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QUASI-VARIATIONAL ANALYSIS

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ABSTRACT. We introduce the concept of quasi-variational principles (QVPs) and present general methods of analysis using QVPs for problems that are not necessarily of Euler–Lagrange type. We also introduce a new class of nonlinear operators named "quasi-subdifferential operators" for the mathematical analysis of variational and quasi-variational inequalities, and derive their existence theorems from QVPs. Furthermore, we explain the notion of quasi-subdifferential evolution equations.

1. Introduction: Variational and quasi-variational principles

A variational principle enables us to derive a physical law or differential equation as a minimization problem of a functional:

$$Au = 0 \quad \underset{Au=\Phi'(u)}{\Longleftrightarrow} \quad \Phi(u) = \min_z \Phi(z).$$

In physics, variational principles are considered "the highest form of physical laws" ([1, p. 1290]) and are one of the most important research themes in mathematical physics. It can be said that mathematical physics originated when Newton described the laws of motion using differential equations. Variational principles, which are ways to derive equations of motion (ordinary differential equations) under finite degrees of freedom (the mechanics of material points and rigid bodies), were developed by Euler, Lagrange, and their peers. In the 19th century, variational principles were also applied to physical laws (partial differential equations) of motion (or equilibrium) with infinite degrees of freedom (continuums, classical fields) by Gauss, Dirichlet, and others. In particular, Riemann named the following idea "Dirichlet's principle" and used it to establish the foundations of complex function theory:

$$(1.1)\quad \begin{cases} -\Delta u = 0 \text{ in } \Omega, \\ u|_{\partial\Omega} = g \end{cases} \quad\Longleftrightarrow\quad \int_\Omega |\nabla u|^2 dx = \min_{z|_{\partial\Omega}=g} \int_\Omega |\nabla z|^2 dx.$$

After Weierstrass's critique and Hilbert's rigorous proof of the existence of a minimization function, the concepts of functional analysis and generalized functions clarified the significance of (1.1). In other words, by considering the functional on a Sobolev space, it became possible to readily derive from the definitions that a unique minimization function exists and satisfies Laplace's equation (cf. Sobolev [49, Chap. II]).

With variational principles, constraints are often placed on admissible functions. An example would be the boundary condition $z|_{\partial\Omega} = g$ in Dirichlet's principle.

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In the 1960s, variational problems conditioned by convex sets were established and named variational inequalities (cf. Lions and Stampacchia [42]). Using the modern analysis tools of functional analysis and generalized functions, variational inequalities can be approached by variational principles. An example would be a problem such as:

$$\text{(1.2)} \quad \begin{cases} u \in K, \\ \displaystyle\int_\Omega \nabla u \cdot \nabla(u-z)dx \le 0 \quad \forall z \in K \end{cases} \Longleftrightarrow \int_\Omega |\nabla u|^2 dx = \min_{z\in K} \int_\Omega |\nabla z|^2 dx.$$

Here, K is a closed convex set of Sobolev space $H^1(\Omega)$. By choosing K appropriately, variational principles relating to common boundary value problems can be expressed by (1.2). For example, Dirichlet's principle (1.1) can be expressed by taking $K = \{z \in H^1(\Omega) |\ z|_{\partial\Omega} = g\}$ in (1.2).

Thus, variational principles are very important in physics and mathematics. However, they also have limitations. First, the types of physical laws and differential equations handled by variational principles are restricted to those that can be expressed as Euler–Lagrange equations (functional derivatives) of functionals. Variational principles are not able to handle many important equations in mathematical physics, such as the stationary Navier–Stokes equations:

$$\begin{cases} -\Delta \boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla p = 0, \\ \nabla\cdot\boldsymbol{u} = 0, \end{cases}$$

or a stationary diffusion equation with a convection term:

$$\text{(1.3)} \qquad -\nabla\cdot[\nabla u + k(u)] = 0.$$

In addition, even for Euler–Lagrange-type equations, variational principles cannot be applied to problems where the convex sets that constrain the variational inequalities depend upon unknown functions (quasi-variational inequalities):

$$\begin{cases} u \in K(u), \\ \displaystyle\int_\Omega \nabla u\cdot\nabla(u-z)dx \le 0 \quad \forall z \in K(u) \end{cases} \Longleftrightarrow \quad ?$$

Here, a family of convex sets $K(v)$ that depends upon function $v \in H^1(\Omega)$ is already given, and the convex set $K(u)$ that gives the constraint $u \in K(u)$ is itself determined by the unknown function u. Therefore, the problem cannot be simply treated as an extreme problem involving a functional.

In this paper, we consider boundary value problems as well as variational and quasi-variational inequalities involving equations that are not necessarily of Euler–Lagrange-type. To capture the existence of a solution by relating to functionals, we consider the following principle:

$$\text{(1.4)} \quad Au = 0 \quad \underset{Au = \delta_z\Phi(u;u)}{\Longleftrightarrow} \quad \begin{cases} u \text{ is a fixed point of } v \mapsto z_v : \\ \Phi(v; z_v) = \min_z \Phi(v; z). \end{cases}$$

We consider the function $\Phi(v; z)$, which depends on parameter v. Here, $\delta_z\Phi$ indicates the functional derivative related to variable z. This strategy was introduced by Joly and Mosco [20] for their studies on quasi-variational inequalities. In this study, we call ideas such as (1.4) a "quasi-variational principle"[1] (QVP) and assert that QVPs are not only applicable to quasi-variational inequalities but also to many non-Euler–Lagrange-type problems. As a tool for mathematical analysis, a class of nonlinear operators called "quasi-subdifferential operators" (QSOs) is introduced, and the existence theorem for solutions of their equations is proved using QVPs. Furthermore, we discuss the concept of a "quasi-subdifferential evolution equation" (QSE) as a class of evolution equations related to QSOs. A concise description of the subject exposed in this paper is given in [35].

Symbols. Throughout this paper, H indicates a real Hilbert space; $|\cdot|_H$, $(\cdot,\cdot)$ are the norm and inner product of H, respectively. The norm of a Banach space E will generally be expressed as $|\cdot|_E$. When $\psi : H \to \mathbf{R}\cup\{+\infty\}$ is a proper (i.e., $\psi \not\equiv +\infty$), lower-semicontinuous, and convex function with effective domain $D(\psi) := \{z \in H|\ \psi(z) < +\infty\}$, an operator on H, $\partial\psi : H \to H$, which generally becomes multivalued, is defined as follows:

$$z^* \in \partial\psi(z) \quad \underset{\text{def.}}{\Longleftrightarrow} \quad \begin{cases} z \in D(\psi), \\ (z^*, y - z) \le \psi(y) - \psi(z) \quad \forall y \in D(\psi). \end{cases}$$

$\partial\psi$ is called the subdifferential operator of ψ, and $D(\partial\psi) := \{z \in H|\ \partial\psi(z) \neq \emptyset\}$ is the domain of definition of $\partial\psi$. For these concepts and their basic properties, we refer to [14, 26, 31] and references therein.

