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ON SPECTRAL ASYMPTOTICS OF THE TENSOR PRODUCT OF OPERATORS WITH ALMOST REGULAR MARGINAL ASYMPTOTICS

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Abstract. The spectral asymptotics of a tensor product of compact operators in Hilbert space with known marginal asymptotics is studied. The methods of A. Karol′, A. Nazarov, and Ya. Nikitin are generalized to operators with almost regular marginal asymptotics. In many (but not all) cases it is shown that the tensor product in question also has almost regular asymptotics. The results are then applied to the theory of small ball probabilities of Gaussian random fields.

§1. Introduction

We consider compact nonnegative selfadjoint operators $\mathcal{T} = \mathcal{T}^* \geq 0$ on a Hilbert space $\mathcal{H}$ and $\widetilde{\mathcal{T}}$ on a Hilbert space $\widetilde{\mathcal{H}}$. Let $\lambda_n = \lambda_n(\mathcal{T})$ denote the eigenvalues of $\mathcal{T}$ arranged in a nondecreasing order and repeated in accordance with their multiplicity. We also consider their counting function

$$\mathcal{N}(t) = \mathcal{N}(t, \mathcal{T}) = \#\{n : \lambda_n(\mathcal{T}) > t\}.$$

Similarly we define $\widetilde{\lambda}_n$ and $\widetilde{\mathcal{N}}(t)$ for $\widetilde{\mathcal{T}}$.

Having known the asymptotics for $\mathcal{N}(t, \mathcal{T})$ and $\mathcal{N}(t, \widetilde{\mathcal{T}})$ as $t \to 0$, we want to determine the asymptotics for $\mathcal{N}(t, \mathcal{T} \otimes \widetilde{\mathcal{T}})$. Our results are easily generalized to the case of a tensor product of several operators.

Known applications of such results can be found in problems concerning asymptotics of quantization for random variables and vectors (see, e.g., [1, 2]), average complexity of linear problems, i.e., problems of approximation of a continuous linear operator (see, e.g., [3]), and also in the rapidly developing theory of small deviations of random processes in L_2-norm (see, e.g., [4, 5]).

The abstract methods of spectral asymptotic analysis for tensor products, generalized in this paper, were developed in [4] and [5]. In the case treated in [4] the eigenvalues of the operator factors have the so-called *regular* asymptotic behavior:

$$\lambda_n \sim \frac{\psi(n)}{n^p}, \quad n \to \infty,$$

where $p > 1$, and ψ is a *slowly varying* function (SVF). In the paper [5], a similar approach was used in the case where the eigenvalue counting function has the asymptotics of a slowly varying function.

In this paper we consider compact operators with almost regular asymptotics,

$$\lambda_n(\mathcal{T}) \sim \frac{\psi(n) \cdot \mathfrak{s}(\ln(n))}{n^p}, \quad n \to +\infty, \tag{1}$$

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where $p > 1$, ψ is an SVF, and $\mathfrak{s}$ is a continuous periodic function. As an example of such an operator, we mention Green integral operators with a singular arithmetically self-similar weight measure (see [6, 7, 8]).

For the asymptotics (1), the following fact holds, which is similar to Lemma 3.1 from [4], so we present it without proof.

Proposition 1. *For any $p > 0$, the spectral asymptotics* (1) *for the operator $\mathcal{T}$ is equivalent to the asymptotics*

$$\mathcal{N}(t,\mathcal{T}) \sim \mathcal{N}_{as}(t) := \frac{\varphi(1/t)\cdot s(\ln(1/t))}{t^{1/p}}, \qquad t \to +0, \tag{2}$$

where φ is an SVF, s is a periodic function (*the period T of s corresponds to the period T/p of the function $\mathfrak{s}$*)*. Moreover, the convergence of the integral $\int_1^\infty \varphi(\tau)\frac{d\tau}{\tau}$ is equivalent to the convergence of the sum $\sum_n \lambda_n^{1/p}(\mathcal{T})$.*

Application of the results obtained in this paper is demonstrated by an example pertaining to the problem of L_2-small deviations of Gaussian random fields.

The study of the small deviation problem was initiated in [9] and was continued by many other authors. The history of the problem and the summary of main results are the subjects of two reviews [10] and [11]. Links to recent results in the field of small deviations of random processes can be found on the web-site [12].

The study of small deviations of Gaussian fields of tensor product type was initiated in the classical paper [13], where the logarithmic asymptotics of L_2-small deviations was obtained for the Brownian sheet

$$\mathbb{W}_d(x_1,\dots,x_d) = W_1(x_1)\otimes W_2(x_2)\otimes\cdots\otimes W_d(x_d)$$

on the unit cube (here the W_k are independent Wiener processes). This result was later generalized in [14] to some other marginal processes. In [4] and [5], certain results on small deviations for wide classes of Gaussian fields of tensor product type were obtained as a consequence of the results on spectral asymptotics of the corresponding operators.

This paper has the following structure. In §2 we give the necessary information about slowly varying functions. In §3 we establish some auxiliary facts related to the asymptotics of convolutions of an almost Mellin type.

Spectral asymptotics of the tensor products of operators with marginal asymptotics of the form (2) are the subject of §4. The main results are that we obtain the main term of the spectral asymptotics of the tensor product for all possible combinations of parameters, imposing only slight technical restrictions in some cases. The results are split into several cases depending on the relations between the parameters of the spectral asymptotics of the operators $\mathcal{T}$ and $\widetilde{\mathcal{T}}$:

1. $\widetilde{p} > p$.
2. $\widetilde{p} = p$.
   - 2.1. $\int_1^\infty \varphi(\sigma)\frac{d\sigma}{\sigma} = \int_1^\infty \widetilde{\varphi}(\sigma)\frac{d\sigma}{\sigma} = \infty$.
     - 2.1.1. The functions s and $\widetilde{s}$ have a common period ($T = \widetilde{T}$).
     - 2.1.2. The periods T and $\widetilde{T}$ of s and $\widetilde{s}$ are incommensurable.
   - 2.2. $\int_1^\infty \varphi(\sigma)\frac{d\sigma}{\sigma} < \infty$, $\quad \int_1^\infty \widetilde{\varphi}(\sigma)\frac{d\sigma}{\sigma} = \infty$.
   - 2.3. $\int_1^\infty \varphi(\sigma)\frac{d\sigma}{\sigma} < \infty$, $\quad \int_1^\infty \widetilde{\varphi}(\sigma)\frac{d\sigma}{\sigma} < \infty$.

In the cases 1 and 2.1.1 the asymptotics of the tensor product is almost regular, in the case 2.1.2 it is regular. In the cases 2.2 and 2.3 we obtain an asymptotics of a more complex form.

In §5 we relate the almost regular spectral asymptotics with the logarithmic asymptotics of small deviations of Gaussian random fields.

Various constants whose values are not essential for this work are denoted by C. The dependence of these constants on parameters is noted in parentheses.

§2. Auxiliary facts about slowly varying functions

We recall that a positive function $\varphi(\tau)$, $\tau > 0$, is said to be *slowly varying* (at infinity) if for any constant $c > 0$ we have

$$\varphi(c\tau)/\varphi(\tau) \to 1 \quad \text{as} \quad \tau \to +\infty. \tag{3}$$

The following simple properties of SVFs are well known (see, e.g., [15] for the proofs).

Proposition 2. *Let φ be an SVF. Then it possesses the following properties.*

1) *Convergence in (3) is uniform for $c \in [a, b]$ for any $0 < a < b < +\infty$.*
2) *The function $\tau \mapsto \tau^p \varphi(\tau)$, $p \neq 0$, is monotone for large values of τ.*
3) *There exists an SVF $\psi \in C^2(\mathbb{R})$ equivalent to φ (i.e., $\varphi(\tau)/\psi(\tau) \to 1$ as $\tau \to \infty$) and such that*

$$\tau \cdot (\ln(\psi))'(\tau) \to 0, \quad \tau^2 \cdot (\ln(\psi))''(\tau) \to 0, \qquad \tau \to \infty.$$

4) *If $\int_1^\infty \varphi(\tau)\frac{d\tau}{\tau} < \infty$, then $\varphi(\tau) \to 0$ as $\tau \to \infty$.*

Following [4], we define the *Mellin convolution* of two SVFs φ and ψ:

$$(\varphi * \psi)(\tau) = \int_1^\tau \varphi(\sigma)\psi(\tau/\sigma)\frac{d\sigma}{\sigma} = h_{\varphi,\psi}(\tau) + h_{\psi,\varphi}(\tau),$$

where

$$h_{\varphi,\psi}(\tau) = \int_1^{\sqrt{\tau}} \varphi(\sigma)\psi(\tau/\sigma)\frac{d\sigma}{\sigma}.$$

Proposition 3 ([4, Theorem 2.2])**.** *We continue the list of properties of SVFs.*

1) *If $\int_1^\infty \varphi(\tau)\frac{d\tau}{\tau} = \infty$, then $\psi(\tau) = o(h_{\varphi,\psi}(\tau))$ as $\tau \to \infty$.*
2) *If $\psi(\tau) = \psi_1(\tau)(1 + o(1))$ as $\tau \to \infty$, then*

$$h_{\varphi,\psi}(\tau) = h_{\varphi,\psi_1}(\tau)(1 + o(1)), \quad \tau \to \infty.$$

If also $\int_1^\infty \psi(\tau)\frac{d\tau}{\tau} = \infty$, then

$$h_{\psi,\varphi}(\tau) = h_{\psi_1,\varphi}(\tau)(1 + o(1)), \quad \tau \to \infty.$$

3) *$h_{\varphi,\psi}$ is an SVF.*
4) *Let $\int_1^\infty \varphi(\tau)\frac{d\tau}{\tau} < \infty$, and let*

$$\int_1^\infty \varphi(\sigma) m_\psi(\sigma)\frac{d\sigma}{\sigma} < \infty,$$

where

$$m_\psi(\sigma) = \sup_{\tau > \sigma^2} \frac{\psi(\tau/\sigma)}{\psi(\tau)}.$$

Then

$$h_{\varphi,\psi}(\tau) = \psi(\tau)\int_1^\infty \varphi(\sigma)\frac{d\sigma}{\sigma} \cdot (1 + o(1)), \qquad \tau \to \infty. \tag{4}$$

§3. Preliminary facts about the asymptotics of almost Mellin convolutions

In this section φ and $\widetilde{\varphi}$ are SVFs, and s and $\widetilde{s}$ are continuous and bounded functions separated away from zero and with periods T and $\widetilde{T}$ (respectively) that have the form

$$s(\tau) = e^{-\tau/p}\varrho(\tau), \qquad \widetilde{s}(\tau) = e^{-\tau/p}\widetilde{\varrho}(\tau),$$

where $p > 0$, ϱ and $\widetilde{\varrho}$ are monotone. In particular, it follows that s and $\widetilde{s}$ are of bounded variation.

We define the *almost Mellin convolution* by the formulas

$$\begin{aligned}
(\varphi s * \widetilde{\varphi}\widetilde{s})(\tau) &= \int_1^\tau \varphi\left(\frac{\tau}{\sigma}\right)\widetilde{\varphi}(\sigma) s\left(\ln\frac{\tau}{\sigma}\right)\widetilde{s}(\ln\sigma)\frac{d(\widetilde{\varrho}(\ln\sigma))}{\widetilde{\varrho}(\ln\sigma)} \\
&= H[\varphi s, \widetilde{\varphi}\widetilde{s}](\tau) + H_1[\varphi s, \widetilde{\varphi}\widetilde{s}](\tau), \\
H[\varphi s, \widetilde{\varphi}\widetilde{s}](\tau) &= \int_1^{\sqrt{\tau}} \varphi\left(\frac{\tau}{\sigma}\right)\widetilde{\varphi}(\sigma) s\left(\ln\frac{\tau}{\sigma}\right)\widetilde{s}(\ln\sigma)\frac{d(\widetilde{\varrho}(\ln\sigma))}{\widetilde{\varrho}(\ln\sigma)}, \\
H_1[\varphi s, \widetilde{\varphi}\widetilde{s}](\tau) &= \int_{\sqrt{\tau}}^\tau \varphi\left(\frac{\tau}{\sigma}\right)\widetilde{\varphi}(\sigma) s\left(\ln\frac{\tau}{\sigma}\right)\widetilde{s}(\ln\sigma)\frac{d(\widetilde{\varrho}(\ln\sigma))}{\widetilde{\varrho}(\ln\sigma)}.
\end{aligned}$$

The integral here should be interpreted as a Lebesgue–Stieltjes integral.

Lemma 1.

$$\begin{aligned}
(\varphi s * \widetilde{\varphi}\widetilde{s})(\tau) &\asymp (\varphi * \widetilde{\varphi})(\tau), \quad \tau \to \infty, \\
H[\varphi s, \widetilde{\varphi}\widetilde{s}](\tau) &\asymp h_{\widetilde{\varphi},\varphi}(\tau), \quad \tau \to \infty, \\
H_1[\varphi s, \widetilde{\varphi}\widetilde{s}](\tau) &\asymp h_{\varphi,\widetilde{\varphi}}(\tau), \quad \tau \to \infty.
\end{aligned}$$

Proof. We prove the upper estimate for the first relation, the other estimates can be obtained similarly. We introduce the operator

$$F_\sigma[\varphi](\xi) = \varphi\big(e^{j\widetilde{T}}\sigma\big) \ \text{ for } \ \xi \in \big[e^{j\widetilde{T}}, e^{(j+1)\widetilde{T}}\big), \tag{5}$$

which transforms a given function φ to a step function.

