcal{O}'')} \leq \|\mathbf{v}\|_{H^\sigma(\mathcal{O}''')}.$$

Proof. By (1.1),

$$\begin{aligned}\|S_\varepsilon \mathbf{v}\|^2_{H^\sigma(\mathcal{O}'')} &= |\Omega|^{-2} \sum_{|\alpha|\le\sigma} \int_{\mathcal{O}''} d\mathbf{x} \left| \int_\Omega \mathbf{D}^\alpha \mathbf{v}(\mathbf{x}-\varepsilon\mathbf{z})\, d\mathbf{z}\right|^2 \\ &\le |\Omega|^{-1} \sum_{|\alpha|\le\sigma} \int_{\mathcal{O}''} d\mathbf{x} \int_\Omega |\mathbf{D}^\alpha \mathbf{v}(\mathbf{x}-\varepsilon\mathbf{z})|^2\, d\mathbf{z}.\end{aligned} \tag{8.1}$$

Since $0 < \varepsilon r_1 \le \delta/4$, for $\mathbf{x} \in \mathcal{O}''$ and $\mathbf{z} \in \Omega$ we have $\mathbf{x} - \varepsilon\mathbf{z} \in \mathcal{O}'''$. Hence, changing the order of integration in (8.1), we obtain the required estimate. $\square$

8.2. The cut-off function $\chi(\mathbf{x})$. We fix a smooth cut-off function $\chi(\mathbf{x})$ such that

$$\begin{aligned}&\chi \in C_0^\infty(\mathbb{R}^d), \quad 0 \le \chi(\mathbf{x}) \le 1; \quad \chi(\mathbf{x}) = 1,\ \mathbf{x} \in \mathcal{O}'; \\ &\operatorname{supp}\chi \subset \mathcal{O}''; \quad |\mathbf{D}^\alpha \chi(\mathbf{x})| \le \kappa_\sigma \delta^{-\sigma}, \quad |\alpha| = \sigma, \quad \sigma \in \mathbb{N}.\end{aligned} \tag{8.2}$$

The constants κ_σ depend only on d, σ, and the domain $\mathcal{O}$.

Lemma 8.2. *Suppose that $\chi(\mathbf{x})$ is a cut-off function satisfying* (8.2). *Let $k \in \mathbb{Z}_+$.*
1°. *For any function $\mathbf{v} \in H^k(\mathbb{R}^d;\mathbb{C}^m)$ we have*

$$\|\chi\mathbf{v}\|_{H^k(\mathbb{R}^d)} \le C_k^{(11)} \sum_{j=0}^k \delta^{-(k-j)} \|\mathbf{v}\|_{H^j(\mathcal{O}'')}. \tag{8.3}$$

2°. *For any function $\mathbf{v} \in H^{k+1}(\mathbb{R}^d;\mathbb{C}^m)$ we have*

$$\|\chi\mathbf{v}\|_{H^{k+1/2}(\mathbb{R}^d)} \le C_{k+1/2}^{(11)} \Bigl(\sum_{j=0}^{k+1} \delta^{-(k+1-j)} \|\mathbf{v}\|_{H^j(\mathcal{O}'')}\Bigr)^{1/2} \Bigl(\sum_{i=0}^{k} \delta^{-(k-i)} \|\mathbf{v}\|_{H^i(\mathcal{O}'')}\Bigr)^{1/2}. \tag{8.4}$$

The constants $C_k^{(11)}$ and $C_{k+1/2}^{(11)}$ depend on d, k, and the domain $\mathcal{O}$.

Proof. Inequality (8.3) follows from the Leibniz formula for the derivatives of the product $\chi\mathbf{v}$ and from estimates for the derivatives of χ (see (8.2)). To check (8.4), we should also take into account that

$$\|\mathbf{w}\|^2_{H^{k+1/2}(\mathbb{R}^d)} \le \|\mathbf{w}\|_{H^{k+1}(\mathbb{R}^d)} \|\mathbf{w}\|_{H^k(\mathbb{R}^d)}, \quad \mathbf{w} \in H^{k+1}(\mathbb{R}^d;\mathbb{C}^m). \qquad \square$$

8.3. Proof of Lemma 2.16. Under the assumptions of Lemma 2.16, let $\mathbf{u}_0$ be given by (2.3) with $\boldsymbol{\varphi} \in L_2(\mathcal{O};\mathbb{C}^n)$. By (1.43) and (2.7), (2.8), we have

$$\|\mathbf{D}\mathbf{u}_0(\cdot,t)\|_{L_2(\mathcal{O})} \le \|\mathbf{u}_0(\cdot,t)\|_{H^1(\mathcal{O})} \le c_3 \|f\|_{L_\infty} t^{-1/2} e^{-c_\flat t/2} \|\boldsymbol{\varphi}\|_{L_2(\mathcal{O})}, \tag{8.5}$$

$$\|\mathbf{D}\mathbf{u}_0(\cdot,t)\|_{H^1(\mathcal{O})} \le \|\mathbf{u}_0(\cdot,t)\|_{H^2(\mathcal{O})} \le \widetilde{c} \|f\|_{L_\infty} t^{-1} e^{-c_\flat t/2} \|\boldsymbol{\varphi}\|_{L_2(\mathcal{O})}. \tag{8.6}$$