2. Abstract Navier–Stokes equations

As a simple example, we will express the proof of the existence of a solution for the stationary Navier–Stokes equations (cf. [41, pp. 116–117]) using a QVP. For comparison with Theorem 4.4 in a later section, the theorem is written in abstracted form.

Theorem 2.1. *$V \subset H \subset V^*$ is posited as a triplet with compact embeddings of real Hilbert spaces. The duality of V^* and V is $\langle\cdot,\cdot\rangle$, and the duality mapping is $F : V \to V^*$. We also posit that, by a compact mapping $B : V \to V^*$, the following holds*:

$$\langle Bz, z\rangle = 0 \quad \forall z \in V. \tag{2.1}$$

Then, for all $f \in H$, there exists $u \in V$ that is a solution to the following equation:[2]

$$Fu + Bu = f \quad \text{in } H. \tag{2.2}$$

Proof. (By QVP) Function $\Phi : H \times H \to \mathbf{R}\cup\{+\infty\}$ is defined with the following equation:

$$\Phi(v; z) := \begin{cases} \frac{1}{2}|z|_V^2 + \langle Bv, z\rangle - (f, z), & \text{if } v, z \in V, \\ +\infty, & \text{otherwise.} \end{cases}$$

[1] In [17], the term "quasi-variational principle" is used to mean a "variational principle that does not use functionals" (cf. [17, (17)]). Mathematically, however, it seems to mean the "fundamental lemma of the calculus of variations".

[2] As can be seen from the proof, Theorem 2.1 is supported by $f \in V^*$, in which case equation (2.2) is an equation in V^*. Here, we keep in mind a QSO (cf. Example 4.3) in H and state the case of $f \in H$.

For each $v \in V$, $\Phi(v;\cdot) : H \to \mathbf{R} \cup \{+\infty\}$ is a strictly convex lower-semicontinuous function bounded from below, and $z_v \in V$, which takes a minimum value, exists uniquely as

$$\Phi(v; z_v) = \min_{z\in V} \Phi(v; z), \quad \text{i.e., } z_v = F^{-1}(f - Bv).$$

We seek a fixed point of mapping $v \mapsto z_v$ using the Leray–Schauder theorem. For each $\lambda \in [0,1]$, a function $\Phi_\lambda : H \times H \to \mathbf{R} \cup \{+\infty\}$ is defined by the following equation:

$$\Phi_\lambda(v; z) := \begin{cases} \frac{1}{2}|z|_V^2 + \lambda\big(\langle Bv, z\rangle - (f, z)\big), & \text{if } v, z \in V, \\ +\infty, & \text{otherwise.} \end{cases}$$

In this case, for each $v \in V$, there is a unique minimization element $z_{v,\lambda} \in V$:

$$\Phi_\lambda(v; z_{v,\lambda}) = \min_{z} \Phi_\lambda(v; z), \quad \text{i.e., } z_{v,\lambda} = \lambda F^{-1}(f - Bv).$$

We can easily see that the mapping $(v, \lambda) \mapsto z_{v,\lambda}$ is compact. Now, when u_λ is taken as any possible fixed point of $v \mapsto z_{v,\lambda}$, then, from (2.1),

$$\langle Fu_\lambda, u_\lambda\rangle = \lambda\langle Fu_\lambda, F^{-1}f\rangle$$

holds. Therefore, the following a priori estimate is obtained ($C > 0$ is a constant):

$$|u_\lambda|_V \le C|f|_H.$$

Thus, according to the Leray–Schauder theorem, a fixed point u (a solution to (2.2)) of the mapping $v \mapsto z_{v,1}$ exists. □

In the standard setting of function spaces, Theorem 2.1 and its proof correspond to the original case of the stationary Navier–Stokes equations (cf. the aforementioned work [41]).

3. Variational inequalities and quasi-variational inequalities

We consider the following elliptic-type variational inequality:

$$\text{(3.1)} \quad \begin{cases} u \in K, \\ \displaystyle\int_\Omega \boldsymbol{a}(u, \nabla u) \cdot \nabla(u - z)dx + \int_\Omega a_0(u)(u - z)dx \le (f, u - z) \quad \forall z \in K. \end{cases}$$

Here, K is a closed convex set of $H^1(\Omega)$, $\Omega \subset \mathbf{R}^N (N \ge 1)$ is a bounded domain, $f \in L^2(\Omega)$ is a given function, and $(\cdot,\cdot)$ is the inner product of $L^2(\Omega)$. The vector field $\boldsymbol{a}$ and function a_0 are supposed to satisfy

$$\exists\ \hat{a} \in C^1(\mathbf{R} \times \mathbf{R}^N) :\ \boldsymbol{a}(r, \boldsymbol{p}) = \partial_{\boldsymbol{p}}\hat{a}(r, \boldsymbol{p}), \qquad a_0 \in C(\mathbf{R})$$

as well as appropriate conditions that involve growth, etc. If the potential $\hat{a}$, function a_0, and vector field $\boldsymbol{a}$ satisfy the following condition:

(3.2) $\hat{a}(r, \boldsymbol{p})$ is jointly convex in $(r, \boldsymbol{p}) \in \mathbf{R} \times \mathbf{R}^N$ and $a_0 = \partial_r \hat{a}$,

then problem (3.1) is equivalent to the following:

$$(f, z - u) \le \psi(z) - \psi(u) \qquad \forall z \in K.$$

Here, $\psi : L^2(\Omega) \to \mathbf{R} \cup \{+\infty\}$ is defined by the following equation:

$$\psi(z) := \begin{cases} \displaystyle\int_\Omega \hat{a}(z, \nabla z)dx, & \text{if } z \in K, \\ +\infty, & \text{otherwise.} \end{cases}$$

Therefore, the variational inequality (3.1) can be rewritten in the form of the following operator equation:

$$\partial\psi(u) \ni f, \tag{3.3}$$

where $\partial\psi$ is the subdifferential operator of ψ.

