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LOCAL RIGIDITY PROBLEM OF SMOOTH GROUP ACTIONS

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ABSTRACT. In this article, we survey results on local rigidity of smooth group actions by focusing on dynamical and cohomological aspects.

1. Rigidity problem of smooth actions of discrete groups

Let M be a smooth manifold and Γ a discrete group. We denote the group of diffeomorphisms of M by $\mathrm{Diff}(M)$. A smooth (left) Γ-*action* is a homomorphism from Γ to $\mathrm{Diff}(M)$. For a Γ-action ρ and an element γ of Γ write ρ^γ for the diffeomorphism $\rho(\gamma)$.

Example 1.1 (The trivial action)**.** Any manifold admits *the trivial action*. It is the homomorphism whose image is the identity map in $\mathrm{Diff}(M)$.

Example 1.2 (Iteration of diffeomorphisms)**.** Denote the set of integers by $\mathbb{Z}$. Any diffeomorphism on a manifold generates a $\mathbb{Z}$-action. More precisely, for any given diffeomorphism f, there exists a unique $\mathbb{Z}$-action ρ_f such that $\rho_f^1 = f$. For $n \geq 1$, diffeomorphisms ρ_f^n and ρ_f^{-n} are the n-times iterations of f and f^{-1} respectively. The map ρ_f^0 is the identity map. We write simply f^n for ρ_f^n.

Example 1.3 (Standard actions on homogeneous spaces)**.** Let G be a Lie group, H a Lie subgroup of G, and Γ a discrete subgroup of G. **The standard Γ-action** ρ_Γ on G/H is defined by $\rho_\Gamma^\gamma(gH) = (\gamma g)H$.

Iteration of a diffeomorphism is one of the main objects in the theory of dynamical systems. The standard action ρ_Γ is called the boundary action on G/H in the case that G is a semisimple Lie group, H is its parabolic subgroup, and Γ is a lattice of G. Boundary actions play an important role in the study of homogeneous spaces.

Let $\mathcal{A}(\Gamma, M)$ be the set of smooth Γ-actions on M. We introduce a topology to this set. The $\mathrm{Diff}(M)$ is endowed with the C^∞ compact-open topology. The set $\mathcal{A}(\Gamma, M)$ is a subset of the set of maps from Γ to $\mathrm{Diff}(M)$, which is identified with a product space of $\mathrm{Diff}(M)$. The topology of $\mathcal{A}(\Gamma, M)$ is induced from the product topology. For a generating set S of Γ, the convergence of a sequence $(\rho_n)_{n\geq 1}$ in $\mathcal{A}(\Gamma, M)$ to an action ρ_* is equivalent to the convergence of ρ_n^γ to ρ_*^γ in $\mathrm{Diff}(M)$ for each $\gamma \in S$. We say two actions ρ_1 and ρ_2 in $\mathcal{A}(\Gamma, M)$ are C^∞-*conjugate* if there exists a diffeomorphism h of M which satisfies $\rho_2^\gamma = h \circ \rho_1^\gamma \circ h^{-1}$ for any $\gamma \in \Gamma$. The diffeomorphism h is called *a conjugacy* between ρ_1 and ρ_2. C^∞-conjugacy is

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an equivalence relation on $\mathcal{A}(\Gamma, M)$. Let $\bar{\mathcal{A}}(\Gamma, M)$ be the quotient space of $\mathcal{A}(\Gamma, M)$ by C^∞-conjugacy.

Problem 1.4 (Deformation space)**.** For a given Γ-action ρ_0 on a manifold M, describe local topology of a small neighborhood of the conjugacy class of ρ_0 in $\bar{\mathcal{A}}(\Gamma, M)/\sim$. For example, is it locally homeomorphic to $\mathbb{R}^n$ for some n? If it is, which invariants of ρ determine the dimension n?

The case with the simplest local topology is that the conjugacy class of ρ is an isolated point of $\bar{\mathcal{A}}(\Gamma, M)$. It is equivalent that there exists a neighborhood of ρ in $\mathcal{A}(\Gamma, M)$ in which any action is C^∞-conjugate to ρ. We say that the action ρ is C^∞ *locally rigid* in this case.

Problem 1.5 (Local rigidity)**.** In which action is C^∞ locally rigid? Is there a criterion for local rigidity?

The deformation problem and the local rigidity problem for actions can be regarded as the special cases of more general problems. Let G and H be topological groups. By $\mathrm{Hom}(G, H)$, we denote the set of continuous homomorphisms from G to H. This set is endowed with the compact-open topology as a subspace of the space $C^0(G, H)$ of a continuous map from G to H. We say two homomorphisms π_1 and p_2 in $\mathrm{Hom}(G, H)$ are *conjugate* if there exists an element h of H such that $pi_2(g) = h \cdot \pi_1(g) \cdot h^{-1}$ for any $g \in G$. We also say that a homomorphism $\pi_0 \in \mathrm{Hom}(G, H)$ is *locally rigid* if it admits a neighborhood in which any homomorphism is conjugate to π_0. As we see in Section 2, the local rigidity problem for the case that H is a finite-dimensional Lie group and G is its lattice has a long history and there are many beautiful results. For example, the problem is completely solved if H is a simple Lie group and G is its irreducible lattice. The concepts of C^∞-conjugacy and C^∞ local rigidity of actions are just specializations of those for homomorphisms to discrete groups $G = \Gamma$ and a diffeomorphism group $H = \mathrm{Diff}(M)$.

In the 1970s, Margulis discovered a strong rigidity, so-called superrigidity, for a homomorphism from lattices of a semisimple Lie group to another semisimple Lie group. He proved that if G and H are good semisimple Lie groups and Γ is a good lattice of G then any homomorphism $\pi \in \mathrm{Hom}(\Gamma, H)$ can be decomposed into homomorphisms in $\mathrm{Hom}(\Gamma, G)$ and $\mathrm{Hom}(G, H)$, i.e., there exist homomorphisms $\alpha \in \mathrm{Hom}(\Gamma, G)$ and $\pi' \in \mathrm{Hom}(G, H)$ such that $\pi = \alpha \circ \pi'$ (see, e.g., [30]). We ask whether group actions admit such a decomposition under some mild conditions or not. Let G be a Lie group, H its closed subgroup, and Γ a discrete subgroup of G. For a homomorphism $\pi : \Gamma \to G$, we define *the homogeneous action* $\rho_\pi \in \mathcal{A}(\Gamma, G/H)$ associated with π by $\rho_\pi^\gamma(gH) = (\pi(\gamma)g)H$. The standard Γ-action ρ_Γ on G/H defined above is the homogeneous action associated with the inclusion from Γ to G. Let α_G be a homomorphism from G to $\mathrm{Diff}(G/H)$ defined by $\alpha_G(g)(g'H) = (gg')H$. Then, the homogeneous action ρ_π has a decomposition $\rho_\pi = \alpha_G \circ \pi$. Recall that if an action ρ is C^∞-conjugate to ρ_π by a diffeomorphism h, then the action ρ also admits a decomposition $\rho = \alpha \circ \pi$ where α is a homomorphism in $\mathrm{Hom}(G, \mathrm{Diff}(G/H))$ such that $\alpha_h(g) = h \circ \alpha_G(g) \circ h^{-1}$.

Problem 1.6 (Local homogeneity)**.** For a given standard action $\rho_\Gamma \in \mathcal{A}(\Gamma, G/H)$, is any perturbation of ρ_Γ decomposed into homomorphisms in $\mathrm{Hom}(\Gamma, G)$ and $\mathrm{Hom}(G, \mathrm{Diff}(G/H))$? Does there exist a neighborhood of ρ_Γ in which any actions are C^∞-conjugate to homogeneous actions?

In this article, we will discuss the problems listed above. It seems that the dynamical method has been the most successful way to attack the problems so far. Hence, we mainly focus on this method in this article. The structure of the article is as follows: The rigidity problem for the case that G is a semisimple Lie group and Γ is a lattice of G has a long history and it is also one of the origins of the rigidity problem of group actions. In the 1960s, Weil solved the local rigidity problem for a large class of semisimple Lie groups by using the deformation complex associated with a homomorphism. We review his theory in Section 2. We also mention some attempts to apply his method to the rigidity problem of group actions in Section 3.

The local rigidity problem of $\mathbb{Z}$-actions has been studied as the stability problem of dynamical systems, which is almost solved now. In Section 4, we review the stability theorem of Anosov systems as typical dynamical systems exhibiting stability.

One of the most powerful methods in the local rigidity problem of group actions is the reduction to the stability problem of foliations by constructing the suspension foliation. In Sections 5 and 6, we explain this method. As an application of this method, we also see that the stability of Anosov systems implies the topological local rigidity of a group action. In Sections 7 and 8, we apply detailed investigations of Anosov systems to the smooth rigidity of certain standard actions associated with $SL_n\mathbb{R}$.

As we see in Section 3, deformation of a group action is closely related to the cohomology of a group by Weil's method. Deformation of a foliation is also related to a cohomology associated with a foliation. We discuss this relationship in Section 9. For instance, we see that a rigidity result for isometric actions is obtained by using the deformation theory of foliations.

In Section 10 and after, we discuss parameter deformation of locally free actions of a Lie group. In Section 10, we formulate the local rigidity problem of locally free actions of a Lie group. In Section 11, we see the equivalence between the parameter rigidity and the vanishing of the leafwise cohomology of the orbit foliation for locally free $\mathbb{R}^n$-actions. As applications, we also see some rigidity results for actions discussed in Section 8. In Section 12, we describe the deformation of certain locally free actions associated with $SL_2\mathbb{R}$ with the help of the so-called 'thermodynamical formalism of dynamical systems'.

Until recently, original papers and surveys (Fisher's survey [12] covers from classical to recent results) have been the only source to access the rigidity problem. But, a few years ago, a good textbook by Katok and Niţică [24] was published The author of this article recommends it to the reader interested in this subject. The author also recommends Matsumoto's survey [32] in *Sugaku Expositions*, in which the relationship between parameter rigidity and the leafwise cohomology is discussed.

2. Local rigidity problem of lattices in Lie groups

Let G be a Lie group. We call a subgroup Γ of G a *lattice* if Γ is a discrete subgroup and the left-invariant volume on G induces a finite volume on $\Gamma\backslash G$. We say a lattice Γ is *uniform* (or *cocompact*) if $\Gamma\backslash G$ is compact. The oldest studies on deformation of lattices are probably those of uniform lattices of $PSL_2\mathbb{R}$. Let Γ be a torsion-free uniform lattice of $PSL_2\mathbb{R}$. It acts on the upper half plane $\mathbb{H} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$ by linear fractional transformations. Since the action is free and properly discontinuous, the quotient space $\Gamma\backslash\mathbb{H}$ is a Riemann surface of genus $g \geq 2$. Deformation of the inclusion $\iota : \Gamma \to PSL_2\mathbb{R}$ in $\mathrm{Hom}(\Gamma, PSL_2\mathbb{R})$ corresponds to deformation of the Riemann surface $\Gamma\backslash\mathbb{H}$, which is parameterized by the Teichmüller space. More precisely, we have the following

Theorem 2.1. *Let $\mathcal{U}$ be the connected component of $\mathrm{Hom}(\Gamma, PSL_2\mathbb{R})$ that contains the inclusion map ι. Then, the quotient space of $\mathcal{U}$ by conjugacy is homeomorphic to $\mathbb{R}^{6g-6}$.*

Contrary to $PSL_2\mathbb{R}$, uniform lattices of $PSL_n\mathbb{R}$ with $n \geq 3$ are rigid.

Theorem 2.2 (Selberg [41])**.** *Let Γ be a uniform lattice of $G = SL_n\mathbb{R}$ or $PSL_n\mathbb{R}$ with $n \geq 3$. Then, the inclusion $\iota : \Gamma \to G$ is locally rigid as an element of $\mathrm{Hom}(\Gamma, G)$.*

Irreducible uniform lattices of a large class of semisimple Lie groups exhibit local rigidity (Calabi [9], Calabi-Vesentini [10], Weil [42, 43, 44]). In the rest of this section, we explain Weil's general method to obtain the local rigidity of lattices, which relates local rigidity and vanishing of the cohomology of the group Γ. In Section 3, we apply his idea to describe deformation of group actions.

Let V be a vector space which admits a linear Γ-action. We call a map $c : \Gamma \to V$ *a cocycle* if $c(\gamma\gamma') = c(\gamma) + \gamma \cdot c(\gamma')$ for any $\gamma, \gamma' \in \Gamma$. For $v \in V$, we define a cocycle c_v by $c_v(\gamma) = v - \gamma \cdot v$. We call a cocyle of this form a *coboundary.* Let $Z(V)$ and $B(V)$ be the set of cocycles and coboundaries, respectively. The set $Z(V)$ admits a natural vector space structure and $B(V)$ is a vector subspace of $Z(V)$. The quotient vector space $Z(V)/B(V)$ is called *the first cohomology group*[1] of Γ valued in V, for which we write $H^1(\Gamma; V)$.

Let Γ be a lattice of a Lie group G. The adjoint action of G on the Lie algebra $\mathfrak{g}$ induces a linear Γ-action on $\mathfrak{g}$.

