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ON TOPOLOGICAL PROPERTIES OF FATOU SETS AND JULIA SETS OF TRANSCENDENTAL ENTIRE FUNCTIONS

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Contents

1. Introduction

The theory of complex dynamical systems was founded by Fatou and Julia around 1920. Since 1980, after a stagnation period, it has been drawing the attention of people again together with the development of computer graphics. In particular, the theory of dynamics of polynomials and rational functions has been explosively developed by adopting classical theories of complex analysis, for example, the theory of quasiconformal maps and the Teichmüller theory. On the other hand, as for the theory of dynamics of transcendental entire functions, it has been magnificently developed after the celebrated paper by Fatou ([Fat]) by the applications of the above theories and, in addition, the theory of covering surfaces by Ahlfors, the value distribution theory (Nevanlinna theory), and the Wiman-Varilon theory. Also it has shed new light on these classical theories. There is a huge

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variety of research on transcendental dynamics, and in this paper we focus on topological properties of various invariant sets, like Fatou sets and Julia sets and present known results, historical remarks, subjects for future research and some open problems which are related to these topics. As most of the readers are non-experts of the theory of complex dynamics, we shall also explain basic notions and results clearly and plainly.

Let f be an entire function, that is, a holomorphic function on the whole complex plane $\mathbb{C}$ and f^n $(n \in \mathbb{N} \cup \{0\})$ be the n-th iterate of f. That is,

$$f^n(z) = \overbrace{f \circ f \circ \cdots \circ f}^{n \text{ times}}(z) \quad (f^0 := \mathrm{Id} \;\text{ i.e. }\; f^0(z) := z).$$

Throughout this paper we assume that $f(z)$ is not of the form $az + b$.[1]

The most fundamental and important invariant sets (see Proposition 1.2) for the the theory of complex dynamics are the **Fatou set** $F(f)$ and the **Julia set** $J(f)$,[2] which are defined as follows:

$$F(f) := \{z \in \mathbb{C} \mid {}^{\exists}\text{a neighborhood } U \text{ of } z \text{ s.t. } \{f^n|_U\}_{n=1}^{\infty} \text{ is a normal family}\},$$
$$J(f) := \mathbb{C} \setminus F(f).$$

In general, the family $\mathcal{F}$ which consists of meromorphic functions on a domain $U \subset \mathbb{C}$ (that is, holomorphic maps from U to the Riemann sphere $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$) is called a **normal family** if any sequence from $\mathcal{F}$ has a subsequence which converges locally uniformly on U. Here we adopt the spherical metric as the metric on $\widehat{\mathbb{C}}$. So if f is entire then we have $f : \mathbb{C} \to \mathbb{C}$ and hence we can rephrase the definition by using the topology of $\mathbb{C}$ as follows: $\{f^n|_U\}_{n=1}^{\infty}$ is a normal family if any sequence from $\{f^n|_U\}_{n=1}^{\infty}$ has a subsequence which converges locally uniformly on U or diverges to ∞ locally uniformly on U. This notion was introduced by Montel ([T, p. 279]). The following Montel's Theorem ([Bea, p. 57, Theorem 3.3.4]) is often used to decide whether a family $\mathcal{F}$ is normal or not. This is the fundamental criterion which is the starting point of the theory of complex dynamics by Fatou and Julia.

[1] $f(z) = az + b$ is a constant function when $a = 0$, and in particular $f^{-1}(z) = \emptyset$ for any $z \neq b$. When $a \neq 0$, it is a linear function and in particular it has a globally defined inverse function on $\mathbb{C}$. So in this aspect, $f(z) = az + b$ is markedly different from other entire functions and we have to treat it separately. But actually we can understand the dynamics of this function completely and we can find no dynamically interesting phenomena. See also Footnote 9.

[2] The term "Julia set" seems to have been used since the 1980s, when complex dynamics began to draw attention again. On the other hand, the term "Fatou set" began to be used a little bit later and it seems that it first appeared in the paper [Bl, p. 90] by Blanchard. ([Bea, p. 50]) (Before this paper in the early 1980s, there were several papers which used "Fatou set" to describe what we call the Julia set nowadays.) There are a lot of cases where some terms which are different from these terms were used in the papers before the 1990s. For example, the Julia set used to be called "the set of non-normality" ([Ba5, p. 277]) or "Fatou-Julia set"([BaRi, p. 49], [Ba8, p. 563]) and the Fatou set used to be called "the set of normality" ([Ba8, p. 563], [EL1, p. 458]) or "the stable set" ([De1, p. 237, Definition 2.2]). Julia denoted the Julia set by E' as the derived set of E, where E is the set of all repelling periodic points ([J, pp. 83, 95]; see Theorem 1.4), and Fatou denoted the Julia set by "F" ([Fat, p. 347]). In particular, among the old papers there are a lot of them which followed the Fatou's notation and denoted the Julia set by "$F(f)$" (or "$\mathfrak{F}(f)$") which we use to denote the Fatou set nowadays (for example, the papers by Baker before 1984). So we should be careful. It seems that Fatou and Julia did not use any specific symbol for the Fatou set. In the papers before 1990 the Fatou set was denoted by, for example, "$E(f)$" (or "$\mathfrak{E}(f)$") ([Ba5, p. 277], [Ba6, p. 173]) or "$N(f)$" (or "$\mathcal{N}(f)$") ([Ba8, p. 563], [EL1, p. 458]).

Theorem 1.1 (Montel). *Let $\mathcal{F}$ be a family of meromorphic functions on a domain U. If there exist three distinct points $p,\ q,\ r \in \widehat{\mathbb{C}}$ such that $f(z) \neq p, q, r$ for every $f \in \mathcal{F}$ and $z \in U$, then $\mathcal{F}$ is a normal family.* □

$F(f)$ and $J(f)$ are invariant sets in the following sense. Here in general $X \subset \mathbb{C}$ is called **forward invariant** if $f(X) \subseteq X$ and **backward invariant** if $f^{-1}(X) := \{y \mid f(y) \in X\} \subseteq X$. Also X is called **completely invariant** if X is both forward and backward invariant.

Proposition 1.2. *(1) $F(f)$ and $J(f)$ are completely invariant. More precisely,*[3]

$$f(F(f)) \subseteq F(f),\ f^{-1}(F(f)) = F(f), \quad f(J(f)) \subseteq J(f),\ f^{-1}(J(f)) = J(f).$$

(2) $F(f^n) = F(f), \quad J(f^n) = J(f), \quad (n \in \mathbb{N}).$ □

From Proposition 1.2 (1), it follows that in order to investigate the dynamics of f we may just consider $f|_{F(f)} : F(f) \to F(f)$ and $f|_{J(f)} : J(f) \to J(f)$ individually. $F(f)$ is the set of all initial points z_0 where the **forward orbit** $O^+(z_0) := \{f^n(z_0)\}_{n=0}^{\infty}$ behaves "tamely". In other words, if $z_0 \in F(f)$ and z_0' is sufficiently close to z_0, then $O^+(z_0')$ behaves similarly as $O^+(z_0)$. On the other hand, $J(f)$ is the set of all initial points z_0 where $O^+(z_0)$ shows so-called "chaotic" behavior. In other words, if $z_0 \in J(f)$, then there exists a point z_0' which is arbitrarily close to z_0 but the behavior of $O^+(z_0)$ and $O^+(z_0')$ are completely different (we say f has **sensitive dependence on initial conditions** at z_0). For example if f is entire, then by Theorem 1.1 we have

$$\bigcup_{n=0}^{\infty} f^n(U) = \mathbb{C} \ \text{ or } \ \mathbb{C} \setminus \{\text{one point}\} \tag{1.1}$$

for every $z_0 \in J(f)$ and its neighborhood U. This is one of the descriptions of sensitive dependence on initial conditions. As we show in Section 2, in fact we can understand the behavior of $O^+(z_0)$ almost completely for $z_0 \in F(f)$. So it is the point in $J(f)$ that shows interesting dynamical behaviors.

The **backward orbit** $O^-(z_0)$ of z_0 is defined by

$$O^-(z_0) := \bigcup_{n=0}^{\infty} \{z \mid f^n(z) = z_0\} = \bigcup_{n=0}^{\infty} f^{-n}(z_0).$$

z_0 is called an **exceptional point** if $O^-(z_0)$ is a finite set. By Picard's theorem transcendental entire function f has at most one exceptional point and if z_0 is exceptional, then it is easy to show that $f(z)$ has the form $f(z) = z_0 + (z - z_0)^m e^{g(z)}$ ($m \in \mathbb{N} \cup \{0\}$, $g(z)$ is entire). The next is one of the characterizations of $J(f)$ and it is also an aspect of sensitive dependence on initial conditions.

Proposition 1.3. *If z_0 is not an exceptional point, then $J(f) \subseteq \overline{O^-(z_0)}$. Moreover, if $z_0 \in J(f)$, then $J(f) = \overline{O^-(z_0)}$.* □

[∵ Let U be any open set with $U \cap J(f) \neq \emptyset$. Then by (1.1), we have $z_0 \in f^{n_0}(U)$ for some n_0.[4] This shows the former assertion and the latter follows by complete invariance of $J(f)$].

[3] Strict inclusion holds in (1) for transcendental entire functions with a finite Picard exceptional value like λe^z, but the difference is only one point by Picard's theorem. Equality always holds for functions without Picard exceptional values.

[4] When we have $\mathbb{C} \setminus \{\text{one point}\}$ in (1.1), then the "one point" is actually an exceptional point.

Periodic points are the most fundamental and important initial points in the theory of dynamical systems. A point z_0 is a **periodic point** if $f^{n_0}(z_0) = z_0$ and the minimal such $n_0 \in \mathbb{N}$ is called the **period** of z_0. In particular when $n_0 = 1$, z_0 is called a **fixed point**. $\mu := (f^{n_0})'(z_0)$ is called the **multiplier** of a periodic point z_0 and z_0 is called **attracting**, **repelling** or **indifferent** according to $|\mu| < 1$, $|\mu| > 1$ or $|\mu| = 1$ (in particular if $\mu = 0$, it is called **super-attracting**). If $|\mu| \neq 0,\ 1$, then f^{n_0} is linearizable in a neighborhood of z_0, that is, there exists an analytic isomorphism φ from a neighborhood of z_0 to a neighborhood of the origin 0 with $\varphi \circ f^{n_0} \circ \varphi^{-1}(z) = \mu z$. If $\mu = 0$, then $\varphi \circ f^{n_0} \circ \varphi^{-1}(z) = z^k$ $(2 \leq {}^{\exists}k \in \mathbb{N})$ holds for a similar φ. Then it follows that (super-)attracting periodic points belong to $F(f)$ and repelling periodic points belong to $J(f)$. If $|\mu| = 1$, then z_0 is called **parabolic** (or **rationally indifferent**) or **irrationally indifferent** according to whether μ is a root of unity or not. This corresponds to $\theta \in \mathbb{Q}$ or $\theta \in \mathbb{R} \setminus \mathbb{Q}$ if we write $\mu = e^{2\pi i\theta}$ $(\theta \in \mathbb{R})$. The former belongs to $J(f)$ but it depends on the cases of whether the latter belongs to $F(f)$ or $J(f)$.[5] The following result is a famous characterization of $J(f)$ ([Ba2, p. 253, Theorem 1]). This is a fundamental result which was proved by Fatou and Julia independently in the case of rational functions. In the case of transcendental entire functions, Baker first proved it by using "Ahlfors Five Islands Theorem", which is a main result of the theory of covering surface by Ahlfors. This is also one of the descriptions of the fact that $f|_{J(f)}$ is chaotic.[6]

Theorem 1.4 (Baker, 1968)**.** *$J(f)$ coincides with the closure of the set of all repelling periodic points of f.* □

Example 1.5. Let $f(z) := z^2$; then $f^n(z) = z^{2^n}$ and 0 and ∞ are super-attracting fixed points and

(1) $f^n(z) \to 0$ $(n \to \infty)$ if $0 \leq |z| < 1$ and this convergence is locally uniform on $\{z \mid 0 \leq |z| < 1\}$.
(2) $f^n(z) \to \infty$ $(n \to \infty)$ if $|z| > 1$ and this convergence is locally uniform on $\{z \mid |z| > 1\}$.
(3) If $|z| = 1$, then $U \cap \{z \mid 0 \leq |z| < 1\} \neq \emptyset$, $U \cap \{z \mid |z| > 1\} \neq \emptyset$ for any neighborhood U of z and no subsequence of $\{f^n|_U\}$ contains a subsequence which converges locally uniformly by (1) and (2).

It follows that $J(f) = \{z \mid |z| = 1\}$. Also it is easy to see that (1.1) with $\mathbb{C}\setminus\{0\}$ holds in this case.[7] Moreover, one can show by direct calculation that all the periodic points z_0 other than 0 and ∞ are repelling and satisfy $|z_0| = 1$ and they are dense in $J(f)$ (i.e., Theorem 1.4 holds for this f). □

By definition, $F(f)$ is open and hence its complement $J(f)$ is closed. In general, it may happen that $F(f) = \emptyset$ but $J(f)$ is always non-empty,[8] and $J(f)$ has an empty

[5] An irrationally indifferent periodic point z_0 belongs to $F(f)$ if f^{n_0} is linearizable in a neighborhood of z_0, and to $J(f)$ if f^{n_0} is not linearizable. z_0 is called a Siegel point or a Cremer point, accordingly. Let $\mu = e^{2\pi i\theta}$ $(\theta \in \mathbb{R})$ be the multiplier of z_0; then f^{n_0} is linearizable for Lebesgue almost all θ, that is, z_0 is a Siegel point ([Bea, p. 134, Theorems 6.6.4, 6.6.5]).

[6] The property of "existence of dense periodic points" is one of the conditions in the definition of the so-called "Devaney's chaos" ([De1, p. 43, Definition 8.5]).

[7] Of course, strictly speaking, this is the case when U is a sufficiently small neighborhood of $z \in J(f)$ with $0 \notin U$. If U is so large that $0 \in U$, then the right hand side of (1.1) becomes $\mathbb{C}$.

[8] If f is a polynomial and $J(f) = \emptyset$, then there exists a subsequence $\{f^{n_k}\}_{k=1}^{\infty}$ of $\{f^n\}_{n=1}^{\infty}$ which converges locally uniformly on $\widehat{\mathbb{C}}$. Let $f^{n_k} \to \varphi$; then φ is also a polynomial. But on

interior unless $J(f) = \mathbb{C}$. [$\because$ Suppose that $J(f)$ contains an open set U; then by the invariance of $J(f)$ (Proposition 1.2 (1)) and Montel's Theorem it follows that $J(f) \supset \bigcup_{n=0}^{\infty} f^n(U) = \mathbb{C}$ or $\mathbb{C} \setminus \{\text{one point}\}$. On the other hand, $J(f)$ is closed and hence $J(f) = \mathbb{C}$]. Also $J(f)$ is a **perfect set** (i.e., it has no isolated points. Hence in particular, it contains uncountably many points).[9] If f is a polynomial, $J(f)$ is a compact set in $\mathbb{C}$. Moreover, define the **escaping set** $I(f)$ of f by

$$I(f) := \{z \in \mathbb{C} \mid f^n(z) \to \infty\}.$$

Then $I(f)$ is the (immediate) attractive basin of ∞, which is a super-attracting fixed point, completely invariant by definition, and it holds that

$$I(f) \neq \emptyset, \quad I(f) \subset F(f), \quad \partial I(f) = J(f). \tag{1.2}$$

By using this fact, one can fairly easily draw the Julia sets of polynomials by computer graphics. With a very few exceptions like the above Example 1.5, they are what is called "fractal sets".[10] A lot of readers may have seen or drawn such graphics. On the other hand, in the case of transcendental entire functions, which we handle mainly in this paper, $J(f)$ is a non-compact closed set in $\mathbb{C}$ and in particular it is unbounded, since ∞ is an essential singularity of f. But in fact it is known that $\partial I(f) = J(f)$ holds also in this case but we always have $I(f) \cap J(f) \neq \emptyset$ (see [E, p. 343, Theorem 2]; see also Theorem 7.1).