There are, however, numerous problems in physics and engineering that do not satisfy condition (3.2). For example, in a permeable flow model that takes into account the advection effect by the gravity, a variational inequality involving (1.3) is present (cf. Alt and Luckhaus [7], Kenmochi and Kubo [29], Kubo and Yamazaki [39], Kubo, Shirakawa and Yamazaki [36]). Actually, the ellipticity of the operator $\nabla \cdot \boldsymbol{a}(u, \nabla u)$ follows from the convexity that relates only to the variable $\boldsymbol{p}$ in $\hat{a}(r, \boldsymbol{p})$. In this case, problem (3.1) is equivalent to the following problem:

$$(f, z-u) \leq \varphi(u; z) - \varphi(u; u) \qquad \forall z \in K.$$

Here, $\varphi : L^2(\Omega) \times L^2(\Omega) \to \mathbf{R} \cup \{+\infty\}$ is defined with the following equation:

$$\varphi(v; z) := \begin{cases} \displaystyle\int_\Omega \hat{a}(v, \nabla z)dx + \int_\Omega a_0(v) z dx, & \text{if } v \in H^1(\Omega) \text{ and } z \in K, \\ +\infty, & \text{otherwise.} \end{cases} \tag{3.4}$$

Consequently, the following operator equation is obtained:

$$\partial\varphi(u; u) \ni f, \tag{3.5}$$

where $\partial\varphi(u; \cdot)$ is the subdifferential of $\varphi(u; \cdot)$. It is noteworthy that, for each v, $\varphi(v; z)$ is a proper lower-semicontinuous convex function on $L^2(\Omega)$ with respect to the variable z.

Thus, a new operator concept is reached. In other words, an operator A is defined by the following equation:

$$Au := \partial\varphi(u; u). \tag{3.6}$$

Function $\varphi(v; z)$ is convex with respect to variable z and is parameterized by variable v. Such an operator A is called a "quasi-subdifferential operator" (QSO) (cf. Definition 4.1, [34, Definition 2.1]). Using the QSO A, equation (3.5) can be written as follows:

$$Au \ni f. \tag{3.7}$$

Another motivation for introducing the concept of a QSO is the following problem:

$$\begin{cases} u \in K(u), \\ \displaystyle\int_\Omega \boldsymbol{a}(u, \nabla u) \cdot \nabla(u - z)dx + \int_\Omega a_0(u)(u - z)dx \leq (f, u - z) \ \forall z \in K(u). \end{cases} \tag{3.8}$$

Compared with (3.1), the convex set $K(u)$, which provides the constraints, depends on u. This type of problem is called a quasi-variational inequality and has been the subject of much research (cf. Baiocchi and Capelo [10], Joly and Mosco [20], Kenmochi [26], Kano, Kenmochi, and Murase [44]). A quasi-variational inequality such as (3.8) cannot be expressed in the form of (3.3) even if the coefficient satisfies condition (3.2). However, using a QSO defined by the following convex function with a parameter:

$$(3.9)\quad \varphi(v;z) := \begin{cases} \displaystyle\int_\Omega \hat{a}(v,\nabla z)dx + \int_\Omega a_0(v)z dx, & \text{if } v \in H^1(\Omega) \text{ and } z \in K(v), \\ +\infty, & \text{otherwise,} \end{cases}$$

where $K(v)$ is a closed convex set of $H^1(\Omega)$ and depends on the parameter $v \in H^1(\Omega)$, the quasi-variational inequality (3.8) can be expressed in the form of an operator equation (3.7).

Variational inequalities (3.1) and quasi-variational inequalities (3.8) have been studied using the theory of pseudomonotone operators (cf. Brézis [12], Browder and Hess [15], Kenmochi [23]) by Kenmochi [26] and Kano, Kenmochi, and Murase [44]. Earlier, Joly and Mosco [20] studied quasi-variational inequalities for linear partial differential operators using QVPs (1.4). Compared with the approach of pseudomonotone operators, the method of QSOs has the distinguishing characteristic of simplifying the argument by maintaining a close relation to variational principles via the use of functionals with parameters such as (3.4) or (3.9).

4. Quasi-subdifferential operators

As for function $\varphi : H \times H \to \mathbf{R} \cup \{+\infty\}$, we postulate that for each $v \in H$, $\varphi(v;\cdot) : H \to \mathbf{R} \cup \{+\infty\}$ is a lower-semicontinuous convex function. In this study, we call such a function φ a "parameterized convex function". When the lower-semicontinuous convex function $\varphi(v;\cdot) : H \to \mathbf{R}\cup\{+\infty\}$ is proper, i.e., not identical to $+\infty$, its subdifferential can be written as $\partial\varphi(v;\cdot)$.

Definition 4.1 ([34, Definition 2.1])**.** An operator $A : H \to H$, which is multi-valued in general, is called a *quasi-subdifferential operator*, or simply QSO, when it is defined as follows:

$$Au = \partial\varphi(u;u) \quad \text{for } u \in D(A),$$

where φ is a parameterized convex function and satisfies the following condition:

$$D(A) := \{v \in H |\ \varphi(v;\cdot) \text{ is proper and } v \in D(\partial\varphi(v;\cdot))\} \neq \emptyset.$$

We write A^φ to clearly express the defining function by calling the parameterized convex function φ a defining function of the QSO A.

Remark 4.2. The defining function φ of a QSO $A = A^\varphi$ is not uniquely determined from A. For example, with regard to an arbitrary real-valued function $\chi(v)$ that has $v \in H$ as a variable, we obtain $A^\varphi = A^{\varphi+\chi}$.

Example 4.3. Based on the symbols of Theorem 2.1, we consider an operator $A : H \to H$ defined by

$$Az = Fz + Bz, \qquad D(A) = \{z \in V |\ Fz + Bz \in H\}.$$

Then, A is a QSO with defining function $\varphi : H \times H \to \mathbf{R} \cup \{+\infty\}$ given by

$$\varphi(v;z) := \begin{cases} \dfrac{1}{2}|z|_V^2 + \langle Bv, z\rangle, & \text{if } v, z \in V, \\ +\infty, & \text{otherwise.} \end{cases}$$

By Theorem 2.1, we have $R(A) = H$. This indicates the existence of a solution for the stationary Navier–Stokes equations by setting up standard function spaces [41, Chapter 5, Theorem 1].

Existence theorems related to QSO equations can be proved by QVPs. Various assumptions are possible, depending upon which fixed-point theorem is used. In Theorem 2.1, the Leray–Schauder theorem was used. In the following theorem, we employ Schauder's fixed-point theorem.