Let $\widetilde{\mathbf{u}}_0 = P_\mathcal{O} \mathbf{u}_0$. Relations (7.1) and (7.2) remain valid. We need to estimate the following quantity:

$$\begin{aligned}&\bigl\|\mathcal{K}_D(t;\varepsilon)\boldsymbol{\varphi} - \mathcal{K}_D^0(t;\varepsilon)\boldsymbol{\varphi}\bigr\|_{H^1(\mathcal{O}')} \\ &\quad\le \bigl\|\Lambda^\varepsilon \chi\bigl((S_\varepsilon - I) b(\mathbf{D})\widetilde{\mathbf{u}}_0\bigr)(\cdot,t)\bigr\|_{H^1(\mathbb{R}^d)} + \bigl\|\widetilde{\Lambda}^\varepsilon \chi\bigl((S_\varepsilon - I)\widetilde{\mathbf{u}}_0\bigr)(\cdot,t)\bigr\|_{H^1(\mathbb{R}^d)}.\end{aligned} \tag{8.7}$$

Recall (cf. Subsection 2.10) that $\mathbf{u}_0(\cdot,t) \in H^\sigma(\mathcal{O}''';\mathbb{C}^n)$ for any $\sigma \in \mathbb{Z}_+$. Then the function $\widetilde{\mathbf{u}}_0(\cdot,t)$ satisfies the assumptions of Lemma 8.1 for any $\sigma \in \mathbb{Z}_+$. Hence, $(S_\varepsilon \widetilde{\mathbf{u}}_0)(\cdot,t) \in H^\sigma(\mathcal{O}'';\mathbb{C}^n)$ for $0 < \varepsilon \le (4r_1)^{-1}\delta$. Then we can apply Lemma 6.3(2°) to estimate the first summand on the right-hand side of (8.7):

$$\begin{aligned}&\bigl\|\Lambda^\varepsilon \chi\bigl((S_\varepsilon - I) b(\mathbf{D})\widetilde{\mathbf{u}}_0\bigr)(\cdot,t)\bigr\|_{H^1(\mathbb{R}^d)} \\ &\le C^{(1)} \varepsilon^{-1} \bigl\|\chi\bigl((S_\varepsilon - I) b(\mathbf{D})\widetilde{\mathbf{u}}_0\bigr)(\cdot,t)\bigr\|_{L_2(\mathbb{R}^d)} + C^{(2)} \bigl\|\chi\bigl((S_\varepsilon - I) b(\mathbf{D})\widetilde{\mathbf{u}}_0\bigr)(\cdot,t)\bigr\|_{H^l(\mathbb{R}^d)},\end{aligned} \tag{8.8}$$

$l = d/2$. The first term on the right-hand side of (8.8) is estimated by using inequality (7.5) (which is valid without additional smoothness assumption on $\partial\mathcal{O}$):

$$\varepsilon^{-1}\big\|\chi\big((S_\varepsilon - I)b(\mathbf{D})\tilde{\mathbf{u}}_0\big)(\,\cdot\,,t)\big\|_{L_2(\mathbb{R}^d)} \le C^{(3)}t^{-1}e^{-c_\flat t/2}\|\boldsymbol{\varphi}\|_{L_2(\mathcal{O})}. \tag{8.9}$$

Now, we consider the second summand on the right-hand side of (8.8). Obviously,

$$\begin{aligned}&\big\|\chi\big((S_\varepsilon - I)b(\mathbf{D})\tilde{\mathbf{u}}_0\big)(\,\cdot\,,t)\big\|_{H^l(\mathbb{R}^d)} \\ &\qquad\le \|\chi(S_\varepsilon b(\mathbf{D})\tilde{\mathbf{u}}_0)(\,\cdot\,,t)\|_{H^l(\mathbb{R}^d)} + \|\chi b(\mathbf{D})\tilde{\mathbf{u}}_0(\,\cdot\,,t)\|_{H^l(\mathbb{R}^d)}.\end{aligned} \tag{8.10}$$

To estimate the second term on the right-hand side of (8.10), we apply Lemma 8.2 and (1.4). If $l = d/2$ is an integer (i.e., the dimension d is even), then

$$\|\chi b(\mathbf{D})\tilde{\mathbf{u}}_0(\,\cdot\,,t)\|_{H^l(\mathbb{R}^d)} \le C_l^{(11)}(d\alpha_1)^{1/2}\sum_{j=0}^{l}\delta^{-(l-j)}\|\mathbf{D}\mathbf{u}_0(\,\cdot\,,t)\|_{H^j(\mathcal{O}'')}. \tag{8.11}$$