Now let us consider the Julia set of an exponential function

$$E_\lambda(z) := \lambda e^z \ (\lambda \in \mathbb{C} \setminus \{0\}),$$

which is probably the simplest transcendental entire function. Then it turns out that its properties are quite different compared with the case of polynomials. For example, the Julia set of e^z, that is, when $\lambda = 1$, is the whole complex plane $\mathbb{C}$ ([Mis, 1981]. In the case of polynomials, since $F(f) \neq \emptyset$ by (1.2), $J(f) = \mathbb{C}$ never occurs.[11] This fact was conjectured by Fatou in 1926 ([Fat, p. 370]) and it had been unsolved for more than 50 years. When $\lambda = 1$, we have $0 \in I(E_\lambda)$, and in general if $0 \in I(E_\lambda)$, that is, $E_\lambda^n(0) \to \infty$ $(n \to \infty)$ (for example, if $\lambda > 1/e$), we have $J(E_\lambda) = \mathbb{C}$ ([BaRi, p. 74, Corollary 1]).[12] When $\lambda > 1/e$, we have $J(E_\lambda) = \mathbb{C}$ and we can define an itinerary of $z \in \mathbb{C}$ by dividing $\mathbb{C}$ into countably many horizontal strips with width 2π. Then it is known that a set of all points with a certain itinerary

the other hand, since $\deg f \geq 2$, we have $\deg f^{n_k} = (\deg f)^{n_k} \to \infty$, which is a contradiction. Unexpectedly, proving this fact is more difficult in the case of transcendental entire functions. Fatou first proved this fact ([Fat]) and also a proof by using Nevanlinna theory ([Ber1, p. 160]) and a more elementary proof without Nevanlinna theory ([Barg1]) are known.

[9] Let $f(z) = az + b$ and if $a \neq 0$, then f is conjugate to either (i) $f_1(z) = az$, or (ii) $f_2(z) = z + c$, according to whether f has a fixed point in $\mathbb{C}$ or not. In the case (i), we have $J(f) = \{0\}$, $\emptyset$, $\{\infty\}$ according to $|a| > 1$, $|a| = 1$, $|a| < 1$. In the case (ii), we have $J(f) = \{\infty\}$. If $a = 0$, then since $f^n(z) \equiv b$, we have $J(f) = \emptyset$. Hence in particular $J(f)$ is not perfect in every case.

[10] When f is a polynomial, $J(f)$ becomes a manifold only when $f(z)$ is conjugate to (i) az^d $(a \neq 0,\ d \geq 2)$, or (ii) a Chebychev polynomial. In this case $J(f)$ is a circle or a closed segment, respectively.

[11] The first example of a transcendental entire function f with $J(f) = \mathbb{C}$ is $f(z) = kze^z$ (k is a certain real with $k > 3e$) by Baker in 1970 ([Ba3, p. 4, Theorem 1, p. 8, Lemma 5, Lemma 6]).

[12] We can understand this fact also by the theorem of non-existence of wandering domains for a function in class $\mathcal{S}$ ([EL3, p. 1004, Theorem 3], [GKe, p. 184, Theorem 4.2]) or by using Theorem 2.8.

becomes an **indecomposable continuum**[13] (i.e., it is a continuum, that is, a connected compact set with more than one point, and which cannot be represented by a union of two connected compact proper subsets) ([DeJ, p. 7, Theorem 5.1]).

When $0 < \lambda < 1/e$, then $J(E_\lambda)$ is the set in Figure 1 (Remark: All figures in this paper are drawn by Professor Shunsuke Morosawa at Kochi University). The darkest part in Figure 1 is $J(E_\lambda)$ (where $\lambda = 0.3$). Since $J(E_\lambda) \neq \mathbb{C}$ in this case, $J(E_\lambda)$ has an empty interior but it looks like it has a non-empty interior in Figure 1. This is due to the difficulty in calculation by computer and so Figure 1 is an approximate picture of $J(E_\lambda)$.[14] The following properties are known for these sets:

(a) $I(E_\lambda) \subset J(E_\lambda)$, hence in particular, $J(E_\lambda) = \overline{I(E_\lambda)}$.

(b) $F(E_\lambda)$ coincides with the immediate attractive basin of a unique attracting fixed point and it is connected and simply connected.

(c) $J(E_\lambda) \cup \{\infty\} \subset \widehat{\mathbb{C}}$ is connected but $J(E_\lambda) \subset \mathbb{C}$ is disconnected and has uncountably many connected components (see Theorems 3.5, 3.13, 3.16, 3.17).

(d) $J(E_\lambda)$ is not locally connected at every point (see Theorem 4.6).

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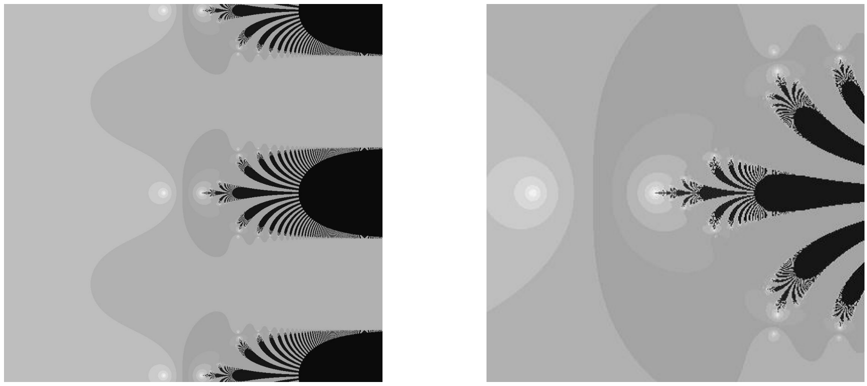
[13] Geometrical objects you can easily think of are probably not indecomposable continua. A typical example is the boundary of a "lake of Wada" ([Y, pp. 60–61]). Also the following "Knaster-like continuum" is such an example ([De3, p. 628]): Consider the Cantor middle-thirds set C in the interval $[0, 1]$ in x-axis in $\mathbb{R}^2$. Then connect two points in the Cantor set which are located symmetrically with respect to $x = 1/2$ by the semi-circle centered at $x = 1/2$ in the upper-half plane. Then consider the points in the right half of C (i.e., in $[2/3, 1] \cap C$) and connect two points among them which are located symmetrically with respect to $x = 5/6$ by the semi-circle centered at $x = 5/6$ in the "lower"-half plane. Next consider the points in the "right half" of the rest (i.e., it is the right half of $[0, 1/3] \cap C$, so it is $[2/9, 1/3] \cap C$) and similarly connect two points among them which are located symmetrically with respect to $x = 5/18$ by the semi-circle centered at $x = 5/18$ in the lower-half plane. By repeating this procedure infinitely many times, what you can get is the Knaster-like continuum, which is indecomposable.

[14] If f is a polynomial, we can find a constant $M > 0$ depending on f which satisfies

$$(**) \qquad |z| \geq M \implies f^n(z) \to \infty.$$

So when we draw $I(f)$, we first set the maximal number of iteration $k_0 \in \mathbb{N}$ and compute $f(z),\ f^2(z),\ \cdots$. Then if $|f^k(z)| \geq M$ for some $k \leq k_0$, then $z \in I(f)$ and display the point z by some color according to the minimal number of such k. If there is no such $k \leq k_0$, then we display z by black. With this method, the larger we set the number k_0, the more accurate picture of $I(f)$ we can draw. In the transcendental case, however, there is no constant $M > 0$ which satisfies the above $(**)$. So we have to change the method according to the function f. For example, for a function f in class $\mathcal{S}$ (see Definition 2.9), which includes exponential functions and sine functions $\lambda \sin z$, with finite order, there is the following method: Let D be a disk centered at origin with $\operatorname{sing}(f^{-1}) \subset D$ (see Definition 2.1). Then $f^{-1}(\mathbb{C} \setminus D)$ consists of finitely many mutually disjoint unbounded domains. We call each domain a logarithmic tract. For example, for $E_\lambda(z)$ this is the single domain $U_1 := \{z \mid \operatorname{Re} z \geq M\}$, and for $\lambda \sin z$ this consists of two disjoint domains which contains the connected components of $U_2 := \{z \mid |\operatorname{Im} z| \geq M\}$. Then set a large $M > 0$ and compute $f(z),\ f^2(z), \cdots$. If there is a k with $f^k(z) \in U_i$, we regard it as $z \in I(f)$. But in general it happens that $f^k(z) \in U_i$ for some k but the orbit of z returns to D after some more iterates, so actually we cannot always tell $z \in I(f)$. So if we draw pictures with this method, some parts appear like a domain with narrow width each of which is, in reality, a curve like a hair. Professor Morosawa kindly let me know the above facts.



(i) $\{z = x+iy | -5 \le x \le 8,\ -6.5 \le y \le 6.5\}$ (ii) $\{z = x + iy \mid 0 \le x \le 4,\ -2 \le y \le 2\}$

FIGURE 1. $J(E_{0.3})$. (ii) is an extension of the center part of (i).

(e) $J(E_\lambda)$ is a union of uncountably many C^∞-curves $h(t) : [0, \infty) \to \mathbb{C}$ with $h(t) \to \infty$ $(t \to \infty)$ and $h((0, \infty)) \subset I(E_\lambda)$ (a **hair**[15]) (it is called a "**Cantor bouquet**") ([DeK], [V]).

(f) Let A_λ be the set of all points $h(0)$, that is, the **end points**[16] of hairs; then $A_\lambda \subset \mathbb{C}$ is **totally disconnected** (i.e., all of its connected components consist of one point), but $A_\lambda \cup \{\infty\} \subset \widehat{\mathbb{C}}$ is connected (i.e., ∞ is an **explosion point**[17] of $A_\lambda \cup \{\infty\} \subset \widehat{\mathbb{C}}$) ([Ma, p. 182, Theorem 3]).

(g) $\dim_H(J(E_\lambda)) = 2$ but $\mathrm{Leb}(J(E_\lambda)) = 0$ ($\dim_H(X)$ and $\mathrm{Leb}(X)$ are the Hausdorff dimension and the Lebesgue measure of X, respectively) ([EL3, p. 1010, Theorem 8], [Mc, p. 329, Theorem 1.2, 1.3]).

(h) The set $J(E_\lambda) \setminus A_\lambda$, that is, the set of all hairs without end points, which seems to occupy almost all of $J(E_\lambda)$, satisfies $\dim_H(J(E_\lambda) \setminus A_\lambda) = 1$. But on the other hand we have $\dim_H(A_\lambda) = 2$ ([Ka1, p. 1041, Theorem 1.1], [Ka2, p. 270, Theorem 1]). This phenomenon is called "dimension paradox".

A set with an explosion point, which is considered to be pathological, has been constructed artificially ([Ma, p. 181]), but it is extremely interesting that such a set arises as the Julia set of an exponential function, which is an invariant set defined naturally in the theory of complex dynamics.

It is possible to consider $J(f)$, which is an unbounded closed set in $\mathbb{C}$ when f is transcendental, as a closed subset of $\widehat{\mathbb{C}}$ by adding ∞. This definition is adopted in the case of transcendental meromorphic functions ([Ber1, p. 154]) and also when one considers convergence problems of Julia sets ([Ki1], [Kr], [KrK]). If you adopt this definition, then $J(f)$ becomes a compact subset of $\widehat{\mathbb{C}}$ and is easier to handle as a topological space (see Section 3.1). But natural phase space for a dynamical

[15] $h(t)$ is called a hair (by Devaney) and also a Devaney hair (by Fagella, [DoFag], [ReRiSt]) or a dynamic ray (by Schleicher, [Sch], [SchZ]).

[16] This is called an "end point" also when $h(t)$ is called a dynamic ray. We think it should be called a "hair root" when one calls $h(t)$ a "hair".

[17] Let X be a topological space. Then $p \in X$ is called an explosion point if X is connected but $X \setminus \{p\}$ is totally disconnected.

system f is the complex plane $\mathbb{C}$, not the Riemann sphere $\widehat{\mathbb{C}}$. It is because it is impossible to extend f naturally to a dynamical system on $\widehat{\mathbb{C}}$ like polynomials (or rational functions), since ∞ is an essential singularity of f. So in this paper we consider $J(f) \subset \mathbb{C}$ as in the first definition, when f is transcendental. In what follows, we occasionally consider polynomial cases to compare with transcendental cases. In that case note that $F(f)$ and $J(f)$ are subsets of $\widehat{\mathbb{C}}$ and $\infty \in F(f)$.

In this paper we concentrate on the Fatou set $F(f)$, the Julia set $J(f)$, the escaping set $I(f)$ and the fast escaping set $A(f)$ which is defined in Section 7, and explain known results, historical remarks, future works and some open problems which are related to these topics. First in Section 2 we explain fundamental notions in the theory of complex dynamics and their properties. We discuss the connectedness of $J(f) \cup \{\infty\} \subset \widehat{\mathbb{C}}$ and $J(f) \subset \mathbb{C}$ in Section 3 and local connectedness of $J(f)$ in Section 4. Next we investigate topological properties of Fatou components in Section 5 and of Julia components in Section 6. In the last Section 7 we explain various properties of $I(f)$ and $A(f)$ which have been revealed recently. We put a table of symbols before the references.

2. Preliminaries

In this section we briefly explain some notions and their fundamental properties which are needed to state results. For the proofs and their further explanation, see [Bea] and [Mil2] for polynomial or rational dynamics, and [Ber1] and [MNTU] for transcendental entire or meromorphic dynamics. Since most of the contents in this section are fundamental, readers who are familiar with the basic theory of complex dynamics can skip this section and go on to the next section.

An important key to understand the dynamics of f is the behavior of the singular values of f, which are defined as follows:

Definition 2.1. Let c be a **critical point** of f, that is, $f'(c) = 0$. Then the forward image $f(c)$ of c is called a **critical value** of f. If there exists a curve $L(t) \subset \mathbb{C}$ $(0 \leq t < \infty)$ with $\lim_{t\to\infty} L(t) = \infty$ and $\lim_{t\to\infty} f(L(t)) = p \in \widehat{\mathbb{C}}$, then we call the point p an **asymptotic value** of f. A point is called a **singular value** if it is either a critical value or an asymptotic value or their accumulation point and we denote the set of all singular values by $\mathrm{sing}(f^{-1})$. This is the set of all points where we cannot define some branches of f^{-1} in their every small neighborhood. We call

$$P(f) := \overline{\bigcup_{n=0}^{\infty} f^n(\mathrm{sing}(f^{-1}))}$$

the **post-singular set** of f.[18] □

[18]In Japanese "forward" and "backward" are usually translated as "zenpou" and "kouhou". For example, "forward orbit" is translated as "zenpou kidou". So at first we thought $P(f)$ should be called "zenpou tokui shuugou" in Japanese, since it is the closure of the set of all the "forward" orbits of singular values. But on the other hand, "pre-" and "post-" are usually translated as "zen-" and "kou- (or go-)". This is because we call the past "zen" and the future "kou (or go)" on the basis of the present. There is a word, for example, "kon-go", which means "from now on" or "in the future". From this point of view, $P(f)$ should be called "kou (or go) tokui shuugou". In this way, the translated term contains "zen" or "kou", which have completely opposite meanings, depending on how you think. As a result of much thought, we translated it as "posuto tokui shuugou". How difficult it is to translate English into Japanese! Incidentally, note that some

If f is transcendental entire, ∞ is always an asymptotic value. But since ∞ is not a point in the phase space $\mathbb{C}$, what is important for transcendental dynamics is the asymptotic value p in $\mathbb{C}$, that is, the finite asymptotic values. Since singular values are closely related to the Fatou components of f, which we explain later, we can understand the dynamics of f by investigating their behaviors. We show the details later in Theorem 2.7.

Example 2.2. (1) $E_\lambda(z) = \lambda e^z$ has no critical values, since $E'_\lambda(z) \neq 0$. But let $L(t) := -t$ $(0 \leq t < \infty)$; then we have $E_\lambda(L(t)) \to 0$ $(t \to \infty)$. Thus 0 is an asymptotic value and this is the unique finite asymptotic value of $E_\lambda(z)$.