Theorem 4.4 ([34, Theorem 2.2]). *Let A be a QSO with a defining function φ. Suppose that*

$$D(\varphi(v;\cdot)) \subset K \quad \forall v \in K$$

holds for a closed convex set K of a reflexive Banach space X that is compactly embedded in H: $X \subset H$. Furthermore, the following conditions (A1)–(A4) *are assumed for some constants $C_1, C_2, C_3 > 0$, $p > q \geq 1$.*

(A1) *For some $z_0 \in H$ and for all $v \in K$, the following holds:*

$$\varphi(v; z_0) \leq C_1 \left(|v|_X^q + 1\right).$$

(A2) *For all $v \in K$ and $z \in X$, the following holds:*

$$\varphi(v; z) \geq C_2 |z|_X^p - C_3 \left(|v|_X^q + 1\right).$$

(A3) *For all $v \in K$, $\varphi(v; z)$ is strictly convex with respect to $z \in D(\varphi(v;\cdot))$.*

(A4) *If $K \ni v_n \to v$ weakly in X, then $\varphi(v_n;\cdot) \to \varphi(v;\cdot)$ in the sense of Mosco.*[3]
Then, for all $f \in H$, there exists a solution $u \in K$ of the following equation:

$$Au \ni f \quad \text{in } H. \tag{4.1}$$

Proof. (By QVP) For an arbitrary $v \in K$, conditions (A2) and (A3) imply there is a unique $z_v \in X$ that satisfies the following:

$$\varphi(v; z_v) - (f, z_v) = \min \left\{ \varphi(v; z) - (f, z) \mid z \in H \right\}.$$

From condition (A1),

$$\varphi(v; z_v) - (f, z_v) \leq \varphi(v; z_0) - (f, z_0) \leq C_1 \left(|v|_X^q + 1\right) + |f|_H |z_0|_H.$$

From condition (A2),

$$\varphi(v; z_v) - (f, z_v) \geq C_2 |z_v|_X^p - C_3 \left(|v|_X^q + 1\right) - |f|_H |z_v|_H.$$

Thus, using Young's inequality, the following equation is obtained:

$$C_4 |z_v|_X^p - C_3 \left(|v|_X^q + 1\right) \leq C_1 \left(|v|_X^q + 1\right) + |f|_H |z_0|_H + C_5 |f|_H^{\frac{p}{p-1}}.$$

Here, constants C_4 and $C_5 > 0$ are determined only by C_2 and p. Therefore, we have

$$|z_v|_X^p \leq C_6 \left(|v|_X^q + 1\right),$$

where the constant $C_6 > 0$ is determined only by $C_1, \ldots, C_5, |f|_H, |z_0|_H$, and p. Thus, taking a constant $M > 0$ to satisfy $M \geq \max\{1, (2C_6)^{1/(p-q)}\}$, we have, for all $v \in K$ with $|v|_X \leq M$,

$$|z_v|_X \leq M.$$

Thus, considering a closed convex set K_M of X as

$$K_M := \{v \in K \mid |v|_X \leq M\},$$

we can define a mapping of $\mathcal{F} : K_M \to K_M$ by the following equation:

$$\mathcal{F}v := z_v.$$

[3] Mosco convergence is considered for a sequence of proper lower-semicontinuous convex functions on H. We refer to [26] regarding Mosco convergence.

K_M is a convex set that is also compact in H. Furthermore, using a standard argument that involves condition (A4) and Mosco convergence, it can be shown that mapping $\mathcal{F}$ is continuous in the topology of H. Thus, using Schauder's fixed-point theorem, a fixed point u of $\mathcal{F}$ is obtained. At this point,

$$\varphi(u;u) - (f,u) = \min\{\varphi(u;z) - (f,z) |\ z \in H\}.$$

This is equivalent to equation (4.1). □

5. Applications

In this section, Theorem 4.4 is applied to specific problems of variational inequalities. First, we present the symbols and assumptions. $\Omega \subset \mathbf{R}^N$ $(N \geq 1)$ is a bounded domain and $H := L^2(\Omega)$, $X := W_0^{1,p}(\Omega)$, or $W^{1,p}(\Omega)$ $(p \geq 2)$ are postulated. $p_N := Np/(N-p)$ (if $p < N$), $p_N :=$ any number $\in (p, \infty)$ (if $p \geq N$) is posited as a Sobolev exponent: $W^{1,p}(\Omega) \subset L^{p_N}(\Omega)$. Furthermore, $\boldsymbol{a} = (a_1, \cdots, a_N) = \partial_{\boldsymbol{p}} \hat{a}$, $\hat{a} \in C^1(\mathbf{R} \times \mathbf{R}^N)$, $a_0 \in C(\mathbf{R})$, We posit $1 < q < p$ and assume the following conditions (a1)–(a4).

(a1) There exists a constant $C_1 > 0$, and the following hold for all $r \in \mathbf{R}$ and $\boldsymbol{p} \in \mathbf{R}^N$:

$$|\hat{a}(r,\boldsymbol{p})| + |a_0(r)|^{p_N/(p_N-1)} \leq C_1\left(|r|^q + |\boldsymbol{p}|^p + 1\right),$$

$$\sum_{i=1}^N |a_i(r,\boldsymbol{p})|^{p/(p-1)} \leq C_1\left(|r|^p + |\boldsymbol{p}|^p + 1\right).$$

(a2) There exist constants C_2 and $C_3 > 0$ such that, for all $r \in \mathbf{R}$ and $\boldsymbol{p} \in \mathbf{R}^N$, the following holds:

$$\hat{a}(r,\boldsymbol{p}) \geq C_2|\boldsymbol{p}|^p - C_3(|r|^q + 1).$$

(a3) Function $\hat{a}(r,\boldsymbol{p})$ is strictly convex with respect to $\boldsymbol{p} \in \mathbf{R}^N$. In addition, for a constant $\nu \geq 0$, we posit the following:

$$a_\nu(r) := a_0(r) + \nu|r|^{p-2}r \qquad \forall r \in \mathbf{R}.$$

When $X = W^{1,p}(\Omega)$, we also assume $\nu > 0$.