Note that

$$F_\sigma[\varphi](\tau) = \varphi(\tau)(1 + o(1)), \quad \tau \to \infty, \tag{6}$$

uniformly over $\sigma \in [1, e^{\widetilde{T}}]$. Let $k \in \mathbb{N}$ be a number such that $e^{(k-1)\widetilde{T}} < \tau \le e^{k\widetilde{T}}$. Then

$$(\varphi s * \widetilde{\varphi}\widetilde{s})(\tau) \le C \int_1^{e^{k\widetilde{T}}} F_{e^{k\widetilde{T}}/\tau}[\varphi]\left(\frac{\tau}{\sigma}\right) F_1[\widetilde{\varphi}](\sigma)\frac{d(\widetilde{\varrho}(\ln\sigma))}{\widetilde{\varrho}(\ln\sigma)}.$$

Observe that the function $F_{e^{k\widetilde{T}}/\tau}[\varphi]\big(\frac{\tau}{\sigma}\big)F_1[\widetilde{\varphi}](\sigma)$ is constant with respect to σ on every interval $\big(e^{j\widetilde{T}}, e^{(j+1)\widetilde{T}}\big)$, $j = 0, \dots, k-1$. The measure $\frac{d(\widetilde{\varrho}(\ln\sigma))}{\widetilde{\varrho}(\ln\sigma)} = d\ln(\widetilde{s}(\ln\sigma)\sigma^{1/p})$ is periodic with respect to $\ln\sigma$, which allows us to replace the integral with a sum. We obtain

$$\begin{aligned}
(\varphi s * \widetilde{\varphi}\widetilde{s})(\tau) &\le C \int_1^{e^{\widetilde{T}}} \frac{d(\widetilde{\varrho}(\ln\sigma))}{\widetilde{\varrho}(\ln\sigma)} \sum_{j=0}^{k-1} F_{e^{k\widetilde{T}}/\tau}[\varphi]\Big(\frac{\tau}{e^{j\widetilde{T}}}\Big) F_1[\widetilde{\varphi}](e^{j\widetilde{T}}) \\
&\le C \int_1^{e^{\widetilde{T}}} \frac{d\sigma}{\sigma} \sum_{j=0}^{k-1} F_{e^{k\widetilde{T}}/\tau}[\varphi]\Big(\frac{\tau}{e^{j\widetilde{T}}}\Big) F_1[\widetilde{\varphi}](e^{j\widetilde{T}}) \\
&= C \int_1^{e^{k\widetilde{T}}} F_{e^{k\widetilde{T}}/\tau}[\varphi]\left(\frac{\tau}{\sigma}\right) F_1[\widetilde{\varphi}](\sigma)\frac{d\sigma}{\sigma} \le C \int_1^\tau \varphi\left(\frac{\tau}{\sigma}\right)\widetilde{\varphi}(\sigma)\frac{d\sigma}{\sigma}. \qquad \square
\end{aligned}$$

The proof of the next proposition is similar to Theorem 2.2 in [4], so we omit it.

Proposition 4. *Let* $\widetilde{\varphi}(\tau) = \psi_1(\tau)(1+o(1))$ *as* $\tau \to \infty$. *Then*

$$H[\varphi s, \widetilde{\varphi}\widetilde{s}](\tau) = H[\varphi s, \psi_1 \widetilde{s}](\tau)(1+o(1)).$$

If, moreover, $\int_1^\infty \widetilde{\varphi}(\tau)\frac{d\tau}{\tau} = \infty$, *then*

$$H_1[\varphi s, \widetilde{\varphi}\widetilde{s}](\tau) = H_1[\varphi s, \psi_1 \widetilde{s}](\tau)(1+o(1)).$$

Lemma 2. *Suppose that* $\int_1^\infty \varphi(\tau)\frac{d\tau}{\tau} = \infty$, $\int_1^\infty \widetilde{\varphi}(\tau)\frac{d\tau}{\tau} = \infty$. *Then*

$$H_1[\varphi s, \widetilde{\varphi}\widetilde{s}](\tau) = H[\widetilde{\varphi}\widetilde{s}, \varphi s](\tau)(1+o(1)), \quad \tau \to \infty,$$

and the almost Mellin convolution is asymptotically symmetric, i.e.,

$$(\varphi s * \widetilde{\varphi}\widetilde{s})(\tau) = (\widetilde{\varphi}\widetilde{s} * \varphi s)(\tau)(1+o(1)), \quad \tau \to \infty.$$

Proof. The second relation follows from the first immediately. In order to prove the first relation, we write

$$H_1[\varphi s, \widetilde{\varphi}\widetilde{s}](\tau) = \tau^{-1/p} \int_{\sqrt{\tau}}^{\tau} \varphi\left(\frac{\tau}{\sigma}\right) \widetilde{\varphi}(\sigma) \rho\left(\ln\frac{\tau}{\sigma}\right) d(\widetilde{\rho}(\ln\sigma)),$$

replace σ with τ/σ and integrate by parts:

$$\begin{aligned}
H_1[\varphi s, \widetilde{\varphi}\widetilde{s}](\tau) &= -\tau^{-1/p} \int_1^{\sqrt{\tau}} \varphi(\sigma)\widetilde{\varphi}\left(\frac{\tau}{\sigma}\right) \rho(\ln\sigma) d\left(\widetilde{\rho}\left(\ln\frac{\tau}{\sigma}\right)\right) \\
&= \tau^{-1/p} \int_1^{\sqrt{\tau}} \varphi(\sigma)\widetilde{\varphi}\left(\frac{\tau}{\sigma}\right) \widetilde{\rho}\left(\ln\frac{\tau}{\sigma}\right) d(\rho(\ln\sigma)) + \varphi(\sigma)\widetilde{\varphi}\left(\frac{\tau}{\sigma}\right) s(\ln\sigma)\widetilde{s}\left(\ln\frac{\tau}{\sigma}\right)\Big|_1^{\sqrt{\tau}} \\
&\quad + \int_1^{\sqrt{\tau}} \Big(\frac{\sigma\varphi'(\sigma)}{\varphi(\sigma)} - \frac{(\tau/\sigma)\widetilde{\varphi}'(\tau/\sigma)}{\widetilde{\varphi}(\tau/\sigma)}\Big) \varphi(\sigma)\widetilde{\varphi}\left(\frac{\tau}{\sigma}\right) \widetilde{s}\left(\ln\frac{\tau}{\sigma}\right) s(\ln\sigma)\frac{d\sigma}{\sigma}.
\end{aligned}$$

The first term equals $H[\widetilde{\varphi}\widetilde{s}, \varphi s](\tau)$. It remains to show that the second and third terms satisfy the estimate $o(H[\widetilde{\varphi}\widetilde{s}, \varphi s](\tau))$. As to the second term, we have

$$\varphi(\sigma)\widetilde{\varphi}\left(\frac{\tau}{\sigma}\right) s(\ln\sigma)\widetilde{s}\left(\ln\frac{\tau}{\sigma}\right)\Big|_1^{\sqrt{\tau}} = \varphi(\sqrt{\tau})\widetilde{\varphi}(\sqrt{\tau})s(\ln\sqrt{\tau})\widetilde{s}(\ln\sqrt{\tau}) - \varphi(1)\widetilde{\varphi}(\tau)s(0)\widetilde{s}(\ln\tau).$$

All periodic components are bounded, and

$$\widetilde{\varphi}(\tau) = o(h_{\varphi,\widetilde{\varphi}}(\tau)) = o(H[\widetilde{\varphi}\widetilde{s}, \varphi s](\tau)), \quad \tau \to \infty,$$

in accordance with Part 1 of Proposition 3, by Lemma 1. Thus, it suffices to estimate

$$\begin{aligned}
\varphi(\sqrt{\tau})\widetilde{\varphi}(\sqrt{\tau}) &= \varphi(1)\widetilde{\varphi}(\tau) + \int_1^{\sqrt{\tau}} \left(\varphi(\sigma)\widetilde{\varphi}\left(\frac{\tau}{\sigma}\right)\right)'_\sigma d\sigma \\
&= \varphi(1)\widetilde{\varphi}(\tau) + \int_1^{\sqrt{\tau}} \Big(\frac{\sigma\varphi'(\sigma)}{\varphi(\sigma)} - \frac{(\tau/\sigma)\widetilde{\varphi}'(\tau/\sigma)}{\widetilde{\varphi}(\tau/\sigma)}\Big)\varphi(\sigma)\widetilde{\varphi}\left(\frac{\tau}{\sigma}\right)\frac{d\sigma}{\sigma} \\
&= \varphi(1)\widetilde{\varphi}(\tau) + \int_1^{\sqrt{\tau}} \Big(1 + \frac{\sigma\varphi'(\sigma)}{\varphi(\sigma)}\Big)\varphi(\sigma)\widetilde{\varphi}\left(\frac{\tau}{\sigma}\right)\frac{d\sigma}{\sigma} \\
&\qquad - \int_1^{\sqrt{\tau}} \varphi(\sigma)\Big(1 + \frac{(\tau/\sigma)\widetilde{\varphi}'(\tau/\sigma)}{\widetilde{\varphi}(\tau/\sigma)}\Big)\widetilde{\varphi}\left(\frac{\tau}{\sigma}\right)\frac{d\sigma}{\sigma} \\
&= o(h_{\varphi,\widetilde{\varphi}}(\tau)) + h_{\varphi,\widetilde{\varphi}}(\tau)(1+o(1)) - h_{\varphi,\widetilde{\varphi}}(\tau)(1+o(1)) \\
&= o(h_{\varphi,\widetilde{\varphi}}(\tau)) = o(H[\widetilde{\varphi}\widetilde{s}, \varphi s](\tau)), \quad \tau \to \infty.
\end{aligned}$$

We have used Part 2 of Proposition 3 and part 3 of Proposition 2, when estimating the integrals above.

Using similar arguments and Proposition 4, we see that

$$\int_1^{\sqrt{\tau}} \Big(\frac{\sigma\varphi'(\sigma)}{\varphi(\sigma)} - \frac{(\tau/\sigma)\widetilde{\varphi}'(\tau/\sigma)}{\widetilde{\varphi}(\tau/\sigma)}\Big)\varphi(\sigma)\widetilde{\varphi}\left(\frac{\tau}{\sigma}\right)\widetilde{s}\left(\ln\frac{\tau}{\sigma}\right)s(\ln\sigma)\frac{d\sigma}{\sigma} = o(H[\widetilde{\varphi}\widetilde{s},\varphi s](\tau))$$

as $\tau\to\infty$, which concludes the proof of the lemma. □

The case of coinciding periods. Now we consider the case where the functions s and $\widetilde{s}$ have a common period ($T=\widetilde{T}$). Denote

$$(s\star\widetilde{s}\,)(\eta) := \frac{1}{T}\int_0^T s(\eta-\lambda)\widetilde{s}(\lambda)\,d\lambda.$$

Note that there exists a continuous derivative

$$(7)\quad (s\star\widetilde{s}\,)'(\eta) = \frac{1}{T}\int_0^T s(\eta-\lambda)d(\widetilde{s}(\lambda)) = -\frac{1}{p}(s\star\widetilde{s}\,)(\eta) + e^{-\eta/p}\frac{1}{T}\int_0^T \varrho(\eta-\lambda)\,d\widetilde{\varrho}(\lambda).$$

The fact that it is continuous follows from the continuity of ϱ and $\widetilde{\varrho}$.