(2) The origin 0 is a critical value of $f(z) = z^2 e^z$, since $f(0) = f'(0) = 0$. It is also an asymptotic value, since $f(-t) \to 0$ $(t \in \mathbb{R},\ t \to \infty)$. □

We can understand the dynamics of f on $F(f)$ almost completely, as we see below. We call a connected component U of $F(f)$ a **Fatou component** of f. Then $f(U)$ is either a Fatou component again or a Fatou component with a single deleted point ([BerR, p. 1857, Theorem], [He, p. 264]). So we classify them as follows from the viewpoint of how they behave under the iterate of f:

Definition 2.3. For a Fatou component U, let U_n be the Fatou component which contains $f^n(U)$.

(1) U is called a **wandering domain** if $U_m \cap U_n = \emptyset$ for every $m, n \in \mathbb{N}$ $(m \neq n)$.

(2) U is called a **periodic component** of period n_0 if there exists an $n_0 \in \mathbb{N}$ with $U_{n_0} \subseteq U$, where n_0 is the minimum of n with $U_n \subseteq U$. In particular, it is called an **invariant component** if $n_0 = 1$.

(3) U is called a **preperiodic component** if U_m is a periodic component for some $m \in \mathbb{N}$. U is called an **eventually periodic component** if it is either a periodic or a preperiodic component. □

While there exist no wandering domains if f is a polynomial (or rational, in general) (Sullivan's No Wandering Domain Theorem [Su, p. 404, Theorem 1]), there are a lot of examples of transcendental entire functions with wandering domains (see, for example, [Ba1], [Ba6], [Ba8], [Ba9], [BaDo2], [Ber1], [BerZ1], [Do], [EL1], [EL2], [KiSh1]). According to the above classification of Fatou components, we can understand the dynamics on $F(f)$ completely, once we can understand the dynamics on each periodic Fatou component. The following holds for the dynamics on periodic Fatou components:

Theorem 2.4 (Classification Theorem of Periodic Components)**.** *Let f be transcendental and U a periodic Fatou component. Then U is either one of the following 4 cases*[19] *(where n_0 is the period of U):*

1. *There exists an attracting periodic point $z_0 \in U$ of period n_0 and $f^{n_0 k}(z) \to z_0$ $(k \to \infty)$ holds for every $z \in U$. U is called an* **immediate attractive basin** *of z_0.*

authors define the post-singular set as the set without the closure in Definition 2.1 (for example, see [Bara, p. 34]).

[19] Here we use terms which are used most commonly now, but there are also the following terms for (1), (2) and (4): (1) Böttcher domain if z_0 is super-attracting, Schröder domain if it is not ([EL2, p. 579]), (2) Leau domain ([EL2, p. 581]), (4) le domaine d'attraction du point $z = \infty$ ([Fat, p. 359]), domain at infinity ([BaKoL, p. 606]). The term "Baker domain" is comparably new. It was first used by Eremenko and Lyubich in 1990 ([EL2, p. 621]) and it has been gradually established since then. Incidentally when the author visited Professor Baker and had discussions with him in March 2000, he himself used the term "Baker domain", which was very impressive.

(2) *There exists a point $z_0 \in \partial U$ with $f^{n_0}(z_0) = z_0$ (Remark: it may happen that $f^{n_1}(z_0) = z_0$ for some n_1 ($< n_0$) with $n_1 | n_0$) and $(f^{n_0})'(z_0) = 1$ and $f^{n_0 k}(z) \to z_0$ ($k \to \infty$) holds for every $z \in U$. z_0 is a parabolic periodic point and U is called a* **parabolic basin**.

(3) *$f^{n_0}|_U$ is analytically conjugate to an irrational rotation $z \mapsto e^{2\pi i\theta} z$ ($\theta \in \mathbb{R} \setminus \mathbb{Q}$) on the unit disk $\mathbb{D} := \{z \mid |z| < 1\}$. That is, there exists an analytic isomorphism $\varphi : U \to \mathbb{D}$ with $\varphi(f^{n_0}(\varphi^{-1}(z))) = e^{2\pi i\theta} z$. $z_0 = \varphi^{-1}(0) \in U$ is an irrationally indifferent periodic point with period n_0 and satisfies $(f^{n_0})'(z_0) = e^{2\pi i\theta}$. U is called a* **Siegel disk**.

(4) *$f^{n_0 k}(z) \to \infty$ ($k \to \infty$) holds for every $z \in U$. U is called a* **Baker domain**. □

Remark 2.5. (1) When U is a periodic component of period n_0, the set $\{U_0, U_1, \cdots, U_{n_0-1}\}$ is called the **periodic cycle** (of a periodic component) (Remark: $U_0 = U$). Also, for example, it is called an attracting cycle, if U is an immediate attractive basin. In the same manner, when z_0 is a periodic point of period n_0, $O^+(z_0) = \{z_0, f(z_0), \cdots, f^{n_0-1}(z_0)\}$ is called a periodic cycle (of z_0) and also, it is called an attracting periodic cycle if z_0 is attracting, and so on.

(2) Since $F(f^{n_0}) = F(f)$ from Proposition 1.2 (2), we usually treat a periodic Fatou component of period n_0 for f as an invariant component for f^{n_0}. That is, we have only to discuss the case of invariant components and the case of periodic components can be reduced to the former case.

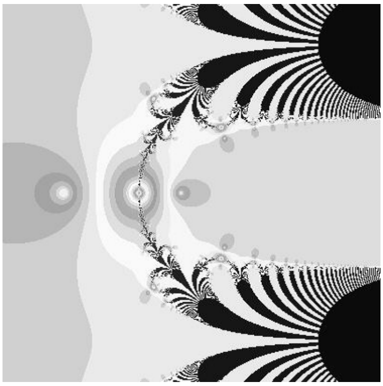
(3) A Fatou component called a Herman ring (i.e., a component U of period n_0 such that there exists an analytic isomorphism $\varphi : U \to A := \{z \mid 1 < |z| < r\}$ ($r > 1$) with $\varphi(f^{n_0}(\varphi^{-1}(z))) = e^{2\pi i\theta} z$ ($\theta \in \mathbb{R} \setminus \mathbb{Q}$)), which can arise for rational functions, does not exist for entire functions [$\because$ Suppose that we have a Herman ring U. We may assume that U is invariant by the above (2). Take an invariant closed Jordan curve γ in U and let D be its interior. Then we have $D \cap J(f) \neq \emptyset$. On the other hand, since $f^n(\gamma) = \gamma$, $\{f^n|_\gamma\}$ is uniformly bounded. Since $\gamma = \partial D$, it follows that $\{f^n|_D\}$ is also uniformly bounded by the maximum principle. Hence this is a normal family by Montel's Theorem (Theorem 1.1) and therefore $D \subset F(f)$, which is a contradiction]. □

We show some examples of transcendental entire functions with each type of Fatou components below (see Figure 2):

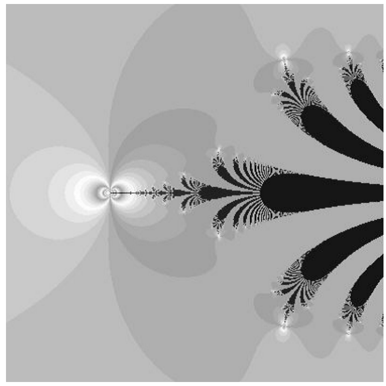
Example 2.6. (1) $E_{-3.4}(z) = -3.4e^z$ has an attracting cycle $\{p_0, p_1\}$ ($p_0 \sim -2.71421$, $p_1 \sim -0.22527$) of period 2 ($(E^2_{-3.4})'(p_i) \sim 0.61144$ ($i = 0, 1$)) with immediate attractive basins $U_0 \cup U_1$. U_0 is the left 1/3 part of Figure 2 (1) and U_1 is the component which contains positive real axis.

(2) $E_{1/e}(z) = 1/e \cdot e^z = e^{z-1}$ has $z = 1$ as a parabolic fixed point with an invariant parabolic basin U. U is the large domain which occupies almost all parts in Figure 2 (2). U is completely invariant and we have $F(E_{1/e}) = U$.

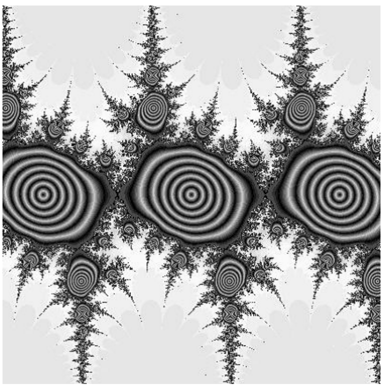
(3) $f(z) = e^{2\pi i\theta} \sin z$ ($\theta = (1 + \sqrt{5})/2$) has $z = 0$ as an irrationally indifferent fixed point with an invariant Siegel disk U ($e^{2\pi i\theta} \sim -(0.73737 + 0.67549i)$). U is the middle part of Figure 2 (3) with a striped pattern which describes the orbits of the irrational rotation. The other components in both sides and other parts are inverse images of U by f.



(1) $E_{-3.4}(z) = -3.4e^z$,
Immediate attractive basin of period 2
$\{z = x + iy \mid -4 \leq x \leq 4,\ -4 \leq y \leq 4\}$

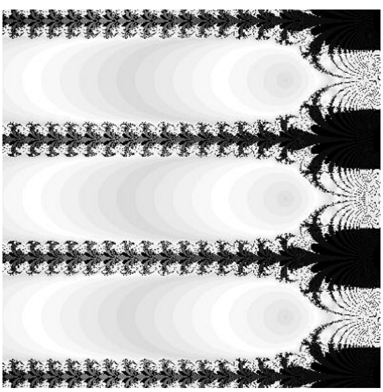


(2) $E_{1/e}(z) = 1/e \cdot e^z$,
Invariant parabolic basin
$\{z = x+iy \mid -0.1 \leq x \leq 3.9,\ -2 \leq y \leq 2\}$

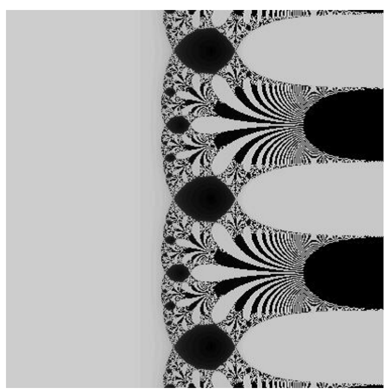


(3) $f(z) = e^{2\pi i\theta} \sin z$,
Invariant Siegel disk
$\{z = x + iy \mid -4 \leq x \leq 4,\ -4 \leq y \leq 4\}$

(4) $f(z) = z - 1 + e^z$,
Invariant Baker domain
$\{z = x + iy \mid -8 \leq x \leq 6,\ -7 \leq y \leq 7\}$



(5) $f(z) = z - e^z + 1 + 2\pi i$,
Wandering domain
$\{z = x+iy \mid -15 \leq x \leq 5,\ -10 \leq y \leq 10\}$



(6) $f(z) = 2 - \log 2 + 2z - e^z$,
Invariant Baker domain, invariant immediate attractive basin, wandering domain
$\{z = x + iy \mid -4 \leq x \leq 4,\ -4 \leq y \leq 4\}$

FIGURE 2. Examples of various kinds of Fatou components.

(4) ([Fat, p. 358, Example I]): $f(z) = z - 1 + e^z$ has an invariant Baker domain U which contains $\{z \mid \text{Re } z < 0\}$.[20] [$\because$ Let $\varepsilon > 0$ and $U_\varepsilon := \{z \mid \text{Re } z \leq -\varepsilon\}$; then we have

$$z \in U_\varepsilon \implies \text{Re } f(z) = \text{Re } z - 1 + \text{Re } e^z \leq \text{Re } z - 1 + e^{-\varepsilon} < \text{Re } z.$$

Hence $f(U_\varepsilon) \subset U_\varepsilon$ and $f^n(U_\varepsilon) \subset U_\varepsilon$. Therefore by Montel's Theorem we have $U_\varepsilon \subset F(f)$. Moreover since

$$z \in U_\varepsilon \implies \text{Re } f^n(z) \leq \text{Re } z - n(1 - e^{-\varepsilon}) \to -\infty \quad (n \to \infty),$$

it follows that the Fatou component U containing U_ε is a Baker domain. Since $\varepsilon > 0$ is arbitrary, U contains $\{z \mid \text{Re } z < 0\}$]. U is completely invariant and $F(f) = U$. In particular, $F(f)$ is connected.

(5) ([Ba8, p. 567, Example 5.1]): $f(z) = z - e^z + 1 + 2\pi i$ has a wandering domain[21] [$\because$ Let $\tilde{f}(z) := z - e^z + 1$; then $2n\pi i$ $(n \in \mathbb{Z})$ is a super-attracting fixed points of $\tilde{f}$. Let U_n be its immediate attractive basin. Since $\tilde{f}$ satisfies $\tilde{f}(z+2\pi i) = \tilde{f}(z) + 2\pi i$, we have $\varphi(F(\tilde{f})) = \varphi^{-1}(F(\tilde{f})) = F(\tilde{f})$, where $\varphi(z) := z + 2\pi i$. By using this we conclude that $F(f) = F(\tilde{f})$ for $f(z) = \tilde{f}(z) + 2\pi i$. With this fact and $f(2n\pi i) = 2(n+1)\pi i$, we have $f(U_n) = U_{n+1}$ and hence U_n is a wandering domain]. Then each unbounded domain expanding horizontally in Figure 2 (5) is the wandering domain. (we can see U_1, U_0 and U_{-1} from above in the figure).

(6) ([Ber2, p. 527 §2]): $f(z) = 2 - \log 2 + 2z - e^z$ has an invariant Baker domain, an invariant immediate attractive basin and a wandering domain [$\because$ Let $h(z) := z^2 e^{2-z}/2$; then we have $\exp(f(z)) = h(\exp(z))$ and $\exp(F(f)) = F(h)$. 0 and 2 are super-attracting fixed points of h and let U, V be each immediate attractive basin. Then $\exp^{-1}(U)$ is an invariant Baker domain of f. Also we have $\exp^{-1}(2) = \{z_n = \log 2 + 2n\pi i \mid n \in \mathbb{Z}\}$ and $\exp^{-1}(V) = \bigcup_{n\in\mathbb{Z}} U_n$, where U_n is the Fatou component which contains z_n. Then U_0 is the immediate basin of a super-attracting fixed point $\log 2$ of f and U_n $(n \neq 0)$ is a wandering domain, since $f(U_n) = U_{2n}$]. In Figure 2 (6), the left half domain with light gray color is the invariant Baker domain, the black domain in the middle is the invariant immediate attractive basin U_0 and the domains located in its upper and lower area with the same shape are U_1 and U_{-1}. □

There is an intimate relationship between periodic components of $F(f)$ and singular values as follows:

Theorem 2.7. *(1) If U is either an immediate attractive basin or a parabolic basin of period n_0, then $\bigcup_{k=0}^{n_0-1} f^k(U)$ contains at least one singular value of f.*

(2) If U is a Siegel disk, then $\partial U \subset P(f)$.

(3) ([Barg2, p. 294, Theorem 4]*): If U is an invariant Baker domain, there exist constants $C > 1$ and $r_0 > 0$ such that the annulus $\{z \mid r/C < |z| < Cr\}$ contains at least one singular value for every $r \geq r_0$.* □

From the Classification Theorem of Periodic Components, we can completely understand the dynamics on eventually periodic components in the sense that we

[20] This example is essentially the same as $g(z) = z + 1 + e^{-z}$, which Fatou considered in [Fat]. In fact let $\varphi(z) := -z - \pi i$; then we have $\varphi^{-1} \circ g \circ \varphi(z) = f(z)$.