(a4) There exist a constant $C_4 > 0$ and an exponent $\hat{p}$: $p \leq \hat{p} < p_N$ (if $p < N$), $p \leq \hat{p} < \infty$ (if $p \geq N$) such that, for all $r_1, r_2 \in \mathbf{R}$ and $\boldsymbol{p} \in \mathbf{R}^N$, the following hold:

$$|\hat{a}(r_1,\boldsymbol{p}) - \hat{a}(r_2,\boldsymbol{p})| \leq C_4|r_1 - r_2|(|r_1|^{p_N} + |r_2|^{p_N} + |\boldsymbol{p}|^p + 1)^{(\hat{p}-1)/\hat{p}},$$

$$|a_0(r_1) - a_0(r_2)| \leq C_4|r_1 - r_2|(|r_1|^{\hat{p}-2} + |r_2|^{\hat{p}-2} + 1).$$

5.1. **Variational inequalities.** Let K be a nonempty closed convex set of X. We consider the following variational inequality:

$$(5.1) \qquad \begin{cases} u \in K, \\ \displaystyle\int_\Omega \boldsymbol{a}(u,\nabla u)\nabla(u-z)dx + \int_\Omega a_\nu(u)(u-z)dx \leq (f, u-z) \quad \forall z \in K. \end{cases}$$

Let us define a parameterized convex function $\varphi : H \times H \to \mathbf{R} \cup \{+\infty\}$ by the following equation:

$$\varphi(v;z) := \begin{cases} \displaystyle\int_\Omega \hat{a}(v,\nabla z)dx + \int_\Omega \left(a_0(v)z + \frac{\nu}{p}|z|^p\right)dx, & \text{if } v \in X \text{ and } z \in K, \\ +\infty, & \text{otherwise.} \end{cases}$$

Using conditions (a1)–(a4), we can show that assumptions (A1)–(A4) of Theorem 4.4 hold. For example, for (A1), an arbitrary $z_0 \in K$ is acceptable. Therefore, we know that a solution exists for (5.1).

Remark 5.1. For examples of a convex set of K, we refer to Brézis [13] and Kenmochi [26].

5.2. **Quasi-variational inequalities.** Suppose K is a nonempty closed convex set in X. For each $v \in K$, we posit that $K(v)$ is a nonempty closed convex set in X that satisfies (K1) and (K2).

(K1) $K(v) \subset K$ $\forall v \in K$, and $\bigcap_{v\in K} K(v) \neq \emptyset$.

(K2) If $K \ni v_n \to v$ weakly in X, then $K(v_n) \to K(v)$ in the sense of Mosco.[4]

Furthermore, we assume the following condition (a5), in addition to conditions (a1)–(a4).

(a5) There exists a constant $C_5 > 0$ such that, for all $r \in \mathbf{R}$, $\boldsymbol{p}_1$, $\boldsymbol{p}_2 \in \mathbf{R}^N$, the following holds:

$$|\hat{a}(r,\boldsymbol{p}_1) - \hat{a}(r,\boldsymbol{p}_2)| \le C_5|\boldsymbol{p}_1 - \boldsymbol{p}_2|(|r| + |\boldsymbol{p}_1| + |\boldsymbol{p}_2| + 1).$$

We now consider the following problem (quasi-variational inequality):

$$\begin{cases} u \in K(u), \\ \displaystyle\int_\Omega \boldsymbol{a}(u,\nabla u)\nabla(u-z)dx + \int_\Omega a_\nu(u)(u-z)dx \le (f, u-z) \quad \forall z \in K(u). \end{cases} \tag{5.2}$$

Let us define a parameterized convex function φ by the following equation:

$$\varphi(v;z) := \begin{cases} \displaystyle\int_\Omega \hat{a}(v,\nabla z)dx + \int_\Omega \left(a_0(v)z + \frac{\nu}{p}|z|^p\right)dx, & \text{if } v \in X \text{ and } z \in K(v), \\ +\infty, & \text{otherwise.} \end{cases}$$

All the conditions of Theorem 4.4 can be confirmed from assumptions (a1)–(a5) and (K1), (K2). In particular, condition (A1) follows from assumption (K1) with an arbitrarily chosen $z_0 \in \bigcap_{v\in K} K(v)$. Moreover, condition (A4) is established by assumption (K2) and also by (a5). Thus, a solution for (5.2) is obtained. Subsequently, we give specific examples of $K(v)$ that satisfy assumptions (K1) and (K2). These have been studied by Kenmochi [26] using the theory of pseudomonotone operators.

Example 5.2. We posit that $k_c : \mathbf{R} \to \mathbf{R}$ is a Lipschitz continuous function that satisfies

$$0 < k_c(r) \le k^* \quad \forall r \in \mathbf{R}$$

with respect to a constant $k^* > 0$. We put $X := W_0^{1,p}(\Omega)$ $(p \ge 2)$ and then set

$$K := \{z \in X |\ |\nabla z| \le k^* \text{ a.e. in } \Omega\}$$

and

$$K(v) := \{z \in X |\ |\nabla z| \le k_c(v) \text{ a.e. in } \Omega\}.$$

At this point, condition (K1) is satisfied by $0 \in \bigcap_{v\in K} K(v)$. Condition (K2) can be confirmed using a result of Kenmochi [26, Lemma 10.1].

[4]Here, Mosco convergence is considered for a sequence of closed convex sets of X.

Example 5.3. We posit $N = 1$, $X := W^{1,p}(0,1)$ $(p \geq 2)$. Suppose that $k_c : \mathbf{R} \to \mathbf{R}$ is a Lipschitz continuous function that, under constants k^* and k_*, satisfies the following:

$$k_* \leq k_c(r) \leq k^* \quad \forall r \in \mathbf{R}.$$

Furthermore, let us put

$$K := \{z \in X | \ z \geq k_* \text{ on } (0,1)\}$$

and

$$K(v) := \{z \in X | \ z \geq k_c(v) \text{ on } (0,1)\} \quad \text{for } v \in K.$$

At this point,

$$K(v) \subset K \quad \forall v \in K \quad \text{and} \quad k^* \in \bigcap_{v \in K} K(v)$$

is obtained; hence, condition (K1) is satisfied. Moreover, condition (K2) can be demonstrated via the compact embedding $X \subset C([0,1])$ (cf. [26, Lemma 10.2]).

6. Quasi-subdifferential evolution equations (I)

We call evolution equations that involve subdifferentials depending on unknown functions *quasi-subdifferential evolution equations* (QSEs). In this study, we consider two types. The first type is equations such as

$$u'(t) + \partial\varphi^t(u(t); u(t)) \ni f(t) \quad \text{in } H, \quad 0 < t < T. \tag{6.1}$$

Here, $\partial\varphi^t(\cdot;\cdot)$ is the subdifferential of $\varphi^t(\cdot;\cdot)$, which is a time-dependent convex function with respect to the second variable, with the first variable as a parameter and $f(t)$ a given H-valued function.