Lemma 3. *Let $\int_1^\infty \widetilde{\varphi}(\tau)\frac{d\tau}{\tau}=\infty$, and let s and $\widetilde{s}$ have a common period T. Then*

$$\int_1^{\sqrt{\tau}} \varphi\left(\frac{\tau}{\sigma}\right)s\left(\ln\frac{\tau}{\sigma}\right)\widetilde{\varphi}(\sigma)\widetilde{s}(\ln\sigma)\frac{d\sigma}{\sigma} \sim h_{\widetilde{\varphi},\varphi}(\tau)(s\star\widetilde{s}\,)(\ln\tau),\quad \tau\to\infty.$$

If, moreover, $\int_1^\infty \varphi(\tau)\frac{d\tau}{\tau}=\infty$, then

$$\int_1^{\tau} \varphi\left(\frac{\tau}{\sigma}\right)s\left(\ln\frac{\tau}{\sigma}\right)\widetilde{\varphi}(\sigma)\widetilde{s}(\ln\sigma)\frac{d\sigma}{\sigma} \sim (\varphi*\widetilde{\varphi})(\tau)(s\star\widetilde{s}\,)(\ln\tau),\quad \tau\to\infty.$$

Proof. For $e^{2(k-1)T}<\tau\le e^{2kT}$ we write

$$\begin{aligned}
\int_1^{\sqrt{\tau}} \varphi\left(\frac{\tau}{\sigma}\right)s\left(\ln\frac{\tau}{\sigma}\right)\widetilde{\varphi}(\sigma)\widetilde{s}(\ln\sigma)\frac{d\sigma}{\sigma} &\sim \int_1^{e^{kT}} \varphi\left(\frac{\tau}{\sigma}\right)s\left(\ln\frac{\tau}{\sigma}\right)\widetilde{\varphi}(\sigma)\widetilde{s}(\ln\sigma)\frac{d\sigma}{\sigma}\\
&= \sum_{j=0}^{k-1}\int_1^{e^T} \varphi(e^{-jT}\cdot\frac{\tau}{\sigma})s\left(\ln\frac{\tau}{\sigma}\right)\widetilde{\varphi}(e^{jT}\sigma)\widetilde{s}(\ln\sigma)\frac{d\sigma}{\sigma}\\
&= \int_1^{e^T} s\left(\ln\frac{\tau}{\sigma}\right)\widetilde{s}(\ln\sigma)\sum_{j=0}^{k-1}\varphi\Big(e^{-jT}\cdot\frac{\tau}{\sigma}\Big)\widetilde{\varphi}(e^{jT}\sigma)\frac{d\sigma}{\sigma}\\
&= \int_1^{e^T} s\left(\ln\frac{\tau}{\sigma}\right)\widetilde{s}(\ln\sigma)T^{-1}\int_1^{e^{kT}} F_{e^{-(k-1)T}\cdot\frac{\tau}{\sigma}}[\varphi](e^{kT}/\xi)F_\sigma[\widetilde{\varphi}](\xi)\frac{d\xi}{\xi}\frac{d\sigma}{\sigma},
\end{aligned}$$

where the operator F was introduced in (5). Employing the asymptotics (6) and Part 2 of Proposition 3, we obtain

$$\int_1^{\sqrt{\tau}} \varphi\left(\frac{\tau}{\sigma}\right)s\left(\ln\frac{\tau}{\sigma}\right)\widetilde{\varphi}(\sigma)\widetilde{s}(\ln\sigma)\frac{d\sigma}{\sigma} \sim h_{\widetilde{\varphi},\varphi}(e^{kT})(s\star\widetilde{s}\,)(\ln\tau) \sim h_{\widetilde{\varphi},\varphi}(\tau)(s\star\widetilde{s}\,)(\ln\tau).$$

The second part of the lemma can be proved similarly, using the fact that

$$\int_1^\infty \varphi(\tau)\frac{d\tau}{\tau}=\infty.$$

□

Lemma 4. *Let $\int_1^\infty \widetilde{\varphi}(\tau)\frac{d\tau}{\tau}=\infty$, and let s and $\widetilde{s}$ have a common period T. Then*

$$H[\varphi s,\widetilde{\varphi}\widetilde{s}](\tau) \sim h_{\widetilde{\varphi},\varphi}(\tau)\Big(\frac{1}{p}(s\star\widetilde{s}\,)+(s\star\widetilde{s}\,)'\Big)(\ln\tau),\quad \tau\to\infty.$$

If, moreover, $\int_1^\infty \varphi(\tau)\frac{d\tau}{\tau} = \infty$, *then*

$$(\varphi s * \widetilde{\varphi}\widetilde{s})(\tau) \sim (\varphi * \widetilde{\varphi})(\tau)\Big(\frac{1}{p}(s \star \widetilde{s}\,) + (s \star \widetilde{s}\,)'\Big)(\ln \tau), \quad \tau \to \infty.$$

Proof. The proof is precisely the same as for the preceding lemma. We only need to verify that

$$\int_1^{e^T} s\left(\ln \frac{\tau}{\sigma}\right) \widetilde{s}(\ln \sigma)\frac{d(\widetilde{\varrho}(\ln \sigma))}{\widetilde{\varrho}(\ln \sigma)} = \tau^{-1/p}\int_0^T \varrho(\ln \tau - \lambda)\, d\widetilde{\varrho}(\lambda),$$

which is clear if we substitute $\lambda = \ln \sigma$ on the left-hand side. □

The case of incommensurable periods. Now, let the functions s and $\widetilde{s}$ have no common period.

Lemma 5. *If the periods* T *and* $\widetilde{T}$ *are incommensurable, then*

$$\int_1^\tau s(\ln(\omega/\sigma))\widetilde{s}(\ln \sigma)\frac{d(\widetilde{\varrho}(\ln \sigma))}{\widetilde{\varrho}(\ln \sigma)} = (\mathfrak{C} + o(1)) \ln \tau, \quad \tau \to +\infty,$$

uniformly over $\omega \in \mathbb{R}$, *where*

$$\mathfrak{C} = \frac{1}{p}\cdot\frac{1}{T}\int_0^T s(t)\,dt \cdot \frac{1}{\widetilde{T}}\int_0^{\widetilde{T}} \widetilde{s}(t)\,dt. \tag{8}$$

Proof.

Step 1. We prove the estimate

$$\int_1^\tau s(\ln(\tau/\sigma))\widetilde{s}(\ln \sigma)\frac{d(\widetilde{\varrho}(\ln \sigma))}{\widetilde{\varrho}(\ln \sigma)} = (\mathfrak{C} + o(1)) \ln \tau, \qquad \tau \to +\infty. \tag{9}$$

Substitute $t = \ln \tau$, $r = \ln \sigma$:

$$\int_1^\tau s(\ln(\tau/\sigma))\widetilde{s}(\ln \sigma)\frac{d(\widetilde{\varrho}(\ln \sigma))}{\widetilde{\varrho}(\ln \sigma)} = \int_0^t s(t-r)e^{-r/p}\, d\widetilde{\varrho}(r) =: Q(t).$$

We define a $\widetilde{T}$-periodic function

$$q(t) := \int_0^T s(r)\widetilde{s}(t+T-r)\,dr = \int_t^{t+T} s(t-r)\widetilde{s}(r)\,dr.$$

This function admits a continuous derivative

$$q'(t) = \int_0^T s(r)\,d\widetilde{s}(t+T-r) = \int_t^{t+T} s(t-r)\,d\widetilde{s}(r) = -\frac{1}{p}\cdot q(t) + \int_t^{t+T} s(t-r)e^{-r/p}\,d\widetilde{\varrho}(r).$$

Therefore,

$$Q(t+T) - Q(t) = \int_t^{t+T} s(t-r)e^{-r/p}\,d\widetilde{\varrho}(r) = q'(t) + \frac{1}{p}\cdot q(t) =: q_1(t),$$

where $q_1(t)$ is a continuous $\widetilde{T}$-periodic function. Hence,

$$Q(t+nT) = Q(t) + \sum_{k=0}^{n-1} q_1(t+kT). \tag{10}$$

Using Oxtoby's ergodic theorem (see [20]), we get

$$\lim_{n\to+\infty}\frac{1}{n}\sum_{k=0}^{n-1} q_1(t+kT) = \frac{1}{\widetilde{T}}\int_0^{\widetilde{T}} q_1(t)\,dt \tag{11}$$

uniformly with respect to t. From (10) and (11) we deduce the estimate

$$Q(t) = (\mathfrak{C} + o(1))t, \qquad t \to +\infty,$$

where

$$\mathfrak{C} = \frac{1}{T\widetilde{T}} \int_0^{\widetilde{T}} q_1(t)\,dt = \frac{1}{p} \cdot \frac{1}{T} \int_0^T s(t)\,dt \cdot \frac{1}{\widetilde{T}} \int_0^{\widetilde{T}} \widetilde{s}(t)\,dt.$$

Substituting $t = \ln \tau$, we complete the proof of formula (9).

Step 2. For every τ there exists $k(\tau) \in \mathbb{Z}$ such that

$$0 \le \tau - \omega - Tk(\tau) < T.$$

Then

$$\begin{aligned}\int_1^\tau s(\ln(\omega/\sigma))\widetilde{s}(\ln\sigma)\frac{d(\widetilde{\varrho}(\ln\sigma))}{\widetilde{\varrho}(\ln\sigma)} &= \int_{\omega+Tk(\tau)}^\tau s(\ln(\omega/\sigma))\widetilde{s}(\ln\sigma)\frac{d(\widetilde{\varrho}(\ln\sigma))}{\widetilde{\varrho}(\ln\sigma)} \\ &\quad + \int_1^{\omega+Tk(\tau)} s(\ln((\omega+Tk(\tau))/\sigma))\widetilde{s}(\ln\sigma)\frac{d(\widetilde{\varrho}(\ln\sigma))}{\widetilde{\varrho}(\ln\sigma)}.\end{aligned}$$

The first term is uniformly bounded, and the second satisfies the estimate

$$(\mathfrak{C} + o(1))\ln(\omega + Tk(\tau)) = (\mathfrak{C} + o(1))\ln\tau, \qquad \tau \to +\infty. \qquad \square$$

§4. Spectral asymptotics of tensor products

Lemma 6. *In formula* (2) *the function s has the form*

$$s(\tau) = e^{-\tau/p}\varrho(\tau),$$

where ϱ is a monotone function; *thus, s is a function of bounded variation.*

Proof. The asymptotics can be reshaped as follows:

$$\frac{s(\ln(1/t))}{t^{1/p}} = \frac{\mathcal{N}(t)}{\varphi(1/t)}(1 + \varepsilon(t)), \quad \varepsilon(t) \to 0 \text{ as } t \to +0.$$

Replacing t with $e^{-kT}t$, we get

$$\frac{s(\ln(1/t))}{(e^{-kT}t)^{1/p}} = \frac{\mathcal{N}(e^{-kT}t)}{\varphi(e^{kT}/t)}(1 + \varepsilon(e^{-kT}t)).$$

Thus,

$$\frac{s(\ln(1/t))}{t^{1/p}} = \lim_{k\to+\infty} e^{-\frac{kT}{p}} \frac{\mathcal{N}(e^{-kT}t)}{\varphi(e^{kT}/t)},$$

where convergence is uniform on $[1, e^T]$. Hence, for a fixed $\varepsilon > 0$ we obtain the relation

$$s(\ln(1/t)) = t^{\frac{1}{p}+\varepsilon} \cdot \lim_{k\to+\infty} \frac{e^{-\frac{kT}{p}-kT\varepsilon}\mathcal{N}(e^{-kT}t)}{(e^{kT}/t)^{-\varepsilon}\varphi(e^{kT}/t)}.$$

Note that the numerator of the fraction above decreases with t, and the denominator grows with t if k is sufficiently large, in accordance with Part 2 of Proposition 2. Denote

$$\varrho_\varepsilon(\ln(1/t)) := \lim_{k\to+\infty} \frac{e^{-\frac{kT}{p}-kT\varepsilon}\mathcal{N}(e^{-kT}t)}{(e^{kT}/t)^{-\varepsilon}\varphi(e^{kT}/t)}.$$

Being a uniform limit of monotone functions, ϱ_ε is monotone. The function s has the form

$$s(\tau) = e^{-(\frac{1}{p}+\varepsilon)\tau}\varrho_\varepsilon(\tau).$$

Passing to the limit as $\varepsilon \to 0$ and denoting $\varrho(\tau) := \lim_{\varepsilon\to 0}\varrho_\varepsilon(\tau)$, we obtain

$$s(\tau) = e^{-\tau/p}\varrho(\tau),$$

where ϱ is also a monotone function. $\square$

Remark 1. For some Green integral operators with singular arithmetically selfsimilar weight measures (see [16, 17, 18]) it is shown that $\varrho(\tau)$ is a continuous purely singular function, i.e., its generalized derivative is a measure singular with respect to Lebesgue measure.

Below we assume that all periodic functions arising in our asymptotics are continuous (thus satisfying all the requirements of §3). Also, Part 3 of Proposition 2 allows us to assume all SVFs to be C^2-smooth.

Theorem 1. *Suppose that an operator $\mathcal{T}$ on a Hilbert space $\mathcal{H}$ has the spectral asymptotics* (2)*, and an operator $\widetilde{\mathcal{T}}$ on a Hilbert space $\widetilde{\mathcal{H}}$ has the spectral asymptotics*

$$\widetilde{\mathcal{N}}(t) := \widetilde{\mathcal{N}}(t, \widetilde{\mathcal{T}}) = O(t^{-1/\widetilde{p}}), \quad t \to 0+, \qquad \widetilde{p} > p.$$

Then the operator $\mathcal{T} \otimes \widetilde{\mathcal{T}}$ on the Hilbert space $\mathcal{H} \otimes \widetilde{\mathcal{H}}$ has the asymptotics

$$\mathcal{N}_\otimes(t) := \mathcal{N}(t, \mathcal{T} \otimes \widetilde{\mathcal{T}}) \sim \frac{\varphi(1/t) \cdot s^*(\ln(1/t))}{t^{1/p}}, \quad t \to +0, \tag{12}$$

where

$$s^*(\tau) := \sum_k s(\tau + \ln(\widetilde{\lambda}_k)) \cdot \widetilde{\lambda}_k^{1/p} \tag{13}$$

is a periodic function with period T (the series converges because $\widetilde{p} > p$).