[21] This example is due to Herman ([Ba8, p. 564, Example 2]). The one in [Ba8] is $g(z) = z + 2\pi i - 1 + e^{-z}$ and let $\tilde{g}(z) := z - 1 + e^{-z}$; then we have $g(z) = \tilde{g}(z) + 2\pi i$ and $\tilde{g} \circ \psi = \psi \circ \tilde{f}$, where $\psi(z) := -z$. Since $\tilde{f}$ and $\tilde{g}$ are conjugate, f and g are the examples by essentially the same idea, although they are not conjugate to each other.

can tell all the behavior of the orbit of each point. On the other hand, the dynamics on wandering domains cannot be understood completely so far. The relationship between wandering domains and singular values is as follows ([BHKMT, p. 370, Theorem]):

Theorem 2.8 (Bergweiler-Haruta-Kriete-Meier-Terglane, 1993)**.** *Let U be a wandering domain of f. Then every limit function of $\{f^n|_U\}$, that is, a limit function defined by a convergent subsequence, is constant and either equal to ∞ or contained in the derived set of $\bigcup_{n=0}^{\infty} f^n(\mathrm{sing}(f^{-1}))$ (i.e., the set of all accumulation points of this set).* □

It is known that there exists an f with a wandering domain with infinitely many limit functions ([EL1, p. 458, Example 1]). ∞ is a limit function for all known examples of wandering domains. It is not known whether this is always the case or not. In other words, we can ask the following ([EL1, p. 463, Problem]):

Problem A (Eremenko-Lyubich, 1987): Is there a transcendental entire function f with a wandering domain U such that the set of all limit functions of $\{f^n|_U\}_{n=1}^{\infty}$ is an infinite bounded set? □

At the end of this section, we define two important classes of transcendental entire functions.

Definition 2.9. We define two classes of transcendental entire functions $\mathcal{S}$ and $\mathcal{B}$ as follows (Remark: by definition, $\mathcal{S} \subset \mathcal{B}$):

$$\mathcal{S} := \{f \mid \mathrm{sing}(f^{-1}) \text{ is finite}\}, \quad \mathcal{B} := \{f \mid \mathrm{sing}(f^{-1}) \text{ is bounded}\}. \quad \square$$

Example 2.10. λe^z and $z^2 e^z$ belong to $\mathcal{S}$. Also $\sin z \in \mathcal{S}$ [$\because$ There are infinitely many critical points $(n + 1/2)\pi$ $(n \in \mathbb{Z})$ of $\sin z$ but there are only two critical values ± 1. Also there exist no finite asymptotic values]. Also $\sin z/z = \sum_{n=0}^{\infty}(-1)^n z^{2n}/(2n+1)! \in \mathcal{B} \setminus \mathcal{S}$ ([EL3, p. 991]). Also ([BHKMT, p. 372∼, §3]),

$$\pi^2 - \alpha\frac{\sin\sqrt{z}}{\sqrt{z}} = \pi^2 - \alpha\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^n \ (\pi^2 < \alpha < 2\pi^2), \quad \frac{\pi^2}{\pi^2 - z^2}\sin z \in \mathcal{B}\setminus\mathcal{S}. \quad \square$$

Class $\mathcal{S}$ is named after "S" of "Speiser class" and $\mathcal{B}$ comes from "B" of "bounded". Nowadays class $\mathcal{B}$ is also called "Eremenko-Lyubich class", since this was first investigated systematically by Eremenko and Lyubich ([EL3]). These two classes have been comparably well understood. In particular for $f \in \mathcal{S}$, this is similar to a polynomial in the sense that $\mathrm{sing}(f^{-1})$ is finite, and several analogies of the results have been proved which hold in the case of polynomials. For example, if $f \in \mathcal{S}$, f has no wandering domains ([EL3, p. 1004, Theorem 3], [GKe, p. 184, Theorem 4.2]), and no Baker domains ([EL3, p. 994, Theorem 1]). f has only finitely many periodic cycles of Fatou components and an analogy of Fatou-Shishikura inequality ([Sh, p. 5, Corollary 2]), which holds for their numbers in the case of rational functions, holds ([EL3, p. 1005, Theorem 5]). If $f \in \mathcal{B}$, then there exists no Fatou components U with $f^n|_U \to \infty$ $(n \to \infty)$. In particular, f has no Baker domains and also no wandering domains with $f^n|_U \to \infty$ $(n \to \infty)$ (i.e., with the simplest possible behavior) ([EL3, p. 994, Theorem 1]). It has been an open problem for a long time whether there exists an $f \in \mathcal{B}$ with a wandering domain or not, but recently Bishop has constructed such an example ([Bi, p. 20, Theorem 4.7]).

3. Connectedness of the Julia sets

In general, a topological space X is called **connected** if $X = U \cup V$, $U \cap V = \emptyset$, where U and V are open; then either $U = \emptyset$ or $V = \emptyset$. The connectedness of $J(f) \cup \{\infty\} \subset \widehat{\mathbb{C}}$ and $J(f) \subset \mathbb{C}$ is different in general.

Definition 3.1. We define the **connectivity** $\mathrm{conn}(U)$ of a domain $U \subset \mathbb{C}$ by

$$\mathrm{conn}(U) := \text{the number of connected components of } \widehat{\mathbb{C}} \setminus U.$$

(Of course, $\mathrm{conn}(U) = \infty$ may happen.) U is called **simply connected** if $\mathrm{conn}(U) = 1$[22], and **multiply connected** if $\mathrm{conn}(U) \geq 2$. □

3.1. Connectedness of $J(f) \cup \{\infty\} \subset \widehat{\mathbb{C}}$.

Since $J(f) \cup \{\infty\}$ is a compact subset of $\widehat{\mathbb{C}}$, it is rather easy to handle as a topological space. In fact, there is the following general criterion for connectedness of a compact set in $\widehat{\mathbb{C}}$ ([Bea, p. 81, Proposition 5.1.5]):

Proposition 3.2. *Let $K \subset \widehat{\mathbb{C}}$ be a compact set. The following two conditions are equivalent:*
(1) K is connected.
(2) Each connected component of the complement K^c is simply connected. □

We apply this for $K = J(f) \cup \{\infty\}$. In order to tell whether K is connected or not, we need to see whether each component of $K^c = F(f)$, that is, Fatou component, is simply connected or not. Here is a known result for the connectedness of the Fatou components ([Ba5, p. 278, Theorem 1], [Ba8, p. 565, Theorem 3.1]):

Theorem 3.3 (Baker, 1975). *Let f be a transcendental entire function. Then all the unbounded Fatou components of f are simply connected.* □

Theorem 3.4 (Baker, 1984). *Let f be a transcendental entire function and U a Fatou component of f. If U is multiply connected, then it is bounded and a wandering domain. Moreover, $f^n|_U \to \infty$ $(n \to \infty)$ and the rotation number of $f^n(\gamma)$ with respect to the origin, where $\gamma \subset U$ is a non-trivial closed Jordan curve, goes to ∞ as $n \to \infty$. Therefore in particular, every eventually periodic component of $F(f)$ is simply connected.*[23] □

By using the above results, we have the following ([Ki3, p. 191, Theorem 1]):

Theorem 3.5 (Kisaka, 1998). *The following two conditions are equivalent:*
(1) $J(f) \cup \{\infty\} \subset \widehat{\mathbb{C}}$ is connected.
(2) f has no multiply connected wandering domains. □

So $J(f) \cup \{\infty\} \subset \widehat{\mathbb{C}}$ is connected under the situation that f has no multiply connected wandering domains. Here are some sufficient conditions for this situation ([Ki3, p. 191, Corollary 1]).

---

[22] In general, the simple connectivity of a topological space X is defined by the triviality of the fundamental group $\pi_1(X)$ of X. It is known that this is equivalent to $\mathrm{conn}(U) = 1$ for a domain $U \subset \mathbb{C}$ (moreover, for a domain $U \subset \widehat{\mathbb{C}}$).

[23] This means that in general, every eventually periodic component of $F(f)$ is simply connected, independent of whether f has a multiply connected Fatou component or not.

Corollary 3.6 (Kisaka, 1998). *$J(f) \cup \{\infty\} \subset \widehat{\mathbb{C}}$ is connected under either one of the following conditions:*

(1) $f \in \mathcal{B}$.

(2) f has an unbounded Fatou component.

(3) There exists a curve $L(t)$ $(0 \leq t < \infty)$ with $\lim_{t\to\infty} L(t) = \infty$ and $f|L$ is bounded. In particular, this is satisfied if f has a finite asymptotic value. □

For example, for all the $J(f)$ in Example 2.6 (1) $\sim$ (6), $J(f) \cup \{\infty\}$ is connected in $\widehat{\mathbb{C}}$.

3.2. **Properties of the boundaries of periodic components and connectedness of $J(f) \subset \mathbb{C}$.** Next we consider the connectedness of $J(f) \subset \mathbb{C}$. In the case that f is a polynomial, there is the following famous criterion ([Bea, p. 202, Theorem 9.5.1]):

Proposition 3.7. *If f is a polynomial, the following two conditions are equivalent:*

(1) $J(f)$ is connected.

(2) Every orbit of a critical value except for ∞ is bounded.[24] □

This criterion, however, does not hold if f is transcendental as the next examples show.

Example 3.8. (1) ([DeG, p. 265, Remark]): If $E_\lambda(z) := \lambda e^z$ has an attracting fixed point p_λ (for example, if $0 < \lambda < 1/e$), the immediate attractive basin U_λ of p_λ is equal to the basin of attraction of p_λ. That is, $U_\lambda = \{z \mid E_\lambda^n(z) \to p_\lambda \ \ (n \to \infty)\}$ and U_λ is unbounded, completely invariant and coincides with $F(E_\lambda)$. In this case $J(E_\lambda) = \partial U_\lambda$ is disconnected. Moreover, the following holds: Since U_λ is simply connected by Theorem 3.3, we can consider a Riemann map $\varphi : \mathbb{D} \to U_\lambda$. Then define

$$\Theta_\infty := \{e^{i\theta} \mid \varphi(e^{i\theta}) := \lim_{r\nearrow 1} \varphi(re^{i\theta}) = \infty\} \subset \partial\mathbb{D}; \tag{3.1}$$

then Θ_∞ is dense in $\partial\mathbb{D}$. This implies that ∂U_λ $(= J(E_\lambda))$ is extremely complicated.
(2) If the unique singular value 0 of an exponential function $E_\lambda(z) = \lambda e^z$ satisfies $E_\lambda^n(0) \to \infty$ $(n \to \infty)$ (for example if $\lambda > 1/e$), then $J(E_\lambda) = \mathbb{C}$ and in particular $J(E_\lambda)$ is connected (see [De2, p. 295]). □

So we have to take a totally different method. We divide the situation into two cases:

(I) The case that f has no unbounded Fatou components: The following holds ([Ki3, p. 192, Theorem 2]):

Theorem 3.9 (Kisaka, 1998). *If all the Fatou components are bounded and simply connected, then $J(f)$ is connected.* □

The next follows immediately from Theorem 3.5 and Theorem 3.9 ([Ki3, p. 192, Corollary 2]).

Corollary 3.10 (Kisaka, 1998). *If all the Fatou components of f are bounded, then the following two conditions are equivalent:*

(1) $J(f) \subset \mathbb{C}$ is connected.

(2) $J(f) \cup \{\infty\} \subset \widehat{\mathbb{C}}$ is connected. □

[24] When f is a polynomial, f has no finite asymptotic values and ∞ is a critical value and $\operatorname{sing}(f^{-1})$ coincides with the set of all critical values.

Various kinds of sufficient conditions are known which guarantee that all the Fatou components of f are bounded:

(i) ([Ba7, p. 484, Theorem 2]): Let $M(r, f) := \sup_{|z|=r} |f(z)|$ and for some $1 < p < 3$

$$\log M(r, f) = O((\log r)^p) \quad (r \to \infty).$$

(ii) ([St, p. 43, Theorem B]): For some $\varepsilon \in (0, 1)$ and for any sufficiently large r,

$$\log\log M(r, f) < \frac{(\log r)^{1/2}}{(\log\log r)^{\varepsilon}}.$$

(iii) ([AH, p. 3245, Theorem 2]): The **order** $\rho(f)$ of f satisfies $\rho(f) < 1/2$ and there exists a constant $c > 0$ such that for any sufficiently large x

$$\frac{\varphi'(x)}{\varphi(x)} \geq \frac{1+c}{x}, \quad \text{where} \quad \varphi(x) := \log M(e^x, f),^{25} \quad \rho(f) := \limsup_{r\to\infty} \frac{\log\log M(r, f)}{\log r}.$$

The condition (ii) is weaker than the condition (i). It is a big problem to decide how much we can weaken this condition and it is conjectured as follows ([Ba7, p. 489]):

Conjecture B (Baker, 1981): If $\rho(f) < 1/2$ or $\rho(f) = 1/2$ and f is minimal type (i.e., $\limsup_{r\to\infty} \log M(r, f)/r^{\rho(f)} = 0$), then all the Fatou components of f are bounded. □

Baker showed that the function

$$f(z) = \cos\left((\varepsilon^2 z + (9/4)\pi^2)^{1/2}\right) \quad (0 < \varepsilon < \sqrt{3}\pi)$$

satisfies $\rho(f) = 1/2$ and is ε-type (i.e., $\limsup_{r\to\infty} \log M(r, f)/r^{\rho(f)} = \varepsilon$) and the immediate attractive basin of the attracting fixed point $z = 0$ is unbounded ([Ba7, p. 484]). This implies that the condition in the above Conjecture B is sharp. Although there is much research on this conjecture, it remains unsolved for now. It is known from the research so far that under the condition of Conjecture B all the Fatou components other than wandering domains are bounded. So the problem is to show the non-existence of unbounded wandering domains. There is a survey article by Hinkkanen which deals with Conjecture B including detailed information on historical remarks ([Hi2]). Incidentally Hinkkanen asserts that this conjecture will be solved by the following approach (in the case of $\rho(f) < 1/2$) ([Hi1, p. 221, Theorem 1, Question]).

Theorem 3.11 (Hinkkanen, 2005)**.** *Assume that f satisfies $\rho(f) < 1/2$. Moreover, assume that there exist positive constants $R_0,\ L,\ \delta,\ C$ with $R_0 > e,\ M(R_0, f) > e,\ L > 1,\ 0 < \delta \leq 1$ and satisfies the following: For any $r > R_0$ there exists $t \in (r, r^L]$ satisfying*

$$\frac{\log m(t, f)}{\log M(r, f)} \geq L\left(1 - \frac{C}{(\log r)^{\delta}}\right), \quad \textit{where} \quad m(t, f) := \min_{|z|=t} |f(z)|.$$

Then all the components of $F(f)$ are bounded. □

[25] The reason why we consider this kind of $\varphi(x)$ is that $\varphi(x)$ is monotone increasing and convex with respect to x by Hadamard three-circle theorem. Recently the function f satisfying the condition (iii) is called log-regular. ([Si, p. 554, Definition], [RiSt5, p. 814]).

Conjecture C (Hinkkanen, 2005): Every transcendental entire function f with $\rho(f) < 1/2$ satisfies the condition in Theorem 3.11. Therefore if $\rho(f) < 1/2$, then Conjecture B is true. □

At the end of this section, we introduce a recent result by Rippon and Stallard which concerns Conjecture B. In general, a function f with $\rho(f) < 1$ can be written as follows by Hadamard's Theorem ([T, p. 486, Theorem XVII.5]):

$$f(z) = cz^{p_0} \prod_{n=1}^{\infty} \left(1 + \frac{z}{a_n}\right)^{p_n}, \quad p_n \in \mathbb{N} \cup \{0\}, \quad c \in \mathbb{C}, \quad a_n \in \mathbb{C}, \tag{3.2}$$

where $|a_n|$ is monotone increasing with $|a_n| \nearrow \infty$ $(n \to \infty)$. In particular, a function f with $\rho(f) < 1/2$ can be written as above and they showed the following ([RiSt4, p. 293, Theorem 1.1, Remarks 1]):

Theorem 3.12 (Rippon-Stallard, 2011). *Let f be a transcendental entire function with $\rho(f) < 1/2$ and assume that $c \in \mathbb{R} \setminus \{0\}$, $a_n > 0$ in the expression* (3.2) *for f. Then all the Fatou components of f are bounded.* □

Theorem 3.12 is a special case of [RiSt4, Theorem 1.1] and in [RiSt4, Theorem 1.1] they give a partial answer for Eremenko's conjecture (Conjecture K) which concerns a structure of $I(f)$ (i.e., they show that $I(f)$ is a spider's web (see Definition 7.11) under the assumption of Theorem 3.12 and in particular it is connected).