Theorem 2.3 (Weil [44])**.** *Suppose that Γ is finitely presented. If $H^1(\Gamma; \mathfrak{g})$ vanishes, then the inclusion $\iota : \Gamma \to G$ is locally rigid in $\mathrm{Hom}(\Gamma, G)$.*

The vanishing of $H^1(\Gamma; \mathfrak{g})$ has been shown for irreducible uniform lattices of a large class of higher rank semisimple Lie groups ([9, 35, 43]). The local rigidity of the lattices follows from these results.

We give an outline of the proof of Theorem 2.3. The idea is to relate $H^1(\Gamma; \mathfrak{g})$ with infinitesimal deformation of a homomorphism and to relate the latter with actual deformation by the implicit function theorem. Let Γ be a lattice of a Lie group G which is finitely presented. Take a generating set $\gamma_1, \ldots, \gamma_k$ and a set of relations $R_1, \ldots, R_l$ which determine Γ. We set $M_0 = G$, $M_1 = G^k$, and $M_2 = G^l$. We also fix base points $x_0 = e$, $x_1 = (\gamma_1, \ldots, \gamma_k)$, and $x_2 = (e, \ldots, e)$ of M_0, M_1, and M_2, respectively. Define a sequence of maps

$$(M_0, x_0) \xrightarrow{d_0} (M_1, x_1) \xrightarrow{d_1} (M_2, x_2) \tag{2.1}$$

by $d_0(g) = (g\gamma_i g^{-1})_{i=1,\ldots,k}$ and $d_1(g_1, \ldots, g_k) = (R_j(g_1, \ldots, g_k))_{j=1,\ldots,l}$. It is easy to check that the image of $d_1 \circ d_0$ is $\{x_2\}$. Each element $(g_1, \ldots, g_k)$ of $d_1^{-1}(x_2)$ corresponds to an element π of $\mathrm{Hom}(\Gamma, G)$ with $\pi(\gamma_i) = g_i$. With this correspondence, we can see that $\iota : \Gamma \to G$ is locally rigid if there exists a neighborhood U of x_1 in

[1] We can define cocycles and coboundaries of higher degree. The quotient space gives the cohomology of higher degree.

M_1 such that $U \cap d_1^{-1}(x_2) \subset \operatorname{Im} d_0$. We call the latter condition "the exactness of the non-linear complex (2.1)".

We identify the tangent space of G at each point with the Lie algebra $\mathfrak{g}$. By taking the differential of d_0 and d_1, we have a (linear) complex

$$\mathfrak{g} \xrightarrow{(Dd_0)_{x_0}} \mathfrak{g}^k \xrightarrow{(Dd_1)_{x_1}} \mathfrak{g}^l. \tag{2.2}$$

This linearization of the complex (2.1) can be regarded as the complex which describes infinitesimal deformation of the inclusion map ι in $\operatorname{Hom}(\Gamma, G)$. We can relate $H^1(\Gamma; \mathfrak{g})$ with the linearized complex by direct computation.

Proposition 2.4. $\operatorname{Ker}(Dd_1)_{x_1} / \operatorname{Im}(Dd_0)_{x_0}$ *is isomorphic to* $H^1(\Gamma; \mathfrak{g})$ *as a linear space.*

The following is a version of the implicit function theorem.

Theorem 2.5 (Weil [44])**.** *If the linear complex* (2.2) *is exact, then so is the non-linear complex* (2.1).

Theorem 2.3 follows from the theorem and the proposition above.

3. The deformation complex of a group action

In this section, we see some attempts to apply Weil's method to the local rigidity problem of group actions. Since group actions are elements of $\operatorname{Hom}(\Gamma, G)$ with $G = \operatorname{Diff}(M)$, we can relate infinitesimal deformation of an action with the first cohomology of Γ valued in a vector space.[2]

Let $\mathfrak{X}(M)$ be the space of smooth vector fields on M. It can be regarded as the Lie algebra of Fréchet Lie group $\operatorname{Diff}(M)$. The group $\operatorname{Diff}(M)$ acts on $\mathfrak{X}(M)$ by $(f \cdot X)(x) = Df(X(f^{-1}(x)))$. This action induces a linear Γ-action on $\mathfrak{X}(M)$. By replacing the Lie group G with $\operatorname{Diff}(M)$ in the non-linear complex (2.1), we obtain an 'infinite-dimensional non-linear complex'

$$(\operatorname{Diff}(M), \operatorname{Id}_M) \xrightarrow{F} (\operatorname{Diff}(M)^k, (\rho^{\gamma_k})_{i=1}^k) \xrightarrow{G} (\operatorname{Diff}(M)^l, (\operatorname{Id}_M, \dots, \operatorname{Id}_M)), \tag{3.1}$$

where Id_M is the identity map of M. As a formal linearization of the complex, we also obtain an analog of the complex (2.2):

$$\mathfrak{X}(M) \xrightarrow{VF} \mathfrak{X}(M)^k \xrightarrow{VG} \mathfrak{X}(M)^l. \tag{3.2}$$

As before, the action ρ is locally rigid if and only if the complex (3.1) is exact. The quotient $\operatorname{Ker} VG / \operatorname{Im} VF$ in the complex (3.2) is isomorphic to $H^1(\Gamma; \mathfrak{X}(M))$ as a vector space. If an analog of Theorem 2.5 holds, then we obtain a criterion for local rigidity, like Theorem 2.3. In fact, Hamilton showed a version of Theorem 2.5 for tame differential maps on Fréchet spaces ([19, Theorem 3.1.1]). This gives the following criterion for local rigidity of actions.

Proposition 3.1 (Fisher [13])**.** *Suppose that the linear complex* (3.2) *admits a tame splitting, i.e., there exist continuous linear maps* $VP : \mathfrak{X}(M)^k \to \mathfrak{X}(M)$*,* $VQ : \mathfrak{X}(M)^l \to \mathfrak{X}(M)^k$*, an integer* $r \geq 0$*, and a sequence* $(C_s)_{s \geq 0}$ *of positive integers which satisfy the following conditions:*

(1) $VF \circ VP + VQ \circ VG = \operatorname{Id}$.

[2] The method presented in the former half of the section is due to Fisher [13].

(2) $\|VP\omega\|_s \leq C_s\|\omega\|_{s+r}$ *and* $\|VQ\eta\|_s \leq C_s\|\eta\|_{s+r}$ *for any* $s \geq 1$, $\omega \in \mathfrak{X}(M)^k$, *and* $\eta \in \mathfrak{X}(M)^l$, *where* $\|\cdot\|_s$ *is the* C^r*-norm on* $\mathfrak{X}(M)^m$ *for each* $m \geq 1$.

Then, the complex (3.1) *is exact. In particular, the action* ρ *is* C^∞ *locally rigid.*

The second condition in the proposition is called *tameness* of the splitting. This proposition gives a general framework on the local rigidity problem of group actions. However, it is very hard to show the tameness of VP and VQ in general. In fact, there are no known applications of the proposition so far.[3]

Another attempt to apply Weil's method to group actions is to reduce the non-linear complex (3.1) to a finite-dimensional one. Once it is done, we can apply Theorem 2.5 directly. For $n, k \geq 1$, let $\Gamma_{n,k}$ be a finitely presented group

$$\Gamma_{n,k} = \langle a, b_1, \dots, b_n \mid ab_ia^{-1} = b_i^k,\ b_ib_j = b_jb_i\ (i, j = 1, \dots, n)\rangle.$$

The group $\Gamma_{1,k}$ is the Baumslag-Solitar group $B(1, k)$, which is an important example in geometric group theory. For a basis $B = (v_1, \dots, v_n)$ of $\mathbb{R}^n$, we define an affine $\Gamma_{n,k}$-action ρ_B on $\mathbb{R}^n$ by $\rho_B^a(x) = kx$ and $\rho_B^{b_i}(x) = x + v_i$. This action extends to a conformal action on $S^n = \mathbb{R}^n \cup \{\infty\}$ naturally. The infinity ∞ is a global fixed point of ρ_B, i.e., $\rho_B^\gamma(\infty) = \infty$ for any $\gamma \in \Gamma_{n,k}$. We also write ρ_B for this conformal action. The following result is a corollary of a classification theorem of $BS(1, k)$-action on the circle.

Theorem 3.2 (Burslem-Wilkinson [8]). *If* $n = 1$, ρ_B *is* C^∞ *locally rigid.*

For $n \geq 2$, the action is not C^∞ locally rigid as follows. We say two Γ-actions ρ_1 and ρ_2 on a manifold M are *topologically conjugate* if there exists a homeomorphism H of M such that $\rho_2^\gamma = H \circ \rho_1^\gamma \circ H^{-1}$ for any $\gamma \in \Gamma$. Let $CO(n)$ be the group of conformal linear transformations of $\mathbb{R}^n$.

Proposition 3.3. *For any bases* B, B' *of* $\mathbb{R}^n$, *two actions* ρ_B *and* ρ'_B *are topologically conjugate. If* ρ_B *and* $\rho_{B'}$ *are* C^∞*-conjugate, then there exists* $C \in CO(n)$ *such that* $B' = CB$. *In particular,* ρ_B *is not* C^∞ *locally rigid if* $n \geq 2$.

Since $CO(n)$ is strcitly contained in $GL_n\mathbb{R}$ if $n \geq 2$, any ρ_B is not C^∞ locally rigid in this case. However, deformation of ρ_B comes from deformaion of the basis B.

Theorem 3.4 (Asaoka [4]). *For any given basis* B *of* $\mathbb{R}^n$, *there exists a neighborhood* $\mathcal{U}$ *of* ρ_B *in* $\mathcal{A}(\Gamma_{n,k}, S^n)$ *such that any* $\rho \in \mathcal{U}$ *is* C^∞*-conjugate to* $\rho_{B'}$ *for some* $B' \in GL_n\mathbb{R}$.

The point of the proof is to reduce the infinite-dimensional non-linear complex (3.1) to a finite one. Once it is done, the theorem follows from elementary computation of the cohomology of the linearized complex and Weil's implicit function theorem (Theorem 2.5). The keys of the reduction are the following properties of the action ρ_B:

(1) The global fixed point ∞ persists under perturbation of the action.

[3] In his preprint [13], Fisher asserted that we can apply the proposition to a group action. But, he found a gap in his proof of the tameness of the splitting later. We possibly apply the proposition to the action in Theorem 9.3. Theorem 9.3 is obtained by a tame estimate of the infinitesimal deformation complex of a foliation associated with the action. See Section 9.

(2) The problem can be localized; if the germ of a perturbed $\Gamma_{n,k}$-action ρ at the global fixed point is conjugate to that of ρ_B by a local diffeomorphism, then the actions ρ and ρ_B are C^∞ conjugate.

(3) A local $\Gamma_{n,k}$-action with a global fixed point is determined by its 2-jet.

All of these properties come from the relation on $\Gamma_{n,k}$, especially, the 'expansion' $ab_i a^{-1} = b_i^k$ by a.

4. Structural stability of Anosov systems

As we mentioned in the previous section, it is hard to show the local rigidity of actions by a general framework like Proposition 3.1. Many of local rigidity results have been obtained from stability of Anosov systems instead of vanishing of the cohomology of the deformation complex.[4] In this section, we introduce stability of dynamical systems and review basic properties of Anosov systems. The results in this section can be found in standard textbooks on dynamical systems, e.g., [38].

An (invertible) dynamical system is a $\mathbb{Z}$- or $\mathbb{R}$-action.[5] We start with the fact that C^∞ local rigidity is almost nonsense for $\mathbb{Z}$-action. Let f be a diffeomorphism on a closed manifold M. We identify f with the $\mathbb{Z}$-action ρ_f generated by f and write f^n for ρ_f^n. Denote the set $\{x \in M \mid f(x) = x\}$ of a fixed point by $\mathrm{Fix}(f)$ and the set $\bigcup_{n\geq 1} \mathrm{Fix}(f^n)$ of periodic points of f by $\mathrm{Per}(f)$. If two diffeomorphisms f and g are conjugate by a homeomorphism h of M, then $\mathrm{Fix}(g) = h(\mathrm{Fix}(f))$ and $\mathrm{Per}(g) = h(\mathrm{Per}(f))$.

Proposition 4.1. *Suppose that M is of at least one dimension. If the action generated by a diffeomorphism f of M is C^∞ locally rigid, then* $\mathrm{Per}(f)$ *is empty.*

Proof. Suppose that two diffeomorphisms f and g are conjugate by $h \in \mathrm{Diff}(M)$. Then, $(Dg^n)_{h(x)} = Dh_x \circ (Df^n)_x \circ (Dh_x)^{-1}$ for any $n \geq 1$ and $x \in \mathrm{Fix}(f^n)$. Hence, the set $\Lambda_n(f) = \{\det Df_x^n \mid x \in \mathrm{Fix}(f^n)\}$ is invariant under C^∞-conjugacy.

By the Kupka-Smale theorem ([38, Theorem XI.1.1]), the set of diffeomorphisms f such that $\mathrm{Fix}(f^n)$ is a finite set for any $n \geq 1$ is dense in $\mathrm{Diff}(M)$. In particular, if the $\mathbb{Z}$-action generated by f is C^∞ locally rigid, then f is an element of this dense subset of $\mathrm{Diff}(M)$. Suppose that $\mathrm{Per}(f)$ is non-empty. Then, $\mathrm{Fix}(f^n)$ is non-empty for some $n \geq 1$. It is easy to change $\Lambda_n(f)$ by perturbing f in a small neighborhood of $\mathrm{Fix}(f^n)$. It contradicts the invariance of $\Lambda_n(f)$ under C^∞-conjugacy and the local rigidity of f. ◻

There are no known example of open subsets of $\mathrm{Diff}(M)$ in which any diffeomorphism has no periodic points.[6] In particular, there is no known example of C^∞ locally rigid actions for manifolds of dimension at least one.