If $l = d/2 = k + 1/2$, then

$$\begin{aligned}\|\chi b(\mathbf{D})\tilde{\mathbf{u}}_0(\,\cdot\,,t)\|_{H^l(\mathbb{R}^d)} \le C_l^{(11)}(d\alpha_1)^{1/2}&\bigg(\sum_{j=0}^{k+1}\delta^{-(k+1-j)}\|\mathbf{D}\mathbf{u}_0(\,\cdot\,,t)\|_{H^j(\mathcal{O}'')}\bigg)^{1/2} \\ &\times\bigg(\sum_{\sigma=0}^{k}\delta^{-(k-\sigma)}\|\mathbf{D}\mathbf{u}_0(\,\cdot\,,t)\|_{H^\sigma(\mathcal{O}'')}\bigg)^{1/2}.\end{aligned} \tag{8.12}$$

The norms of $\mathbf{D}\mathbf{u}_0(\,\cdot\,,t)$ in $L_2(\mathcal{O};\mathbb{C}^n)$ and in $H^1(\mathcal{O};\mathbb{C}^n)$ were estimated in (8.5) and (8.6). By (1.43), (2.3), and (2.45) (with $\mathcal{O}'$ replaced by $\mathcal{O}''$), we have

$$\|\mathbf{D}\mathbf{u}_0(\,\cdot\,,t)\|_{H^\sigma(\mathcal{O}'')} \le \mathrm{C}'_{\sigma+1}\|f\|^2_{L_\infty}2^\sigma t^{-1/2}(\delta^{-2}+t^{-1})^{\sigma/2}e^{-c_\flat t/2}\|\boldsymbol{\varphi}\|_{L_2(\mathcal{O})}, \tag{8.13}$$

$\sigma \ge 2$. Using (8.5), (8.6), and (8.11)–(8.13), we arrive at the inequality

$$\|\chi b(\mathbf{D})\tilde{\mathbf{u}}_0(\,\cdot\,,t)\|_{H^l(\mathbb{R}^d)} \le C^{(12)}t^{-1/2}(\delta^{-2}+t^{-1})^{d/4}e^{-c_\flat t/2}\|\boldsymbol{\varphi}\|_{L_2(\mathcal{O})}. \tag{8.14}$$

The constant $C^{(12)}$ depends only on the problem data (1.9).

To estimate the first term on the right-hand side of (8.10), we apply Lemmas 8.1 and 8.2. Assume that $0 < \varepsilon \le (4r_1)^{-1}\delta$. By (1.4), in the case where l is an integer, we have

$$\|\chi(S_\varepsilon b(\mathbf{D})\tilde{\mathbf{u}}_0)(\,\cdot\,,t)\|_{H^l(\mathbb{R}^d)} \le C_l^{(11)}(d\alpha_1)^{1/2}\sum_{\sigma=0}^{l}\delta^{-(l-\sigma)}\|\mathbf{D}\mathbf{u}_0(\,\cdot\,,t)\|_{H^\sigma(\mathcal{O}''')}. \tag{8.15}$$

The norms of $\mathbf{D}\mathbf{u}_0(\,\cdot\,,t)$ in $L_2(\mathcal{O};\mathbb{C}^n)$ and in $H^1(\mathcal{O};\mathbb{C}^n)$ were estimated in (8.5) and (8.6). By (1.43), (2.3), and (2.45) (with $\mathcal{O}'$ replaced by $\mathcal{O}'''$),

$$\|\mathbf{D}\mathbf{u}_0(\,\cdot\,,t)\|_{H^\sigma(\mathcal{O}''')} \le \mathrm{C}'_{\sigma+1}\|f\|^2_{L_\infty}4^\sigma t^{-1/2}(\delta^{-2}+t^{-1})^{\sigma/2}e^{-c_\flat t/2}\|\boldsymbol{\varphi}\|_{L_2(\mathcal{O})}, \tag{8.16}$$

$\sigma \ge 2$. From (8.5), (8.6), (8.15), and (8.16) it follows that

$$\|\chi(S_\varepsilon b(\mathbf{D})\tilde{\mathbf{u}}_0)(\,\cdot\,,t)\|_{H^l(\mathbb{R}^d)} \le C^{(13)}t^{-1/2}(\delta^{-2}+t^{-1})^{d/4}e^{-c_\flat t/2}\|\boldsymbol{\varphi}\|_{L_2(\mathcal{O})}. \tag{8.17}$$