Proof. Since the eigenvalues of a tensor product of operators are equal to the products of their eigenvalues, we have

$$\mathcal{N}_\otimes(t) = \#\{k, j \,:\, \lambda_k \widetilde{\lambda}_j > t\} = \sum_k \#\{j \,:\, \lambda_j > t/\widetilde{\lambda}_k\} = \sum_k \mathcal{N}(t/\widetilde{\lambda}_k).$$

Thus,

$$\frac{t^{1/p}}{\varphi(1/t)} \sum_k \mathcal{N}(t/\widetilde{\lambda}_k) = \sum_k \left(\frac{(t/\widetilde{\lambda}_k)^{1/p}\mathcal{N}(t/\widetilde{\lambda}_k)}{\varphi(\widetilde{\lambda}_k/t)s(\ln(\widetilde{\lambda}_k/t))} \right) \left(\frac{\varphi(\widetilde{\lambda}_k/t)}{\varphi(1/t)} \right) s(\ln(\widetilde{\lambda}_k/t))\widetilde{\lambda}_k^{1/p}.$$

The first factor is uniformly bounded and tends to 1 as $t \to 0+$, see (2). The second factor also tends to 1. Also, since for every ε the function $\tau^\varepsilon\varphi(\tau)$ grows when $\tau > \tau_0(\varepsilon)$ by Part 2 of Proposition 2, we have

$$\frac{\lambda^\varepsilon \varphi(\lambda\tau)}{\varphi(\tau)} = \frac{(\lambda\tau)^\varepsilon \varphi(\lambda\tau)}{\tau^\varepsilon \varphi(\tau)} \le 1 \quad \text{for } \lambda\tau > \tau_0(\varepsilon), \ \lambda < 1.$$

Thus, for every $\varepsilon > 0$ we have the following estimate uniform for $t < 1$:

$$\frac{\varphi(\widetilde{\lambda}_k/t)}{\varphi(1/t)} \le C(\varepsilon)\widetilde{\lambda}_k^{-\varepsilon},$$

whence

$$\frac{\varphi(\widetilde{\lambda}_k/t)}{\varphi(1/t)} \widetilde{\lambda}_k^{1/p} \le C(\varepsilon) \cdot k^{-\widetilde{p}(1/p-\varepsilon)}.$$

For sufficiently small ε (such that $\widetilde{p}(1/p - \varepsilon) > 1$), this gives us an estimate sufficient for using Lebesgue's dominated convergence theorem. Passing to the limit, we obtain (12). $\square$

Remark 2. Generally speaking, for an arbitrary function s and an operator $\widetilde{\mathcal{T}}$ the function $s^*(\tau)$ can degenerate to a constant. We could, for example, demand that $s(\tau) + s(\tau + T/2) = 1$, $T = 2p\ln 2$, and choose a finite-rank operator $\widetilde{\mathcal{T}}$ with three eigenvalues: 2^p, 2^p, and 2^{2p}. Then

$$\begin{aligned} s^*(\tau) &= s(\tau + p\ln 2)\cdot 2 + s(\tau + p\ln 2)\cdot 2 + s(\tau + 2p\ln 2)\cdot 2^2 \\ &= 4(s(\tau) + s(\tau + T/2)) = \text{const}\,. \end{aligned}$$

However, if $s(\tau) = \exp(-\tau/p)\varrho(\tau)$, where $\varrho(\tau)$ is a monotone nondecreasing purely singular function (like in Remark 1), then no linear combination of shifts will be constant. Moreover, we observe that in this case the function $s^*(\tau)$ also has the form

$$s^*(\tau) = \exp(-\tau/p)\varrho^*(\tau), \quad \varrho^*(\tau) = \sum_k \varrho(\tau + \ln\widetilde{\lambda}_k),$$

and $\varrho^*(\tau)$ is a purely singular function, because $\varrho(\tau)$ is monotone.

Now, we consider the case where the operators have coinciding power exponents in their spectral asymptotics.

Theorem 2. *Let $\mathcal{T}$ have the spectral asymptotics* (2), *and let $\widetilde{\mathcal{T}}$ have the asymptotics*

$$\mathcal{N}(t, \widetilde{\mathcal{T}}) \sim \widetilde{\mathcal{N}}_{as}(t) := \frac{\widetilde{\varphi}(1/t)\cdot\widetilde{s}(\ln(1/t))}{t^{1/p}}, \qquad t \to +0. \tag{14}$$

Here $\widetilde{\varphi}$ is an SVF, $\widetilde{s}$ has period $\widetilde{T}$. Then for every $\varepsilon > 0$ we have the estimates

$$\mathcal{N}_\otimes(t) \lesseqgtr \frac{\alpha_\pm(\varepsilon)}{t^{1/p}}\cdot\left[S(t,\varepsilon) + \widetilde{S}(t,\varepsilon) + \int_{\alpha_\mp(\varepsilon)/\varepsilon}^{\varepsilon\tau} \varphi\left(\frac{\tau}{\sigma}\right)\widetilde{\varphi}(\sigma) s\left(\ln\frac{\tau}{\sigma}\right)\widetilde{s}(\ln\sigma)\frac{d(\widetilde{\varrho}(\ln\sigma))}{\widetilde{\varrho}(\ln\sigma)}\right]$$

uniformly for $t > 0$. Here the integral should be interpreted as a Lebesgue–Sieltjes integral, $\tau = \alpha_\pm(\varepsilon)/t$. When $\varepsilon\tau < a_\mp(\varepsilon)/\varepsilon$, the integral is assumed to be zero. The coefficients $\alpha_\pm(\varepsilon)$ tend to 1 as $\varepsilon \to 0$, and the functions $S(t,\varepsilon)$ and $\widetilde{S}(t,\varepsilon)$ have the following asymptotics as $t \to +0$:

$$\begin{aligned} S(t,\varepsilon) &\sim \varphi(1/t)\cdot\sum_{\widetilde{\lambda}_k \geq \varepsilon} s(\ln(1/t) + \ln(\widetilde{\lambda}_k))\widetilde{\lambda}_k^{1/p}, \\ \widetilde{S}(t,\varepsilon) &\sim \widetilde{\varphi}(1/t)\cdot\Bigg(\sum_{\lambda_k \geq \varepsilon} \widetilde{s}(\ln(\tau) + \ln(\lambda_k))\lambda_k^{1/p} + \varphi(1/\varepsilon)s(\ln(1/\varepsilon))\widetilde{s}(\ln(\tau\varepsilon))\Bigg). \end{aligned} \tag{15}$$

Proof. The proof follows the lines of Theorem 3.3 in [4]. We prove the upper estimate, the lower estimate can be obtained similarly. We have

$$t^{1/p}\mathcal{N}_\otimes(t) = t^{1/p}\sum_{\widetilde{\lambda}_k < \varepsilon} \mathcal{N}(t/\widetilde{\lambda}_k) + S(t,\varepsilon),$$

where

$$S(t,\varepsilon) = t^{1/p}\sum_{\widetilde{\lambda}_k \geq \varepsilon} \mathcal{N}(t/\widetilde{\lambda}_k).$$

The asymptotics for $S(t,\varepsilon)$ can be obtained from Theorem 1 for a finite-rank operator $\widetilde{\mathcal{T}}$.

Denote by $\widetilde{\mu}$ the function inverse to $\widetilde{\mathcal{N}}_{as}$. Then $\widetilde{\lambda}_k/\widetilde{\mu}(k) \to 1$ as $k \to \infty$, so that

$$\alpha_-(\varepsilon)\widetilde{\mu}(k) \leq \widetilde{\lambda}_k \leq \alpha_+(\varepsilon)\widetilde{\mu}(k) \quad \text{ for } \widetilde{\lambda}_k < \varepsilon$$

for certain values of $\alpha_\pm(\varepsilon)$ such that $\alpha_\pm(\varepsilon) \to 1$ as $\varepsilon \to 0$.

Since the function $\mathcal{N}$ is monotone, we have

$$\sum_{\widetilde{\lambda}_k<\varepsilon} \mathcal{N}(t/\widetilde{\lambda}_k) \le \sum_{\widetilde{\mu}(k)<\alpha_-^{-1}(\varepsilon)\varepsilon} \mathcal{N}\Big(\frac{t}{\alpha_+(\varepsilon)\widetilde{\mu}(k)}\Big).$$

The function $k \mapsto \mathcal{N}\big(\frac{t}{\alpha_+(\varepsilon)\widetilde{\mu}(k)}\big)$ is also monotone, which yields

$$t^{1/p}\sum_{\widetilde{\lambda}_k<\varepsilon} \mathcal{N}(t/\widetilde{\lambda}_k) \le t^{1/p}\mathcal{N}\Big(\frac{\alpha_-(\varepsilon)t}{\alpha_+(\varepsilon)\varepsilon}\Big) + t^{1/p}\int_0^{\varepsilon\alpha_-^{-1}(\varepsilon)} \mathcal{N}\Big(\frac{t}{\alpha_+(\varepsilon)\mu}\Big)(-d\widetilde{\mathcal{N}}_{as}(\mu)). \tag{16}$$

The first term can be estimated as $O(\varepsilon^{1/p}\varphi(1/t))$; thus, adding it to the term $S(t,\varepsilon)$, we obtain $\alpha_+(\varepsilon)S(t,\varepsilon)$. Next, viewing $-d\widetilde{\mathcal{N}}_{as}(\mu)$ as a Lebesgue–Stieltjes measure, we obtain

$$-d\widetilde{\mathcal{N}}_{as}(\mu) = \frac{1}{\mu}\widetilde{\varphi}(1/\mu)\widetilde{\varrho}(\ln(1/\mu))\Big(\frac{-\mu d(\widetilde{\varrho}(\ln(1/\mu)))}{\widetilde{\varrho}(\ln(1/\mu))} + \frac{\widetilde{\varphi}\,'(1/\mu)}{\mu\widetilde{\varphi}(1/\mu)}d\mu\Big). \tag{17}$$

The density of the second term tends to zero as $\mu \to 0$, while the first term

$$\frac{-\mu d(\widetilde{\varrho}(\ln(1/\mu)))}{\widetilde{\varrho}(\ln(1/\mu))} = \frac{d\widetilde{\varrho}(\ln(1/\mu))}{\widetilde{\varrho}(\ln(1/\mu))} = d(\ln(\widetilde{\varrho}(\lambda)))$$

is a positive periodic measure (here $\lambda = \ln(1/\mu)$), because

$$\ln(\varrho(\tau+T)) = \ln(\varrho(\tau)) + \frac{T}{p}.$$

Hence, for sufficiently small ε the contribution of the second term of (17) to the integral in (16) is negligible, and this integral is dominated by

$$\alpha_+(\varepsilon)t^{1/p}\int_0^{\varepsilon\alpha_-^{-1}(\varepsilon)} \mathcal{N}\Big(\frac{t}{\alpha_+(\varepsilon)\mu}\Big)\widetilde{\varphi}(1/\mu)\big(-d(\widetilde{\varrho}(\ln(1/\mu)))\big).$$

Splitting the integral into two parts and changing variables, we obtain the estimate

$$\alpha_+(\varepsilon)t^{1/p}\int_\varepsilon^{+\infty} \mathcal{N}(s)\widetilde{\varphi}(\tau s)d(\widetilde{\varrho}(\ln(\tau s))) + \alpha_+(\varepsilon)t^{1/p}\int_{\alpha_-(\varepsilon)/\varepsilon}^{\varepsilon\tau} \mathcal{N}(\sigma/\tau)\widetilde{\varphi}(\sigma)d(\widetilde{\varrho}(\ln\sigma)).$$

Replacing in the second integral $\mathcal{N}$ with $\alpha_+(\varepsilon)\mathcal{N}_{as}$, we obtain exactly the third term of the estimate we are proving. The first integral gives us the term $\widetilde{S}(t,\varepsilon)$. Next, we have

$$\frac{\widetilde{\varphi}(\alpha_+(\varepsilon)s/t)}{\widetilde{\varphi}(1/t)} \to 1 \quad \text{as } t \to 0$$

uniformly for $s \in [\varepsilon, \lambda_1(\mathcal{T})]$. Thus,

$$\widetilde{S}(t,\varepsilon) \sim \widetilde{\varphi}(1/t)\int_\varepsilon^{+\infty} \mathcal{N}(s)d(\widetilde{s}(\ln(\tau s))s^{1/p}).$$

It is clear that $\mathcal{N}(s) = 0$ for $s > \lambda_1(\mathcal{T})$. Integrating by parts, we arrive at the asymptotics (15). $\square$

In Theorems 3–5 it is assumed that

$$\int_1^\infty \varphi(\tau)\frac{d\tau}{\tau} = \int_1^\infty \widetilde{\varphi}(\tau)\frac{d\tau}{\tau} = \infty. \tag{18}$$

Theorem 3. *Let operators $\mathcal{T}$ and $\widetilde{\mathcal{T}}$ satisfy the conditions of Theorem* 2. *Suppose also that condition* (18) *is fulfilled, and the periods of s and $\widetilde{s}$ coincide and are equal to T. Then*

$$\mathcal{N}_\otimes(t) \sim \frac{\phi(1/t)\cdot s_\otimes(\ln(1/t))}{t^{1/p}}, \quad t \to +0,$$

where $\phi(s) := (\varphi * \widetilde{\varphi})(s)$ *is an SVF, and*

$$s_\otimes(\eta) = \frac{(s \star \widetilde{s}\,)(\eta)}{p} + (s \star \widetilde{s}\,)'(\eta) = e^{-\eta/p} \frac{1}{T} \int_0^T \varrho(\eta - \sigma)\, d\widetilde{\varrho}(\sigma) \tag{19}$$

is a continuous positive T*-periodic function.*

Proof. Fix $\varepsilon > 0$ and consider the estimate obtained in Theorem 2. By Part 1 of Proposition 3, we have

$$S(t, \varepsilon) = o(\phi(1/t)), \quad \widetilde{S}(t, \varepsilon) = o(\phi(1/t)), \qquad t \to +0.$$