Topological conjugacy and structural stability are more suitable than C^∞ conjugacy and C^∞ local rigidity when we discuss the rigidity of $\mathbb{Z}$-action. Recall that two Γ-actions ρ_1 and ρ_2 on a manifold M are topologically conjugate if ρ_1 and ρ_2 are conjugate by a homeomorphism h of M. We say that a Γ-action ρ in $\mathcal{A}(\Gamma, M)$ is *structurally stable* if it admits a neighborhood consisting of actions which are topologically conjugate to ρ. We define topological conjugacy and structural stability of diffeomorphisms by those of corresponding $\mathbb{Z}$-actions. Structural stability is an

[4]It seems interesting to find a connection between two different methods.

[5]Group actions are considered as a generalized dynamical system.

[6]For C^1 topology, instead of C^∞ topology, such an open subset of $\mathrm{Diff}(M)$ does not exist by Pugh's C^1 closing lemma [37]. The C^r closing lemma for $2 \leq r \leq \infty$ is still open.

important concept not only in rigidity theory but also from the viewpoint of applications to engineering since it means the topological nature of the time-evolution persists under perturbation.

Structural stability is not abstract nonsense. The following is an example of a structurally stable diffeomorphism.

Theorem 4.2 (Hyperbolic total automorphisms)**.** *Let A be an element of $SL_2\mathbb{Z}$ without eigenvalues of absolute value one. We define a diffeomorphism f_A on the two-dimensional torus $\mathbb{T}^2\mathbb{R}^n/\mathbb{Z}^2$ by $f_A(x+\mathbb{Z}^2) = (Ax)+\mathbb{Z}^2$. Then, f_A is structurally stable.*

This map is a typical example of an Anosov system, which plays an important role in rigidity theory. Let f be a diffeomorphism of a closed Riemannian manifold M. We call a continuous splitting $TM = E^s \oplus E^u$ of TM an *Anosov splitting* for f if $Df(E^s) = E^s$, $Df(E^u) = E^u$, and there exist $C > 0$ and $\lambda > 0$ such that

$$\|Df_x^n(v_s)\| \leq Ce^{-\lambda n}\|v_s\|,$$
$$\|Df_x^{-n}(v_u)\| \leq Ce^{-\lambda n}\|v_u\|,$$

for any $x \in M$, $v_s \in E^s(x)$, $v_u \in E^u(x)$, and $n \geq 1$. In other words, an Anosov splitting is a Df-invariant splitting such that Df is exponentially expanding on E^u and exponentially contracting on E^s. We remark that the definition of an Anosov splitting does not depend on the choice of a Riemannian metric on M. We call a diffeomorphism *an Anosov diffeomorphism* if it admits an Anosov splitting. The hyperbolic toral automorphism f_A is an Anosov diffeomorphism. Its Anosov splitting is given by the decomposition into eigenspaces of A. It is known that Anosov systems exhibit stability in several senses.

Proposition 4.3. *The set of Anosov diffeomorphisms is an open subset of* $\mathrm{Diff}(M)$.[7]

Theorem 4.4. *Any Anosov diffeomorphism is structurally stable. Moreover, if a diffeomorphism g is close to an Anosov diffeomorphism f in the C^1 topology, then we can take a conjugacy map $h \in \mathrm{Homeo}(M)$ which is close to the identity map in the C^0 topology.*

For an Anosov diffeomorphism f and $x \in M$, set

$$W^s(x) = \{y \in M \mid d(f^n(x), f^n(y)) \longrightarrow 0 \ (n \to +\infty)\},$$
$$W^u(x) = \{y \in M \mid d(f^n(x), f^n(y)) \longrightarrow 0 \ (n \to -\infty)\}.$$

By the stable manifold theorem, they are injectively immersed manifold tangent to E^s and E^u respectively. We call them *the stable manifold* and *the unstable manifold* of x for f. Since they are defined by the topological property of dynamics, we have the following

Proposition 4.5. *Any topological conjugacy h between two Anosov diffeomorphisms f and g preserves the stable and unstable manifolds, i.e., $W^s(h(x); g) = h(W^s(x; f))$, $W^u(h(x); g) = h(W^u(x; f))$ for any $x \in M$.*

A flow on a manifold is an $\mathbb{R}$-action. We define the concepts of structural stability and Anosov systems for flows. Let $\mathcal{A}(M,\mathbb{R})$ be the set of a smooth flow on a manifold M. This set admits the C^∞ compact-open topology as a subset

[7]There are manifolds which admit no Anosov diffeomorphisms (e.g., S^1, S^2).

of $C^\infty(M \times \mathbb{R}, M)$. Let $\Phi = (\Phi^t)$ be a flow on M. For $x \in M$, we call the set $\{\Phi^t(x) \mid t \in \mathbb{R}\}$ the *orbit* of x and denote it by $\mathcal{O}_\Phi(x)$. We say that two flows Φ and Ψ on M are *topologically equivalent* by a homeomorphism h if $h(\mathcal{O}_\Phi(x)) = \mathcal{O}_\Psi(h(x))$ for any $x \in M$ and the homeomorphism h preserves the orientations of the orbits induced by the flows. We also say that a flow Φ is *structurally stable* if it admits a neighborhood in $\mathcal{A}(M, \mathbb{R})$ where any flow is topologically equivalent to Φ. Suppose that M is a closed Riemannian manifold and a flow Φ on M has no stationary point. We call a continuous splitting $TM = E^{ss} \oplus E^c \oplus E^{uu}$ an *Anosov splitting* for Φ if $E^c(x) = T_x\mathcal{O}_\Phi(x)$, the splitting $TM = E^{ss} \oplus E^c \oplus E^{uu}$ is $D\Phi^t$-invariant, and there exist constants $C > 0$ and $\lambda > 0$ such that

$$\|D\Phi^t_x(v_s)\| \le Ce^{-\lambda t}\|v_s\|,$$
$$\|D\Phi^{-t}_x(v_u)\| \le Ce^{-\lambda t}\|v_u\|,$$

for any $x \in M$, $v_s \in E^s(x)$, $v_u \in E^u(x)$, and $t > 0$. As before, the definition of Anosov splitting does not depend on the choice of a Riemannian metric on M. We say that a flow is *Anosov* if it admits an Anosov splitting. We call the subbundles E^{ss}, E^{uu}, $E^{ss} \oplus E^c$, $E^c \oplus E^{uu}$ the *strong stable*, *strong unstable*, *weak stable*, and *weak unstable* subbundles of TM.

Example 4.6 (Geodesic flows on hyperbolic closed surfaces)**.** It is known that the geodesic flow of a hyperbolic closed manifold is an Anosov flow. We see this fact for the case of a closed surface. Take a basis of the Lie algebra $\mathfrak{sl}_2\mathbb{R}$

$$X = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix},\ Y^s = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ Y^u = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{4.1}$$

We identify elements of $\mathfrak{sl}_2\mathbb{R}$ with left-invariant vector fields on $PSL_2\mathbb{R}$. Recall that the upper half plane $\mathbb{H} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$ admits a $PSL_2\mathbb{R}$-action by linear fractional transformations. The space $\mathbb{H}$ admits a Riemannian metric of curvature -1 which is invariant under this action. Since the $PSL_2\mathbb{R}$-action is free and transitive on the unit tangent bundle $S\mathbb{H}$ of $\mathbb{H}$, we can identify $S\mathbb{H}$ with $PSL_2\mathbb{R}$. Under the identification, the geodesic flow $\tilde{\Phi}$ on $S\mathbb{H}$ is given by $\tilde{\Phi}^t(g) = g\exp(tX)$. A hyperbolic closed oriented surface is isometric to $\Gamma\backslash\mathbb{H}$ with a torsion-free uniform lattice Γ of $PSL_2\mathbb{R}$. The identification between $S\mathbb{H}$ and $PSL_2\mathbb{R}$ induces an identification between the unit tangent bundle $S(\Gamma\backslash\mathbb{H})$ of $\Gamma\backslash\mathbb{H}$ and $\Gamma\backslash PSL_2\mathbb{R}$. Let $\bar{X}$, $\bar{Y}^s$, $\bar{Y}^u$ be vector fields on $\Gamma\backslash PSL_2\mathbb{R}$ induced from left-invariant vector fields X, Y^s, Y^u on $PSL_2\mathbb{R}$. Then, under the identification between $S(\Gamma\backslash\mathbb{H})$ and $\Gamma\backslash PSL_2\mathbb{R}$, the geodesic flow Φ_Γ on $S(\Gamma\backslash\mathbb{H})$ coincides with the flow generated by $\bar{X}$. Since $[Y^s, X] = -Y^s$, $[Y^u, X] = Y^u$, we have $D\Phi^t_\Gamma(\bar{Y}^s(x)) = e^{-t} \cdot \bar{Y}^s(\Phi^t_\Gamma(x))$ and $D\Phi^t_\Gamma(\bar{Y}^u(x)) = e^t \cdot \bar{Y}^u(\Phi^t_\Gamma(x))$. These equations imply that the frame $(\bar{Y}^s, \bar{X}, \bar{Y}^u)$ gives an Anosov splitting for the geodesic flow Φ_Γ.

Similar to the case of diffeomorphisms, the sets

$$W^{ss}(x) = \{y \in M \mid d(\Phi^t(x), \Phi^t(y)) \to 0 (t \to +\infty)\},$$
$$W^{uu}(x) = \{y \in M \mid d(\Phi^t(x), \Phi^t(y)) \to 0 (t \to -\infty)\},$$

are injectively immersed manifolds tangent to E^{ss}, E^{uu} (*the strong stable manifold* and *strong unstable manifold*). *The weak stable manifold* $W^s(x) = \bigcup_{y \in \mathcal{O}_\Phi(x)} W^{ss}(y)$ and *the weak unstable manifold* $W^u(x) = \bigcup_{y \in \mathcal{O}_\Phi(x)} W^{uu}(y)$ are also injectively immersed submanifolds of M. It is known that the family $\mathcal{F}^\sigma = \{W^\sigma(x) \mid x \in M\}$

is a topological foliation[8] for each $\sigma = ss, uu, s, u$. We call the foliations $\mathcal{F}^{ss}$, $\mathcal{F}^{uu}$, $\mathcal{F}^{s}$, $\mathcal{F}^{u}$ the *strong stable*, *strong unstable*, *weak stable*, and *weak unstable* foliations of the Anosov flow Φ. Leaves of these foliations are characterized as the maximal path-connected injectively immersed manifolds that are tangent to the corresponding subbundle.

The analogy of Proposition 4.3, Theorem 4.4 and Proposition 4.5 holds.

Proposition 4.7. *The set of Anosov flows on a closed manifold M is an open subset of $\mathcal{A}(M, \mathbb{R})$.*

Theorem 4.8. *Any Anosov flow is structurally stable. Moreover, if Ψ is a flow close to an Anosov flow Φ in the C^1 topology, then we can take a topological conjugacy between them which is close to the identity map in the C^0 topology.*

Proposition 4.9. *Any topological conjugacy h between two Anosov flows Φ and Ψ preserves the weak stable and unstable manifolds, i.e., $W^s(h(x); \Psi) = h(W^s(x; \Phi))$ and $W^u(h(x); \Psi) = h(W^u(x; \Phi))$ for any $x \in M$.*

5. Stability of foliations

One of the most powerful methods in the study of local rigidity of smooth actions of a discrete group is to reduce the problem to the rigidity problem of foliations. We formulate stability of foliations in this section. We will see how it connects with the local rigidity problem of group actions in the next section.

We call a partition $\mathcal{F} = \{\mathcal{F}_\lambda \mid \lambda \in \Lambda\}$ of an n-dimensional manifold of M an m-dimensional *topological foliation* if each $\mathcal{F}_\lambda$ is a path-connected subset of M and there exists a family $(\varphi_x)_{x \in M}$ such that each φ_x is a homeomorphism from an open neighborhood U_x of x to $(0,1)^n$, and $\varphi_x^{-1}((0,1)^m \times \{t\})$ is a connected component of $\mathcal{F}_\lambda \cap U_x$ if $\lambda \in \Lambda$ and $(s,t) \in (0,1)^m \times (0,1)^{n-m}$ satisfy $\varphi_x^{-1}(s,t) \in \mathcal{F}_\lambda$. The local coordinate φ_x is called a coordinate adapted to the foliation $\mathcal{F}$. We remark that the coordinate change $\psi \circ \varphi^{-1}$ between two coordinates φ and ψ adapted to $\mathcal{F}$ has the form $(s,t) \mapsto (f(s,t), g(t))$ with smooth maps f and g. We call each element of the partition $\mathcal{F}$ a *leaf* of $\mathcal{F}$. Each leaf is an injectively immersed m-dimensional submanifold of M. We write $\mathcal{F}(x)$ for the leaf containing $x \in M$. If all coordinates in the definition are smooth embedding, we say that the foliation $\mathcal{F}$ is smooth. Below, 'a foliation' means a smooth foliation unless we emphasize that it is a 'topological' one.