Problems of type (6.1) have been studied by Kenmochi and Kubo [29], Kubo and Yamazaki [37, 38], Yamazaki [58], and Kubo, Shirakawa, and Yamazaki [36] with respect to elliptic-parabolic variational inequalities (cf. (8.1)) that appear in permeable flow models. Using the concept of QSOs, equation (6.1) can be simply written as follows:

$$u'(t) + A(t)u(t) \ni f(t) \quad \text{in } H, \quad 0 < t < T. \tag{6.2}$$

Here, $A(t)$ is the QSO defined by the parameterized convex function φ^t, that is, $A(t) := A^{\varphi^t}$. The second type of QSE is discussed in the next section.

Using the ideas cited in [29, 36, 37, 38, 58] in relation to equation (6.2), we can prove the following existence theorem. For the sake of simplicity, we consider the case where $f = 0$. Even in the case where a nonhomogenous term $f \in L^2(0,T;H)$ is given on the right-hand side, the same theorem holds.

Theorem 6.1. ([34, Theorem 4.1]) *We posit that $A(t) = A^{\varphi^t}$ is a QSO defined by a time-dependent parameterized convex function $\varphi^t : H \times H \to \mathbf{R} \cup \{+\infty\}$ $(0 \leq t \leq T)$. We assume the following conditions* $(\Phi1)$ *and* $(\Phi2)$.

$(\Phi1)$ *A Banach space X is compactly embedded in $H : X \subset H$, and $D(\varphi^t(v;\cdot)) \subset X$ for all $v \in H$ and $0 \leq t \leq T$. In addition, the following holds*:

$$\varphi^t(v; z) \geq G\left(|z|_X\right) \qquad \forall (v, z) \in H \times X \text{ and } 0 \leq t \leq T,$$

where $G : \mathbf{R}_+ (:= [0,\infty)) \to \mathbf{R}_+$ is monotonically increasing and satisfies $\lim_{r\to+\infty} G(r) = +\infty$.

$(\Phi2)$ *There exist functions $\alpha \in W^{1,2}(0,T), \beta \in W^{1,1}(0,T)$ and a constant $C_0 > 0$ such that the following condition* $(*)$ *holds.*

(∗) *For all* $v, w \in H$, $0 \le s \le t \le T$ *and* $z \in D(\varphi^s(v;\cdot))$, *there exists* $\tilde{z} \in D(\varphi^t(w;\cdot))$ *such that the following inequality holds*:

$$\begin{cases} |\tilde{z} - z|_H \le |\alpha(t) - \alpha(s)| \left(\varphi^s(v; z)\right)^{1/2}, \\ \varphi^t(w; \tilde{z}) - \varphi^s(v; z) \le |\beta(t) - \beta(s)| \varphi^s(v; z) + C_0 |w - v|_H \left(\varphi^s(v; z)\right)^{1/2}. \end{cases}$$

Furthermore, we put $K(t) := \{z \in H | \ \varphi^t(z; z) < +\infty\}$.

Then, for each $u_0 \in K(0)$, *a solution* $u \in W^{1,2}(0, T; H)$ *exists for the initial value problem*:

$$\begin{cases} u'(t) + A(t)u(t) \ni 0 \quad \textit{in } H \quad \textit{a.e. } t \in (0, T), \\ u(0) = u_0. \end{cases} \tag{6.3}$$

Furthermore, u *satisfies the following energy inequality for all* $0 \le s \le t \le T$ *and* $0 < \varepsilon < 1$:

$$\begin{aligned} &\varphi^t(u(t); u(t)) - \varphi^s(u(s); u(s)) + (1 - \varepsilon) \int_s^t |u'(\tau)|_H^2 d\tau \\ &\le \int_s^t \Big\{ \left(\varepsilon^{-1} |\alpha'(\tau)|^2 + |\beta'(\tau)|\right) \varphi^\tau(u(\tau); u(\tau)) \\ &\qquad\qquad + C_0 |u'(\tau)|_H \left(\varphi^\tau(u(\tau); u(\tau))\right)^{1/2} \Big\} d\tau. \end{aligned} \tag{6.4}$$

Proof. For each $v \in W^{1,2}(0, T; H)$, we can use (Φ2) to show that the family of time-dependent convex functions $t \mapsto \Phi_v^t(\cdot) := \varphi^t(v(t); \cdot)$ satisfies the solvability condition (cf. Remark 6.2) for time-dependent subdifferential evolution equations. Thus, there exists $u = u_v \in W^{1,2}(0, T; H)$, uniquely determined with respect to v, such that

$$\begin{cases} u'(t) + \partial \Phi_v^t(u(t)) \ni 0 \quad \text{in } H \quad \text{a.e. } t \in (0, T), \\ u(0) = u_0, \end{cases} \tag{6.5}$$

and the following energy inequality holds for all $0 \le s \le t \le T$:

$$F(t) - F(s) + C_\varepsilon \int_s^t |u'(\tau)|_H^2 d\tau \le \int_s^t \left\{ \gamma(\tau) F(\tau) + C_0 |v'(\tau)|_H F(\tau)^{1/2} \right\} d\tau. \tag{6.6}$$

Here, we have put

$$F(t) := \Phi_v^t(u(t)), \quad C_\varepsilon := 1 - \varepsilon, \quad \gamma(t) := \varepsilon^{-1} |\alpha'(t)|^2 + |\beta'(t)|.$$

From (6.6), we obtain

$$\tilde{F}'(t) + C_1 |u'(t)|_H^2 \le C_2 |v'(t)|_H \tilde{F}(t)^{1/2} \le |v'(t)|_H^2 + C_2^2 \tilde{F}(t), \tag{6.7}$$

where we put

$$\tilde{F}(t) := \exp\left\{ -\int_0^t |\gamma(s)| ds \right\} F(t), \quad C_1 := \exp\left\{ -|\gamma|_{L^1(0,T)} \right\} C_\varepsilon$$

and

$$C_2 := C_0 \exp\left\{ \int_0^T |\gamma(s)| ds \right\}.$$

From inequality (6.7) and Gronwall's lemma, we obtain

$$\tilde{F}(t) \le C_3 \left(\int_0^t |v'(s)|_H^2 ds + \tilde{F}(0) \right), \quad C_3 := \exp\left(C_2^2 T\right).$$

Thus, once again, from inequality (6.7), we obtain

$$\tilde{F}'(t) + C_1|u'(t)|_H^2 \le C_4|v'(t)|_H \left(\int_0^t |v'(s)|_H^2 ds + \tilde{F}(0)\right)^{1/2}, \quad C_4 := C_2C_3.$$

Integrating, we find that

$$\tilde{F}(t) + C_1\int_0^t |u'(s)|_H^2 ds \le \tilde{F}(0) + \sqrt{T}C_4\left(\int_0^t |v'(s)|_H^2 ds + \tilde{F}(0)\right).$$