Next, we can extend the integration interval because, considering $\tau = \alpha_\pm(\varepsilon)/t$ and using Part 1 of Proposition 3, we have

$$\begin{aligned}
&\int_{\varepsilon\tau}^{\tau} \varphi\left(\frac{\tau}{\sigma}\right) \widetilde{\varphi}(\sigma) s\left(\ln \frac{\tau}{\sigma}\right) \widetilde{s}(\ln \sigma) \frac{d(\widetilde{\varrho}(\ln \sigma))}{\widetilde{\varrho}(\ln \sigma)} \\
&\qquad \sim \widetilde{\varphi}(\tau) \int_1^{1/\varepsilon} \varphi(\sigma) s(\ln \sigma) \widetilde{s}(\ln(\tau/\sigma)) \frac{d(\widetilde{\varrho}(\ln(\tau/\sigma)))}{\widetilde{\varrho}(\ln(\tau/\sigma))} = o(\phi(1/t)), \quad t \to +0, \\
&\int_1^{\alpha_\mp(\varepsilon)/\varepsilon} \varphi\left(\frac{\tau}{\sigma}\right) \widetilde{\varphi}(\sigma) s\left(\ln \frac{\tau}{\sigma}\right) \widetilde{s}(\ln \sigma) \frac{d(\widetilde{\varrho}(\ln \sigma))}{\widetilde{\varrho}(\ln \sigma)} \\
&\qquad \sim \varphi(\tau) \int_1^{\alpha_\mp(\varepsilon)/\varepsilon} \widetilde{\varphi}(\sigma) s\left(\ln \frac{\tau}{\sigma}\right) \widetilde{s}(\ln \sigma) \frac{d(\widetilde{\varrho}(\ln \sigma))}{\widetilde{\varrho}(\ln \sigma)} = o(\phi(1/t)), \quad t \to +0.
\end{aligned}$$

Thus,

$$\mathcal{N}_\otimes(t) \lesseqgtr \frac{\alpha_\pm(\varepsilon)}{t^{1/p}} (\varphi s * \widetilde{\varphi}\widetilde{s}\,)(\tau)(1 + o(1)). \tag{20}$$

Using Lemma 4, we obtain

$$\mathcal{N}_\otimes(t) \lesseqgtr \alpha_\pm(\varepsilon) \frac{\phi(\tau)}{t^{1/p}} \Big(\frac{(s \star \widetilde{s}\,)}{p} + (s \star \widetilde{s}\,)' \Big)(\ln(\tau))(1 + o(1)), \quad t \to +0.$$

Observe also that $\phi(\tau) = \phi(1/t)(1 + o(1))$ as $t \to +0$. Hence,

$$\begin{aligned}
&\limsup_{t \to +0} \mathcal{N}_\otimes(t) \Big(\frac{\phi(1/t) \cdot s_\otimes(\ln(1/t))}{t^{1/p}} \Big)^{-1} \le \alpha_+(\varepsilon) \cdot \sup_{t \in [1, e^T]} \frac{s_\otimes(\ln(\alpha_+(\varepsilon)) + \ln(1/t))}{s_\otimes(\ln(1/t))}, \\
&\liminf_{t \to +0} \mathcal{N}_\otimes(t) \Big(\frac{\phi(1/t) \cdot s_\otimes(\ln(1/t))}{t^{1/p}} \Big)^{-1} \ge \alpha_-(\varepsilon) \cdot \inf_{t \in [1, e^T]} \frac{s_\otimes(\ln(\alpha_-(\varepsilon)) + \ln(1/t))}{s_\otimes(\ln(1/t))}.
\end{aligned} \tag{21}$$

Since the function $s_\otimes$ is uniformly continuous on a segment, the supremum and infimum on the right-hand sides of (21) tend to 1 as $\varepsilon \to +0$. Passing to the limit as $\varepsilon \to +0$, we complete the proof. $\square$

Remark 3. The question of the nonconstancy of the $s_\otimes$ remains open. Even if we assume that $s(\tau) = \exp(-\tau/p)\varrho(\tau)$, $\widetilde{s}(\tau) = \exp(-\tau/p)\widetilde{\varrho}(\tau)$, and the functions ϱ and $\widetilde{\varrho}$ are purely singular, we have $s_\otimes(\tau) = \exp(-\tau/p)\varrho_\otimes(\tau)$, where

$$\varrho_\otimes(\tau) = \frac{1}{T} \int_0^T \varrho(\tau - \lambda) d\widetilde{\varrho}(\lambda).$$

It is clear that $\varrho_\otimes' = \varrho' \star \widetilde{\varrho}\,'$ is a convolution of singular measures. However, the convolution of singular measures often turns out to be absolutely continuous (see, e.g., [19]).

Theorem 4. *Let operators $\mathcal{T}$ and $\widetilde{\mathcal{T}}$ satisfy the conditions of Theorem 2. Suppose also that condition (18) is fulfilled and the functions s and $\widetilde{s}$ have no common period, i.e., their periods T and $\widetilde{T}$ are incommensurable. Then*

$$\mathcal{N}_{\otimes}(t) \sim \frac{\psi(1/t)\phi(1/t)}{t^{1/p}}, \quad t \to +0,$$

where $\phi(s) = (\varphi * \widetilde{\varphi})(s)$, $\psi(t)$ is a certain SVF bounded and separated away from zero.

Proof. We repeat the proof of Theorem 3 until we obtain relation (20). Next we obtain an estimate capable to replace Lemma 4.

We introduce a function $r(\tau)$ defined in accordance with the relation

$$(\varphi s * \widetilde{\varphi}\widetilde{s})(\tau) = \phi(\tau) r(\ln \tau).$$

By Lemma 1, the function r is bounded and separated away from zero. We need to prove that it is uniformly continuous. We have

$$\begin{aligned} r(\ln\tau + \delta) - r(\ln \tau) &= r(\ln \tau + \delta)\Big(\frac{\phi(\tau e^{\delta})}{\phi(\tau)} - 1\Big) \\ &+ \frac{1}{\phi(\tau)} \cdot \int_1^{\tau} \left(\frac{\varphi(\frac{\tau e^{\delta}}{\sigma}) s(\ln \frac{\tau e^{\delta}}{\sigma})}{\varphi\left(\frac{\tau}{\sigma}\right) s\left(\ln \frac{\tau}{\sigma}\right)} - 1\right) \varphi\left(\frac{\tau}{\sigma}\right) s\left(\ln\frac{\tau}{\sigma}\right) \widetilde{\varphi}(\sigma)\widetilde{s}(\ln\sigma) \frac{d(\widetilde{\varrho}(\ln\sigma))}{\widetilde{\varrho}(\ln\sigma)} \\ &+ \frac{1}{\phi(\tau)} \cdot \int_{\tau}^{\tau e^{\delta}} \varphi\left(\frac{\tau}{\sigma}\right) s\left(\ln\frac{\tau}{\sigma}\right) \widetilde{\varphi}(\sigma)\widetilde{s}(\ln\sigma) \frac{d(\widetilde{\varrho}(\ln\sigma))}{\widetilde{\varrho}(\ln\sigma)}. \end{aligned}$$

We show that each term here tends to zero as $\delta \to 0$ uniformly with respect to τ. Without loss of generality we may assume that $0 < \delta \le \delta_0$ for some δ_0. For the first term we use the mean value theorem:

$$\frac{\phi(\tau e^{\delta}) - \phi(\tau)}{\phi(\tau)} = (\tau e^{\delta} - \tau)\frac{\phi'(\zeta)}{\phi(\tau)} = (e^{\delta} - 1) \cdot \frac{\tau}{\zeta} \cdot \frac{\phi(\zeta)}{\phi(\tau)} \cdot \frac{\zeta\phi'(\zeta)}{\phi(\zeta)},$$

where $\zeta \in [\tau, \tau e^{\delta}]$. The factor $\frac{\tau}{\zeta}$ is bounded. For the last two factors, observe the existence of the limits

$$\frac{\phi(\zeta)}{\phi(\tau)} \to 1, \quad \frac{\zeta\phi'(\zeta)}{\phi(\zeta)} \to 0, \quad \tau \to \infty,$$

which means that they are also bounded. Thus,

$$\left|\frac{\phi(\tau e^{\delta})}{\phi(\tau)} - 1\right| \le C(e^{\delta} - 1) \to 0, \quad \delta \to 0,$$

uniformly for $\tau \in \mathbb{R}_+$.

Similarly, we can show that in the second term the expression

$$\frac{\varphi(\frac{\tau e^{\delta}}{\sigma}) s(\ln \frac{\tau e^{\delta}}{\sigma})}{\varphi\left(\frac{\tau}{\sigma}\right) s\left(\ln \frac{\tau}{\sigma}\right)} - 1$$

tends to zero uniformly, because φ is an SVF, and s is continuous, periodic, bounded, and separated away from zero. The other factors in the second term are bounded by Lemma 1.

For the third term, as in Lemma 1, we obtain the estimate

$$\int_{\tau}^{\tau e^{\delta}} \varphi\left(\frac{\tau}{\sigma}\right) s\left(\ln\frac{\tau}{\sigma}\right) \widetilde{\varphi}(\sigma)\widetilde{s}(\ln\sigma) \frac{d(\widetilde{\varrho}(\ln\sigma))}{\widetilde{\varrho}(\ln\sigma)} = O(\phi(\tau e^{\delta}) - \phi(\tau)), \quad \tau \to \infty,$$

so that it tends to zero like the first term. Thereby, the uniform continuity in question is proved.

Now we need to prove that $r(\ln\tau)$ is an SVF. By definition, we have

$$
\begin{aligned}
&r(\ln\tau+T)\phi(\tau e^T)-r(\ln\tau)\phi(\tau)\\
&\quad=\int_\tau^{\tau e^T}\varphi\left(e^T\cdot\frac{\tau}{\sigma}\right)s\left(\ln\frac{\tau}{\sigma}\right)\widetilde{\varphi}(\sigma)\widetilde{s}(\ln\sigma)\frac{d(\widetilde{\varrho}(\ln\sigma))}{\widetilde{\varrho}(\ln\sigma)}\\
&\qquad+\int_1^\tau\left(\varphi\left(e^T\cdot\frac{\tau}{\sigma}\right)-\varphi\left(\frac{\tau}{\sigma}\right)\right)s\left(\ln\frac{\tau}{\sigma}\right)\widetilde{\varphi}(\sigma)\widetilde{s}(\ln\sigma)\frac{d(\widetilde{\varrho}(\ln\sigma))}{\widetilde{\varrho}(\ln\sigma)}.
\end{aligned}\tag{22}
$$

Using the relation $\phi(\tau e^T)=\phi(\tau)(1+o(1))$, we rewrite the left-hand side of (22) as follows:

$$
r(\ln\tau+T)\phi(\tau e^T)-r(\ln\tau)\phi(\tau)=\big(r(\ln\tau+T)-r(\ln\tau)\big)\phi(\tau)+o(\phi(\tau)),\quad\tau\to\infty.\tag{23}
$$

On the right-hand side of (22), as $\tau\to\infty$, the first integral satisfies the estimate

$$
\begin{aligned}
&\int_\tau^{\tau e^T}\varphi\left(e^T\cdot\frac{\tau}{\sigma}\right)s\left(\ln\frac{\tau}{\sigma}\right)\widetilde{\varphi}(\sigma)\widetilde{s}(\ln\sigma)\frac{d(\widetilde{\varrho}(\ln\sigma))}{\widetilde{\varrho}(\ln\sigma)}\\
&\quad\sim\widetilde{\varphi}(\tau)\int_1^{e^T}\varphi(\frac{e^T}{\sigma})s\left(\ln\frac{1}{\sigma}\right)\widetilde{s}(\ln(\tau\sigma))\frac{d(\widetilde{\varrho}(\ln(\tau\sigma)))}{\widetilde{\varrho}(\ln(\tau\sigma))}=o(\phi(\tau)).
\end{aligned}\tag{24}
$$

To estimate the second integral we use Proposition 4. Since $\varphi(\tau e^T)=\varphi(\tau)(1+o(1))$ as $\tau\to\infty$, we have

$$
\begin{aligned}
&\int_1^\tau\varphi\left(e^T\cdot\frac{\tau}{\sigma}\right)s\left(\ln\frac{\tau}{\sigma}\right)\widetilde{\varphi}(\sigma)\widetilde{s}(\ln\sigma)\frac{d(\widetilde{\varrho}(\ln\sigma))}{\widetilde{\varrho}(\ln\sigma)}\\
&\quad=\int_1^\tau\varphi\left(\frac{\tau}{\sigma}\right)s\left(\ln\frac{\tau}{\sigma}\right)\widetilde{\varphi}(\sigma)\widetilde{s}(\ln\sigma)\frac{d(\widetilde{\varrho}(\ln\sigma))}{\widetilde{\varrho}(\ln\sigma)}(1+o(1)).
\end{aligned}
$$

By Lemma 1, the integral on the right-hand side is $O(\phi(\tau))$, whence

$$
\int_1^\tau\left(\varphi\left(e^T\cdot\frac{\tau}{\sigma}\right)-\varphi\left(\frac{\tau}{\sigma}\right)\right)s\left(\ln\frac{\tau}{\sigma}\right)\widetilde{\varphi}(\sigma)\widetilde{s}(\ln\sigma)\frac{d(\widetilde{\varrho}(\ln\sigma))}{\widetilde{\varrho}(\ln\sigma)}=o(\phi(\tau)),\quad\tau\to\infty.\tag{25}
$$

From (23), (24), and (25) it follows that

$$
r(\ln\tau+T)-r(\ln\tau)=o(1),\quad\tau\to\infty.
$$

By the same argument, we obtain

$$
r(\ln\tau+\widetilde{T})-r(\ln\tau)=o(1),\quad\tau\to\infty.
$$

Hence, for arbitrary $z_1,z_2\in\mathbb{Z}$ we have

$$
r(\ln\tau+z_1T+z_2\widetilde{T})-r(\ln\tau)=o(1),\quad\tau\to\infty.
$$

Since the periods are incommensurable, the set $\{z_1T+z_2\widetilde{T}\mid z_1,z_2\in\mathbb{Z}\}$ is dense in $\mathbb{R}$. Therefore, the uniform continuity of r implies that for every $c\in\mathbb{R}$ we have

$$
r(\ln\tau+c)-r(\ln\tau)=o(1),\quad\tau\to\infty.
$$

Hence, recalling that r is bounded and separated away from zero, we conclude that the function $\psi(\tau):=r(\ln(\tau))$ is an SVF. □

Under certain additional conditions it is possible to demonstrate that $\psi=\text{const}$.