For foliations $\mathcal{F}$ and $\mathcal{F}$ on a manifold M, we say that a homeomorphism $h : M \to M$ maps $\mathcal{F}$ to $\mathcal{F}'$ if $h(\mathcal{F}(x)) = \mathcal{F}'(h(x))$ for any $x \in M$. We also say that $\mathcal{F}$ and $\mathcal{F}'$ are homeomorphic in this case. If h is a diffeomorphism in addition, we say that $\mathcal{F}$ and $\mathcal{F}'$ are diffeomorphic.

Example 5.1 (The orbits of a flow)**.** The orbits of a flow without stationary points form a one-dimensional smooth foliation.

Example 5.2 (The orbit foliation of the standard action of a Lie group)**.** Let G be a Lie group, H its Lie subgroup, and Γ a uniform lattice of G. We define a (right) H-action $\rho_H : \Gamma\backslash G \times H \to \Gamma\backslash G$ on $\Gamma\backslash G$ by $\rho_H(\Gamma g, h) = \Gamma(gh)$. We call this action *the standard H-action* on $\Gamma\backslash G$. Define the orbit $\mathcal{O}(x)$ of $x \in \Gamma\backslash G$ by $\mathcal{O}(x) = \{\rho_H(x,h) \mid h \in H\}$. Then, the partition $\mathcal{O} = \{\mathcal{O}(x) \mid x \in \Gamma\backslash G\}$ of $\Gamma\backslash G$ is a smooth foliation on $\Gamma\backslash G$.

[8]The definition of a foliation is given in Section 5. The foliation $\mathcal{F}^\sigma$ is not smooth in general.

We introduce stability of foliations. Let $\mathcal{F}$ be a smooth m-dimensional foliation. Each leaf has its tangent space as a subspace of the tangent space of M at each point of M. By gathering them, we can define the tangent bundle $T\mathcal{F}$ of $\mathcal{F}$ as a smooth subbundle of TM. The subbundle $T\mathcal{F}$ can be identified with a section of the Grassmanian bundle over M consisting of m-dimensional planes. The C^∞ topology on the space of sections induces a topology on the space of m-dimensional foliations. We say that a foliation $\mathcal{F}$ is C^∞ *stable* if any foliation in a (C^∞-)neighborhood of $\mathcal{F}$ is diffeomorphic to $\mathcal{F}$ by a diffeomorphism which is close to the identity in the C^∞ topology.[9] Similarly, we say that a foliation $\mathcal{F}$ is *topologically stable* if any foliation in a (C^∞-)neighborhood of $\mathcal{F}$ is homeomorphic to $\mathcal{F}$ by a homeomorphism which is close to the identity in the C^0 topology.

We apply the structural stability of Anosov flows to the stability problem of foliations. Notice that if a foliation is close to the orbit foliation of a flow Φ, then it is the orbit foliation of some flow close to Φ. Stability of the orbit foliation of an Anosov flow follows from this fact and Theorem 4.8.

Theorem 5.3. *The orbit foliation of any Anosov flow is topologically stable.*

Stability of an Anosov flow also implies stability of the weak stable foliation of an Anosov flow.

Theorem 5.4. *The weak stable foliation of an Anosov flow is topologically stable.*

Proof. Let Φ be an Anosov flow and $\mathcal{F}$ the weak stable foliation of Φ. Suppose that a foliation $\mathcal{F}'$ is sufficiently close to $\mathcal{F}$. We will find a homeomorphism between $\mathcal{F}$ and $\mathcal{F}'$ which is close to the identity map in C^0 topology. By taking the orthogonal projection of the vector field generating Φ to $T\mathcal{F}'$, we obtain a flow Ψ which is close to Φ and preserves each leaf of $\mathcal{F}'$. By Proposition 4.7 and Theorem 4.8, the flow Ψ is Anosov and it is topologically equivalent to Φ by a homeomorphism close to the identity map. Since the foliation $\mathcal{F}'$ is under the flow Ψ, we can see that $\mathcal{F}'$ is tangent to the weak stable bundle of Ψ.[10] Hence, $\mathcal{F}'$ is the weak stable foliation of Ψ. By Proposition 4.9, the homeomorphism h maps $\mathcal{F}$ to $\mathcal{F}'$. $\square$

6. Foliated bundles and holonomy actions

Let $\pi : E \to B$ be a smooth fiber bundle with a fiber M and $\mathcal{F}$ a smooth foliation on E. We denote the fiber of E containing $p \in E$ by $M(p)$. We call the pair $(E, \mathcal{F})$ a *foliated bundle* if $T_pE = T_pM(p) \oplus T_p\mathcal{F}$ for any $p \in E$ and the restriction $\pi|_{\mathcal{F}} : \mathcal{F} \to B$ of the projection π to any leaf $\mathcal{F}$ is a covering map. Fix a base point $b_* \in B$ and identify the fiber $p^{-1}(b_*)$ at b_* with M. For a closed curve $l : [0,1] \to B$ with $l(0) = l(1) = b_*$ and a point $x \in M$, there exists a unique lift $l_x : [0,1] \to \mathcal{F}(x)$ of l to $\mathcal{F}(x)$ such that $l_x(1) = x$. Moreover, the point $l_x(0) \in M$ is determined only by the homotopy class of l. We define a smooth action $\rho_{\mathcal{F}}$ of the fundamental group $\pi_1(B, b_*)$ of B on M by $\rho_{\mathcal{F}}^\gamma(x) = l_x(0)$ for $\gamma = [l] \in \pi_1(B)$ and $x \in M$.[11] We call the action $\rho_{\mathcal{F}}$ the *holonomy action* of the foliated bundle $(E, \mathcal{F})$.

Conversely, we can construct a foliated bundle from an action of a discrete group. Let Γ be a discrete group and ρ a smooth Γ-action on a closed manifold M. Suppose

[9] The condition that h is close to the identity map is excluded in some literatures.

[10] We use the fact that the Anosov splitting for Ψ is close to that for Φ.

[11] In this article, we concatenate curves from left to right, i.e., the concatenation $l_1 * l_2$ for two curves $l_1, l_2 : [0,1] \to B$ is taken when $l_1(1) = l_2(0)$. This rule of concatenation is the reason why we define the action $\rho_{\mathcal{F}}$ in this way, not taking the end point $l_x(1)$.

that there exists a closed manifold B whose fundamental group is Γ. The group Γ acts on the universal covering $\tilde{B}$ of B as the covering transformation group. Define a Γ-action on $\tilde{B} \times M$ by $\gamma \cdot (b, x) = (\gamma \cdot b, \rho^{\gamma}(x))$. This action preserves the horizontal foliation $\mathcal{H} = \{\tilde{B} \times \{x\} \mid x \in M\}$ on $\tilde{B} \times M$. Hence, the fiber bundle $E_\rho = \Gamma \backslash (\tilde{B} \times M)$ over B with a fiber M admits a foliation $\mathcal{F}_\rho$ induced from $\mathcal{H}$. We call the foliation $\mathcal{F}_\rho$ *the suspension foliation* of ρ. It is easy to check that $(E_\rho, \mathcal{F}_\rho)$ is a foliated bundle and its holonomy action is ρ.

Let $(E, \mathcal{F})$ be a foliated bundle and ρ_0 its holonomy action. If a foliation $\mathcal{F}'$ is sufficiently close to $\mathcal{F}$, then $(E, \mathcal{F}')$ is also a foliated bundle and its holonomy action is close to ρ_0. Conversely, for any action ρ close to ρ_0, it is known that there exists a foliation $\mathcal{F}_\rho$ close to $\mathcal{F}$ such that the holonomy action of the foliated bundle $(E, \mathcal{F}_\rho)$ is ρ.[12] If two foliated bundles $(E, \mathcal{F})$ and $(E, \mathcal{F}')$ are diffeomorphic by a diffeomorphism which preserves each fiber of $\pi : E \to B$, then this diffeomorphism induces a C^∞-conjugacy between the holonomy actions of $(E, \mathcal{F})$ and $(E, \mathcal{F}')$. With these facts, we can relate stability of foliations to local rigidity of actions.

Theorem 6.1. *Let $(E, \mathcal{F})$ be a foliated bundle. If $\mathcal{F}$ is C^∞ stable, then the holonomy action $\rho_{\mathcal{F}}$ of $(E, \mathcal{F})$ is C^∞ locally rigid. If $\mathcal{F}$ is topologically stable, then the holonomy action $\rho_{\mathcal{F}}$ of $(E, \mathcal{F})$ is structurally stable.*

We apply the theorem to show the local rigidity of the standard action of a uniform lattice of $SL_n\mathbb{R}$ on a homogeneous space. Since the example we explain here will appear several times in this article, we fix notation. Let P be the set of upper triangular matrices in $SL_n\mathbb{R}$ whose diagonal entries are positive and A be the set of diagonal matrices belonging to P. Then, P is a connected solvable closed subgroup of $SL_n\mathbb{R}$ and A is a closed subgroup of $SL_n\mathbb{R}$ which is isomorphic to $\mathbb{R}^{n-1}$ as a Lie group. Fix a uniform lattice Γ of $SL_n\mathbb{R}$. The quotient spaces $M_P = SL_n\mathbb{R}/P$ and $M_\Gamma = \Gamma \backslash SL_n\mathbb{R}$ are closed manifolds. We write ρ_Γ for the standard Γ-action on M_P, ρ_P and ρ_A for the standard (right) P- and A-actions on M_Γ. Let $\mathcal{F}_P$ and $\mathcal{F}_A$ be the orbit foliations of these actions. When Γ is torsion-free,[13] M_Γ is a fiber bundle over $\Gamma \backslash SL_n\mathbb{R}/SO(n)$ with a fiber $SO(n)$ and $(M_\Gamma, \mathcal{F}_P)$ is a foliated bundle. We can identify M_P with $SO(n)$ by the decomposition $SL_n\mathbb{R} = SO(n) \cdot P$ given by Schmidt's orthogonalization. Under this identification, the holonomy action of $(M_\Gamma, \mathcal{F}_P)$ coincides with ρ_Γ.

For the case $n = 2$, the foliation $\mathcal{F}_P$ is the weak stable foliation of an Anosov flow. In fact, let X, Y^s, Y^u be the basis of $\mathfrak{sl}_2\mathbb{R}$ given in (4.1) and define a flow Φ_Γ on M_γ by $\Phi_\Gamma(\Gamma g) = \Gamma(g \exp(tX))$. Same as in Example 4.6, the flow Φ_Γ is an Anosov flow and its Anosov splitting is the decomposition of TM associated with the basis (X, Y^s, Y^u). In particular, $\mathcal{F}_P$ is the weak stable foliation of Φ_Γ. The following is an immediate corollary of Theorem 5.4.

Theorem 6.2. *If $n = 2$, then the foliation $\mathcal{F}_P$ is topologically stable.*

Since ρ_Γ is the holonomy action of the foliated bundle $(M_\Gamma, \mathcal{F}_P)$, Theorem 6.1 implies

Theorem 6.3. *If $n = 2$, then ρ_Γ is structurally stable.*

[12] This fact is well known for experts, but it is non-trivial. The reader can find a detailed proof of these facts in [7].

[13] By Selberg's lemma, any lattice contains a torsion-free and finite-index subgroup.

The structural stability of Anosov flows can be generalized to *Anosov actions*. We apply this to the structural stability of ρ_Γ for $n \geq 3$. Suppose $n \geq 3$ and Γ is a uniform lattice of $SL_n\mathbb{R}$. Let N and N' be the subgroup of $SL_n\mathbb{R}$ consisting of upper and lower matrices whose diagonal entries are one, respectively. The subgroups N, A, N' of $SL_n\mathbb{R}$ induces a splitting $\mathfrak{sl}_n\mathbb{R} = \mathfrak{n} \oplus \mathfrak{a} \oplus \mathfrak{n}'$ of the Lie algebra of $SL_n\mathbb{R}$. Let $TM_\Gamma = E^{ss} \oplus E^c \oplus E^{uu}$ be the associated splitting of TM_Γ. The subbundles E^c and $E^{ss} \oplus E^c$ coincide with the tangent bundle of the orbit foliations of the standard A- and P-action, respectively.

Take an element X of the Lie algebra $\mathfrak{a}$ of A whose diagonal entries are $\lambda_1 > \lambda_2 > \cdots > \lambda_n$. Let Φ be a flow on M_Γ given by $\Phi^t(\Gamma g) = \Gamma(g \exp(tX))$. The splitting $TM_\Gamma = E^{ss} \oplus E^c \oplus E^{uu}$ satisfies all conditions in the definition of Anosov flows for the flow Φ except E^c is not the tangent bundle of the orbit foliation of Φ but $\mathcal{F}_A$. Such an $\mathbb{R}^m$-action is called an *Anosov action*. For Anosov $\mathbb{R}^m$-actions, analogies of Theorems 5.3 and 5.4 can be shown by the stability theorem of normally hyperbolic foliation [21, Theorem 7.1]. In particular, the following holds in our setting.

Theorem 6.4 (cf. [27, Section 2.2.3, Step 1])**.** *The foliations $\mathcal{F}_A$ and $\mathcal{F}_P$ are topologically stable.*

Recall that the holonomy action of the foliated bundle $(M_\Gamma, \mathcal{F}_P)$ is ρ_Γ. The stability of ρ_Γ follows from the stability of $\mathcal{F}_P$.