The constant $C^{(13)}$ depends only on the problem data (1.9). Estimate (8.17) in the case of half-integral l is checked similarly. Combining (8.8)–(8.10), (8.14), and (8.17), we estimate the first summand on the right-hand side of (8.7):

$$\begin{aligned}&\big\|\Lambda^\varepsilon\chi\big((S_\varepsilon - I)b(\mathbf{D})\tilde{\mathbf{u}}_0\big)(\,\cdot\,,t)\big\|_{H^1(\mathbb{R}^d)} \\ &\qquad\le C^{(14)}\big(t^{-1}+t^{-1/2}(\delta^{-2}+t^{-1})^{d/4}\big)e^{-c_\flat t/2}\|\boldsymbol{\varphi}\|_{L_2(\mathcal{O})}.\end{aligned} \tag{8.18}$$

Here $C^{(14)} := \max\{C^{(1)}C^{(3)}; C^{(2)}(C^{(12)}+C^{(13)})\}$.

The second summand on the right-hand side of (8.7) is estimated with the help of Lemma 6.5(2°):

$$\begin{aligned} &\big\|\widetilde{\Lambda}^\varepsilon \chi\big((S_\varepsilon - I)\widetilde{\mathbf{u}}_0\big)(\,\cdot\,,t)\big\|_{H^1(\mathbb{R}^d)} \\ &\qquad \le \widetilde{C}^{(1)}\varepsilon^{-1}\big\|\chi\big((S_\varepsilon - I)\widetilde{\mathbf{u}}_0\big)(\,\cdot\,,t)\big\|_{H^1(\mathbb{R}^d)} + \widetilde{C}^{(2)}\big\|\chi\big((S_\varepsilon - I)\widetilde{\mathbf{u}}_0\big)(\,\cdot\,,t)\big\|_{H^l(\mathbb{R}^d)}, \end{aligned} \tag{8.19}$$

where $l = d/2$. To estimate the first summand on the right-hand side of (8.19), we use (8.2) and inequality (7.10) (which is true without additional smoothness assumptions on the boundary):

$$\begin{aligned} &\varepsilon^{-1}\big\|\chi\big((S_\varepsilon - I)\widetilde{\mathbf{u}}_0\big)(\,\cdot\,,t)\big\|_{H^1(\mathbb{R}^d)} \\ &\qquad \le \varepsilon^{-1}\big\|\big((S_\varepsilon - I)\widetilde{\mathbf{u}}_0\big)(\,\cdot\,,t)\big\|_{H^1(\mathbb{R}^d)} + \varepsilon^{-1}\big\|(\mathbf{D}\chi)\big((S_\varepsilon - I)\widetilde{\mathbf{u}}_0\big)(\,\cdot\,,t)\big\|_{L_2(\mathbb{R}^d)} \\ &\qquad \le C^{(5)} t^{-1} e^{-c_\flat t/2}\|\boldsymbol{\varphi}\|_{L_2(\mathcal{O})} + \varepsilon^{-1}\kappa_1\delta^{-1}\|(S_\varepsilon - I)\widetilde{\mathbf{u}}_0(\,\cdot\,,t)\|_{L_2(\mathbb{R}^d)}. \end{aligned}$$

Combining this with Proposition 1.1 and relations (1.43), (1.46), (2.3), and (2.7), we obtain

$$\varepsilon^{-1}\big\|\chi\big((S_\varepsilon - I)\widetilde{\mathbf{u}}_0\big)(\,\cdot\,,t)\big\|_{H^1(\mathbb{R}^d)} \le C^{(15)}(\delta^{-1}t^{-1/2} + t^{-1})e^{-c_\flat t/2}\|\boldsymbol{\varphi}\|_{L_2(\mathcal{O})}, \tag{8.20}$$

where $C^{(15)} := \max\{C^{(5)}; \kappa_1 r_1 C^{(1)}_{\mathcal{O}} c_3 \|f\|_{L_\infty}\}$.

If $l = d/2$ is an integer, the second summand on the right-hand side of (8.19) is estimated by analogy with (8.15):

$$\big\|\chi\big((S_\varepsilon - I)\widetilde{\mathbf{u}}_0\big)(\,\cdot\,,t)\big\|_{H^l(\mathbb{R}^d)} \le 2C_l^{(11)} \sum_{\sigma=0}^{l} \delta^{-(l-\sigma)}\|\mathbf{u}_0(\,\cdot\,,t)\|_{H^\sigma(\mathcal{O}''')}, \tag{8.21}$$