(II) The case that f has an unbounded Fatou component U: Since from Theorem 3.3 U is simply connected, we can take a Riemann map $\varphi : \mathbb{D} \to U$. Then for this φ define Θ_∞ by (3.1) in Example 3.8. If $^\sharp\Theta_\infty \geq 2$, then

$$\varphi\big(\{re^{i\theta_1} \mid 0 \leq r < 1\} \cup \{re^{i\theta_2} \mid 0 \leq r < 1\}\big) \subset U, \quad (\theta_1, \theta_2 \in \Theta_\infty,\ \theta_1 \neq \theta_2),$$

is a Jordan curve in $F(f)$ by the conformality of φ and this divides $J(f)$ into two relatively open sets in $\mathbb{C}$. Therefore $J(f)$ is disconnected. The following result ([Ki3, p. 192, Main Theorem]) is for Θ_∞, and disconnectedness of $J(f)$ follows as its corollary.

Theorem 3.13 (Kisaka, 1998). *Let f be a transcendental entire function and U be an unbounded periodic Fatou component of period n_0. Also let $\varphi : \mathbb{D} \to U$ be a Riemann map of U and define Θ_∞ by* (3.1). *Consider the following conditions:*

(A) $\infty \in \partial U$ is accessible from U, that is, there exists a continuous curve $\gamma(t)$ $(0 \leq t < \infty) \subset U$ with $\lim_{t\to\infty} \gamma(t) = \infty$.

(B) There exist a point $q \in \partial U$ with $q \notin P(f^{n_0})$ and a curve $C(t)$ $(0 \leq t \leq 1)$ with $C(t) \subset U$ $(0 \leq t < 1)$, $C(1) = q$ which satisfies $f^{m_0}(C) \supset C$ for some $m_0 \in \mathbb{N}$. Then assume one of the following:

(1) U is an immediate attractive basin with (A) and (B).

(2) U is a parabolic basin with (A) and (B).

(3) U is a Siegel disk with (A).

(4) U is a Baker domain with (B) and $f^{n_0}|U$ is not univalent.

Then in the case of (1), (2) and (3), Θ_∞ is dense in $\partial\mathbb{D}$. In the case of (4), Θ_∞ is either dense in $\partial\mathbb{D}$ or $\overline{\Theta_\infty}$ contains a perfect set in $\partial\mathbb{D}$. In particular, $J(f)$ is disconnected in all cases. □

Remark 3.14. Θ_∞ depends on the choice of a Riemann map φ, but the properties "$\Theta_\infty \subset \partial\mathbb{D}$ is dense" and "$\overline{\Theta_\infty}$ contains a perfect set in $\partial\mathbb{D}$" do not depend on the choice of φ. That is, these properties depends only on U. □

On the other hand, it can occur that f has an unbounded Fatou component but $J(f)$ is connected ([Ki3, p. 194, Theorem 4]).

Example 3.15 (Kisaka, 1998)**.** $f(z) = 2 - \log 2 + 2z - e^z$ has an invariant Baker domain U such that $f|U$ is univalent ([Ber2, p. 526, Theorem 1]),[26] and $J(f)$ is connected (see Example 2.6 (6), Figure 2 (6)). □

Later Theorem 3.13 was improved as follows ([BaDo1, p. 439, Theorem 1.1, 1.2, Corollary 1.3]):

Theorem 3.16 (Baker-Domínguez, 1999)**.** *Theorem* 3.13 *holds without the condition (B).* □

Moreover, Baker and Domínguez proved the following general criterion ([BaDo2, p. 372, Theorem B]):

Theorem 3.17 (Baker-Domínguez, 2000)**.** $J(f)$ *is either connected or has uncountably many connected components.* □

Therefore in particular, if f has an unbounded periodic component U with the above condition, then it follows that $J(f)$ has uncountably many connected components.

It is not known whether there exists an unbounded periodic Fatou component which does not satisfy the condition (A). It is known that every Baker domain satisfies (A) ([Ba10, p. 503, Theorem 2]). Also if $f \in \mathcal{S}$ and U is a completely invariant component (that is, $f^{-1}(U) \subset U$) of f, then (A) is satisfied ([BaDo2, p. 374, the comment after Theorem D]). So ([Ki3, p. 204, Conjecture 1, 2]),

Conjecture D (Kisaka, 1998): For every unbounded periodic Fatou component U, ∞ is accessible in U. In particular, Theorem 3.13 holds without the conditions (A) and (B). □

In the case of Baker domains, only a little weaker conclusion has been shown in Theorems 3.13 and 3.16 compared with other cases. For the set Θ_∞ in the case of a Baker domain, there is an example such that it is dense in $\partial\mathbb{D}$ (for example, $f(z) = z + 1 + e^{-z}$ ([BaDo1, §7]; see Fatou's example, Example 2.6 (4))). Quite recently an example such that Θ_∞ is not dense in $\partial\mathbb{D}$ has been constructed by Bergweiler and Zheng ([BerZ2, p. 1035, Theorem 1.2]).

Thus boundaries of unbounded periodic Fatou components can be understood to some extent by investigating the boundary behavior of their Riemann maps and as a result, disconnectedness of $J(f) \subset \mathbb{C}$ follows. On the other hand for unbounded

[26]Bergweiler showed not only that $f|U$ is univalent for this Baker domain U (that is, $U \cap \mathrm{sing}^{-1}(f) = \emptyset$) but also that U satisfies $\mathrm{dist}_{\mathbb{C}}(P(f), U) > 0$. An example of Baker domain U such that $f|U$ is univalent was already constructed by Eremenko and Lyubich before this example ([EL1, p. 459, Example 3]), but the relation between the boundary of the Baker domain and $P(f)$ was not clear for their example.

wandering domains, although we can say something for some very concrete special examples (for example, Example 2.6 (5)),[27] there are no general results. So

Problem E: Investigate boundary behavior of the Riemann maps of unbounded wandering domains. □

4. Local connectedness of the boundaries of unbounded periodic Fatou components and local connectedness of Julia sets

In this section we discuss local connectedness of $J(f)$. In general, a topological space X is called **locally connected at a point** $x \in X$ if $x \in X$ has arbitrary small connected neighborhoods. If X is locally connected at every point $x \in X$, X is called **locally connected**. When f is a polynomial (or a rational function, in general), there is the following famous result ([Mil2, p. 207, Theorem 19.2]).

Theorem 4.1. *Let f be a polynomial (or rational function) and assume that $J(f)$ is connected. If f is hyperbolic,*[28] *then $J(f)$ is locally connected.* □

If f is transcendental, however, we cannot expect this result.[29]

In general the following criterion is known for local connectedness of a compact set in $\widehat{\mathbb{C}}$ ([Wh1], [Mil2, p. 217]).

Theorem 4.2. *A compact set $K \subset \widehat{\mathbb{C}}$ is locally connected if and only if the following two conditions hold:*

(i) The boundary of each connected component of K^c (= the complement of K) is locally connected and

(ii) for every $\varepsilon > 0$ the number of connected components of K^c with diameter larger than ε is finite. □

By applying this result to $K = J(f) \cup \{\infty\}$, we can see that local connectedness of $J(f)$ has a deep relation to the local connectedness of the boundaries of Fatou components. On the other hand for a general simply connected domain U, ∂U is locally connected if and only if its Riemann map $\varphi : \mathbb{D} \to U$ can be extended to a continuous map on $\overline{\mathbb{D}}$ by the theory of Carathéodory ([Mil2, p. 183, Theorem 17.14]). Here we consider the case that f has an unbounded periodic Fatou component U. The following implies that ∂U is extremely complicated unless U is a Baker domain ([BaW, p. 413, Theorem 1]).

Theorem 4.3 (Baker-Weinreich, 1991)**.** *One of the following holds for an unbounded periodic Fatou component U:*

(i) U is a Baker domain.

(ii) ∞ belongs to every impression of a prime end of U.[30] □

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[27] f in Example 2.6 (5) satisfies $F(f) = F(\tilde{f})$, so in order to investigate the wandering domain U_n of f it suffices to investigate the immediate attractive basin U_n of $\tilde{f}$. Since U_n is an unbounded immediate attractive basin of $\tilde{f}$ and ∞ is accessible, from Theorem 3.13 and Theorem 3.16, ∂U_n has a complicated structure and $J(f)$ is disconnected.

[28] A polynomial (or rational function) f is called hyperbolic if $P(f) \subset F(f)$. It seems that the definition of hyperbolicity for transcendental entire functions has not been established yet. A definition of hyperbolicity for the transcendental case requires the compactness of $P(f)$ in addition to the above property (see [Bara, p. 34]). Also in [Mc] and [St], they use "expanding" instead of "hyperbolic" under the above same definition.

[29] For example, $E_\lambda(z)$ $(0 < \lambda < 1/e)$ in Example 3.8 (1) is hyperbolic (see Footnote 28), but $J(E_\lambda)$ is not locally connected, as we explain in the sequel.

[30] For concepts of the prime end and the impression, see, for example, [Mil2, pp. 178–179].

From this we can show the following ([Ki2, p. 115, Theorem A]):

Theorem 4.4 (Kisaka, 1997). *Assume that f has an unbounded periodic Fatou component U and if U is a Baker domain assume that $f^{n_0}|U$ is a d to 1 map ($2 \leq d < \infty$, n_0 is the period of U). Then $\partial U \cup \{\infty\} \subset \widehat{\mathbb{C}}$ is not locally connected. Also $\partial U \subset \mathbb{C}$ is not locally connected either.* □

This is trivial from Theorem 4.3 and the theory of Carathéodory, unless U is a Baker domain. So the case of Baker domains is the main problem. From this result and Theorem 4.2 and with some arguments, the next theorem follows ([Ki2, p. 115, Theorem B]).

Theorem 4.5 (Kisaka, 1997). *Assume that f has an unbounded periodic Fatou component U and if U is a Baker domain assume that $f^{n_0}|U$ is a d to 1 map ($1 \leq d < \infty$, n_0 is the period of U). Then $J(f) \cup \{\infty\} \subset \widehat{\mathbb{C}}$ is not locally connected. Also $J(f) \subset \mathbb{C}$ is not locally connected either.* □

In Theorem 4.5, other than the case of Baker domain U with $d = 1$, the conclusion follows, since from Theorem 4.4 the condition (i) of Theorem 4.2 does not hold. If U is a Baker domain with $d = 1$, ∂U can be a Jordan curve (in particular, it is locally connected) so we have to present a different argument. Actually in this case it turns out that the condition (ii) of Theorem 4.2 does not hold by an elementary argument using Proposition 1.3.

Later Theorem 4.5 was generalized to the following stronger result ([BaDo2, p. 374,Theorem E]).

Theorem 4.6 (Baker-Domínguez, 2000). *Assume that f has an unbounded periodic Fatou component U and if U is a Baker domain, assume that $f^{n_0}|_U$ is not univalent (n_0 is the period of U). Then $J(f) \subset \mathbb{C}$ is not locally connected* **at every point**.□

We briefly show the outline of the proof. Suppose that $J(f)$ is locally connected at $z_0 \in J(f)$; then it follows that $J(f)$ is connected from Montel's Theorem and some elementary arguments. On the other hand if U is a Baker domain such that $f^{n_0}|U$ is not univalent, from Theorem 3.16 it follows that $J(f)$ is not connected, which is a contradiction. In all the other cases, we get a contradiction from Theorem 4.3.

Summarizing the above, we have

Theorem 4.7. *If f has an unbounded periodic Fatou component, then $J(f) \subset \mathbb{C}$ is not locally connected.* □

On the other hand, in the case that f has no unbounded Fatou components, $J(f) \subset \mathbb{C}$ can be locally connected. Actually Morosawa showed the following ([Mo1, p. 268, Theorem 2, p. 270, Theorem 5], [O1, p. 1158, Lemma 6.3]).[31]

[31] The following Theorem 4.8 (1) is the same as [Mo1, p. 268, Theorem2] but the condition "every Fatou component contains at most finitely many critical points" is overlooked in the statement of [Mo1, p. 268, Theorem 2]. If we do not assume this condition, we cannot deny the possibility that some preperiodic Fatou component U may be unbounded, which is a preimage of a bounded periodic Fatou component (Remark: If this kind of U exists, it follows that $J(f)$ is not locally connected by the similar argument as in right after Theorem 4.5). According to Professor Morosawa, this was pointed out by Bergweiler around 2000, but since [Mo1] is a paper in a proceedings, there was no chance to announce the correction. In the paper by Osborne the theorem with the

Theorem 4.8 (Morosawa, 1999). *(1) Let $f \in \mathcal{S}$ and assume that f satisfies the following:*

(i) If $\zeta \in F(f) \cap \operatorname{sing}(f^{-1})$, then ζ is a critical value and $O^+(\zeta)$ is absorbed by an attracting periodic cycle.

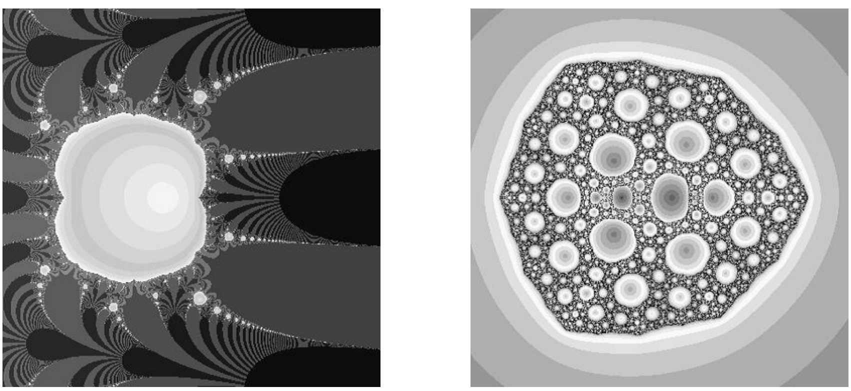
(ii) If $\zeta \in J(f) \cap \operatorname{sing}(f^{-1})$, then $\overline{O^+(\zeta)} \cap \partial U = \emptyset$ for every Fatou component U.

Further if every Fatou component contains at most finitely many critical points and all the periodic components of $F(f)$ are bounded, then $J(f)$ is locally connected.

(2) Let $f_\lambda(z) = \lambda z e^z$ and $\lambda \in \mathbb{C} \setminus \{0\}$ satisfy $|\operatorname{Im} \lambda| \geq e \cdot \operatorname{Arg} \lambda$. If f_λ has an attracting periodic cycle with period greater than 1, then every Fatou component of f_λ is bounded and $J(f_\lambda)$ is locally connected. □

For example, let $\lambda = 9.54894\cdots$, then $f_\lambda(z)$ has an attracting periodic cycle of period 2, so $J(f_\lambda)$ is locally connected from Theorem 4.8 (2). Theorem 4.8 (2) is an example of an application of Theorem 4.8 (1). Also Theorem 4.8 (1) is an analogy of the result for polynomials (or rational functions) (Theorem 4.1). Furthermore Morosawa constructed the following interesting example ([Mo1, p. 272, Theorem 7]). Here in general, a compact set $K \subset \widehat{\mathbb{C}}$ is called a **Sierpiński carpet** if K is connected, locally connected, nowhere dense and the boundaries of each component of the complement of K are mutually disjoint Jordan curves.

Theorem 4.9 ([Mo1]). *Let $g_a(z) := ae^a\{z - (1-a)\}e^z$ $(a > 1)$; then $J(g_a)$ is locally connected and moreover $J(g_a) \cup \{\infty\} \subset \widehat{\mathbb{C}}$ is a Sierpiński carpet. (see Figure 3 (1) (where $a = 1.2$)).* □



(1) $g_a(z) := ae^a\{z-(1-a)\}e^z$ $(a = 1.2)$ (2) $f(z) := 27z^2(z-1)/\{(3z-2)^2(3z+1)\}$

$\{z = x+iy \,|\, -5.5 \leq x \leq 4.5,\ -5 \leq y \leq 5\}$ $\{z = x+iy \,|\, -2 \leq x \leq 3,\ -2.5 \leq y \leq 2.5\}$

FIGURE 3. Examples of Julia sets which are Sierpiński carpets.