From this and assumption (Φ1) and taking a large value for the constant $M_0 > 0$ and a sufficiently small value for $T_0 > 0$, we obtain

$$u(= u_v) \in K_{M_0,T_0} \quad \forall v \in K_{M_0,T_0},$$

where we put

$$K_{M_0,T_0} := \{v \in E|\ |v'|_{L^2(0,T_0;H)} \le M_0 \text{ and } |v|_{L^\infty(0,T_0;X)} \le M_0\},$$
$$E := W^{1,2}(0,T_0;H) \cap L^\infty(0,T_0;X).$$

From assumption (Φ1), K_{M_0,T_0} is a compact set of $L^2(0,T_0;H)$. In addition, from the theory of time-dependent subdifferential evolution equations, the mapping $v \mapsto u$ is continuous in the topology of $L^2(0,T;H)$ (cf. [25, Theorem 2.8.5], [36, Proposition 4.2]). Therefore, by Schauder's fixed-point theorem, the mapping $v \mapsto u$ has a fixed point in K_{M_0,T_0}. This is a solution to (6.3) on the time-interval $[0,T_0]$. The energy inequality (6.4) can be obtained by putting $v = u$ in (6.6).

The existence of a global solution is established as follows. First, put

$$T^* := \sup\{0 < t \le T|\ \text{a solution of (6.3) exists on } [0,t]\}.$$

The existence of a local solution has already been proven, hence $T^* > 0$. Using the energy inequality (6.4) and the argument used in the proof of existence for a local solution, we show that the following limit exists in H:

$$u(T^*) = \lim_{t\uparrow T^*} u(t) \in K(T^*).$$

Thus, by taking $u(T^*)$ as the initial value for time $t = T^*$, the solution can be extended beyond time T^*. Therefore, a solution exists in the entirety of the interval $[0,T]$. □

Remark 6.2. Regarding the solvability condition for the initial value problem of time-dependent subdifferential evolution equations (6.5), we refer to Watanabe [54], Attouch, Bénilan, Damlamian, and Picard [9], Biroli [11], Maruo [43], Kenmochi [24], Yamada [57], Yotsutani [59], Kenmochi [25], Ôtani [45], Kubo [32], Ôtani [46].[5]

The idea for the proof of Theorem 6.1 is based on [29, 36, 37, 38, 58]. The abstract result of the theorem can be applied to the following parabolic variational inequality:

$$\begin{cases} u \in K(t), \\ (u' - f, u - z) + \displaystyle\int_\Omega \boldsymbol{a}(u,\nabla u)\cdot\nabla(u-z)dx + \int_\Omega a_0(u)(u-z)dx \le 0 \quad \forall z \in K(t), \\ u(0) = u_0. \end{cases}$$

[5] The essence of the solvability condition of the theory of time-dependent subdifferential evolution equations is that it is 'necessary and sufficient' for the existence of a solution satisfying an energy inequality such as (6.6), as shown in a previous paper by the author [32].

Conditions on the data, such as the time-dependent convex set $K(t)$, are presented in the above-cited papers.

7. Quasi-subdifferential evolution equations (II)

In this section, we discuss the following problems:

$$u'(t) + \partial\varphi^t(u; u(t)) \ni f(t), \quad 0 < t < T. \tag{7.1}$$

Here, $\partial\varphi^t(v;\cdot)$ is the subdifferential of the convex function $\varphi^t(v;\cdot)$. We are concerned with a convex function depending on time $t \in [0,T]$ having an H-valued function v as a parameter. Compared with equation (6.1), equation (7.1) has an added complexity, because nonlocal dependency involving the parameter v requires special consideration. Problems such as equation (7.1) are being studied in various mathematical models in physics and engineering. For example, the mathematical model for hysteresis can be written in the form of an equation such as (7.1). For models of hysteresis, we refer to Visintin [53], Kenmochi, Koyama, and Meyer [28], Colli, Kenmochi, and Kubo [16], Kubo [33], Aiki [2], and references therein. In Stefanelli [50], Kano, Murase, and Kenmochi [22], and Kano, Kenmochi, and Murase [21], equations of type (7.1) are abstractly studied in relation to problems that are nonlocally dependent on the parameter. Above all, Kenmochi [27] mentions the importance of systematic studies thereof. The method used in [50] is based on the monotonicity of the parameter, whereas [21, 22] use the theory of time-dependent subdifferential evolution equations. In Fukao and Kenmochi [18, 19], the theory in [22] is applied to the study of variational inequalities for the Navier–Stokes equations.

We show a typical example of $\varphi^t(\cdot;\cdot)$. The following convex function is used to mathematically model the hysteresis phenomenon studied in [16]:

$$\varphi^t(v;z) := \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega |\nabla z|^2 dx, & \text{if } z \in K^t(v), \\ +\infty, & \text{otherwise.} \end{cases}$$

The convex set $K^t(v)$ is defined, for $v \in W^{1,2}(0,T;L^2(\Omega))$, by

$$K^t(v) := \{z \in H^1(\Omega) |\ f_a((Gv)(t)) \le z \le f_d((Gv)(t)) \text{ in } \Omega\},$$

and $G : W^{1,2}(0,T;L^2(\Omega)) \to W^{1,2}(0,T;L^2(\Omega))$ is the operator defined in the following equation:

$$Gv := w \text{ is the solution of } \begin{cases} w' + v' - \Delta_N w = 0 \quad \text{in } L^2(\Omega),\ 0 < t < T, \\ w(0) = w_0, \end{cases}$$

where Δ_N is the Laplacian with Neumann boundary condition, w_0 is the initial value given to w, and $f_a, f_d : \mathbf{R} \to \mathbf{R}$ are given functions that characterize the hysteresis.