Theorem 6.5. *The standard Γ-action ρ_Γ on $M_P = SL_n\mathbb{R}/P$ is structurally stable.*

7. Ghys' rigidity theorems

As we saw in the previous section, the standard action ρ_Γ on $M_P = SL_n\mathbb{R}/P$ is structurally stable for $n = 2$. However, it is not C^∞ locally rigid. In fact, for any homomorphisms $\pi, \pi' \in \mathrm{Hom}(\Gamma, SL_2\mathbb{R})$ which are sufficiently close to the inclusion map $\iota : \Gamma \to SL_2\mathbb{R}$, the homogeneous actions ρ_π and $\rho_{\pi'}$ associated with them are C^∞-conjugate if and only if homomorphisms π and π' are conjugate as elements of $\mathrm{Hom}(\Gamma, SL_2\mathbb{R})$. By the Teichmüller theory, the inclusion ι admits a non-trivial deformation in $\mathrm{Hom}(\Gamma, SL_2\mathbb{R})$. Hence, the action ρ_Γ has a non-trivial deformation by homogeneous actions. Ghys proved that the only possible deformation of ρ_Γ is deformation by homogeneous actions.

Theorem 7.1 (Ghys [16], cf. Kononenko-Yue [28])**.** *Any Γ-action close to ρ_Γ is C^∞-conjugate to a homogeneous action.*

Before sketching the proof, we prepare two notions, holonomy along a curve and the transverse projective structure of a foliation. Let M be an n-dimensional manifold and $\mathcal{F}$ an m-dimensional foliation on M. Fix a curve $l : [a, b] \to \mathcal{F}(x)$ in a leaf $\mathcal{F}(x)$ and $(n-m)$-dimensional submanifolds I_a, I_b of M which are transverse to each leaf of $\mathcal{F}$ and satisfy $l(s) \in I_s$ for $s = a, b$. We take $a = t_0 < t_1 < \cdots < t_k = b$ and a family $(\varphi_j : U_j \to (0,1)^n)_{j=0}^{k-1}$ of local coordinates adapted to $\mathcal{F}$ such that $l([t_j, t_{j+1}]) \subset U_j$. Then, there exists a neighborhood U_a of x_0 in I_a and a continuous map $H : [a, b] \times U_a \to M$ such that for any $x \in U_a$, $H(0, x) = x$, $H(1, x) \in I_b$, $H(t, x) \in U_j$ and $\varphi(H(t, x))$ is contained in $(0, 1) \times \{y_j(x)\}$ for some $y_j(x) \in (0, 1)^{n-m}$ if $t \in [t_j, t_{j+1}]$. We can see that $H(1, x) \in I_b$ does not depend on the choice of (t_j), (φ_j), and H. We call the map $h_l : U_a \to I_b$ defined by $h_l(x) = H(1, x)$ *the holonomy map* of $\mathcal{F}$ along the curve l. For a foliated bundle $(E, \mathcal{F})$ and a closed curve γ in the base space, the holonomy map of $\mathcal{F}$ along a lift of $-\gamma$ coincides with the holonomy map of the foliated bundle which was defined in the previous section.

A projective structure on a manifold M is a family $(\varphi_\lambda)_{\lambda\in\Lambda}$ of local coordinate which covers M and in which all coordinate changes are projective transformations. We say that a diffeomorphism between two manifolds with projective structures is projective if it is a projective transformation in local coordinates. *A transversely projective structure* of a foliation $\mathcal{F}$ is an attachment of a projective structure to each transversal of $\mathcal{F}$ which is preserved by any holonomy map. For a foliated bundle $(E,\mathcal{F})$ with a fiber M, a transverse projective structure on $\mathcal{F}$ induces a projective structure on the fiber M and the holonomy action preserves this structure. Conversely, if the holonomy action of $(E,\mathcal{F})$ preserves a projective structure on M, then this structure induces a transverse projective structure of $\mathcal{F}$. A typical example of a foliation with a transverse projective structure is the orbit foliation $\mathcal{F}_P$ of the standard P-action on $M_\Gamma = \Gamma\backslash SL_n\mathbb{R}$ defined in the previous section.

Now, we sketch the proof of Theorem 7.1.[14] Let $(M_\Gamma,\mathcal{F}_P)$ be the foliated bundle in the previous section with $n=2$. Suppose that a Γ-action ρ is sufficiently close to ρ_Γ. Then, there exists a foliation $\mathcal{F}$ close to $\mathcal{F}_P$ such that the holonomy action of $(E,\mathcal{F})$ coincides with ρ. We will show that $\mathcal{F}$ admits a transverse projective structure. Once it is done, the action ρ preserves a projective structure on M_P. Since ρ is topologically conjugate to ρ_Γ by Theorem 6.3, we can see that the projective structure preserved by ρ is equivalent to the standard projective structure which is preserved by ρ_Γ. It implies that ρ is C^∞-conjugate to a homogeneous action induced by a homomorphism in $\mathrm{Hom}(\Gamma, SL_2\mathbb{R})$. Therefore, it is sufficient to show that $\mathcal{F}$ admits a transverse projective structure.

Let X, Y^s, Y^u be the basis of the Lie algebra $\mathfrak{sl}_2\mathbb{R}$ given in (4.1), Φ_Γ the Anosov flow in the previous section, i.e., $\Phi_\Gamma^t = \rho_A^{\exp(tX)}$. Recall that the foliation $\mathcal{F}_P$ is the weak stable foliation of Φ_Γ. Let $\mathcal{F}^{uu}$ be the strong unstable foliation of Φ_Γ. Then, $\mathcal{F}^{uu}(x) = \{\rho_P^{\exp(tY^s)}(x) \mid t\in\mathbb{R}\}$ for any $x\in M_\Gamma$. Fix a foliation close to $\mathcal{F}_P$. We may assume that $\mathcal{F}$ is transverse to each leaf of $\mathcal{F}^{uu}$. As in Theorem 5.4, there exists an Anosov flow Φ close to Φ_Γ whose weak stable foliation is $\mathcal{F}$. By the structural stability of an Anosov flow, the flows Φ_Γ and Φ are topologically equivalent by a homeomorphism h close to the identity map. Put $\Phi_h^t = h\circ\Phi_\Gamma^t\circ h^{-1}$. Then, Φ_h is a C^0 flow whose orbits coincide with those of Φ. For $x\in M_\Gamma$ and $t\in\mathbb{R}$, define a curve $l_x:\mathbb{R}\to\mathcal{F}(x)$ by $l_x(t)=\Phi_h^t(x)$. Let $h_x^t : I_x^t \to \mathcal{F}^{uu}(\Phi^t(x))$ be the holonomy map of $\mathcal{F}$ along $(l_x)|_{[0,t]}$, where I_x^t is an open interval in $\mathcal{F}^{uu}(x)$ which contains x. The family $(h_x^t)_{t\in\mathbb{R}}$ of holonomy maps satisfies $h_x^{s+t} = h^s_{\Phi_h^t(x)}\circ h_x^t$ in a neighborhood of x. Since Φ is an Anosov flow, there exists $t_*>0$ such that $\sup_{x\in M_\Gamma}|Dh_x^{-t}| < 1/2$ for any $x\in M$ and $t\geq t_*$. We apply the 'non-stationary normal form' theory by Guisinsky-Katok [18] to $(h_x^t)_{t\in\mathbb{R},x\in M_\Gamma}$ and obtain a C^∞ linearization of the family, i.e., a continuous family $(\psi_x)_{x\in M_\Gamma}$ of smooth embeddings from $[-1,1]$ to M_Γ such that $\psi_x(0)=x$, $\psi_x([-1,1])\subset\mathcal{F}(x)$, and the map $\alpha_x^t := \psi^{-1}_{\Phi_h^t(x)}\circ h_x^t\circ\psi_x$ is a linear map for any $x\in M_\Gamma$ and $t\in\mathbb{R}$.[15] Since $d(h_x^{-t}(x), h_x^{-t}(x'))$ and $|D\alpha_x^{-t}|$ converge to zero as t goes to infinity for any $x\in M_\Gamma$ and $x'\in\mathcal{F}^{uu}(x)$, the second derivative of

[14] The sketch we give here is based on the proof of a more global result in [17] and another proof by Kononenko and Yue [28] which we mention later.

[15] In [17], Ghys linearized holonomy maps in another way.

the 'coordinate change'

$$\psi_{x'}^{-1} \circ \psi_x = (\alpha_{x'}^{-t})^{-1} \circ (\psi_{\Phi_h^{-t}(x')}^{-1} \circ \psi_{\Phi_{h^{-t}(x)}}) \circ \alpha_x^{-t}$$

vanishes, and hence, $\psi_{x'}^{-1} \circ \psi_x$ is linear. The argument above is summarized as follows: We can attach an affine structure on each leaf of $\mathcal{F}^{uu}$ which is invariant under holonomy along any orbit of Φ_h^t. The affine structure on each leaf induces a projective structure on it.

Define a C^0 flow Ψ_h on M_Γ by $\Psi_h^t = h \circ \rho_P^{\exp(tY^s)} \circ h^{-1}$. Let g_x^t be the holonomy map of $\mathcal{F}$ between $\mathcal{F}^{uu}(x)$ and $\mathcal{F}^{uu}(\Psi_h^t(x))$ along the orbit of x for Ψ_h. Since $[X, Y^s] = Y^s$, the identity $\Psi_h^t = \Phi_h^s \circ \Psi_h^{e^s t} \circ \Phi_h^{-s}$ holds. Set $\hat{g}_x^t = \psi_{\Psi_h^t(x)}^{-1} \circ g_x^t \circ \psi_x$ (the presentation of g_x^t in the 'local coordinate'). We denote the Schwarzian derivative of a one-dimensional map F by $S(F)$. Then, we have

$$\hat{g}_x^t = \alpha^s_{\Psi_h^{e^s t} \circ \Phi_h^{-s}(x)} \circ \hat{g}_{\Phi_h^{-s}(x)} \circ \alpha_x^{-s},$$

and hence

$$S(\hat{g}_x^t) = S(\hat{g}^{e^s t}_{\Phi_h^{-s}(x)}) \cdot (D\alpha_x^{-s})^2.$$

The growth of the C^3 distance between $\hat{g}^{e^s t}_{\Phi_h^{-s}(x)}$ and the identity map is comparable to e^s. If the Anosov flow Φ is sufficiently close to Φ_Γ, then $D\alpha_x^{-s}$ is bounded by a function comparable to $\exp(-3s/4)$. From these estimates, the Schwarzian derivative of $\hat{g}_x^t$ must vanish. Therefore, g_x^t preserves the projective structures of leaves of $\mathcal{F}^{uu}$. Since two C^0 flows Φ_h and Ψ_h generate a C^0 P-action whose orbit foliation is $\mathcal{F}$, any holonomy of $\mathcal{F}$ preserves the projective structures on leaves. In particular, we obtain a transverse projective structure. As mentioned above, this completes the proof of Theorem 7.1

Ghys proved a global version of Theorem 7.1.

Theorem 7.2 ([17])**.** *Suppose that Γ is a uniform lattice of $SL_2\mathbb{R}$. Let U be the connected component of* $\mathrm{Hom}(\Gamma, SL_2\mathbb{R})$ *containing the inclusion map $\iota_\Gamma : \Gamma \to SL_2\mathbb{R}$ and $\mathcal{U}$ the connected components of $\mathcal{A}(\Gamma, M_P)$ containing the standard action ρ_Γ. Then, any action ρ in $\mathcal{U}$ is C^∞-conjugate to a homogeneous action ρ_π with some homomorphism π in U.*

The theorem is obtained from a refinement of the proof above and the following global structural stability theorem.

Theorem 7.3 (Matsumoto [31])**.** *Any action in the connected component $\mathcal{U}$ in the above theorem is topologically conjugate to ρ_Γ.*

This remarkable theorem was proved by using the bounded cohomology of the homeomorphism group of S^1.

8. Katok and Spatzier's rigidity theorems

In this section, we discuss the local rigidity of the standard action ρ_Γ in Section 6 for $n \geq 3$. As we saw in Theorem 6.4, ρ_Γ is structurally stable for $n \geq 3$, like the case $n = 2$. For $n = 2$, the inclusion map $\iota : \Gamma \to SL_2\mathbb{R}$ admits non-trivial deformation in $SL_2\mathbb{R}$, and it induces non-trivial deformation of the action ρ_Γ. By Theorem 7.1, there exists no other deformation up to C^∞-conjugacy. For $n \geq 3$, by Theorem 2.2, the inclusion map $\iota : \Gamma \to SL_n\mathbb{R}$ is locally rigid in $\mathrm{Hom}(\Gamma, SL_n\mathbb{R})$, and hence, ρ_Γ does not admit any non-trivial *homogeneous* deformation. Katok and Spatzier proved that there exists no deformation of ρ_Γ up to C^∞-conjugacy.

Theorem 8.1 (Katok-Spatzier [27]). *For $n \geq 3$ and a uniform lattice Γ, the standard Γ-action ρ_Γ on $M_P = SL_n\mathbb{R}/P$ is C^∞ locally rigid.*

As before, the theorem is reduced to the following result by taking the suspension foliation.

Theorem 8.2 (Katok-Spatzier [27]). *For $n \geq 3$, the orbit foliation $\mathcal{F}_P$ of the standard P-action ρ_P on $M_\Gamma = SL_n\mathbb{R}/\Gamma$ is C^∞ stable.*

They proved the theorem for a large class of higher rank semisimple Lie groups G, their parabolic subgroups P, and irreducible uniform lattices Γ. For these groups, the standard Γ-action on G/P is also C^∞ locally rigid.