An example of a rational function whose Julia set is a Sierpiński carpet was already known before the above example ($f(z) := a(z+z^{-1})+b$, ($a \sim -0.138115091$, $b \sim -0.303108805$) [Mil1, p. 78, Theorem F.1]). The example in Figure 3 (2) is

above condition is stated ([O1, p. 1158, Lemma 6.3, Remark 2]). Incidentally, all the functions which appear in the following Theorem 4.8 (2) and Theorem 4.9 have only finitely many critical points and so this condition is automatically satisfied.

due to Morosawa ([Mo2, p. 161, Example 9]). On the other hand, since it is known that all Sierpiński carpets are mutually homeomorphic ([Wh2, p. 322, Theorem]), from Theorem 4.9 in particular it follows that there exists a transcendental entire function f such that $J(f) \cup \{\infty\}$ is homeomorphic to the Julia set of some rational function. This is rather an unexpected result, of course excluding the trivial case that the Julia set equals $\mathbb{C}$. Later Bergweiler and Morosawa showed a new sufficient condition for $J(f)$ to be locally connected and a new example ($f(z) = az\cos\sqrt{z}/(\pi^2 - 4z)$, $(\pi^2 < a < 43.90495\cdots)$) in [BerMo, p. 1675, Theorem 3, p. 1679, Example 2]. Also Osborne recently showed that if $J(f)$ is locally connected, then $J(f)$ is a spider's web (see Definition 7.11) ([O1, p. 1147, Theorem 1.1]).

At the end of this section, let us consider the case that f has an unbounded wandering domain U. In this case, we can show that $J(f)$ is not locally connected by the similar argument as in the case that f has a Baker domain V such that $f^{n_0}|_V$ is univalent (see the part right after Theorem 4.5). On the other hand for the local connectedness of ∂U, there is an example, like Example 2.6 (5), such that ∂U is not locally connected (see Footnote 27), but we do not have a general theory so far. So here we raise the following problem which is related to the Problem E in Section 3.

Problem F: When f has an unbounded wandering domain U, investigate the local connectedness of ∂U. □

5. Connectivity of Fatou components

As we saw in Section 3.1, a multiply connected Fatou component is a bounded wandering domain. This kind of example was first constructed by Baker (this follows from the next two results [Ba1, p. 206 Statement (A), p. 210 Theorem 1], [Ba6, p. 174, Theorem]).

Theorem 5.1 (Baker, 1963). *Define a transcendental entire function $g(z)$ as follows:*

$$g(z) = Cz^2 \prod_{n=1}^{\infty}\left(1 + \frac{z}{r_n}\right),$$

where $C > 0$, $r_1 > 1$ are constants with $C\exp(2/r_1) < 1$, $Cr_1 > 1$, and r_n are defined by the recursive formula

$$r_{n+1} = Cr_n^2\left(1 + \frac{r_n}{r_1}\right)\left(1 + \frac{r_n}{r_2}\right)\cdots\left(1 + \frac{r_n}{r_n}\right), \quad n = 1, 2, \cdots.$$

Also let A_n be an annulus

$$A_n := \{z \mid r_n^2 < |z| < r_{n+1}^{1/2}\}.$$

Then there exists an $N \in \mathbb{N}$ such that for any $n > N$ we have $g(A_n) \subset A_{n+1}$ and $g^m(z) \to \infty$ $(m \to \infty)$ uniformly on A_n. Also for any $n > N$, A_n is contained in a multiply connected Fatou component $G_n \subset F(g)$. □

Theorem 5.2 (Baker, 1976). *The Fatou components G_n in Theorem* 5.1 *are mutually different and each of them are wandering domains.* □

In Theorem 5.1 Baker was not able to deny the possibility that all G_n $(n > N)$ are equal, that is, the union of these consists of a single Baker domain. Ten years later, he showed Theorem 5.2 and finally G_n was determined to be a wandering domain. Later he showed that in general an unbounded Fatou component is simply connected (Theorem 3.3).[32] This is the first example of a wandering domain. In view of this, a multiply connected wandering domain is sometimes called a **Baker wandering domain**.[33] The connectivity of the wandering domain in this example was not clear but later Baker constructed a wandering domain U with infinite connectivity (that is, $\mathrm{conn}(U) = \infty$) by the similar method ([Ba9, p. 164, Theorem 2]). At this point, the following problem remained unsolved ([Ba9]):

Problem 5.3 (Baker, 1985)**.** Does there exist a wandering domain with finite connectivity? More precisely, for a given $p \in \mathbb{N}$, is there a wandering domain U with $\mathrm{conn}(U) = p$? □

This problem was raised also by Bergweiler in [Ber1, p. 167] as "Question 7".

In general let U be a wandering domain; then for $\mathrm{conn}(f^n(U))$ we can show the following ([KiSh1, p. 219, Theorem A]):

Theorem 5.4 (Shishikura-Kisaka, 2008)**.** *If a transcendental entire function f has a wandering domain U, then $\mathrm{conn}(f^n(U))$ is constant for every sufficiently large n, and it is either 1, 2 or ∞ (we call this the* **eventual connectivity** *of U). If the eventual connectivity of U is 1, then $\mathrm{conn}(U) = 1$. If it is 2, then $f : f^n(U) \to f^{n+1}(U)$ is a covering map between annuli for every sufficiently large n.* □

The answer for Problem 5.3 is given by the following ([KiSh1, p. 219, Theorem B, Theorem C]):

Theorem 5.5 (Shishikura-Kisaka, 2008)**.** *(1) There exists a transcendental entire function f with a wandering domain U such that $f^n(U)$ is doubly connected for $n \geq 0$.*
(2) For every $p \in \mathbb{N}$ there exists a transcendental entire function f with a wandering domain U such that $\mathrm{conn}(U) = p$ and $\mathrm{conn}(f^n(U)) = 2$ for $n \geq 1$. □

We briefly show the outline of the proof. For (1), let

$$A_n := \{z \mid R_n \leq |z| \leq R_{n+1}\}, \quad (0 < R_n < R_{n+1},\ n \in \mathbb{N},\ \text{and } R_0 = 0)$$

and arrange $a_n \in \mathbb{C}^*$ and R_n so that $f_0 : \mathbb{C} \to \mathbb{C}$ satisfies

$$f_0(z) = a_n z^{n+1}, \quad z \in A_n \setminus \partial_{\text{outer}} A_n = \{z \mid R_n \leq |z| < R_{n+1}\}$$

and $f_0|_{A_n} : A_n \to A_{n+1}$ becomes a covering map of degree $n + 1$ for every n (Remark: Here, only A_0 is not an annulus but a disk and assume $f_0|_{A_0} : A_0 \to A_0 \cup A_1$). Next we change f_0 on an appropriate concentric annulus containing the circle $\{z \mid |z| = R_n\}$ so that it interpolates $a_{n-1}z^n$ and $a_n z^{n+1}$, and the new map f_1

[32] It seems strange that the published year of [Ba5], which states a general principle, is older than that of [Ba6], which is a special case, but in fact, [Ba6] is "Received 1 November 1974" and [Ba5] is "Received 26 May 1975".

[33] This term was first introduced by Rippon and Stallard in [RiSt1, p. 529]. More precisely, a Baker wandering domain is a Fatou component U of a transcendental meromorphic function f with the following property: there exists an $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ $f^n(U)$ is contained in a bounded multiply connected Fatou component U_n with $U_n \to \infty$ $(n \to \infty)$. In particular when f is entire and U is a multiply connected Fatou component, the above condition is always satisfied. Note that a "Baker wandering domain" is a "wandering domain" and it is **never** a "Baker domain".

is a quasiregular map so that it satisfies $f_1(A_n^-) \subset A_{n+1}^-$, where $A_n^- \subset A_n$ is a concentric annulus which is obtained by subtracting a concentric annulus on which we applied interpolation from A_n. For this f_1 take an appropriate quasiconformal map φ; then by the Measurable Riemann Mapping Theorem $f := \varphi \circ f_1 \circ \varphi^{-1}$ becomes a transcendental entire function, which is the desired function. Of course, the Fatou component containing $\varphi(A_n^-)$ is a wandering domain with the desired property. The example in (2) is constructed by applying quasiconformal surgery to the example in (1). Thus the construction by quasiconformal surgery ([Sh]) has an advantage in that we can construct the quasiregular map f_1 so that we can understand its dynamical behavior. However, as we can see from the construction, we have no concrete expression for f like in Theorem 5.1.

The connectivity of the wandering domain in Theorem 5.1 was not known for a long time but recently Bergweiler and Zheng showed that it is infinitely connected ([BerZ1, p. 309, §6]).

6. Topological properties of Julia components

From Theorem 3.9 in Section 3, it follows that if $J(f)$ is disconnected, then f has either a multiply connected Fatou component or an unbounded Fatou component. In this section we consider connected components of $J(f)$ in each case.

Definition 6.1. (1) We call a connected component of the Julia set a **Julia component**.

(2) $z \in J(f)$ is called a **buried point** if $z \notin \partial U$ for every Fatou component U.

(3) The set of all buried points of $J(f)$ is called the **residual Julia set** and is denoted by $J_0(f)$.

(4) A Julia component C is called a **buried component** if it satisfies $C \subset J_0(f)$. Furthermore if C consists of a single point, it is called a **buried singleton component**. □

6.1. **The case where f has a multiply connected Fatou component.** When f has a multiply connected Fatou component U, from Theorem 3.4 U is a wandering domain and all the Julia components are bounded [∵ Suppose that there exists an unbounded Julia component; then from Theorem 3.4 it intersects with $f^n(U)$, where n is sufficiently large, which contradicts the invariance of $F(f)$]. As in the case of Fatou components, for a Julia component C we denote the Julia component which contains $f^n(C)$ by C_n. If the orbit $\{C_n\}_{n=0}^{\infty}$ of C is bounded, we can show the following ([Ki6, p. 3, Theorem A, Corollary B]; see [Ki4], [Ki5] for partial results).

Theorem 6.2 (Kisaka, 2011). *Let f be a transcendental entire function with a multiply connected wandering domain and C be a Julia component with a bounded orbit $\{C_n\}_{n=0}^{\infty}$.*
(1) There exists a polynomial g such that C is quasiconformally homeomorphic to some Julia component of $J(g)$.
(2) If C is full (that is, the complement $C^c = \mathbb{C} \setminus C$ is connected), then C is a buried component.
(3) If C is a wandering Julia component (i.e., $C_m \cap C_n = \emptyset$, ${}^\forall m \neq n$), then C is a buried singleton component.
(4) For a repelling periodic point p, denote the Julia component containing p by $C(p)$. If $C(p)$ is full, then $C(p)$ is a buried component. If it is not full, then connected components of its complement $C(p)^c$ consist of some immediate attractive

basins, parabolic basins and Siegel disks and their preimages. p is a buried point, unless it is on the boundary of some immediate attractive basins, parabolic basins or Siegel disks. □

Incidentally, it was known that for an f with a multiply connected wandering domain there always exist buried singleton components ([Do, p. 213, Theorem 8.1]). Also they are dense in $J(f)$, of course. See also Theorem 7.18 (7) in Section 7.

When f is a polynomial, $I(f)$ coincides with the immediate attractive basin of the fixed point ∞ and $I(f)$ is a completely invariant Fatou component and $\partial I(f) = J(f)$. Hence in particular we have $J_0(f) = \emptyset$, that is, there are no buried points. On the other hand if f is transcendental, it is still not completely clear when $J_0(f) \neq \emptyset$ holds, that is, when there exists a buried point. For rational functions, the following conjecture is famous ([EL2, p. 578]):
Conjecture G (Makienko, 1990): Let f be rational; then $J_0(f) \neq \emptyset$ if and only if $F(f^2)$ has no completely invariant components for f^2.[34] □

The condition in the statement of Conjecture G is for the dynamics of f^2 and not for f, because there exists the following example:

Example 6.3. Let $f(z) = 1/z^d$ $(d \geq 2)$ and $U_0 := \mathbb{D}$, $U_1 := \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$. Then it follows that $f(U_0) = U_1,\ f(U_1) = U_0,\ J(f) = \partial U_0 = \partial U_1 = \{z \mid |z| = 1\}$ and U_0 and U_1 are not completely invariant by $f(z)$ but so by $f^2(z) = z^{d^2}$. □

Since for a transcendental f the number of completely invariant Fatou components is at most one ([Ba4, p. 33, Theorem]), it seems that the conjecture is as follows:

Conjecture G' (Makienko, 1990): Let f be a transcendental entire function. Then $J_0(f) \neq \emptyset$ if and only if $F(f)$ has no completely invariant components. □

On the other hand, there is a conjecture for completely invariant components of $F(f)$ ([Ba5, p. 278], [EL2, p. 622]).

Conjecture H (Baker, 1975): If a transcendental entire function f has a completely invariant Fatou component U, then $F(f) = U$, that is, $F(f)$ is connected.[35]
□

Eremenko and Lyubich showed that Conjecture H is true if $f \in \mathcal{S}$ ([EL3, p. 1008, Theorem 6]). Taking the above two conjectures into consideration, we come to the next conjecture:

[34] In [EL2, p. 578] they mentioned: "Let D_k be the components of $F(f)$. If $J(f) = \bigcup_k \partial D_k$, then among these components there is a completely invariant component." This is false because of Example 6.3. So we show a corrected version in Conjecture G. There are some papers which overlooked this point (for example, [DoFag, p. 144, Conjecture 3.6]). If $F(f)$ has no completely invariant component but $F(f^2)$ has, then f is equal to the function $f(z)$ in Example 6.3 up to conjugacy so we can restate the conjecture as follows: "$J_0(f) = \emptyset$ if and only if either $F(f)$ has a completely invariant component or $F(f)$ consists of two components."

[35] Conjecture H is raised in [EL2, p. 622] as the "Baker Conjecture" but there is no description like Conjecture H in [Ba5]. In [Ba5, p. 278] Baker mentioned "It is not clear whether the existence of completely invariant component of $F(f)$ precludes the existence of other components or not" and then showed a theorem which states that "If f has a completely invariant component, then f is univalent on every other Fatou component" ([Ba5, p. 278, Theorem 2]). Here we denoted "Baker Conjecture" following Eremenko and Lyubich, but it seems that this conjecture is in fact due to Eremenko and Lyubich.

Conjecture G”: Let f be transcendental. Then $J_0(f) = \emptyset$ if and only if $F(f)$ is connected. $\square$

For problems related to residual Julia sets, see the article by Domínguez and Fagella which includes detailed information and historical remarks ([DoFag]).

Incidentally there seems to be no results on topological properties of a Julia component C with unbounded orbit. So

Problem I: Let f be a transcendental entire function with a multiply connected Fatou component. Then investigate topological properties of Julia components with unbounded orbits. $\square$

6.2. **The case where f has an unbounded Fatou component.** The following is one of the results on Julia components for the case that $J(f)$ is disconnected and f has an unbounded Fatou component ([DoFag, p. 159, Proposition 6.7]). Here $h : (0, \infty) \to \mathbb{C}$ is called a **hair** of f if $h(t) \to \infty$ $(t \to \infty)$ and $h(t) \in J(f)$, $f^n(h(t)) \to \infty$ $(n \to \infty)$ for every $t \in (0, \infty)$. If $\lim_{t \to +0} h(t) =: z_0$ exists, we say h **lands** at z_0 and we call z_0 the **end point** of a hair h.

Theorem 6.4 (Domínguez-Fagella, 2008)**.** *If $E_\lambda(z) = \lambda e^z$ has an attracting periodic point of period greater than* 1*, then all but at most countably many hairs are buried components.*[36] $\square$

It seems that general properties are not known apart from this result for a very special function. So

Problem J: Investigate topological properties of Julia components for the case that $J(f)$ is disconnected and f has an unbounded Fatou component. $\square$

7. Topological properties of escaping sets and fast escaping sets

The escaping set $I(f)$ is completely invariant by the definition and $I(f^n) = I(f)$ $(n \in \mathbb{N})$ [$\because$ $I(f) \subset I(f^n)$ is obvious. Suppose that $z_0 \in I(f^n) \setminus I(f)$; then $\{f^{n_k}(z_0)\}$ is bounded for some $n_1 < n_2 < \cdots$. Since $\{n_k\} \subset \mathbb{N}$ is an infinite set, there exists an l with $0 \leq l < n$ such that $n_k \equiv l \pmod{n}$ for infinitely many n_k. Let us write these n_k as $np_k + l$ $(p_k \nearrow \infty)$; then $\{f^{np_k+l}(z_0)\}$ is bounded. Applying f^{n-l} we get $f^{n-l}(f^{np_k+l}(z_0)) = f^{n(p_k+1)}(z_0)$ and so $\{f^{n(p_k+1)}(z_0)\}$ is also bounded. This contradicts $z_0 \in I(f^n)$]. If f is a polynomial, $I(f)$ coincides with the immediate attractive basin of ∞ and hence it is open and it is either simply connected or infinitely connected according to $(\mathrm{sing}(f^{-1}) \setminus \{\infty\}) \cap I(f) = \emptyset$ or $\neq \emptyset$. (see Propositions 3.2, 3.7). If $J(f)$ is connected, then $I(f)$ is simply connected and in this case external rays are defined by using a Riemann map of $I(f)$. With this clue to go on, one can understand the dynamics on $J(f)$ ([Mil2, p. 188$\sim$, §18]).