$0 < \varepsilon \le (4r_1)^{-1}\delta$. The norms of $\mathbf{u}_0$ in $L_2(\mathcal{O};\mathbb{C}^n)$, $H^1(\mathcal{O};\mathbb{C}^n)$, and $H^2(\mathcal{O};\mathbb{C}^n)$ are estimated with the help of Lemma 2.1 and relations (1.43), (2.3). For $\sigma \ge 3$, the norm $\|\mathbf{u}_0(\,\cdot\,,t)\|_{H^\sigma(\mathcal{O}''')}$ is estimated by using (2.45) (with $\mathcal{O}'$ replaced by $\mathcal{O}'''$):

$$\|\mathbf{u}_0(\,\cdot\,,t)\|_{H^\sigma(\mathcal{O}''')} \le \mathrm{C}'_{\sigma+1}\|f\|^2_{L_\infty} 4^\sigma t^{-1/2}(\delta^{-2} + t^{-1})^{\sigma/2} e^{-c_\flat t/2}\|\boldsymbol{\varphi}\|_{L_2(\mathcal{O})}.$$

Combining these arguments with (8.21), we deduce that

$$\big\|\chi\big((S_\varepsilon - I)\widetilde{\mathbf{u}}_0\big)(\,\cdot\,,t)\big\|_{H^l(\mathbb{R}^d)} \le C^{(16)} t^{-1/2}(\delta^{-2} + t^{-1})^{d/4} e^{-c_\flat t/2}\|\boldsymbol{\varphi}\|_{L_2(\mathcal{O})}, \tag{8.22}$$

with a constant $C^{(16)}$ depending only on the problem data (1.9). For the case of half-integral l, estimate (8.22) is checked similarly. As a result, relations (8.19), (8.20), and (8.22) imply the following estimate for the second summand on the right-hand side of (8.7):

$$\begin{aligned} \big\|\widetilde{\Lambda}^\varepsilon \chi\big((S_\varepsilon - I)\widetilde{\mathbf{u}}_0\big)(\,\cdot\,,t)\big\|_{H^1(\mathbb{R}^d)} &\le \widetilde{C}^{(1)} C^{(15)}(\delta^{-1}t^{-1/2} + t^{-1})e^{-c_\flat t/2}\|\boldsymbol{\varphi}\|_{L_2(\mathcal{O})} \\ &\quad + \widetilde{C}^{(2)} C^{(16)} t^{-1/2}(\delta^{-2} + t^{-1})^{d/4} e^{-c_\flat t/2}\|\boldsymbol{\varphi}\|_{L_2(\mathcal{O})}. \end{aligned}$$

Together with (8.7) and (8.18), this implies inequality (2.47) with the constant $\mathrm{C}''_d := C^{(14)} + \widetilde{C}^{(1)}C^{(15)} + \widetilde{C}^{(2)}C^{(16)}$. We have taken into account that the term $\delta^{-1}t^{-1/2}$ does not exceed $t^{-1/2}(\delta^{-2} + t^{-1})^{d/4}$. $\square$

8.4. Proof of Theorem 2.17. Inequality (2.48) follows directly from (2.44) and (2.47). Here, $\mathrm{C}_d := \max\{C_{20}; C_{21}\} + \mathrm{C}''_d$.

We check (2.49). From (1.4), (2.32), and (2.48) it follows that

$$\begin{aligned} &\big\|g^\varepsilon b(\mathbf{D})\big(f^\varepsilon e^{-\widetilde{B}_{D,\varepsilon} t}(f^\varepsilon)^* - (I + \varepsilon\Lambda^\varepsilon b(\mathbf{D}) + \varepsilon\widetilde{\Lambda}^\varepsilon) f_0 e^{-\widetilde{B}^0_D t} f_0\big)\big\|_{L_2(\mathcal{O})\to L_2(\mathcal{O}')} \\ &\qquad \le \|g\|_{L_\infty}(d\alpha_1)^{1/2}\mathrm{C}_d \varepsilon h_d(\delta;t) e^{-c_\flat t/2}. \end{aligned} \tag{8.23}$$

We apply identity (7.13). The norm of the second summand on the right-hand side of (7.13) is estimated with the help of (1.4), (8.2), and Lemma 6.3(1°):

$$\begin{aligned}(8.24)\quad &\varepsilon\Big\|\sum_{k,j=1}^{d} g^\varepsilon b_k \Lambda^\varepsilon b_j D_k D_j f_0 e^{-\widetilde{B}_D^0 t} f_0\Big\|_{L_2(\mathcal{O})\to L_2(\mathcal{O}')} \\ &\quad\le \varepsilon\alpha_1 \|g\|_{L_\infty} C^{(0)} \sum_{k,j=1}^{d} \big\|\chi D_k D_j f_0 e^{-\widetilde{B}_D^0 t} f_0\big\|_{L_2(\mathcal{O})\to H^{l-1}(\mathbb{R}^d)}, \quad l = d/2.\end{aligned}$$