Theorem 5. *Let operators $\mathcal{T}$ and $\widetilde{\mathcal{T}}$ satisfy the conditions of Theorem 4. Suppose also that the functions φ and $\widetilde{\varphi}$ satisfy the following estimates:*

$$
\left|\frac{\sigma\ln(\sigma)\varphi'(\sigma)}{\varphi(\sigma)}\right|\le C,\quad\left|\frac{\sigma\ln(\sigma)\widetilde{\varphi}'(\sigma)}{\widetilde{\varphi}(\sigma)}\right|\le C,\qquad\sigma\ge1.\tag{26}
$$

Then

$$\mathcal{N}_\otimes(t) \sim \frac{\mathfrak{C}\phi(1/t)}{t^{1/p}}, \quad t \to +0,$$

where $\phi(s) = (\varphi * \widetilde{\varphi})(s)$*, and the constant* $\mathfrak{C}$ *is as defined in* (8).

Proof. We want to prove the estimate

$$(\varphi s * \widetilde{\varphi s})(\tau) \sim \mathfrak{C}\phi(\tau), \quad \tau \to \infty. \tag{27}$$

First, we estimate $H[\varphi s, \widetilde{\varphi s}](\tau)$. For that, we integrate by parts and use Lemma 5:

$$\begin{aligned} H[\varphi s, \widetilde{\varphi s}](\tau) &= \int_1^{\sqrt{\tau}} \varphi\left(\frac{\tau}{\sigma}\right)\widetilde{\varphi}(\sigma)\, d\Big(\int_1^{\sigma} s\big(\ln\frac{\tau}{\xi}\big)\widetilde{s}\big(\ln\xi\big)\frac{d(\widetilde{\varrho}(\ln\xi))}{\widetilde{\varrho}(\ln\xi)}\Big) \\ &= \varphi(\sqrt{\tau})\widetilde{\varphi}(\sqrt{\tau})\int_1^{\sqrt{\tau}} s\big(\ln\frac{\tau}{\xi}\big)\widetilde{s}\big(\ln\xi\big)\frac{d(\widetilde{\varrho}(\ln\xi))}{\widetilde{\varrho}(\ln\xi)} \\ &\quad - \int_1^{\sqrt{\tau}} \Big(\varphi\left(\frac{\tau}{\sigma}\right)\widetilde{\varphi}(\sigma)\Big)'_\sigma \int_1^{\sigma} s\big(\ln\frac{\tau}{\xi}\big)\widetilde{s}\big(\ln\xi\big)\frac{d(\widetilde{\varrho}(\ln\xi))}{\widetilde{\varrho}(\ln\xi)}\, d\sigma \\ &= \varphi(\sqrt{\tau})\widetilde{\varphi}(\sqrt{\tau})(\mathfrak{C}+o(1))\ln(\sqrt{\tau}) \\ &\quad - \int_1^{\sqrt{\tau}} \Big(\varphi\left(\frac{\tau}{\sigma}\right)\widetilde{\varphi}(\sigma)\Big)'_\sigma (\mathfrak{C}+o(1))\ln\sigma\, d\sigma. \end{aligned}$$

We transform the leading term of the asymptotics by reversing integration by parts:

$$\mathfrak{C}\Big(\varphi(\sqrt{\tau})\widetilde{\varphi}(\sqrt{\tau})\ln(\sqrt{\tau}) - \int_1^{\sqrt{\tau}} \Big(\varphi\left(\frac{\tau}{\sigma}\right)\widetilde{\varphi}(\sigma)\Big)'_\sigma \ln\sigma\, d\sigma\Big) = \mathfrak{C}h_{\widetilde{\varphi},\varphi}(\tau).$$

Now, we estimate the contribution of each $o(1)$:

$$\begin{aligned} &\mathfrak{C}h_{\widetilde{\varphi},\varphi}(\tau) + \int_1^{\sqrt{\tau}} \Big(\varphi\left(\frac{\tau}{\sigma}\right)\widetilde{\varphi}(\sigma)\Big)'_\sigma \ln\sigma \cdot o(1)\, d\sigma \\ &\quad = \mathfrak{C}\int_1^{\sqrt{\tau}} \Big[1 + \Big(\frac{\sigma\ln(\sigma)\widetilde{\varphi}'(\sigma)}{\widetilde{\varphi}(\sigma)} + \frac{\ln(1/\sigma)}{\ln(\tau/\sigma)}\cdot\frac{(\tau/\sigma)\ln(\tau/\sigma)\varphi'(\tau/\sigma)}{\varphi(\tau/\sigma)}\Big)o(1)\Big]\cdot\varphi\Big(\frac{\tau}{\sigma}\Big)\widetilde{\varphi}(\sigma)\frac{d\sigma}{\sigma} \\ &\quad = (\mathfrak{C}+o(1))h_{\widetilde{\varphi},\varphi}(\tau). \end{aligned}$$

by Part 2 of Proposition 3, because the expression in the parentheses is bounded due to the additional conditions (26). By the same argument we have

$$\varphi(\sqrt{\tau})\widetilde{\varphi}(\sqrt{\tau})\ln(\sqrt{\tau})\cdot o(1) = o(1)\cdot\Big(h_{\widetilde{\varphi},\varphi}(\tau) + \int_1^{\sqrt{\tau}}\Big(\varphi\left(\frac{\tau}{\sigma}\right)\widetilde{\varphi}(\sigma)\Big)'_\sigma \ln\sigma\, d\sigma\Big) = o(h_{\widetilde{\varphi},\varphi}(\tau)).$$

Thus, we have obtained the estimate

$$H[\varphi s, \widetilde{\varphi s}](\tau) = (\mathfrak{C}+o(1))h_{\widetilde{\varphi},\varphi}(\tau). \tag{28}$$

Similarly, using Lemma 2, we obtain

$$H_1[\varphi s, \widetilde{\varphi s}](\tau) = H[\widetilde{\varphi s}, \varphi s](\tau)(1+o(1)) = (\mathfrak{C}+o(1))h_{\varphi,\widetilde{\varphi}}(\tau). \tag{29}$$

From the asymptotics (28) and (29) we obtain the required asymptotics (27). □

Remark 4. From the additional conditions (26) it follows that for some $C > 0$ the estimates

$$\varphi(e)(\ln\sigma)^{-C} \le \varphi(\sigma) \le \varphi(e)(\ln\sigma)^{C}, \qquad \widetilde{\varphi}(e)(\ln\sigma)^{-C} \le \widetilde{\varphi}(\sigma) \le \widetilde{\varphi}(e)(\ln\sigma)^{C}$$

are true for $\sigma \ge e$. The additional conditions are clearly satisfied for the SVFs of the form $(1+\ln(\tau))^{\varkappa}$. In the general case, the question of the constancy of the function ψ in Theorem 4 remains open.

Now we consider the cases where one or both of the integrals of the SVFs are finite.

Theorem 6. *Let operators $\mathcal{T}$ and $\widetilde{\mathcal{T}}$ satisfy the conditions of Theorem* 2. *Suppose that*

$$\int_1^\infty \varphi(\tau)\frac{d\tau}{\tau} < \infty, \quad \int_1^\infty \widetilde{\varphi}(\tau)\frac{d\tau}{\tau} = \infty,$$

and the periods of the functions s and $\widetilde{s}$ coincide and are equal to T. Suppose also that the pair $(\varphi, \widetilde{\varphi})$ satisfies Part 4 *of Proposition* 3. *Then*

$$\mathcal{N}_\otimes(t) \sim \frac{h_{\widetilde{\varphi},\varphi}(1/t)\cdot s_\otimes(\ln(1/t)) + \widetilde{\varphi}(1/t)\cdot \widetilde{s}^*(\ln(1/t))}{t^{1/p}},$$

where $s_\otimes$ is defined in (19), *and*

$$\widetilde{s}^*(\tau) = \sum_n \widetilde{s}(\tau + \ln(\lambda_n))\lambda_n^{1/p} \tag{30}$$

(*cf.* (13)).

Remark 5. The sum (30) converges in accordance with Proposition 1.

Proof. Fix $\varepsilon > 0$. By Part 3 of Proposition 3, we have

$$S(t,\varepsilon) = o(h_{\widetilde{\varphi},\varphi}(1/t)), \quad t \to +0.$$

Part 4 of Proposition 2 yields

$$\widetilde{S}(t,\varepsilon) \sim \widetilde{\varphi}(1/t)\cdot\Big(\sum_n \widetilde{s}(\ln\tau + \ln\lambda_k)\lambda_k^{1/p} + \nu(\varepsilon)\Big),$$

where $\nu(\varepsilon) \to 0$ as $\varepsilon \to +0$. It remains to estimate the integral term:

$$\begin{aligned}
&\int_{\alpha_\mp(\varepsilon)/\varepsilon}^{\varepsilon\tau} \varphi\left(\frac{\tau}{\sigma}\right)\widetilde{\varphi}(\sigma)s\left(\ln\frac{\tau}{\sigma}\right)\widetilde{s}(\ln\sigma)\frac{d(\widetilde{\varrho}(\ln\sigma))}{\widetilde{\varrho}(\ln\sigma)} \\
&\quad= \int_1^{\sqrt{\tau}} \varphi\left(\frac{\tau}{\sigma}\right)\widetilde{\varphi}(\sigma)s\left(\ln\frac{\tau}{\sigma}\right)\widetilde{s}(\ln\sigma)\frac{d(\widetilde{\varrho}(\ln\sigma))}{\widetilde{\varrho}(\ln\sigma)} \\
&\qquad- \int_1^{\alpha_\mp(\varepsilon)/\varepsilon} \varphi\left(\frac{\tau}{\sigma}\right)\widetilde{\varphi}(\sigma)s\left(\ln\frac{\tau}{\sigma}\right)\widetilde{s}(\ln\sigma)\frac{d(\widetilde{\varrho}(\ln\sigma))}{\widetilde{\varrho}(\ln\sigma)} \\
&\qquad+ \int_{1/\varepsilon}^{\sqrt{\tau}} \varphi(\sigma)\widetilde{\varphi}\left(\frac{\tau}{\sigma}\right)s(\ln\sigma)\widetilde{s}(\ln(\tau/\sigma))\frac{d(\widetilde{\varrho}(\ln(\tau/\sigma)))}{\widetilde{\varrho}(\ln(\tau/\sigma))}.
\end{aligned}$$

We estimate the first term by Lemma 4. The second term is $O(\varphi(\tau)) = o(h_{\widetilde{\varphi},\varphi}(\tau))$ as $\tau \to \infty$. As to the third term, we estimate it as in Part 4 of Proposition 3:

$$\int_{1/\varepsilon}^{\sqrt{\tau}} \varphi(\sigma)\widetilde{\varphi}\left(\frac{\tau}{\sigma}\right)s(\ln\sigma)\widetilde{s}(\ln(\tau/\sigma))\frac{d(\widetilde{\varrho}(\ln(\tau/\sigma)))}{\widetilde{\varrho}(\ln(\tau/\sigma))} \le C\widetilde{\varphi}(\tau)\int_{1/\varepsilon}^\infty \varphi(\sigma)\frac{d\sigma}{\sigma}, \quad \tau \to \infty.$$

Here

$$\int_{1/\varepsilon}^\infty \varphi(\sigma)\frac{d\sigma}{\sigma} \to 0, \quad \varepsilon \to 0,$$

showing that this term's contribution to the asymptotics is negligible. □

Remark 6. Like in Theorem 4, if the periods T and $\widetilde{T}$ are incommensurable, then instead of $s_\otimes(\ln(\tau))$ in the asymptotics we obtain an SVF bounded and separated away from zero, which degenerates to a constant in the same particular cases as in Theorem 5.