For $n = 2$, the standard A-action ρ_A is essentially the same as the geodesic flow described in Example 4.6. There is a natural one-to-one correspondence between periodic orbits of this flow and non-trivial conjugacy classes in Γ. In particular, the flow has infinitely many periodic orbits. By an analog of Proposition 4.1, we can see that the orbit foliation of ρ_A is not C^∞ stable.

For $n \geq 3$, the situation is totally different.

Theorem 8.3 (Katok-Spatzier [27]). *For $n \geq 3$, the orbit foliation of the standard A-action on M_Γ is C^∞ stable.*

Recall that the orbit foliation of ρ_P is tangent to 'the weak stable subbundle' of the Anosov $\mathbb{R}^{n-1}$-action ρ_A. Similar to obtaining Theorem 5.4 from Theorem 5.3, Theorem 8.2 follows from Theorem 8.3.

In the rest of the section, we sketch the proof of Theorem 8.3. Let Y_{ij} be a square matrix of size n whose (i,j)-entry is 1 and the other entries are 0. Set $X_i = Y_{ii} - (1/n)\sum_{j=1}^n Y_{jj}$. Then, $\{X_i \mid i = 1, \ldots, n-1\} \cup \{Y_{ij} \mid i \neq j\}$ is a basis of the Lie algebra $\mathfrak{sl}_n\mathbb{R}$ of $SL_n\mathbb{R}$. For each $Z \in \mathfrak{sl}_n\mathbb{R}$, we define a flow $\bar{\Phi}_Z$ on M_Γ by $\bar{\Phi}_Z(\Gamma g) = \Gamma g \exp(tZ)$. Let $\mathcal{F}_{ij}$ be the orbit foliation of the flow $\bar{\Phi}_{Y_{ij}}$. For mutually distinct $i, j, k = 1, \ldots, n$, we have $[Y_{ij}, X_i] = Y_{ij}$, $[Y_{ij}, X_j] = -Y_{ij}$, and $[Y_{ij}, X_k] = 0$. Hence, the restriction of $\bar{\Phi}^t_{X_k}$ to each leaf of $\mathcal{F}_{ij}$ is expanding, contracting, or isometric for any $t > 0$ for the case $k = i$, $k = j$, or $k \neq i, j$, respectively. Isometries $(\bar{\Phi}^t_{X_k})$ between leaves of $\mathcal{F}_{ij}$ play an important role in the proof. Recall that we cannot choose such k for $n = 2$.

Let $\mathcal{F}$ be a foliation sufficiently close to the orbit foliation $\mathcal{F}_A$ of ρ_A. By the topological stability of $\mathcal{F}_A$, there exists a homeomorphism h which is close to the identity map and which maps leaves of $\mathcal{F}_A$ to those of $\mathcal{F}$. Since the Lie bracket of the tangent bundles of $\mathcal{F}_{ij}$'s generates the total tangent bundle TM, we can apply a general theory on smoothness of functions which are smooth along a family of foliations (Katok-Spatzier [26]). In fact, if h is smooth along each leaf of $\mathcal{F}_{ij}$ for any i, j, then h is a smooth map. Therefore, it is sufficient to show the smoothness of the restriction of h to each leaf of $\mathcal{F}_{ij}$.

To simplify the argument, we assume that h preserves each leaf of $\mathcal{F}_{ij}$.[16] Define a C^0 flow Φ_k by $\Phi^t_k = h \circ \bar{\Phi}^t_{X_k} \circ h^{-1}$ and let $h^t_{x,k} : \mathcal{F}_{ij}(x) \to M_\Gamma$ be the restriction Φ^t_k to $\mathcal{F}_{ij}(x)$. The flow Φ_k preserves the foliation $\mathcal{F}_{ij}$ and the restriction $h^t_{x,k}$ is a smooth map to $\mathcal{F}_{ij}(\Phi^t_k(x))$. As in the previous section, we apply the Guisinsky-Katok theorem and obtain coordinates of leaves of $\mathcal{F}_{ij}$ which linearize all $h^t_{x,i}$'s.

[16] It does not hold in general. For the general case, we need to use holonomy maps as we did in Section 7.

Moreover, the commutativity of Φ_i and Φ_k implies that the coordinates linearize $h^t_{x,k}$ even for $k \neq i$. For $k \neq i, j$, $h^t_{x,k}$ is equicontinuous with respect to t, x since

$$h^t_{x,k} = h \circ \bar{\Phi}^t_{X_k} \circ h^{-1} \tag{8.1}$$

and the map $\bar{\Phi}_{X_k}$ is isometric on each leaf of $\mathcal{F}_{ij}$. Using the equicontinuity, we can find new coordinates of leaves of $\mathcal{F}_{ij}$ such that $h^t_{x,k}$ is isometric with respect to the coordinates. We also construct a smooth family $(\mu^s)_{s\in\mathbb{R}}$ of isometries of $\mathcal{F}_{ij}(x)$ from the limits of isometries in $\{h^t_{x,k} \mid x \in M_\Gamma, t \in \mathbb{R}\}$. Then, the smoothness of h on $\mathcal{F}_{ij}(x)$ follows from the identity (8.1) and the smoothness of μ^s and $\bar{\Phi}^s_{Y_{ij}}(x)$ with respect to s. As we mentioned above, the smoothness of h follows from the smoothness of h along leaves of $\mathcal{F}_{ij}$ for any i, j. Hence, the conjugacy map h is smooth.

9. Leafwise cohomology and local rigidity of actions

In this section, we discuss the deformation complex of foliated bundles and its application to the local rigidity problem of actions. In Section 3, we saw that infinitesimal deformation of a group action is described by the cohomology of the group. Similarly, infinitesimal deformation of a foliation can be described by a cohomology associated with the foliation.

Let $\mathcal{F}$ be a foliation on a manifold M and $T^*\mathcal{F}$ be the dual bundle of the tangent bundle $T\mathcal{F}$ of the foliation. By $\Omega^k(\mathcal{F})$ we denote the set of smooth sections of the exterior product $\wedge^k T^*\mathcal{F}$. We call an element of $\Omega^k(\mathcal{F})$ a *leafwise k-form.* Let $\mathfrak{X}(\mathcal{F})$ be the set of sections of $T\mathcal{F}$, i.e., vector fields tangent to $\mathcal{F}$. By Frobenius' theorem, we have $[X, Y] \in \mathfrak{X}(\mathcal{F})$ for any $X, Y \in \mathfrak{X}(\mathcal{F})$. This implies that *the leafwise exterior derivative* $d_\mathcal{F} : \Omega^k(\mathcal{F}) \to \Omega^{k+1}(\mathcal{F})$ given by

$$\begin{aligned} d_\mathcal{F}\omega(X_0, \dots, X_k) = &\sum_{0\le i\le k} (-1)^i X_i \omega(X_0, \dots, \check{X}_i, \dots, X_k) \\ &+ \sum_{0\le i<j\le k} (-1)^{i+j} \omega([X_i, X_j], X_0, \dots, \check{X}_i, \dots, \check{X}_j \dots, X_k) \end{aligned}$$

is well defined. Similar to the usual exterior derivative, $d_\mathcal{F}$ satisfies $d_\mathcal{F} \circ d_\mathcal{F} = 0$. We call the cohomology of the complex $(\Omega^*(\mathcal{F}), d_\mathcal{F})$ *the leafwise cohomology* of $\mathcal{F}$. We write $H^*(\mathcal{F})$ for it.

Since the complex $(\Omega^*(\mathcal{F}), d_\mathcal{F})$ is not an elliptic complex, the cohomology $H^*(\mathcal{F})$ is not finite-dimensional in general. For example, when $\mathcal{F}$ is a foliation on $M = B \times F$ given by $\mathcal{F}(x, y) = B \times \{y\}$, then we can obtain $H^*(\mathcal{F}) \simeq H^k(B) \otimes C^\infty(F)$ from easy computations. In the rest of this section, we see the relationship between a version of leafwise cohomology and deformation of foliation for the case of foliated bundles. The leafwise cohomology $H^*(\mathcal{F})$ is also closely related to parameter deformation of locally free actions of a Lie group. We discuss this relationship in Section 11.

Let $p : E \to B$ be a fiber bundle with a fiber M, $\mathcal{F}$ a foliation on M such that $(E, \mathcal{F})$ is a foliated bundle. The tangent bundle TE of E is decomposed into the vertical subbundle $N\mathcal{F}$ tangent to fibers and the tangent bundle $T\mathcal{F}$ of the foliation $\mathcal{F}$. Let $\pi_H : TE \to N\mathcal{F}$ be the projection to $N\mathcal{F}$ associated with this decomposition. By $\Omega^k(\mathcal{F}; N\mathcal{F})$, we denote the set of $N\mathcal{F}$-valued leafwise k-forms.

Define the leafwise exterior-derivative $d_{\mathcal{F}} : \Omega^k(\mathcal{F}; N\mathcal{F}) \to \Omega^{k+1}(\mathcal{F}; N\mathcal{F})$ by

$$\begin{aligned} d_{\mathcal{F}}\omega(X_0, \ldots, X_k) = & \sum_{0 \le i \le k} (-1)^i \pi_H \left([X_i, \omega(X_0, \ldots, \check{X}_i, \ldots, X_k)]\right) \\ & + \sum_{0 \le i < j \le k} (-1)^{i+j} \omega([X_i, X_j], X_0, \ldots, \check{X}_i, \ldots, \check{X}_j, \ldots, X_k). \end{aligned}$$

This exterior derivative also satisfies $d_{\mathcal{F}} \circ d_{\mathcal{F}} = 0$. We call the cohomology of the complex $(\Omega^*(\mathcal{F}; N\mathcal{F}), d_{\mathcal{F}})$ *the $N\mathcal{F}$-valued leafwise cohomology.* We denote it by $H^*(\mathcal{F}; N\mathcal{F})$.

We can regard the first cohomology $H^1(\mathcal{F}; N\mathcal{F})$ as the space of infinitesimal deformations of the foliation $\mathcal{F}$ as follows. Let m be the dimension of the foliation $\mathcal{F}$. For $\omega \in \Omega^1(\mathcal{F}; N\mathcal{F})$, we define an m-dimensional plane field E_ω of E by $E_\omega(x) = \{v + \omega(v) \mid v \in T_x\mathcal{F}\}$. Then, the map $\omega \mapsto E_\omega$ is a bijection from $\Omega^1(\mathcal{F}; N\mathcal{F})$ to the set of m-dimensional plane fields of E which are transverse to $N\mathcal{F}$. We can obtain the following characterization of the integrability of E_ω by computation in local coordinates.

Proposition 9.1. *The plane field E_ω is the tangent bundle of some m-dimensional foliation if and only if ω satisfies*

$$d_{\mathcal{F}}\omega + \frac{1}{2}(\omega \wedge \omega) = 0, \tag{9.1}$$

where $(\omega \wedge \eta)_x(X, Y) = [\omega_x(X), \eta_x(Y)] - [\omega_x(Y), \eta_x(X)]$ for $\omega, \eta \in \Omega^1(\mathcal{F}; N\mathcal{F})$.

Suppose that $\beta \in \Omega^0(\mathcal{F}; N\mathcal{F})$ generates a flow Φ^t as a vector field tangent to $N\mathcal{F}$. We define a family $(\omega_t)_{t \in (-\epsilon, \epsilon)}$ of elements of $\Omega^1(\mathcal{F}; N\mathcal{F})$ by $E_{\omega_t} = D\Phi^t(T\mathcal{F})$. By computation in local coordinates, we obtain

$$\lim_{t \to 0} \frac{1}{t}\omega_t = d_{\mathcal{F}}\beta. \tag{9.2}$$

Since the equation $d_{\mathcal{F}}\omega = 0$ is the formal lineralization of equation (9.1), we can regard $H^1(\mathcal{F}; N\mathcal{F})$ as the space of infinitesimal deformations of $\mathcal{F}$.

Similar to the discussion in Section 3, we need a version of the implicit function theorem in order to obtain C^∞ stability of a foliation $\mathcal{F}$ from the vanishing of $H^1(\mathcal{F}; N\mathcal{F})$. Hamilton's implicit function theorem gives the following criterion to C^∞ stability.

Theorem 9.2 (Hamilton [20], cf. [11])**.** *The foliation is C^∞ stable if the complex $\Omega^0(\mathcal{F}; N\mathcal{F}) \xrightarrow{d_{\mathcal{F}}} \Omega^1(\mathcal{F}; N\mathcal{F}) \xrightarrow{d_{\mathcal{F}}} \Omega^2(\mathcal{F}; N\mathcal{F})$ admits a tame splitting, i.e, there exists a continuous operator $\delta^k : \Omega^{k+1}(\mathcal{F}; N\mathcal{F}) \to \Omega^k(\mathcal{F}; N\mathcal{F})$ for each $k = 0, 1$, an integer $r \ge 1$, and a sequence $(C_s)_{s \ge 0}$ of positive numbers with the following properties:*

1. $d_{\mathcal{F}} \circ \delta^0 + \delta^1 \circ d_{\mathcal{F}} = \mathrm{Id}$.
2. $\|\delta^0\omega\|_s \le C_s\|\omega\|_{s+r}$ *and* $\|\delta^1\eta\|_s \le C_s\|\eta\|_{s+r}$ *for any* $s \ge 0$, $\omega \in \Omega^1(\mathcal{F}; N\mathcal{F})$, *and* $\eta \in \Omega^2(\mathcal{F}; N\mathcal{F})$, *where* $\| \cdot \|_s$ *is the* C^s *norm on* $\Omega^k(\mathcal{F}; N\mathcal{F})$.