Next, we apply Lemma 8.2. If l is an integer, then (1.43) yields

$$\begin{aligned}(8.25)\quad &\sum_{k,j=1}^{d} \big\|\chi D_k D_j f_0 e^{-\widetilde{B}_D^0 t} f_0\big\|_{L_2(\mathcal{O})\to H^{l-1}(\mathbb{R}^d)} \\ &\quad\le d C_{l-1}^{(11)} \|f\|_{L_\infty} \sum_{i=0}^{l-1} \delta^{-(l-1-i)} \big\|f_0 e^{-\widetilde{B}_D^0 t}\big\|_{L_2(\mathcal{O})\to H^{i+2}(\mathcal{O}'')}.\end{aligned}$$

The norm $\|f_0 e^{-\widetilde{B}_D^0 t}\|_{L_2(\mathcal{O})\to H^2(\mathcal{O})}$ satisfies (2.8). If $i \ge 1$, relations (1.43) and (2.45) (with $\mathcal{O}'$ replaced by $\mathcal{O}''$) imply that

$$\big\|f_0 e^{-\widetilde{B}_D^0 t}\big\|_{L_2(\mathcal{O})\to H^{i+2}(\mathcal{O}'')} \le \mathrm{C}'_{i+2} \|f\|_{L_\infty} 2^{i+1} t^{-1/2} (\delta^{-2} + t^{-1})^{(i+1)/2} e^{-c_\flat t/2}.$$

Combining this with (2.8), (8.24), and (8.25), we obtain

$$(8.26)\quad \varepsilon\Big\|\sum_{k,j=1}^{d} g^\varepsilon b_k \Lambda^\varepsilon b_j D_k D_j f_0 e^{-\widetilde{B}_D^0 t} f_0\Big\|_{L_2(\mathcal{O})\to L_2(\mathcal{O}')} \le C^{(17)} \varepsilon t^{-1/2} (\delta^{-2} + t^{-1})^{d/4} e^{-c_\flat t/2},$$

where the constant $C^{(17)}$ depends only on the problem data (1.9). If l is half-integral, inequality (8.26) is checked by using Lemma 8.2(2°).

The third summand on the right-hand side of (7.13) is estimated similarly by using (1.4), (8.2), Lemma 6.5(1°), and Lemma 8.2. As a result, we obtain

$$(8.27)\quad \varepsilon \sum_{j=1}^{d} \|g^\varepsilon b_j \widetilde{\Lambda}^\varepsilon D_j f_0 e^{-\widetilde{B}_D^0 t} f_0\|_{L_2(\mathcal{O})\to L_2(\mathcal{O}')} \le C^{(18)} \varepsilon t^{-1/2} (\delta^{-2} + t^{-1})^{d/4} e^{-c_\flat t/2}.$$

Here the constant $C^{(18)}$ depends only on the problem data (1.9).

Finally, relations (1.27), (7.13), (8.23), (8.26), and (8.27) imply inequality (2.49) with the constant $\widetilde{\mathrm{C}}_d := \|g\|_{L_\infty} (d\alpha_1)^{1/2} \mathrm{C}_d + C^{(17)} + C^{(18)}$. □

References

[AlCPiSiVa] G. Allaire, Y. Capdeboscq, A. Piatnitski, V. Siess, and M. Vanninathan, *Homogenization of periodic systems with large potentials*, Arch. Rational Mech. Anal. **174** (2004), no. 2, 179–220. MR2098106

[BaPa] N. S. Bakhvalov and G. P. Panasenko, *Homogenization: averaging processes in periodic media. Mathematical problems in mechanics of composite materials*, Nauka, Moscow, 1984; English transl., Math. Appl. (Soviet Ser.), vol. 36, Kluwer Acad. Publ. Group, Dordrecht, 1989. MR1112788

[BeLPap] A. Bensoussan, J.-L. Lions, and G. Papanicolaou, *Asymptotic analysis for periodic structures*, Stud. Math. Appl., vol. 5, North-Holland Publ. Co., Amsterdam–New York, 1978. MR503330

[BSu1] M. Sh. Birman and T. Suslina, *Threshold effects near the lower edge of the spectrum for periodic differential operators of mathematical physics*, Systems, Approximation, Singular Integral Operators, and Related Topics (Bordeaux, 2000), Oper. Theory Adv. Appl., vol. 129, Birkhäuser, Basel, 2001, pp. 71–107. MR1882692

[BSu2] ———, *Periodic second-order differential operators. Threshold properties and averaging*, Algebra i Analiz **15** (2003), no. 5, 1–108; English transl., St. Petersburg Math. J. **15** (2004), no. 5, 639–714. MR2068790

[BSu3] ______, *Homogenization with corrector term for periodic elliptic differential operators*, Algebra i Analiz **17** (2005), no. 6, 1–104; English transl., St. Petersburg Math. J. **17** (2006), no. 6, 897–973. MR2202045