Theorem 7. *Let operators $\mathcal{T}$ and $\widetilde{\mathcal{T}}$ satisfy the conditions of Theorem 2. Suppose that*

$$\int_1^\infty \varphi(\tau)\frac{d\tau}{\tau} < \infty, \quad \int_1^\infty \widetilde{\varphi}(\tau)\frac{d\tau}{\tau} < \infty,$$

and $(\varphi, \widetilde{\varphi})$ and $(\widetilde{\varphi}, \varphi)$ satisfy Part 4 of Poposition 3. Then

$$\mathcal{N}_\otimes(t) \sim \frac{\varphi(1/t)\cdot s^*(\ln(1/t)) + \widetilde{\varphi}(1/t)\cdot \widetilde{s}^*(\ln(1/t))}{t^{1/p}},$$

where s^* is defined in (13), $\widetilde{s}^*$ is defined in (30).

The proof of this theorem is similar to that of Theorem 6.

Remark 7. In contrast to the preseding theorems, the asymptotics in the last two cases contain two terms. One of them might be majorized by the other, in which case the asymptotics is again almost regular. However, in the general case, it is impossible to predict their behavior, and it is possible that neither one prevails. In that case the asymptotics may fail to be almost regular.

Example 1. Let

$$\mathcal{N}(t,\mathcal{T}) \sim \frac{\ln^{\varkappa_1}(1/t)\cdot s(\ln(1/t))}{t^{1/p}}, \qquad \mathcal{N}(t,\widetilde{\mathcal{T}}) \sim \frac{\ln^{\varkappa_2}(1/t)\cdot \widetilde{s}(\ln(1/t))}{t^{1/p}}$$

as $t \to +0$. Without loss of generality we assume that $\varphi(\tau) = (1+\ln(\tau))^{\varkappa_1}$, $\widetilde{\varphi}(\tau) = (1+\ln(\tau))^{\varkappa_2}$. In this case, the asymptotics of the Mellin convolution was calculated in [4, Example 1]. We consider all possible cases.

Case 1. $\varkappa_1 \geq -1, \varkappa_2 \geq -1$. In this case Theorem 3 is applicable when the periodic functions have a common period, and Theorem 5 is applicable otherwise.

If the functions s and $\widetilde{s}$ have a common period T, then

$$\mathcal{N}_\otimes(t) \sim \frac{\phi(1/t)\cdot s_\otimes(\ln(1/t))}{t^{1/p}}, \quad t \to +0,$$

where the function $s_\otimes$ is as defined in (19), and

$$\phi(\tau) = \begin{cases} \mathbf{B}(\varkappa_1+1,\varkappa_2+1)(1+\ln(\tau))^{\varkappa_1+\varkappa_2+1}, & \varkappa_1 > -1, \varkappa_2 > -1, \\ \ln(\ln(\tau))\cdot(1+\ln(\tau))^{\varkappa_2}, & \varkappa_1 = -1, \varkappa_2 > -1, \\ 2\ln(\ln(\tau))\cdot(1+\ln(\tau))^{-1}, & \varkappa_1 = \varkappa_2 = -1, \end{cases}$$

where $\mathbf{B}$ is the Euler beta function. Note that the resulting asymptotics is again almost regular.

If the periods T and $\widetilde{T}$ are incommensurable, then

$$\mathcal{N}_\otimes(t) \sim \frac{\mathfrak{C}\phi(1/t)}{t^{1/p}}, \quad t \to +0,$$

where the constant $\mathfrak{C}$ is as defined in (8), and the resulting asymptotics is regular.

Case 2. $\varkappa_1 < -1 \leq \varkappa_2$. In this case, Theorem 6 is applicable, and direct calculations show that

$$h_{\widetilde{\varphi},\varphi}(\tau) = o(\widetilde{\varphi}(\tau)), \quad \tau \to \infty,$$

which means that

$$\mathcal{N}_\otimes(t) \sim \frac{\ln^{\varkappa_2}(1/t)\cdot \widetilde{s}^*(\ln(1/t))}{t^{1/p}}, \tag{31}$$

where $\widetilde{s}^*$ is as defined in (30), and the resulting asymptotics is again almost regular.

Case 3. $\varkappa_1 < \varkappa_2 < -1$. In this case Theorem 7 is applicable, and

$$\varphi(\tau) = o(\widetilde{\varphi}(\tau)), \quad \tau \to \infty,$$

so that, again, we have the asymptotics (31).

Case 4. $\varkappa_1 = \varkappa_2 < -1$. In this case Theorem 7 is applicable, and the two terms of the asymptotics have the same order of growth, so that

$$\mathcal{N}_\otimes(t) \sim \frac{\ln^{\varkappa_1}(1/t)\big(s^*(\ln(1/t)) + \widetilde{s}^*(\ln(1/t))\big)}{t^{1/p}},$$

where s^* is as in (13), $\widetilde{s}^*$ is as in (30). In the case, when the functions s and $\widetilde{s}$ have a common period, this asymptotics turns out to be almost regular, but whenever the periods are incommensurable, we have an almost regular asymptotics with a quasi-periodic component.

§5. Small deviations asymptotics

We recall some facts from the theory of L_2-small deviations of Gaussian random functions.

Suppose we have a Gaussian random function $X(x)$, $x \in \mathcal{O} \subseteq \mathbb{R}^m$, with zero mean and a covariation function $G_X(x,u)$, $x,u \in \mathcal{O}$. Let μ be a finite measure on $\mathcal{O}$. Denote

$$\|X\|_\mu = \Big(\int_{\mathcal{O}} X^2(x)d\mu(x)\Big)^{1/2}.$$

By the logarithmic asymptotics of small deviations in L_2 we mean the asymptotics of $\ln \mathbf{P}\{\|X\|_\mu \le \varepsilon\}$ as $\varepsilon \to 0$.

In accordance with the well-known Karhunen–Loève expansion, we have in distribution

$$\|X(x)\|_\mu^2 \overset{d}{=} \sum_{n=1}^\infty \lambda_n \xi_n^2,$$

where the ξ_n, $n \in \mathbb{N}$, are independent standard normal random variables, and $\lambda_n > 0$, $n \in \mathbb{N}$, $\sum_n \lambda_n < \infty$, are the eigenvalues of the integral equation

$$\lambda f(x) = \int_{\mathcal{O}} G_X(x,u) f(u) d\mu(u). \tag{32}$$

Thus we arrive at the equivalent problem of studying the asymptotic behavior as $\varepsilon \to 0$ of $\ln \mathbf{P}\{\sum_{n=1}^\infty \lambda_n \xi_n^2 \le \varepsilon^2\}$. By the results of [21], the answer only depends on the leading term of the asymptotics of the sequence λ_n.

The case of the purely power asymptotics $\lambda_n \sim Cn^{-p}$, $p > 1$, was considered in [22, 23, 24, 25]. In [4], the case of regular asymptotics was treated, and in [8] the case of almost power asymptotics with a periodic component was explored.

Consider a more general case. Suppose

$$\lambda_n(\mathcal{T}) = \phi(n) := \frac{\psi(n) \cdot \theta(\ln(n))}{n^p}, \tag{33}$$

where $p > 1$, and the function θ is bounded, uniformly continuous on $\mathbb{R}$, and separated away from zero, and the function $\phi(t)$ is monotone on $\mathbb{R}$.

The function $\phi(n)$ satisfies the conditions of Theorem 2 in [26], which in our case has the following form.

Proposition 5.

$$\mathbf{P}\Big\{\sum_{n=1}^\infty \phi(n)\xi_n^2 \le r\Big\} \sim \frac{\exp(L(u) + ur)}{\sqrt{2\pi u^2 L''(u)}}, \quad r \to 0, \tag{34}$$

where

$$L(u) = \sum_{n=1}^\infty \ln f(u\phi(n)), \quad f(t) := (1+2t)^{-1/2},$$

and $u = u(r)$ *is an arbitrary function satisfying*

$$\lim_{r\to 0} \frac{L'(u)+r}{\sqrt{L''(u)}} = 0.$$

First, we analyse the asymptotics of $L'(u)$ as $u \to +\infty$. In our case

$$uL'(u) = -\sum_{n=1}^{\infty} \frac{u\psi(n)\theta(\ln(n))}{n^p + 2u\psi(n)\theta(\ln(n))} \to -\infty, \quad u \to +\infty.$$

Since $\phi(t)$ is a monotone decreasing function, we can write

$$\sum_{n=1}^{\infty} \frac{u\psi(n)\theta(\ln(n))}{n^p + 2u\psi(n)\theta(\ln(n))} \geq \int_1^{\infty} \frac{u\psi(t)\theta(\ln(t))\,dt}{t^p + 2u\psi(t)\theta(\ln(t))}$$

$$\geq \sum_{n=2}^{\infty} \frac{u\psi(n)\theta(\ln(n))}{n^p + 2u\psi(n)\theta(\ln(n))} \sim -uL'(u),$$

whence

$$uL'(u) \sim I_1(u) := -\int_1^{\infty} \frac{u\psi(t)\theta(\ln(t))\,dt}{t^p + 2u\psi(t)\theta(\ln(t))}.$$

Replacing the integration interval with $(0,\infty)$ and substituting

$$t = t(z) := z\phi^{-1}(1/u) = z\gamma(u),$$

$$\gamma(u) := \phi^{-1}(1/u) \sim u^{1/p}\varphi(u)\vartheta(\ln(u)), \quad u \to \infty,$$

where φ is an SVF and ϑ is a function that is uniformly continuous, bounded, and separated away from zero, we obtain

$$I_1(u) = -\gamma(u) \cdot \int_0^{\infty} \frac{dz}{2 + z^p \cdot \dfrac{(\gamma(u))^p}{u\psi(t(z))\theta(\ln(t(z)))}} + O(1), \quad u \to \infty.$$

Since $1/u = \phi(\gamma(u))$, we get the relation

$$(\gamma(u))^p/u = \psi(t(z)/z)\theta(\ln(t(z)/z)).$$

Substituting this in the integral and recalling the definition of $\gamma(u)$, we see that

$$I_1(u) = -\gamma(u) \cdot \int_0^{\infty} \frac{dz}{2 + z^p \cdot \dfrac{\psi(\gamma(u))\theta(\ln(\gamma(u)))}{\psi(z\gamma(u))\theta(\ln(z\gamma(u)))}} + O(1), \quad u \to \infty.$$

Clearly,

$$\theta(\ln(z\gamma(u))) = \theta\Big(\frac{\ln(u)}{p} + \ln(z)\Big)(1 + o(1)), \quad u \to \infty.$$

Observe also that, by Part 2 of Proposition 2, for every $\varepsilon > 0$ the ratio $\psi(t)/t^{\varepsilon}$ decreases at large values of t; thus, for $z > 1$ we have

$$\frac{\psi(t)}{\psi(zt)} = \frac{1}{z^{\varepsilon}} \cdot \frac{\psi(t)}{t^{\varepsilon}} \cdot \frac{(zt)^{\varepsilon}}{\psi(zt)} \geq \frac{C(\varepsilon)}{z^{\varepsilon}}.$$

This majorant allows us to to use the Lebesgue theorem. As a result, we have

$$I_1(u) = -u^{1/p}\vartheta(u) \cdot \int_0^{\infty} \frac{dz}{2 + z^p \cdot \dfrac{\theta(\ln(u)/p)}{\theta(\ln(u)/p + \ln(z))}} + O(1), \quad u \to \infty. \tag{35}$$

Since the integral is a uniformly continuous and bounded function of $\ln(u)$ separated away from zero, we obtain

$$L'(u) \sim -u^{-\frac{p-1}{p}}\varphi(u)\vartheta_1(\ln(u)), \quad u \to \infty, \tag{36}$$

where φ is an SVF from the asymptotics of γ, and the function ϑ_1 is uniformly continuous, bounded, and separated away from zero.

Similarly,

$$u^2 L''(u) \sim 2\int_1^\infty \frac{(u\psi(t)\theta(\ln(t)))^2\,dt}{(t^p + 2u\psi(t)\theta(\ln(t)))^2} \asymp u^{1/p}\varphi(u), \tag{37}$$

$$L(u) \sim -\frac{1}{2}u^{1/p}\varphi(u)\vartheta(\ln(u)) \cdot \int_0^\infty \ln\left(1 + \frac{2\theta(\ln(u)/p + \ln(z))}{z^p\theta(\ln(u)/p)}\right)\,dt.$$

Since $L''(u) > 0$, for sufficiently small r the equation $L'(u) + r = 0$ has a unique solution $u(r)$ such that $u(r) \to \infty$ as $r \to 0$. Moreover, relation (36) yields

$$u(r) \sim r^{-\frac{p}{p-1}}\eta(1/r)\vartheta_2(\ln(1/r)), \quad r \to 0, \tag{38}$$

where η is an SVF, and the function ϑ_2 is uniformly continuous, bounded, and separated away from zero.