Similar to the case of group actions, it is very hard to show the tameness in general. However, Benveniste showed the tamenessfor the suspension foliations of isometric actions on Riemannian manifolds by investigating leafwise diffusion and he proved the following

Theorem 9.3 (Benveniste [7]). *Let G be a simple Lie group of finite center and of real rank at least two, and Γ its uniform lattice. If a Γ-action on a closed Riemannian manifold preserves the metric, then it is C^∞ locally rigid.*

Fisher and Margulis extended this rigidity result to actions of a wider class of discrete groups.[17]

Theorem 9.4 (Fisher-Margulis [14]). *Let Γ be a finitely generated group with Property (T). If a Γ-action on a closed Riemannian manifold M preserves the metric, then the action is C^∞ locally rigid.*

We say that a finitely generated group Γ has *Property (T)* if $H^1(\Gamma; V)$ vanishes for any Hilbert space with a unitary Γ-action. Typical examples are lattices of a simple Lie group of higher rank.

Kanai [22] also obtained a local rigidity result by reducing the problem to the vanishing of a cohomology associated with the action. Prior to Katok-Spatzier's and Benveniste's rigidity results, he proved the C^∞ local rigidity of the standard Γ-action on G/P in the case that G is $SL_n\mathbb{R}$ with $n \geq 26$ and P is a parabolic subgroup such that G/P is the $(n-1)$-dimensional sphere. As Ghys' theorem (Theorem 7.1), he reduced this theorem to the existence of a projective structure invariant under the action. Kanai obtained it from the vanishing of the leafwise cohomology valued in a bundle and he also proved the vanishing by investigation of leafwise diffusion.[18] As we saw in Section 7, Ghys proved his rigidity theorem from the expansion and contraction of Anosov systems. Kononenko-Yue [28] gave another proof of Ghys' result which emphasized the relationship to the vanishing of the leafwise cohomology.

10. Local rigidity problem of Lie group actions

As we saw above, the rigidity of the standard Γ-action on the homogeneous space G/P is closely related to the stability of the orbit foliation of the standard P-action on $\Gamma\backslash G$. How rigid is the P-action itself when its orbit foliation is rigid? Deformation of an action which preserves the orbit foliation is called a *parameter deformation.* Such a deformation is interesting from the viewpoint of deformation of geometric structures since parameter deformation can be regarded as deformation of a leafwise P-structure. In the rest of the article, we discuss parameter deformation of Lie group actions.

Let G be a Lie group and M a manifold. We say that a map $\rho : M \times G \to M$ is a smooth (right) G-action on M if ρ is a smooth map such that $\rho(x, e) = x$, $\rho(x, gh) = \rho(\rho(x, g), h)$ for any $x \in M$ and $g, h \in G$. By $\mathcal{A}(M, G)$, we denote the set of smooth G-actions on M. The set $\mathcal{A}(M, G)$ is endowed with the C^∞ compact-open topology as a subset of $C^\infty(M \times G, M)$. We set $\rho^g(x) = \rho(x, g)$ for $\rho \in \mathcal{A}(M, G)$, $g \in G$, and $x \in M$. Then, ρ^g is a diffeomorphism of M and the map $g \mapsto \rho^g$ is an anti-homomorphism from G to $\mathrm{Diff}(M)$. For $x \in M$ we call the set $\{\rho(x, g) \mid g \in G\}$ *the orbit* of x and denote it by $\mathcal{O}_\rho(x)$. Let G_x be *the isotropy*

[17] Their method was different from Benveniste's.

[18] It seems interesting to compare Kanai's method and Benveniste's.

subgroup $\{g \in G \mid \rho(x,g) = x\}$ of x. Then, the map $I_x : G_x\backslash G \to M$ defined by $I_x(G_x g) = \rho(x,g)$ is an injective immersion whose image is $\mathcal{O}_\rho(x)$.

We say that an action is *locally free* if the isotropy subgroup G_x is a discrete subgroup of G for any $x \in M$. A locally free $\mathbb{R}$-action is just a flow with no stationary points. The standard H-action on $\Gamma\backslash G$ with a Lie group G, its Lie subgroup H and a uniform lattice Γ of G is another typical example of a locally free action. It is known that the orbits of a flow with no stationary points form a foliation. Locally free actions have the same property.

Proposition 10.1. *The orbits of a smooth locally free action ρ form a smooth foliation.*

By $\mathcal{A}_{LF}(M,G)$, we denote the set of locally free G-actions on M. Recall that this set may be empty e.g. if $G = \mathbb{R}$ and the Euler number of M is non-zero.

Unlike actions of discrete groups, there are trivial deformations of an action other than taking a conjugation, which comes from automorphisms of the Lie group. We say that two actions ρ_1 and ρ_2 in $\mathcal{A}(M,G)$ are C^∞*-conjugate* if there exists a diffeomorphism h of M and an automorphism θ of G such that $\rho_2(h(x),\theta(g)) = h(\rho_1(x,g))$ for any $x \in M$ and $g \in G$. Like the discrete case, we say that $\rho_0 \in \mathcal{A}(M,G)$ is C^∞ *locally rigid* if there exists a neighborhood of ρ_0 in $\mathcal{A}(M,G)$ in which any action is C^∞-conjugate to ρ_0

Another difference from the discrete case is that there may exist a non-trivial deformation in which the orbit foliation does not change. For example, if we multiply a given vector field by a positive smooth function, then the flow generated by the new vector field has the same orbit foliation as the original one, but in many cases the new flow is not C^∞-conjugate to the original.[19] Deformation preserving the orbit foliation and rigidity under such deformation are formulated as follows. Let G be a Lie group and $\mathcal{F}$ a foliation on a manifold M. By $\mathcal{A}_{LF}(\mathcal{F},G)$, we denote the set of locally free G-action on M whose orbit foliation is $\mathcal{F}$. Let $\mathrm{Diff}(\mathcal{F})$ be the group of diffeomorphisms of M which preserve each leaf of $\mathcal{F}$, and $\mathrm{Diff}_0(\mathcal{F})$ its path-connected component which contains the identity map. We say that two actions ρ_1 and ρ_2 in $\mathcal{A}_{LF}(\mathcal{F},G)$ are *parameter equivalent* if the actions ρ_1 and ρ_2 are conjugate by an element of $\mathrm{Diff}_0(\mathcal{F})$, i.e., there exist a diffeomorphism $h \in \mathrm{Diff}_0(\mathcal{F})$ and an automorphism θ of G such that $\rho_2(h(x),\theta(g)) = h(\rho_1(x,g))$ for any $x \in M$ and $g \in G$. We also say that an action $\rho_0 \in \mathcal{A}_{LF}(\mathcal{F},M)$ is *parameter rigid* if *any* action in $\mathcal{A}_{LF}(\mathcal{F},M)$ is parameter equivalent to ρ_0. We can define 'local parameter rigidity'. However, there is no known interesting example which is 'locally parameter rigid' but is not parameter rigid.

The local rigidity problem of a locally free action can be decomposed into the problems of stability of the orbit foliation and parameter rigidity of the action.

Proposition 10.2. *If a locally free action is parameter rigid and the orbit foliation is C^∞ stable, then the action is C^∞ locally rigid.*[20]

[19] The ratio of the period of two periodic orbits is invariant under C^∞-conjugacy.

[20] Does there exists a locally free action which is C^∞ locally rigid but whose orbit foliation is not C^∞ stable?

11. Parameter deformation and leafwise cohomology

In this section, we discuss the relation between parameter deformation of actions and the leafwise cohomology. S. Matsumoto's survey [32] is a good reference for this topic.

Let M be a closed manifold, G a Lie group, and $\mathfrak{g}$ the Lie algebra of G. We denote the space of smooth vector fields on M by $\mathfrak{X}(M)$. It is an infinite-dimensional Lie algebra with the bracket $[X,Y] = XY - YX$. Let $\mathrm{Hom}(\mathfrak{g}, \mathfrak{X}(M))$ be the set of Lie algebra homomorphisms from $\mathfrak{g}$ to $\mathfrak{X}(M)$. This set has the C^∞ compact open topology as a subset of $\mathfrak{X}(M)$. Let $\mathcal{F}$ be a foliation on M. For $\mathfrak{g}$-valued leafwise one-forms $\omega, \eta \in \Omega^1(\mathcal{F}) \otimes \mathfrak{g}$, we define a $\mathfrak{g}$-valued leafwise two-form $\omega \wedge \eta \in \Omega^2(\mathcal{F}) \otimes \mathfrak{g}$ by

$$(\omega \wedge \eta)_x(X,Y) = [\omega_x(X), \eta_x(Y)] - [\omega_x(Y), \eta_x(X)].$$

We say a $\mathfrak{g}$-valued leafwise one-form ω is *regular* if $\omega(x) : T_x\mathcal{F} \to \mathfrak{g}$ is a linear isomorphism for any x. For an action $\rho \in \mathcal{A}_{LF}(\mathcal{F}, G)$, let ϕ be an element of $\mathrm{Hom}(\mathfrak{g}, \mathfrak{X}(M))$ given by $\phi(\xi)(x) = (d/dt)\rho(x, \exp(t\xi))|_{t=0}$. Since ρ is locally free, the map $\phi(\cdot)(x) : \mathfrak{g} \to T_x\mathcal{F}$ is a linear isomorphism for any $x \in M'$. We define a regular leafwise one-form $\omega_\rho \in \Omega^1(\mathcal{F}) \otimes \xi$ by $\omega_\rho(x) = [\phi(\cdot)(x)]^{-1}$. We can see that ω_ρ satisfies *the leafwise Maurer-Cartan equation*

$$d_{\mathcal{F}}\omega_\rho + \frac{1}{2}(\omega_\rho \wedge \omega_\rho) = 0. \tag{11.1}$$

We can also see that if G is connected and simply connected, then the map $\rho \mapsto \omega_\rho$ is a bijective between the set $\mathcal{A}_{LF}(\mathcal{F}, g)$ the subset of $\Omega^1(\mathcal{F}) \otimes \mathfrak{g}$ consisting of one-forms which satisfy the above equation. This correspondence allows us to interpret parameter equivalence in terms of leafwise forms.

Proposition 11.1 (Matsumoto-Mitsumatsu [33]). *Let ρ_1 and ρ_2 be actions in $\mathcal{A}_{LF}(\mathcal{F}, G)$, and ω_1 and ω_2 be $\mathfrak{g}$-valued leafwise one-forms corresponding to actions. The actions ρ_1 and ρ_2 are parameter equivalent if and only if there exists a smooth map $b : M' \to G$ homotopic to the identity and an automorphism Θ of G such that*

$$\omega_2 = b^{-1} \cdot \Theta_*\omega_1 \cdot b + b^{-1} d_{\mathcal{F}} b, \tag{11.2}$$

where Θ_* is an automorphism of $\mathfrak{g}$ induced from Θ.

We denote the dual space of $\mathfrak{g}$ by $\mathfrak{g}^*$ and the cohomology of $\mathfrak{g}$ by $H^1_{Lie}(\mathfrak{g})$. Let ρ be an action in $\mathcal{A}_{LF}(\mathcal{F}, G)$ and ω_ρ be the leafwise one-form corresponding to ρ. We define a homomorphism $\iota_\rho : \mathfrak{g}^* \to \Omega^1(\mathcal{F})$ by $\iota_\rho(\alpha) = \alpha \circ \omega_\rho$. It is known that the map ι_ρ induces an injective homomorphism from $H^1_{Lie}(\mathfrak{g})$ to $H^1(\mathcal{F})$. In particular, the rank of $H^1(\mathcal{F})$ is not less than that of $H^1_{Lie}(\mathfrak{g})$ and they are equal if and only if $(\iota_\rho)_*$ is an isomorphism between $H^1_{Lie}(\mathfrak{g})$ and $H^1(\mathcal{F})$.

The first cohomology $H^1_{Lie}(\mathbb{R}^n)$ is isomorphic to $\mathbb{R}^n$. For $G = \mathbb{R}^n$, equations (11.1) and (11.2) are

$$d_{\mathcal{F}}\omega = 0, \ \omega_2 = \Theta_*\omega_1 + d_{\mathcal{F}} b.$$

If we ignore Θ_*, they are just equations defining the first cohomology $H^1(\mathcal{F}) \otimes \mathbb{R}^n$. These facts imply the following

Theorem 11.2 (Arraut-dos Santos [1], cf. Matsumoto-Mitsumatsu [33]). *An action $\rho \in \mathcal{A}_{LF}(M, \mathbb{R}^n)$ is parameter rigid if and only if the first cohomology $H^1(\mathcal{F})$ of the orbit foliation $\mathcal{F}$ is isomorphic to $\mathbb{R}^n$.*

Let A be an abelian subgroup of $SL_n\mathbb{R}$ defined in Section 6. As mentioned in Section 8, Katok and Spatzier proved the C^∞ stability of the orbit foliation $\mathcal{F}_A$ of the standard A-action ρ_A on $\Gamma\backslash SL_n\mathbb{R}$ for $n \geq 3$ (Theorem 8.2). They also proved the following result by using the fact that ρ_A is an Anosov action.