[BSu4] ______, *Homogenization with corrector for periodic differential operators. Approximation of solutions in the Sobolev class $H^1(\mathbb{R}^d)$*, Algebra i Analiz **18** (2006), no. 6, 1–130; English transl., St. Petersburg Math. J. **18** (2007), no. 6, 857–955. MR2307356

[Bo] D. I. Borisov, *Asymptotics of solutions of elliptic systems with rapidly oscillating coefficients*, Algebra i Analiz **20** (2008), no. 2, 19–42; English transl., St. Petersburg Math. J. **20** (2009), no. 2, 175–191. MR2423995

[ChKonLe] J. H. Choe, K.-B. Kong, and Ch.-O. Lee, *Convergence in L^p space for the homogenization problems of elliptic and parabolic equations in the plane*, J. Math. Anal. Appl. **287** (2003), no. 2, 321–336. MR2022721

[GeS] J. Geng and Zh. Shen, *Convergence rates in parabolic homogenization with time-dependent periodic coefficients*, J. Funct. Anal. **272** (2017), no. 5, 2092–2113. MR3596717

[Gr1] G. Griso, *Error estimate and unfolding for periodic homogenization*, Asymptot. Anal. **40** (2004), no. 3–4, 269–286. MR2107633

[Gr2] ______, *Interior error estimate for periodic homogenization*, Anal. Appl. (Singap.) **4** (2006), no. 1, 61–79. MR2199793

[ZhKO] V. V. Zhikov, S. M. Kozlov, and O. A. Oleinik, *Homogenization of differential operators*, Nauka, Moscow, 1993; English transl., Homogenization of differential operators and integral functionals, Springer-Verlag, Berlin, 1994. MR1329546

[Zh1] V. V. Zhikov, *Asymptotic behavior and stabilization of solutions of a second-order parabolic equation with lowest terms*, Tr. Moskov. Mat. Obshch. **46** (1983), 69–98; English transl., Trans. Mosc. Math. Soc. **1984**, no. 2, Amer. Math. Soc., Providence, RI, pp. 69–99. MR737901

[Zh2] ______, *On operator estimates in homogenization theory*, Dokl. Akad. Nauk **403** (2005), no. 3, 305–308; English transl., Dokl. Math. **72** (2005), no. 1, 534–538. MR2164541

[ZhPas1] V. V. Zhikov and S. E. Pastukhova, *On operator estimates for some problems in homogenization theory*, Russ. J. Math. Phys. **12** (2005), no. 4, 515–524. MR2201316

[ZhPas2] ______, *Estimates of homogenization for a parabolic equation with periodic coefficients*, Russ. J. Math. Phys. **13** (2006), no. 2, 224–237. MR2262826

[ZhPas3] ______, *On operator estimates on homogenization theory*, Uspekhi Mat. Nauk **71** (2016), no. 3, 27–122; English transl., Russian Math. Surveys **71** (2016), no. 3, 471–511. MR3535364

[Ka] T. Kato, *Perturbation theory for linear operators*, Grundlehren Math. Wiss, Bd. 132, Springer-Verlag, New York, 1966. MR0203473

[KeLiS] C. E. Kenig, F. Lin, and Z. Shen, *Convergence rates in L^2 for elliptic homogenization problems*, Arch. Rational Mech. Anal. **203** (2012), no. 3, 1009–1036. MR2928140

[K] S. M. Kozlov, *Reducibility of quasiperiodic differential operators and averaging*, Tr. Moskov. Mat. Obshch. **46** (1983), 99–123; English transl., Trans. Mosc. Math. Soc. **1984**, no. 2, Amer. Math. Soc., Providence, RI, pp. 101–126. MR737902

[KoE] V. A. Kondrat′ev and S. D. Eidel′man, *Boundary-surface conditions in the theory of elliptic boundary value problems*, Dokl. Akad. Nauk SSSR **246** (1979), no. 4, 812–815; English transl., Soviet. Math. Dokl. **20** (1979), 561–563. MR543538

[LaSoU] O. A. Ladyzhenskaya, V. A. Solonnikov, and N. N. Yraltseva, *Linear and quasilinear equations of parabolic type*, Nauka, Moscow, 1967; English transl., Transl. Math. Monogr., vol. 23, Amer. Math. Soc., Providence, RI, 1968. MR0241822

[LaU] O. A. Ladyzhenskaya and N. N. Yraltseva, *Linear and quasi-linear equations of elliptic type*, Nauka, Moscow, 1964; English transl., Acad. Press, New York, 1968. MR0211073

[MaSh] V. G. Maz′ya and T. O. Shaposhnikova, *Theory of multipliers in spaces of differentiable functions*, Leningrad. Univ., Leningrad, 1986; English transl., Monogr. Stud. Math., vol. 23, Pitman Publ., Co., Boston, MA, 1985. MR785568