Substituting (37) in (34), we conclude that

$$\begin{aligned}\ln \mathbf{P}\Bigg\{\sum_{n=1}^\infty \phi(n)\xi_n^2 \le r\Bigg\} &\sim L(u) + ur = L(u) - uL'(u)\\ &\sim -u^{1/p}\varphi(u)\vartheta(\ln(u)) \cdot \int_0^\infty \left[\frac{1}{2}\ln\left(1 + \frac{2\theta(\ln(u)/p + \ln(z))}{z^p\theta(\ln(u)/p)}\right)\right.\\ &\qquad \left. - \frac{1}{2 + z^p \cdot \frac{\theta(\ln(u)/p)}{\theta(\ln(u)/p + \ln(z))}}\right] dz.\end{aligned} \tag{39}$$

It remains to note that the integrand

$$\frac{1}{2}\ln(1 + 2x) - \frac{x}{2x+1}$$

is positive, so that the integral is a uniformly continuous function of $\ln(u)$, bounded and separated away from zero. Substituting the asymptotics of u obtained above and replacing r with ε^2, we formulate the following theorem.

Theorem 8. *Let the eigenvalues of* (32) *have the form* (33). *Then, as* $\varepsilon \to 0$,

$$\ln \mathbf{P}\{\|X\|_\mu \le \varepsilon\} \sim -\varepsilon^{-\frac{2}{p-1}}\xi(1/\varepsilon)\zeta(\ln(1/\varepsilon)), \tag{40}$$

where ξ *is an SVF and the function* ζ *is uniformly continuous, bounded, and separated away from zero. Moreover, if the function* θ *in* (33) *is asymptotically* $\frac{T}{p}$*-periodic, then the function* ζ *can be chosen to be* $\frac{T(p-1)}{2p}$*-periodic.*

Proof. The first statement of the theorem follows from (39) and (38) if we replace r with ε^2. Next, if θ is asymptotically $\frac{T}{p}$-periodic, then the function ϑ is asymptotically T-periodic, and using the Lebesgue theorem it is easy to verify that the integrals in (35) and (39) are also asymptotically T-periodic functions of $\ln(u)$. Thus, the function ϑ_1 in (36) is asymptotically T-periodic, whence ϑ_2 in (38) is asymptotically $\frac{T(p-1)}{p}$-periodic. It remains to note that (38) implies

$$\ln(u) = \frac{p}{p-1}\ln(1/r)(1 + o(1)), \quad r \to 0.$$

Thus, the integral in (39) and the function $\vartheta(\ln(u))$ are asymptotically $\frac{T(p-1)}{p}$-periodic functions of $\ln(1/r)$, and the second statement is proved. $\square$

Now, suppose we have two Gaussian processes $X(x)$, $x \in \mathcal{O}_1 \subseteq \mathbb{R}^{m_1}$, and $Y(y)$, $y \in \mathcal{O}_2 \subseteq \mathbb{R}^{m_2}$, with zero mean and covariation functions $G_X(x,u)$, $x,u \in \mathcal{O}_1$, and $G_Y(y,v)$, $y,v \in \mathcal{O}_2$, respectively. Consider a new Gaussian function $Z(x,y)$, $x \in \mathcal{O}_1$, $y \in \mathcal{O}_2$, with zero mean and the covariation $G_Z((x,y),(u,v)) = G_X(x,u)G_Y(y,v)$. Such a Gaussian function obviously exists, and the integral operator with the kernel G_Z is the tensor product of the operators with the kernels G_X and G_Y. Therefore, we use the notation $Z = X \otimes Y$ and we call the process Z the tensor product of the processes X and Y. Generalization to the tensor product $\bigotimes_{j=1}^d X_j$ of several factors is straightforward.

Example 2. We demonstrate application of theorems from §4 by the example of the Brownian sheet

$$\mathbb{W}_d(x_1,\dots,x_d) = W_1(x_1) \otimes W_2(x_2) \otimes \cdots \otimes W_d(x_d)$$

in the unit cube with the norm of $L_2(\mu)$, where $\mu = \bigotimes_{j=1}^d \mu_j$, and every measure μ_j is a self-similar measure of generalized Cantor type. The spectral asymptotics of the operators-factors in this case are known from [6] and [7]:

$$\mathcal{N}_j(t) \sim \frac{s_j(\ln(1/t))}{t^{1/p_j}}, \quad t \to 0+,$$

where the s_j are continuous and T_j-periodic, $p_j > 1$. This power asymptotics was considered in Example 1 and corresponds to the case where $\varkappa_1 = \varkappa_2 = 0$.

For certain measures μ_j the functions s_j can be constant, but in [16, 18] we find description of wide classes of measures for which the nonconstancy of the periodic component is proved.

Let $\mathfrak{p} := p_1 = \min p_j$. First, we use Theorem 1 for each operator with $p_j > \mathfrak{p}$, multiplying it by the first one. As a result, without loss of generality we may assume that all operator's asymptotics have the same power exponent.

If among the remaining operators at least one has a degenerate periodic component, then the tensor product will also have a degenerate periodic component. If at least two periods are incommensurable, then the periodic component of their tensor product will degenerate to a constant as in Example 1, and as a result, the periodic component of the entire tensor product will also degenerate to a constant.

If all power exponents coincide and all periods are commensurable, then we can use Example 1 to obtain

$$\mathcal{N}_\otimes(t) \sim \frac{C \ln^{\mathfrak{d}-1}(1/t) s_\otimes^{(\mathfrak{d})}(\ln(1/t))}{t^{1/\mathfrak{p}}}, \quad t \to 0+,$$

where $\mathfrak{d}$ is the number of the power exponents equal to $\mathfrak{p}$, and $s_\otimes^{(\mathfrak{d})}$ is obtained by iterating formula (19) a required number of times. This allows us to use Theorem 8 for this Gaussian field. Moreover, by direct calculation we discover that, in (38) and (40),

$$\eta(1/r) \sim \ln^{\frac{(\mathfrak{d}-1)\mathfrak{p}}{\mathfrak{p}-1}}(1/r), \quad r \to 0,$$

$$\xi(1/r) \sim \ln^{\frac{(\mathfrak{d}-1)\mathfrak{p}}{\mathfrak{p}-1}}(1/r), \quad r \to 0.$$

Thus, as $\varepsilon \to 0$ we have

$$\ln \mathbf{P}\left\{\|\mathbb{W}_d\|_\mu \le \varepsilon\right\} \sim -\varepsilon^{-\frac{2}{\mathfrak{p}-1}} \ln^{\frac{(\mathfrak{d}-1)\mathfrak{p}}{\mathfrak{p}-1}}(1/\varepsilon)\zeta(\ln(1/\varepsilon)),$$

where ζ is a $\frac{T(\mathfrak{p}-1)}{2\mathfrak{p}}$-periodic function.

Consider the simplest case where all measures are the classical Cantor measures. In this case we know that

$$p = \log_2 6, \quad T = \ln 6.$$

Substituting this in the asymptotics, we see that

$$\ln \mathbf{P}\left\{\|\mathbb{W}_d\|_\mu \le \varepsilon\right\} \sim -\varepsilon^{-2\log_3 2} \ln^{(d-1)\log_3 6}(1/\varepsilon)\zeta(\ln(1/\varepsilon)) \quad \text{as} \quad \varepsilon \to 0,$$

where ζ is a $\frac{\ln 3}{2}$-periodic function.

Remark 8. Similar results hold true if instead of the Wiener process we consider different independent Green Gaussian processes. Some examples of well-known Green Gaussian processes can be found in [8].

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References

[1] S. Graf, H. Luschgy, and G. Pagès, *Functional quantization and small ball probabilities for Gaussian processes*, J. Theoret. Probab. **16** (2003), no. 4, 1047–1062. MR2033197

[2] H. Luschgy and G. Pagès, *Sharp asymptotics of the functional quantization problem for Gaussian processes*, Ann. Probab. **32** (2004), no. 2, 1574–1599. MR2060310

[3] A. Papageorgiou and G. W. Wasilkowski, *On the average complexity of multivariate problems*, J. Complexity **6** (1990), no. 1, 1–23. MR1048027

[4] A. Karol′, A. Nazarov, and Ya. Nikitin, *Small ball probabilities for Gaussian random fields and tensor products of compact operators*, Trans. Amer. Math. Soc. **360** (2008), no. 3, 1443–1474. MR2357702

[5] A. Karol′ and A. Nazarov, *Small ball probabilities for smooth Gaussian fields and tensor products of compact operators*, Math. Nachr. **287** (2014), no. 5–6, 595–609. MR3193939

[6] J. Kigami and M. L. Lapidus, *Weyl's problem for the spectral distributions of Laplacians on p.c.f. self-similar fractals*, Comm. Math. Phys. **158** (1991), no. 1, 93–125. MR1243717

[7] M. Solomyak and E. Verbitsky, *On a spectral problem related to self-similar measures*, Bull. London Math. Soc. **27** (1995), no. 3, 242–248. MR1328700

[8] A. I. Nazarov, *Logarithmic asymptotics of small deviations for some Gaussian processes in the L_2-norm respect to a self-similar measure*, Zap. Nauchn. Sem. S.-Peterburg. Mat. Inst. Steklov. (POMI) **311** (2004), 190–213; English transl., J. Math. Sci. (N.Y.) **133** (2006), no. 3, 1314–1327. MR2092208

[9] G. N. Sytaya, *Certain asymptotic representations for a Gaussian measure in Hilbert space*, Theory of Random Processed, No. 2, Naukova Dumka, Kiev, 1974, pp. 93–104. (Russian). MR0362471

[10] M. A. Lifshits, *Asymptotic behavior of small ball probabilities*, Prob. Theory and Math. Stat., VSP/TEV, Vilnius, 1999, pp. 453–468.

[11] W. V. Li and Q. M. Shao, *Gaussian processes: inequalities, small ball probabilities and applications*, Stochastic Processes: Theory and Methods, Handbook of Statistics, vol. 19, North–Holland, Amsterdam, 2001, pp. 533–597. MR1861734

[12] Small deviations for stochastic processes and related topics, Internet site, `www.proba.jussieu.fr/pageperso/smalldev/`.

[13] E. Csàki, *On small values of the square integral of a multiparameter Wiener process*, Statistics and Probability (Visegraid, 1982), Reidel, Dordrecht, 1984, pp. 19–26. MR758997

[14] W. V. Li, *Comparison results for the lower tail of Gaussian seminorms*, J. Theor. Probab. **5** (1992), no. 1, 1–31. MR1144725

[15] E. Seneta, *Regularly varying functions*, Lecture Notes in Math., vol. 508, Springer-Verlag, Berlin, 1976. MR0453936

[16] A. A. Vladimirov and I. A. Sheĭpak, *On the Neumann problem for the Sturm–Liouville equation with cantor-type self-similar weight*, Funktsional. Anal. i Prilozhen. **47** (2013), no. 4, 18–29; English transl., Funct. Anal. Appl. **47** (2013), no. 4, 261–270. MR3185121

[17] ———, *An oscillation method in a problem on the spectrum of a fourth-order differential operator with self-similar weight*, Algebra i Analiz **27** (2015), no. 2, 83–95; English transl., St. Petersburg Math. J. **27** (2016), no. 2, 237–244. MR3444462

[18] N. B. Rastegaev, *On spectral asymptotics of the Neumann problem for the Sturm–Liouville equation with self-similar weight of generalized Cantor type*, Zap. Nauchn. Sem. S.-Peterburg. Mat. Inst. Steklov. (POMI) **425** (2014), 86–98; English transl., J. Math. Sci. **210** (2015), no. 6, 814–821. MR3407794

[19] D. Damanik, A. Gorodetski, and B. Solomyak, *Absolutely continuous convolutions of singular measures and an application to the square Fibonacci Hamiltonian*, Duke Math. J. **164** (2015), no. 8, 1603–1640. MR3352042

[20] J. C. Oxtoby, *Ergodic sets*, Bull. Amer. Math. Soc. **58** (1952), no. 2, 116–136. MR0047262

[21] A. I. Nazarov, *Log-level comparison principle for small ball probabilities*, Statist. Probab. Lett. **79** (2009), no. 4, 481–486. MR2494639

[22] V. M. Zolotarev, *Gaussian measure asymptotic in l_2 on a set of centered spheres with radii tending to zero*, Proc. 12th Europ. Meeting of Statisticians, Varna, 1979, pp.254.

[23] ———, *Asymptotics of a Gaussian measure in l_2*, Stability Problems for Stochastic Models (Moscow, 1984), Vsesoyuz. Nauchno-Issled. Inst. Sistem. Issled., Moscow, 1984, pp. 54–58. (Russian) MR0859211

[24] R. M. Dudley, J. Hoffmann-Jørgensen, and L. A. Shepp, *On the lower tail of Gaussian seminorms*, Ann. Probab. **7** (1979), no. 2, 319–342. MR525057

[25] I. A. Ibragimov, *The probability of a Gaussian vector with values in a Hilbert space hitting a ball of small radius*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **85** (1979), 75–93; English transl., J. Soviet. Math. **20** (1982), no. 3, 2164–2175. MR535459

[26] M. A. Lifshits, *On the lower tail probabilities of some random series*, Ann. Probab. **25** (1997), no. 1, 424–442. MR1428515

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