We now discuss how equation (7.1) is expressed using a QSO. We put $\mathcal{H} := L^2(0,T;H)$. For a given initial value u_0, we consider equation (7.1) together with the initial condition:

$$u(0) = u_0. \tag{7.2}$$

An operator $\mathcal{L}_{u_0}$ on the Hilbert space $\mathcal{H}$ is defined as follows:

$$\mathcal{L}_{u_0} v := v' \quad \text{for} \quad v \in D(\mathcal{L}_{u_0}) := \{w \in W^{1,2}(0,T;H) |\ w(0) = u_0\}.$$

For each $t \in [0, T]$ and $v \in \mathcal{H}$, $\varphi^t : \mathcal{H} \times H \to \mathbf{R} \cup \{+\infty\}$ is such that $\varphi^t(v; \cdot) : H \to \mathbf{R} \cup \{+\infty\}$ is a lower-semicontinuous convex function, and $v_0 \in \mathcal{H}$ exists to make $\varphi^t(v_0; \cdot)$ proper. The dependency of the convex function $\varphi^t(v; \cdot)$ on t and v is appropriately assumed. For example, $\varphi^t(v; \cdot)$ is measurable in terms of t and continuous in an appropriate topology in terms of v (cf. Attouch [8]). $\Phi : \mathcal{H} \times \mathcal{H} \to \mathbf{R} \cup \{+\infty\}$ is defined by

$$\Phi(v; w) := \int_0^T \varphi^t(v; w(t))dt \quad \text{for } w \in \mathcal{H}.$$

Then, $\Phi(v; \cdot)$ is a lower-semicontinuous convex function on $\mathcal{H}$, and Φ is a parameterized convex function on $\mathcal{H}$, in terms of the meaning given in this paper. We assume that there is some $v \in \mathcal{H}$ such that $v \in D(\partial\Phi(v; \cdot))$, and let $\mathcal{A}$ be the QSO defined by Φ. With respect to a given $f \in \mathcal{H}$, the problem (7.1)–(7.2) can now be written in the form of the following operator equation:

$$\mathcal{L}_{u_0} u + \mathcal{A}u \ni f \quad \text{in } \mathcal{H}. \tag{7.3}$$

In Brézis [13], parabolic-type variational inequalities are studied using abstract equations in a similar form to (7.3). There, an abstract theorem [13, Théorème II.1] is used, and when $\mathcal{A}$ is a pseudomonotone operator, the surjectivity of the sum of the operators $\mathcal{L}_{u_0} + \mathcal{A}$ is derived under appropriate assumptions (e.g., coercivity). Currently, we cannot expect to construct an abstract general theory on the surjectivity of the sum of operators $\mathcal{L}_{u_0} + \mathcal{A}$ with respect to a QSO $\mathcal{A}$. However, because many specific examples already exist, such an approach might provide a valuable perspective on abstract theories.

8. Afterword

In this study, we have defined the concepts of quasi-variational principles, quasi-subdifferential operators, and quasi-subdifferential evolution equations and stated some basic existence theorems. These approaches have already been used in previous studies. In fact, without naming the concepts, QVPs were introduced by Joly and Mosco [20], and QSOs and QSEs were introduced by Kenmochi and Kubo [29]. The prefix 'quasi' was taken from examples such as 'quasi-variational inequality' and 'quasi-linear'. Particularly, for QSOs, in the case where $\partial\varphi(v; \cdot)$ is linear (that is, positive self-adjoint), for each v, the operator $Au := \partial\varphi(u; u)$ becomes a quasi-linear operator with respect to u. With regard to quasi-linear parabolic-type evolution equations, a detailed abstract theory exists (cf. Yagi [56]), and QSEs related to this particular type QSO A would be a special class of that.

In this paper, we explained the basic existence theorems (Theorem 4.4 and Theorem 6.1) by following the work of [34]; however, it is possible to make generalizations in various directions. For Theorem 4.4, we used Schauder's fixed-point theorem, but, as in Joly and Mosco [20], it is also possible to give a condition where the fixed-point theory of Kakutani-Ky Fan could be used. In this case, condition (A3), which guarantees the uniqueness of the minimization point z_v, would not be required. Theorem 6.1 is based on the theory of time-dependent subdifferential evolution equations, and condition (Φ2) can be weakened using this theory to the maximum possible degree. Theorem 4.4 and Theorem 6.1 aim to simplify the arguments and clarify the ideas.

Condition (Φ2) used in Theorem 6.1 and the condition given in Kano, Murase, and Kenmochi [22] are variations, developed for the case of parameter dependency, of the conditions introduced in the theory of time-dependent subdifferential evolution equations (cf. Kenmochi [24, 25], Yamada [57]). In [22], nonlocal dependency, as it relates to the parameters, is considered as in Section 7; therefore, a more detailed argument is required than that given here for Theorem 6.1. A characteristic of the theory of time-dependent subdifferential evolution equations is that the domain of definition (or even its closure) of the operators is allowed to change in time.[6] The result is that application to various nonlinear problems becomes possible, such as noncylindrical domain problems (cf. Ôtani and Yamada [47], Kenmochi and Pawłow [30]). As for the history of the theory, please refer to Watanabe [55] and Ôtani [46, Introduction]. The theory of subdifferential evolution equations continues to evolve as an abstract theory of nonlinear evolution equations and to be applied to various problems in physics and engineering. For example, in Akagi [3, 4], Akagi and Ôtani [5, 6], an abstract theory is presented in a V-V^* form, and its application to doubly nonlinear evolution equations and degenerate parabolic-type equations is discussed. The theory of QSEs and their application to variational and quasi-variational inequalities, as discussed in this paper, and the approaches of Kenmochi [27] and Kano, Murase, and Kenmochi [22] are possible future directions.

With regard to (both stationary and evolutionary) quasi-subdifferential equations (4.1) and (6.2), to argue beyond the existence of a solution and consider uniqueness, regularity, asymptotic behavior, etc., the discussion needs to conform to a specific problem rather than to an abstract general theory. With regard to the elliptic-parabolic-type QSE

$$\rho(u)'(t) + \partial\varphi^t(u(t); u(t)) \ni f(t), \quad 0 < t < T, \tag{8.1}$$

as relates to the quasi-subdifferential equation (6.1), the existence of a periodic solution, its uniqueness, and asymptotic stability are discussed in Kenmochi and Kubo [29], Kubo and Yamazaki [39, 40], and Shirakawa, Kubo, and Yamazaki [48].

In this paper, we discussed a method for modifying the proof of existence theorem via the direct variational method to a form that can be applied to variational inequalities or quasi-variational inequalities involving non-Euler–Lagrange-type differential equations. The subject of variational analysis is vast.[7] Hence it would be greatly appreciated if the reader can perceive the range of applications of the idea to regulate various problems of differential equations by using functionals.

Note added in proof

We generalize Theorem 4.4 according to the suggestion given in the afterword and give further applications thereof in the following paper:

M. Kubo and Y. Murase, *Quasi-subdifferential operator approach to elliptic variational and quasi-variational inequalities*, Math. Methods Appl. Sci. **39** (2016), 5626–5635.

[6] In Watanabe [55, p. 530], it is stated with regard to the solvability conditions introduced by Kenmochi [24] that "The assumption ... is of great originality."

[7] For example, we refer to Suzuki [51] and Tanaka [52] for the min-max method for semi-linear problems.

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