Theorem 11.3 (Katok-Spatzier [25])**.** *$H^1(\mathcal{F}_A)$ is isomorphic to $\mathbb{R}^{n-1}$ for $n \geq 3$. In particular, ρ_A is parameter rigid in this case.*

We remark that the theorem does not hold for $n = 2$ by an analog of Proposition 4.1. In the main part of the proof, we need the connectivity of $\mathbb{R}^{n-1} \setminus \{0\}$ for $n \geq 3$, which is false for $n = 2$. By the parameter rigidity of ρ_A and the C^∞ stability of $\mathcal{F}_A$, we obtain the following

Theorem 11.4. *The action ρ_A is C^∞ locally rigid if $n \geq 3$.*

Recently, Maruhashi established the relationship between parameter rigidity of actions of nilpotent groups and the first cohomology of the orbit foliation.[21]

Theorem 11.5 (Maruhashi [34])**.** *Let G be a nilpotent Lie group which is connected and simply connected, $\mathfrak{g}$ the Lie algebra of G, ρ a locally free action in $\mathcal{A}_{LF}(M, G)$, and $\mathcal{F}$ the orbit foliation of ρ. If $H^1(\mathcal{F})$ is isomorphic to $H^1_{Lie}(\mathfrak{g})$, then ρ is parameter rigid. Conversely, if ρ is parameter rigid and $H^0(\mathcal{F}) = \{0\}$, then $H^1(\mathcal{F})$ is isomorphic to $H^1_{Lie}(\mathfrak{g})$.*

12. Parameter deformation of the standard P-action on $\Gamma \setminus SL_n\mathbb{R}$

Fix $n \geq 2$. Let P be the parabolic subgroup of $SL_n\mathbb{R}$ defined in Section 6 and $\mathfrak{p}$ the Lie algebra of P. We also fix a uniform lattice Γ of $SL_n\mathbb{R}$. Let M_Γ be the quotient space $\Gamma\backslash SL_n\mathbb{R}$, ρ_P the standard P-action on M_Γ, and $\mathcal{F}_P$ the orbit foliation of ρ_P. For $n \geq 3$, it is not so hard to compute $H^1(\mathcal{F}_P)$ and show the parameter rigidity of ρ_P from Theorem 11.3.

Theorem 12.1 (e.g. [6, Theorem 4.13])**.** *If $n \geq 3$, then $H^1(\mathcal{F}_P)$ is isomorphic to $H^1_{Lie}(\mathfrak{p})$. Moreover, ρ_P is parameter rigid.*

Combined with the C^∞ stability of $\mathcal{F}_P$ (Theorem 8.2), we obtain the rigidity of the action ρ_P.

Theorem 12.2. *The standard P-action ρ_P is C^∞ locally rigid.*

We discuss the case $n = 2$. The cohomology group $H^1(\mathcal{F}_P)$ is also computed for this case. We denote the de Rham cohomology of M by $H^*_{dR}(M)$.

Theorem 12.3 (Matsumoto-Mitsumatsu [33], cf. Kanai [23])**.** *If $n = 2$, then $H^1(\mathcal{F}_P)$ is isomorphic to $H^1_{Lie}(\mathfrak{p}) \oplus H^1_{dR}(M_\Gamma)$.*

Ghys proved the following global rigidity theorem of the action ρ_P for the volume-preserving case.[22]

Theorem 12.4 (Ghys [15])**.** *Suppose $n = 2$. If a locally free P-action ρ on a closed three-dimensional manifold preserves a volume, then there exists a uniform lattice Γ_ρ of $SL_2\mathbb{R}$ such that the action ρ is C^∞-conjugate to the standard P-action on $\Gamma_\rho\backslash SL_2\mathbb{R}$.*

[21] For the case of the Heisenberg group, dos Santos [40] proved the first half of Theorem 11.5 in 2007.

[22] This is one of the earliest rigidity results for group actions.

Ghys also proved that any P-action must preserve a volume when the manifold satisfies a topological condition.

Theorem 12.5 (Ghys [15]). *If a closed three-dimensional manifold M satisfies $H^1_{dR}(M) = \{0\}$, then any locally free P-action on M preserves a volume. In particular, any locally free P-action on such a manifold is C^∞-conjugate to a standard one.*

It can be shown that any action ρ in $\mathcal{A}_{LF}(\mathcal{F}_P, P)$ which is C^∞-conjugate to ρ_P is parameter equivalent to ρ_P (cf. [3, Proposition 3.7]). Combined with this fact, we obtain the following

Corollary 12.6. *If $H^1_{dR}(M) = \{0\}$, then ρ_P is parameter rigid.*

By Theorem 12.3, the condition $H^1_{dR}(M) = \{0\}$ is equivalent to that $H^1(\mathcal{F}_P)$ is isomorphic to $H^1_{Lie}(\mathfrak{p})$. The quotient space $\Gamma\backslash SL_2\mathbb{R}$ satisfies the condition for several lattices Γ. However, there are many important lattices for which the condition does not hold. e.g., the case that M_Γ is diffeomorphic to the unit tangent bundle of a closed hyperbolic surface.

From Theorems 12.3 and 12.4, it is natural to expect that 'the dimension of the deformation space of ρ_P' is equal to the rank of $H^1_{dR}(M_\Gamma)$ when $H^1_{dR}(M_\Gamma)$ is non-trivial. In fact, we can compute the rank of 'the space of infinitesimal deformations' by taking formal linearizations of equations (11.1) and (11.2), and show that it is equal to the rank of $H^1_{dR}(M_\Gamma)$ (see [6]). However, it is very hard to recover information on deformation of the action from infinitesimal deformation because of difficulties coming from the non-linearity of equations (11.1) and (11.2).[23]

In [15], Ghys associated a cohomology class in $H^1_{dR}(M)$ with a P-action in $\mathcal{A}_{LF}(\mathcal{F}, P)$ and characterized parameter equivalence to the standard action ρ_P by the vanishing of the cohomology class. Let X, Y^s, Y^u be the basis of $\mathfrak{sl}_2\mathbb{R}$ given in (4.1). For a P-action ρ on $M_\Gamma = \Gamma\backslash SL_2\mathbb{R}$, we define a flow Φ_ρ on M_Γ by $\Phi^t_\rho(x) = \rho^{\exp(tX)}(x)$. For a flow Φ, let $\mathrm{Per}(\Phi)$ be the set of periodic points of Φ. For a periodic point x of Φ, let $\tau(x; \Phi)$ be its period and $C(x; \Phi)$ its orbit as an oriented closed curve.

Proposition 12.7 (Ghys [15], Asaoka [3]). *For each $\rho \in \mathcal{A}_{LF}(\mathcal{F}_P, P)$, there exist a closed one-form $\omega \in \Omega^1(M_\Gamma)$ and a constant $\lambda > 0$ such that*

$$\log\det(D\Phi_\rho^{\tau(x;\Phi_\rho)})_x = (1-\lambda)\tau(x;\Phi_\rho) - \int_{C(x;\Phi_\rho)} \omega \tag{12.1}$$

for any $x \in \mathrm{Per}(\Phi_\rho)$. The de Rham cohomology class $\alpha(\rho) = [\omega]$ and the constant $\lambda > 0$ are uniquely determined by Φ_ρ. Moreover, two actions ρ_1 and ρ_2 in $\mathcal{A}_{LF}(\mathcal{F}_P, P)$ are parameter equivalent if and only if $\alpha(\rho_1) = \alpha(\rho_2)$.

Recall that $\alpha(\rho) = 0$ if and only if ρ is parameter equivalent to ρ_Γ. The proposition reduces the problem of parameter deformation of ρ_P to describing the image of the map $\alpha : \mathcal{A}_{LF}(\mathcal{F}_P, P) \to H^1_{dR}(M_\Gamma)$. Until a few years ago, nothing had been known about the image of α for the case $H^1_{dR}(M)$ being non-trivial. The author of this article determined this completely as an application of 'Thermodynamical formalism for Anosov systems'.

[23] As a general case, we have no tame estimate so far. Moreover, we do not have enough information on $H^2(\mathcal{F})$, which we need if we mimic the deformation theory of complex manifolds.

Theorem 12.8 (Asaoka [3]). *Let Φ_Γ be an Anosov flow as defined in Section 6, i.e., $\Phi_\Gamma^t(\Gamma g) = \Gamma(g \exp(tX))$. The image of the map α is*

$$\Delta = \left\{ a \in H^1(M_\Gamma) \;\middle|\; \sup_{x \in \mathrm{Per}(\Phi_\Gamma)} \frac{|\langle a, C(x;\Phi_\Gamma)\rangle|}{\tau(x;\Phi_\Gamma)} < 1 \right\}.$$

This set is a convex subset of $H^1_{dR}(M_\Gamma)$ which contains the origin. In particular, if $H^1_{dR}(M_\Gamma)$ does not vanish, then ρ_P is not parameter rigid.

The linear space $H_1(M_\Gamma)$ admits the so-called stable norm. The linear space $H^1_{dR}(M_\Gamma)$ admits the operator norm of this norm. It can be checked that the above set Δ is the unit open ball for this norm. The set Δ can be regarded as the deformation space of an anti-de Sitter structure ($(SL_2\mathbb{R} \times SL_2\mathbb{R}, SL_2\mathbb{R})$-structure) on M_Γ (Salein [39]). It is not accidental. In fact, the first step of the construction of an action ρ with $\alpha(\rho) = a$ for any given cohomology class $a = [\omega] \in \Delta$ is to deform the Anosov flow Φ_Γ by the deformation of an anti-de Sitter structure associated with the class a. The obtained Anosov flow admits a smooth Anosov splitting $TM = E^{ss} \oplus E^c \oplus E^{uu}$ and it satisfies

$$\log\det D\Phi^{\tau(x;\Phi)}|_{E^{uu}(x)} = -\log\det D\Phi^{\tau(x;\Phi)}|_{E^{ss}(x)} = \tau(x;\Phi) - \int_{C(x;\Phi)} \omega$$

for any $x \in \mathrm{Per}(\Phi)$.

In the second step, we deform Φ along the strong stable foliation $\mathcal{F}^{ss}$ of Φ. Suppose that there exists a homeomorphism h which preserves each leaf of $\mathcal{F}^{ss}$ such that the flow $\Psi^t = h \circ \Phi^t \circ h^{-1}$ is smooth and $\det D\Psi^t|_{E^{ss}(x)} = e^{-\lambda t}$ for any $x \in M_\Gamma$ and $t \in \mathbb{R}$. In this case, it is easy to find a locally free P-action such that $\rho^{\exp(\lambda t X)} = \Psi^t$ and the orbit foliation of the flow $\rho^{\exp(tY^s)}$ coincides with $\mathcal{F}^{ss}$. We can also see that h is smooth on each leaf of the weak unstable foliation of Φ, and hence,

$$\log\det D\Psi^{\tau(x;\Psi)}|_{E^{uu}(x)} = \tau(x;\Psi) - \int_{C(x;\Psi)} \omega.$$

Combined with $\det D\Psi^t|_{E^s(x)} = e^{-\lambda t}$, we can obtain $\alpha(\rho) = [\omega] = a$. Hence, all we need to do is to construct the homeomorphism h as above.

The construction of h is done by using 'Thermodynamical formalism for Anosov systems'. For a given dynamical system, we can define a function called *entropy* on the set of invariant probability measures of the flow. It is known that entropy attains its maximum for any given Anosov flow and the maximum λ is positive. By disintegrating the measure maximizing the entropy along the foliation $\mathcal{F}^{ss}$, we can obtain the measure μ_x on each leaf $\mathcal{F}^{ss}(x)$. If we take a good disintegration, the family $(\mu_x)_{x\in M}$ of measures satisfies

$$\frac{d(\mu_{\Phi^t(x)} \circ \Phi^t)}{d\mu_x} = e^{-\lambda t}$$

for any $x \in M_\Gamma$ and $t \geq 0$.[24] We can define the desired homomorphism h by "remeasureing the length on each leaf of $\mathcal{F}^{ss}(x)$ by the measure μ_x". It is non-trivial that the flow $\Psi^t = h \circ \Phi^t \circ h^{-1}$ is smooth. In our setting, the smoothness of Ψ follows from a (trivial) fact:

constant functions are smooth for any smooth structure of a manifold.

[24] We call the family $(\mu_x)_{x\in M_\Gamma}$ *the Margulis measure.*

There are analogous results for other real-rank one Lie groups. By $SO_+(n,1)$, we denote the connected component of $SO(n,1)$ containing the unit. Let P be the Borel subgroup of $SO_+(n,1)$ and Γ a uniform $SO_+(n,1)$. As before, set $M_\Gamma = \Gamma\backslash SO_+(n,1)$. Let ρ_P be the standard P-action on M_Γ and $\mathcal{F}$ the orbit foliation of ρ_P. For $n=2$, the action ρ_P is just the standard P-action on $\Gamma\backslash SL_2\mathbb{R}$ we discussed above.

Theorem 12.9 (Kanai [23]). *$H^1(\mathcal{F}_P)$ is isomorphic to $H^1_{Lie}(\mathfrak{so}(n,1)) \oplus H^1_{dR}(M_\Gamma)$.*

Similar to the $SL_2\mathbb{R}$ case, we expect the action admits non-trivial parameter deformation when $H^1_{dR}(M_\Gamma)$ is non-trivial. However, this is not true for $n \geq 3$ at least locally.

Theorem 12.10 (Asaoka [5]). *For $n \geq 3$, ρ_P is always C^∞ locally rigid.*[25]

This is proved by using de la Llave's rigidity result [29] on conformal Anosov flows.

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