[McL] W. McLean, *Strongly elliptic systems and boundary integral equations*, Cambridge Univ. Press, Cambridge, 2000. MR1742312

[M] Yu. M. Meshkova, *Homogenization of the Cauchy problem for parabolic systems with periodic coefficients*, Algebra i Analiz **25** (2013), no. 6, 125–177; English transl., St. Petersburg Math. J. **25** (2014), no. 6, 981–1019. MR3234842

[MSu1] Yu. M. Meshkova and T. A. Suslina, *Homogenization of initial boundary value problems for parabolic systems with periodic coefficients*, Appl. Anal. **95** (2016), no. 8, 1736–1775. MR3505416

[MSu2] ______, *Two-parametric error estimates in homogenization of second order elliptic systems in $\mathbb{R}^d$*, Appl. Anal. **95** (2016), no. 7, 1413–1448. MR3499668

[MSu3] ______, *Homogenization of the Dirichlet problem for elliptic systems: Two-parametric error estimates*, `arXiv:1702.00550v4` (2017).

[MSu4] ______, *Homogenization of the Dirichlet problem for elliptic and parabolic systems with periodic coefficients*, Functsional. Anal. i Prilozhen. **51** (2017), no. 3, 87–93; English transl., Funct. Anal. Appl. **51** (2017), no. 3, 230–235. MR3685305

[MoV] Sh. Moskow and M. Vogelius, *First-order corrections to the homogenised eigenvalues of a periodic composite medium. A convergence proof*, Proc. Roy. Soc. Edinburgh Sect. A **127** (1997), no. 6, 1263–1299. MR1489436

[PSu] M. A. Pakhnin and T. A. Suslina, *Operator error estimates for homogenization of the elliptic Dirichlet problem in a bounded domain*, Algebra i Analiz **24** (2012), no. 6, 139–177; English transl., St. Petersburg Math. J. **24** (2013), no. 6, 949–976. MR3097556

[R] V. S. Rychkov, *On restrictions and extensions of the Besov and Triebel–Lizorkin spaces with respect to Lipschitz domains*, J. London Math. Soc. **60** (1999), no. 1, 237–257. MR1721827

[Sa] E. Sanchez-Palencia, *Nonhomogeneous media and vibration theory*, Lecture Notes in Phys., vol. 127, Springer-Verlag, Berlin, 1980. MR578345

[Su1] T. A. Suslina, *On homogenization of periodic parabolic systems*, Functsional. Anal. i Prilozhen. **38** (2004), no. 4, 86–90; English transl., Funct. Anal. Appl. **38** (2004), no. 4, 309–312. MR2117512

[Su2] ______, *Homogenization of a periodic parabolic Cauchy problem*, Amer. Math. Soc. Transl. (2), vol. 220, Amer. Math. Soc., Providence, RI, 2007, pp. 201–233. MR2343612

[Su3] ______, *Homogenization of a periodic parabolic Cauchy problem in the Sobolev space $H^1(\mathbb{R}^d)$*, Math. Model. Nat. Phenom. **5** (2010), no. 4, 390–447. MR2662463

[Su4] ______, *Homogenization in the Sobolev class $H^1(\mathbb{R}^d)$ for second-order periodic elliptic operators with the inclusion of first-order terms*, Algebra i Analiz **22** (2010), no. 1, 108–222; English transl., St. Petersburg Math. J. **22** (2011), no. 1, 81–162. MR2641084

[Su5] ______, *Homogenization of the Dirichlet problem for elliptic systems: L_2-operator error estimates*, Mathematika **59** (2013), no. 2, 463–476. MR3081781

[Su6] ______, *Homogenization of the Neumann problem for elliptic systems with periodic coefficients*, SIAM J. Math. Anal. **45** (2013), no. 6, 3453–3493. MR3131481

[Su7] ______, *Homogenization of elliptic operators with periodic coefficients in dependence on the spectral parametric*, Algebra i Analiz **27** (2015), no. 4, 87–166; English transl., St. Petersburg Math. J. **27** (2016), no. 4, 651–708. MR3580194

[Xu1] Q. Xu, *Uniform regularity estimates in homogenization theory of elliptic system with lower order terms*, J. Math. Anal. Appl. **438** (2016), no. 2, 1066–1107. MR3466080

[Xu2] ______, *Uniform regularity estimates in homogenization theory of elliptic systems with lower order terms on the Neumann boundary problem*, J. Differential Equations **261** (2016), no. 8, 4368–4423. MR3537832

[Xu3] ______, *Convergence rates for general elliptic homogenization problems in Lipschitz domains*, SIAM J. Math. Anal. **48** (2016), no. 6, 3742–3788. MR3566906

[XuZ] Q. Xu and Sh. Zhou, *Quantitative estimates in homogenization of parabolic systems of elasticity in Lipschitz cylinders*, `arXiv:1705.01479` (2017).

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