For the transcendental case, as we mentioned in Section 1, the next result is the most fundamental and this is the starting point for the investigation of $I(f)$ ([E, p. 339, Theorem 1, p. 343, Theorem 2, Theorem 3]).

Theorem 7.1 (Eremenko, 1989)**.** *Let f be a transcendental entire function. Then*

$$I(f) \neq \emptyset, \quad I(f) \cap J(f) \neq \emptyset, \quad \partial I(f) = J(f).$$

[36]In [DoFag, p. 159, Proposition 6.7] they stated “Let $\lambda \in \mathbb{C}$ be such that $E_\lambda(z) = \lambda e^z$ has an attracting periodic orbit. Then $\cdots$” but if the period is one, that is, if it is a fixed point, then obviously the conclusion does not hold (see Example 3.8). They implicitly assume that the period is greater than one in their proof, so it seems that they just forgot to mention it.

Moreover, all connected components of $\overline{I(f)}$ *are unbounded.* □

Showing that $I(f) \neq \emptyset$ is essential for the proof of this theorem and Eremenko applied Wiman-Varilon theory ([Ha]) for this purpose. Once you proved $I(f) \neq \emptyset$, it is rather easy to show that $\partial I(f) = J(f)$ [$\because$ $\partial I(f) \subset J(f)$ can be easily seen by considering the classification of Fatou components. Next since $I(f) \neq \emptyset$, $I(f)$ is an infinite set and so there exists $z_0 \in I(f)$ which is non-exceptional. Then from Proposition 1.3, there exist preimages of z_0 (Remark: these belong to $I(f)$) in every neighborhood of every point $z \in J(f)$. On the other hand if $\mathrm{int}(I(f)) \neq \emptyset$, then $\{f^n|_{\mathrm{int}(I(f))}\}$ omits $J(f)$ so it is normal by Montel's Theorem. Hence $\mathrm{int}(I(f)) \subset F(f)$. Therefore we have $J(f) \subset \partial I(f)$]. Although $I(f)$ is the set of initial points with simple behavior, unlike the case of polynomials, $I(f)$ is in general neither open nor closed and it may have interior (for example, when there exists a Baker domain) or it may not. So it is not so easy to investigate its topological properties. Eremenko asserted the following in the paper ([E, p. 343, 344]).

Conjecture K (Eremenko, 1989): (1) All connected components of $I(f)$ are unbounded. Moreover,
(2) every $z_0 \in I(f)$ can be connected to ∞ by a curve in $I(f)$.[37] □

Before this conjecture, it was known that $I(f)$ consists of curves which extends to ∞ (hairs) in the case that the real exponential function $E_\lambda(z)$ has an attracting fixed point (that is, $0 < \lambda < 1/e$)[38] ([DeK, p. 50, Theorem] (1984)[39]). Also Devaney and Tangerman showed for the same exponential functions that $J(E_\lambda)$ is a Cantor bouquet ([DeT, p. 492$\sim$, §2]) and moreover in general $J(f)$ contains a Cantor bouquet if $f \in \mathcal{S}$ and it has a hyperbolic exponential tract[40] ([DeT, p. 497, Theorem 3.3] (1986)[41]). Later for $E_\lambda(z)$ a complete classification of points in $I(f)$ independent of the parameter λ was accomplished and as a result Schleicher and Zimmer solved Conjecture K for $E_\lambda(z)$ affirmatively ([SchZ, p. 396, Corollary 6.9]). Also this result was generalized to cosine family $ae^z + be^{-z}$ ([RoSch, p. 419, Theorem 6.4]). These results are all for functions $f \in \mathcal{S}$ and there is a result for functions in much wider class $\mathcal{B}$ ([Bara, p. 35, Theorem C]). Note that $f \in \mathcal{B}$ implies $I(f) \subset J(f)$, as we mentioned right after Example 2.10.

Theorem 7.2 (Barański, 2007)**.** *Let f be of finite order and* $\mathrm{sing}(f^{-1})$ *be contained in a compact subset of the immediate attractive basin U of an attracting fixed point. Then $J(f)$ consists of disjoint hairs homeomorphic to the half-line* $[0, \infty)$*. The end points of the hairs are the unique accessible points from U.* □

[37] In [E] Eremenko mentioned: "It is plausible that the set $I(f)$ has no bounded connected components" ([E, p. 343]). "It is plausible that the set $I(f)$ always has the following property: every point $z \in I(f)$ can be joined with ∞ by a curve in $I(f)$" ([E, p. 344]). Later some researchers started to call these statements "Eremenko's conjecture" and Eremenko himself did not clearly indicate his "conjecture".

[38] These curves were conjectured to be smooth ([DeK, p. 42]) and in fact it was proved to be of class C^∞ by Viana ([V, p. 1180, Theorem B]). This result also holds, for instance, for hairs which appear in the Julia set of $P(z)e^{Q(z)}$ (P, Q are polynomials, $\deg Q \geq 1$) ([KiSh2, p. 37$\sim$, §5]). On the other hand, it is an open problem to decide whether these hairs are always of class C^ω or not.

[39] In this paper there is only an outline of the proof.

[40] This is the one which is nowadays called a logarithmic tract (see Footnote 14) with some assumption on expandingness.

[41] In [DeT] they assumed $f \in \mathcal{S}$ but they used only boundedness of $\mathrm{sing}(f^{-1})$ in the discussion and did not use its finiteness. So the same results hold also for $f \in \mathcal{B}$.

Note that f in Theorem 7.2 satisfies $f \in \mathcal{B}$ from the condition and $F(f) = U$ holds by taking $P(f) \subset U$ into consideration.[42] Hence in particular, Conjecture K is true under the situation of Theorem 7.2. Also Rempe showed that at least Conjecture K (1) is true for a certain function $f \in \mathcal{B}$ ([Re1, p. 661, Theorem 1.1]):

Theorem 7.3 (Rempe, 2007)**.** *If $f \in \mathcal{B}$ and $P(f)$ is bounded, then every component of $I(f)$ is unbounded.* $\square$

The following is one of the highest achievements for this direction ([RRRS, p. 79, Theorem 1.2]):

Theorem 7.4 (Rottenfusser-Rückert-Rempe-Schleicher, 2011)**.** *Let $f \in \mathcal{B}$ be a function of finite order, or more generally a finite composition of such functions. Then $I(f)$ consists of hairs together with (if any) their end points which are in $I(f)$.* $\square$

Therefore in particular in this case, Conjecture K is true. Simultaneously in [RRRS] they constructed a counterexample for Conjecture K (2) ([RRRS, p. 78, Theorem 1.1, p. 117, Theorem 8.4] (Remark: $f \in \mathcal{B}$ implies $I(f) \subset J(f)$)).

Theorem 7.5 (Rottenfusser-Rückert-Rempe-Schleicher, 2011)**.** *There exists a hyperbolic*[43] *transcendental entire function $f \in \mathcal{B}$ such that every path-connected component of $J(f)$ is bounded. Moreover, there exists this kind of f such that $J(f)$ contains no non-trivial curves.* $\square$

Now the next result for Conjecture K by Rippon and Stallard seems the first general one which has no restrictions on classes of functions. Before stating the result, we define the first escaping set $A(f)$, which is a subset of $I(f)$.

Definition 7.6. Let $R > 0$ be such that $M(r, f) > r$ for $r \geq R$ and define

$$A_R(f) := \{z \mid |f^n(z)| \geq M^n(R, f) \text{ for } {}^\forall n \in \mathbb{N}\},$$

where $M^n(r, f)$ is the n-th iterate of the function $M(r, f) := \sup_{|z|=r} |f(z)|$ with variable r, that is, $M^n(r, f) = M(M^{n-1}(r, f), f)$. Then the next $A(f)$ is called the **fast escaping set** of f:

$$A(f) := \bigcup_{l=0}^{\infty} f^{-l}(A_R(f)) = \{z \mid {}^\exists l \in \mathbb{N} \cup \{0\},\ {}^\forall n \in \mathbb{N}\ \ |f^{n+l}(z)| \geq M^n(R, f)\}. \quad \square$$

$A(f)$ was first defined by Bergweiler and Hinkkanen ([BerHi, p. 566]).[44] $A(f)$ is independent of the choice of $R > 0$ ([RiSt5, p. 793, Theorem 2.2]).[45] For the $R > 0$ in the definition we have only to take $R > 0$ with $R > \min_{z \in J(f)} |z|$ [$\because$ Let $r > \min_{z \in J(f)} |z|$ and suppose that $M(r, f) \leq r$; then $f^n(B(r, 0)) \subset B(r, 0)$ where $B(r, 0) := \{z \mid |z| < r\}$ and by Montel's Theorem (Theorem 1.1) we have

[42] Hence under the situation of Theorem 7.2 we have $J_0(f) = \emptyset$, which is the situation we discussed in Section 6. Moreover in this case every hair $h(t)$ is a Julia component and since $h(t) \subset \partial U$, this is not a buried component. But the only point in $h(t)$ which is accessible in U is $h(0)$. This shows that all the points other than the end points of $h(t)$'s are "buried", in a sense.

[43] See Footnote 28.

[44] The definition adopted here is by Rippon and Stallard and this is a little different from the original definition by Bergweiler and Hinkkanen but they are equivalent ([RiSt5, p. 798, Corollary 2.7]). The term "first escaping set" is by Rippon and Stallard.

[45] In [BerHi, p. 570], they mentioned this fact without proof.

$B(r, 0) \subset F(f)$. This contradicts $B(r, 0) \cap J(f) \neq \emptyset$]. Also the condition for $R > 0$ in the definition is equivalent to the condition $M^n(R, f) \to \infty$ $(n \to \infty)$. $A(f)$ is a subset of $I(f)$ and satisfies the similar properties as $I(f)$ ([BerHi, p. 570], [RiSt5, p. 798, Theorem 2.8]).

Theorem 7.7. *$A(f)$ is completely invariant and*

$$A(f) \neq \emptyset, \quad A(f) \cap J(f) \neq \emptyset, \quad \partial A(f) = J(f).$$

Also we have $A(f^n) = A(f)$ $(n \in \mathbb{N})$. □

The results by Rippon and Stallard are as follows ([RiSt2, p. 1120, Theorem 1, 2]). (1) is a rather weak result for Conjecture K but is very general and (2) shows that if f has a multiply connected Fatou component, then $I(f)$ is connected and unbounded, which asserts a much stronger conclusion than the conjecture.

Theorem 7.8 (Rippon-Stallard, 2005)**.** *(1) Every component of $A(f)$ is unbounded. In particular, $I(f)$ contains at least one unbounded component.*

(2) If f has a multiply connected Fatou component, then

- (a) *$A(f)$ is connected and unbounded and $\overline{U} \subset A(f)$ holds for every multiply connected Fatou component U.*
- (b) *$I(f)$ is connected and unbounded.* □

Recently research on $A(f)$ has been widely developed by Rippon and Stallard and as a result, various properties of $I(f)$ have been revealed. We explain this in what follows. There are two key ideas. One is dividing $A(f)$ into "levels" and considering each level individually.

Definition 7.9. Let $R > 0$ be such that $M(r, f) > r$ for $r \geq R$. For each $l \in \mathbb{Z}$, the ***l*-th level** $A_R^l(f)$ of $A(f)$ (with respect to R) is defined by

$$A_R^l(f) := \{z \mid |f^n(z)| \geq M^{n+l}(R, f) \text{ for } n \in \mathbb{N} \text{ with } n + l \in \mathbb{N}\}.$$

(In particular, $A_R^0(f) := A_R(f)$.) That is, the $(-l)$-th level is defined by

$$A_R^{-l}(f) := \{z \mid |f^{n+l}(z)| \geq M^n(R, f) \text{ for } n \in \mathbb{N} \text{ with } n + l \in \mathbb{N}\}.$$ □

Each $A_R^l(f)$ is closed and satisfies

$$f(A_R^l(f)) \subset A_R^{l+1}(f) \subset A_R^l(f) \ (l \in \mathbb{Z}).$$

Also by the definition

$$A(f) = \bigcup_{l=0}^{\infty} A_R^{-l}(f) \quad \text{and} \quad A_R^{-l}(f) \subset A_R^{-(l+1)}(f),$$

that is, $A(f)$ can be represented by an increasing union of closed sets. Now their first result is a refinement of Theorem 7.8 ([RiSt5, p. 788, Theorem 1.1, p. 789, Theorem 1.2, 1.3]).

Theorem 7.10 (Rippon-Stallard, 2012)**.** *Let $R > 0$ be such that $M(r, f) > r$ for $r \geq R$ and let $l \in \mathbb{Z}$.*

(1) Each component of $A_R^l(f)$ is closed and unbounded; in particular, each component of $A(f)$ is unbounded.

(2) Let U be a Fatou component with $U \cap A_R^l(f) \neq \emptyset$. Then

(i) $\overline{U} \subset A_R^{l-1}(f)$, (ii) if, in addition, U is simply connected, then $\overline{U} \subset A_R^l(f)$.

(3) The following two conditions are equivalent:

(a) All the components of $A^l_R(f) \cap J(f)$ are unbounded.
(b) f has no multiply connected Fatou components.
In particular if f has no multiply connected Fatou components, then all the components of $A(f) \cap J(f)$ are unbounded. □

The second key is the concept of "spider's web":

Definition 7.11. $E \subset \mathbb{C}$ is called an **(infinite) spider's web** if the following two conditions are satisfied: (1) E is connected. (2) There exists a sequence of bounded simply connected domains G_n $(n \in \mathbb{N})$ such that

$$G_n \subset G_{n+1}, \quad \partial G_n \subset E, \quad \bigcup_{n=1}^{\infty} G_n = \mathbb{C}.$$

Remark 7.12. A set extending unboundedly to ∞ with the shape of a spider's web in nature is an example of an (infinite) spider's web by the above definition, but a set with a different shape, that is, a set with non-empty interior as, for example,

$$\Big(\bigcup_{n=1}^{\infty} \{z \mid 2n-1 < |z| < 2n\} \Big) \cup \mathbb{R} \cup i\mathbb{R}$$

is also a spider's web. Moreover, $\mathbb{C}$ itself is obviously a spider's web. □

Now if $I(f)$ is a spider's web, then since $I(f)$ is connected, it follows in particular that Eremenko's conjecture (1) is true. Indeed in Theorem 7.8 (2), what they really showed is that if f has a multiply connected Fatou component, then $I(f)$ (and $A(f)$) is a spider's web.[46] The next is a little refinement of this result ([RiSt5, p. 790, Theorem 1.4]).

Theorem 7.13 (Rippon-Stallard, 2012)**.** *Let $R > 0$ be such that $M(r, f) > r$ for $r \geq R$. If $A(f)^c$ has a bounded component, then $A_R(f)$, $A(f)$ and $I(f)$ are spider's webs.* □

Hence in particular if $A_R(f)$ is a spider's web, then a repelling periodic point p belongs to $A_R(f)^c$ and the component of $A_R(f)^c$ containing p is bounded and it follows from the above theorem that $A(f)$ and $I(f)$ are spider's webs. So next we raise some sufficient conditions for $A_R(f)$ to be a spider's web ([RiSt5, a part of the Theorem 1.9]).

Theorem 7.14 (Rippon-Stallard, 2012)**.** *Let $R > 0$ be such that $M(r, f) > r$ for $r \geq R$. Under one of the following conditions, $A_R(f)$ is a spider's web:*
(a) f has a multiply connected Fatou component.
(b) There exist $m \in \mathbb{N}$, $m \geq 2$ and $r_0 > 0$ such that

$$\log\log M(r, f) < \frac{\log r}{\log^m r}, \quad {}^{\forall} r > r_0 \text{ (where } \log^m r \text{ is the } m\text{-th iterate of } \log r\text{)}.$$

(c) $\rho(f) < 1/2$ and f has regular growth, that is, there exist $m \in \mathbb{R}$, $m > 1$ and $\{r_n\}_{n=0}^{\infty}$ such that

$$r_n \geq M^n(R, f), \quad M(r_n, f) \geq r_{n+1}^m.$$

□

[46] In [RiSt2], the concept and the term "spider's web" had not been defined yet.

We show some examples of f such that $A_R(f)$ (and hence also $A(f)$, $I(f)$) is a spider's web by using this theorem.

Example 7.15. (1) Since the function f we show in Section 5 has a multiply connected wandering domain, $A_R(f)$ is a spider's web by Theorem 7.14 (a). Also for the same f, $A_R(f)$, $A(f)$ and $I(f)$ are spider's webs with non-empty interior.

(2) Bergweiler and Eremenko constructed an f with arbitrarily small (slow) growth[47] which satisfies $J(f) = \mathbb{C}$ ([BerE, p. 1577, Theorem 1]), and Baker and Boyd independently constructed an f with arbitrarily small growth such that $f^n(z) \to 0$ for every $z \in F(f)$ ([Ba11, p. 369, Theorem 1], [Bo, p. 317, Theorem 1]). In particular, these f can be constructed so that they satisfy the condition (b) in Theorem 7.14, so $A_R(f)$ are spider's webs for these f. Moreover, obviously $F(f) \cap I(f) = \emptyset$ holds for these f, so $A_R(f)$, $A(f)$ and $I(f)$ are spider's webs with an empty interior ([RiSt5, p. 815]).

(3) Since $f(z) = (\cos z^{1/4} + \cosh z^{1/4})/2 = \sum_{n=0}^{\infty} z^n/(4n)! = 1 + z/4! + z^2/8! + z^3/12! + \cdots$ satisfies the condition (c), $A_R(f)$ (and hence also $A(f)$ and $I(f)$) is a spider's web ([RiSt5, p. 29]). □

Other than these, there are various kinds of sufficient conditions for $A_R(f)$ to be a spider's web and a lot of examples ([Si], [MihP]).

Next we see some properties which hold when $A_R(f)$ is a spider's web ([RiSt5, Theorem 1.5∼1.8]).

Theorem 7.16 (Rippon-Stallard, 2012). *Let $R > 0$ be such that $M(r, f) > r$ for $r \geq R$ and let $A_R(f)$ be a spider's web.*

(1) If f has no multiply connected Fatou components, then each of the following is a spider's web:

$$A_R(f) \cap J(f), \quad A(f) \cap J(f), \quad I(f) \cap J(f), \quad J(f).$$

(2) f has no unbounded Fatou components.

(3) Every component of $A(f)^c$ is compact.

(4) For every $z_0 \in J(f)$, there exist $z_n \in A(f)^c$ such that $z_n \to z_0$ $(n \to \infty)$ and each z_n belongs to a different component of $A(f)^c$. □

From Theorem 7.16 (2), we can see that there is a relation between the property that $A_R(f)$ is a spider's web and the Baker's conjecture (Conjecture B). For example, the condition (c) in Theorem 7.14 gives a new partial answer to Conjecture B.[48]

Theorem 7.17 (Rippon-Stallard, 2012). *Let $R > 0$ be such that $M(r, f) > r$ for $r \geq R$ and let $A_R(f)$ be a spider's web.*

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[47] In general, the assertion that "there exists a transcendental entire function f with arbitrarily small (slow) growth which satisfies a property P" means the following: "Let $\phi(r)$ $(r > 0)$ be an arbitrary monotone increasing function with $\phi(r) > 0$ and $\lim_{r\to\infty} \phi(r) = +\infty$. Then there exists a transcendental entire function f which satisfies property P and $\log M(r, f) < \phi(r) \log r$". If an entire function f satisfies $\log M(r, f) = O(\log r)$, then f is a polynomial. Taking this fact into account, the above definition seems natural.

[48] Of course, if (a) or (b) in Theorem 7.14 holds, then we can see that Conjecture B is true via Theorem 7.16. But instead of using Theorem 7.16, if (a) holds, then we can see this from Theorem 3.4. Also the condition (b) is almost the same as or is a little bit stronger condition than the condition (ii) which we stated right after Corollary 3.10 in Section 3.2.

(1) Every $z_0 \in I(f)$ belongs to an unbounded continuum C[49] *of $I(f)$ and $f^n(z) \to \infty$ uniformly on C for every $z \in C$.*

(2) Let K be a component of $A(f)^c$; then either $K \cap I(f) = \emptyset$ or $f^n(z) \to \infty$ uniformly on K.

(3) There is no curve $\gamma(t)$ such that $\gamma(t) \to \infty$ $(t \to \infty)$ and $f|\gamma$ is bounded. In particular, $f \notin \mathcal{B}$. □

As we mentioned in Theorem 7.4, if $f \in \mathcal{B}$ is of finite order then Eremenko's conjecture (Conjecture K) is true. On the other hand, Theorem 7.17 (3) shows that an f such that $A_R(f)$ is a spider's web does not belong to class $\mathcal{B}$. Nevertheless Theorem 7.17 (1) shows that the similar result as Conjecture K (2) holds for this type of f.

Moreover, Osborne showed various kinds of results on the components of $A(f)^c$ when $A_R(f)$ is a spider's web ([O2]). This is a further refinement of Theorem 7.17 (2).

Theorem 7.18 (Osborne, 2012)**.** *Let $R > 0$ be such that $M(r, f) > r$ for $r \geq R$ and let $A_R(f)$ be a spider's web. Let K be a component of $A(f)^c$. Then the following holds:*

(1) $\partial K \subset J(f)$, $\mathrm{int}(K) \subset F(f)$. In particular, $\overline{U} \subset K$ for every Fatou component U with $K \cap \overline{U} \neq \emptyset$.

(2) Every neighborhood of K contains a closed subset L of $A(f) \cap J(f)$ surrounding K (i.e., K is contained in a bounded component of L^c). Also if $\mathrm{int}(K) = \emptyset$, then K consists of buried points of $J(f)$.

(3) If f has a multiply connected Fatou component, every neighborhood of K contains a multiply connected Fatou component surrounding K. If, in addition, $\mathrm{int}(K) = \emptyset$, then K is a buried component of $J(f)$.

(4) $A(f)^c$ has uncountably many components K with (a) (resp. (b), (c)):

(a) $\{f^n(K)\}_{n=0}^{\infty}$ is bounded;

(b) $\{f^n(K)\}_{n=0}^{\infty}$ is unbounded but contains a bounded suborbit;

(c) $f^n(K) \to \infty \quad (n \to \infty)$.

(5) $J_0(f) \neq \emptyset$.

(6) If $\{f^n(K)\}_{n=0}^{\infty}$ is bounded, then the following holds:

(i) There exists a polynomial g with $\deg g \geq 2$ such that each component of $A(f)^c$ in the orbit of K is quasiconformally homeomorphic to a component of the filled Julia set[50] *of g.*

(ii) K is a single point if and only if $\{f^n(K)\}_{n=0}^{\infty}$ contains no periodic component of $A(f)^c$ containing a critical point.

(iii) If $\mathrm{int}(K) \neq \emptyset$, then $\mathrm{int}(K)$ consists of Fatou components other than wandering domains. Also if all these Fatou components are not Siegel disks, then all these are Jordan domains.

(iv) All but countably many components of $A(f)^c$ with bounded orbits consist of a single point.

[49] A continuum is a connected compact set with more than one point. Here we stated exactly the same as in [RiSt5, p. 791, Theorem 1.7] but strictly speaking, this is a little bit strange. Of course, what this means is that "$C \cup \{\infty\} \subset \widehat{\mathbb{C}}$ is a continuum".

[50] The filled Julia set of a polynomial f (it is called the filled-in Julia set in old papers) is defined by

$$I(f)^c = \{z \mid f^n(z) \not\to \infty\} = \{z \mid O^+(z) \text{ is bounded}\}$$

and is usually denoted by $K(f)$.

(7) $A(f)^c$ *has periodic singleton components and these are dense in* $J(f)$*. Also if* f *has a multiply connected Fatou component, then these are buried singleton components.* □

We conclude this section with a result by Rippon and Stallard concerning topological properties of $I(f)$ ([RiSt3, p. 2814, Theorem 4.1]).

Theorem 7.19 (Rippon-Stallard, 2011)**.** *(1) Every bounded component of* $I(f)$ *intersects* $J(f)$*.*

(2) $I(f) \cup \{\infty\} \subset \widehat{\mathbb{C}}$ *is connected.* □

8. Concluding remarks

In this paper we mainly explained topological properties of invariant sets, like Fatou sets or Julia sets of transcendental entire functions. Of course, what we showed here are only a small part of known results. Other than topological properties, as we introduced in Section 1 by raising an example like exponential functions, measure theoretic properties of $J(f)$ or metric properties like Hausdorff dimension have been widely investigated. A paper by Schleicher ([Sch]) is an article which reviews the research of transcendental dynamics including these topics. Although there are some overlaps with this paper, it provides a lot of interesting and useful information including what we were not able to mention in this paper, so we recommend it. Topics which are investigated in the research of dynamics of transcendental entire functions are also investigated in the research of dynamics of transcendental meromorphic functions. A major difference is that the former is a dynamical system on $\mathbb{C}$ but for the latter, we can define the iterate of f only finitely many times at points which are eventually mapped to a pole, so it cannot be considered to be a dynamical system on $\mathbb{C}$. However, we can define basic invariant sets like Fatou sets or Julia sets and similar results as in Section 2 hold (their proofs may be different, though). See a very useful and valuable paper by Bergweiler ([Ber1]).

Speaking of the research on complex dynamics, it seems that the dynamics of polynomials or rational functions is the main stream compared to that of transcendental entire functions. The crucial differences between the two are that the phase space is compact ($\widehat{\mathbb{C}}$) and the map is finite to one for the former, whereas it is non-compact ($\mathbb{C}$) and it is infinite to one for the latter. In our opinion, we think that in mathematics, in general, researchers consider problems quite often under some finiteness conditions (for example, compactness). It is true that it often happens that one becomes unable to do anything without any finiteness assumptions. Class $\mathcal{S}$ or class $\mathcal{B}$ have been extensively investigated in the research of transcendental entire dynamics, because one can get results by assuming some finiteness conditions. This is why polynomial dynamics is the main stream for now and it seems that a lot of researchers think they should investigate polynomial dynamics first. Nevertheless by applying various kinds of theories of transcendental entire (or meromorphic) functions effectively, a variety of facts we explained in this paper, especially very interesting phenomena which never occur for polynomial dynamics have become clear. We expect a lot of people especially of the younger generation to join this very fascinating and challenging research field.

We conclude this paper by introducing an intriguing application of transcendental entire dynamics. There is a so-called "$(3n+1)$-problem",[51] which is one of the famous open problems in number theory:

$(3n+1)$-Problem: For $n \in \mathbb{N}$ let

$$\varphi(n) := \begin{cases} 3n+1 & n : \text{odd}, \\ n/2 & n : \text{even}. \end{cases}$$

Then the orbit of every point $n \in \mathbb{N}$ eventually mapped to the periodic cycle $1 \mapsto 4 \mapsto 2 \mapsto 1$? □

For this φ, for example, it is not even known whether there exists an unbounded orbit (that is, an orbit with $\varphi^k(n) \to \infty$ $(k \to \infty)$) or not.[52]

For this problem Schleicher et al. consider the following transcendental entire function:

$$f(z) := \frac{z}{2} + \left(z + \frac{1}{2}\right)\frac{1 - \cos \pi z}{2} + \frac{1}{\pi}\left(\frac{1}{2} - \cos \pi z\right)\sin \pi z + h(z)\sin^2 \pi z,$$

where $h(z)$ is an arbitrary entire function. This f coincides with the following $\widetilde{\varphi}$ on $\mathbb{N}$:

$$\widetilde{\varphi}(n) := \begin{cases} (3n+1)/2 & n : \text{odd}, \\ n/2 & n : \text{even}. \end{cases}$$

Note that if n is odd, then $3n+1$ is even, so it is equivalent to consider $\widetilde{\varphi}$ instead of φ. They showed that even if this f has a Baker domain U, $U \cap \mathbb{N} = \emptyset$ holds ([LSchW, p. 247, Proposition 3.6]), and furthermore they conjectured as follows ([LSchW, p. 247, Conjecture 3.7]):

Conjecture L (Letherman-Schleicher-Wood, 1999): For a choice of $h(z)$, the Fatou set of the corresponding f contains $\mathbb{N}$ and every wandering domain (if any) does not intersect $\mathbb{N}$. □

If Conjecture L is true, then there are no unbounded orbits for $\widetilde{\varphi}$, and hence also for φ, and therefore it follows that every orbit is eventually periodic. Furthermore if you choose a suitable $h(z)$ ingeniously, it may be possible to solve the original $(3n+1)$-Problem affirmatively.

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[51]This problem is also called the "Collatz problem", "Kakutani problem", "Ulam problem", etc. See [Wi] and [La] which are more recent for the history, present and future of this problem.

[52]For a dynamical system f on $\mathbb{N}$, which we should call a 0-dimensional dynamic, we can easily see the following, since there are only finitely many points in a bounded subset of $\mathbb{N}$:

(1) If $O^+(n)$ is bounded, then n is an eventually periodic point.

(2) If $O^+(n)$ is unbounded, then $f^k(n) \to \infty$ $(k \to \infty)$.

Thus the dynamics of f are completely understood from the viewpoint of the general theory. But it is sometimes extremely difficult to tell what kind of orbits actually appear for a given specific f, like in this case of φ.

suggestions to improve this article. The author would like to really thank all of them.

Table of Notation

- $F(f),\ J(f)$: Fatou set, Julia set (beginning of Section 1)
- $O^{\pm}(z_0)$: forward and backward orbit of z_0 (after Proposition 1.2)
- $I(f)$: escaping set (after Example 1.5)
- $\mathrm{sing}(f^{-1})$: the set of all singular values (Definition 2.1)
- $P(f)$: post-singular set (Definition 2.1)
- $\mathcal{S},\ \mathcal{B}$: class $\mathcal{S},\ \mathcal{B}$ (Definition 2.9)
- $\mathrm{conn}(U)$: connectivity of a domain U (Definition 3.1)
- $M(r, f)$: maximal modulus function of f (after Corollary 3.10)
- $\rho(f)$: the order of f (after Corollary 3.10)
- $J_0(f)$: residual Julia set (Definition 6.1)
- $A_R(f),\ A(f)$: fast escaping set (Definition 7.6)
- $A_R^l(f)$: l-the level of $A(f)$ (Definition7.9)

Added in Proof (July 2017): This paper was submitted in May 2014. Since then a lot of developments have been achieved in this research area. The following are only some of them and especially (1) has big news on the results by Baker ([Ba4, p.33, Theorem], [Ba5, p.278, Theorem 2]). I heard this news from David Marti-Pete in March, 2017 and learned the details from the paper by Lasse Rempe-Gillen and Dave Sixsmith [ReSi2], which David kindly asked them to send me. Also Dave Sixsmith kindly read through this paper and gave me very valuable comments on the recent developments. I would like to thank all of them.

(1) Around the end of 2016, Julien Duval pointed out that there is a serious gap in the proofs of [Ba4, p.33, Theorem] and [Ba5, p.278, Theorem 2], which were proved by a similar argument. So the two assertions "An entire function f can have at most one completely invariant Fatou component" together with "If f has a completely invariant component, then f is univalent on every other Fatou component" seem to be still open now. Unfortunately there are several papers which use these results to show other theorems, some of which can be recovered and some of which are considered still open. For the details, see [ReSi2].

(2) Concerning Footnote 28, Rempe-Gillen and Sixsmith gave a definitive definition of hyperbolicity for transcendental entire functions in [ReSi1, p.786, Theorem and Definition 1.3]. In this paper they also compared and discussed their definition and several other previously given definitions of hyperbolicity.

(3) Concerning the Problem J, Rempe-Gillen investigated hyperbolic entire functions (in the sense of the one in [ReSi1]) with connected Fatou set and under an additional assumption, he gave a complete topological classification of Julia components ([Re2]).

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