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THE STATIONARY MEASURE OF THE STOCHASTIC TRANSPORT PROCESS WITH REFLECTING BARRIERS IN A SEMI-MARKOV ENVIRONMENT

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Abstract. The stationary distribution is studied for the process described by stochastic evolution differential equations with reflecting barriers in a semi-Markov environment.

The transport process in a Markov environment is described by the equation [1, 2]

$$\frac{dv(t)}{dt} = C(\kappa(t), v(t)) \tag{1}$$

where $\kappa(t)$ is a semi-Markov process. Let

$$G = X \cup Y, \qquad X = \{x_1, x_2, \dots, x_n\}, \qquad Y = \{y_1, y_2, \dots, y_m\}$$

be the phase space of the semi-Markov process $\kappa(t)$, let $P = \{p_{\alpha\beta}, \alpha, \beta \in G\}$ be the matrix of transition probabilities of the embedded (into $\kappa(t)$) ergodic Markov chain κ_l, $l \in N$, that is, $p_{\alpha\beta} = \mathsf{P}\left\{\kappa_{l+1} = \alpha \,/\, \kappa_l = \beta\right\}$, and let τ_α be the time spent by the chain κ_l at a state $\alpha \in G$. We assume that τ_α is a random variable with a general distribution function $F_\alpha(t)$. We also assume the following.

C1. The distribution F_α is absolute continuous with the density $f_\alpha(t) = dF_\alpha(t)/dt$ such that the first two moments

$$m_\alpha = \int_0^\infty t f_\alpha(t)\,dt, \qquad m_\alpha^{(2)} = \int_0^\infty t^2 f_\alpha(t)\,dt$$

are finite for all $\alpha \in G$.

Let V_0, V_1 and $a_i, b_i \in \mathbf{R}$ be such that $V_0 < V_1$ and $a_i > 0$, $b_i > 0$ for $i = 1, \dots, n$. Assume that the function C on the right hand side of equation (1) is such that

$$\text{for } x_i \in X,\ i = 1, \dots, n, \qquad C(x_i, v) = \begin{cases} -a_i, & V_0 < v \le V_1, \\ 0, & v = V_0, \end{cases}$$

$$\text{for } y_j \in Y,\ j = 1, \dots, m, \qquad C(y_j, v) = \begin{cases} b_j, & V_0 \le v < V_1, \\ 0, & v = V_1. \end{cases}$$

Functions C of this type appear in problems of the efficiency of multiphase systems with bunkers [3, 4].

We introduce a three component process ξ on the phase space $Z = [0, \infty) \times G \times [V_0, V_1]$:

$$\xi(t) = (\tau(t), \kappa(t), v(t)),$$

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where

$$\tau(t) = t - \sup\{u \le t \colon \kappa(t) \ne \kappa(u)\}.$$

Our aim is to study the stationary measure of the process $\xi(t)$. Note that $\xi(t)$ is a Markov process [5, 6]. Its infinitesimal operator is given by [3, 7]

$$A\varphi(\tau,\alpha,v) = \frac{\partial}{\partial\tau}\varphi(\tau,\alpha,v) + r_\alpha(\tau)[P\varphi(0,\alpha,v) - \varphi(\tau,\alpha,v)] + C(\alpha,v)\frac{\partial}{\partial v}\varphi(\tau,\alpha,v)$$

and the boundary conditions are

$$\varphi'_\tau(\tau,x,V_1) = \varphi'_\tau(\tau,y,V_0) = 0, \qquad x \in X,\ y \in Y,$$

where $r_\alpha(\tau) = f_\alpha(\tau)/(1 - F_\alpha(\tau))$ and

$$P\varphi(0,\alpha,v) = \sum_{\beta\in G} p_{\alpha\beta}\varphi(0,\beta,v).$$

If the stationary distribution $\rho(\cdot)$ exists for the process $\xi(t)$, then

$$\int_Z A\,\varphi(dz)\,\rho(dz) = 0 \tag{2}$$

for all functions $\varphi(\cdot)$ of the domain of the operator A. It follows from the properties of the process $\xi(t)$ that the points (τ,x,V_0), $x \in X$, and (τ,y,V_1), $y \in Y$, of the phase space Z are the atoms of the stationary measure $\rho(\cdot)$. In what follows, we denote these atoms by $\rho[\tau,x,V_0]$ and $\rho[\tau,y,V_1]$, respectively. The density of the measure is denoted by $\rho(\tau,x,v)$. Changing the order of integration in the integral on the left hand side of (2), we obtain $A^*\rho = 0$ where A^* is the conjugate operator, namely

$$\begin{aligned}
&\int_Z A\varphi(z)\,\rho(dz)\\
&\quad= \sum_{\alpha\in G}\Big[\int_0^\infty\int_{V_0}^{V_1} \rho(\tau,\alpha,v)\frac{\partial}{\partial\tau}\varphi(\tau,\alpha,v)\,dv\,d\tau + \int_0^\infty\int_{V_0}^{V_1} r_\alpha(\tau)\rho(\tau,\alpha,v)P\varphi(0,\alpha,v)\\
&\qquad\quad - \int_0^\infty\int_{V_0}^{V_1} r_\alpha(\tau)\rho(\tau,\alpha,v)\varphi(\tau,\alpha,v) + \int_0^\infty\int_{V_0}^{V_1} \rho(\tau,\alpha,v)C(\alpha,v)\frac{\partial}{\partial v}\varphi(\tau,\alpha,v)\Big]\\
&\qquad + \sum_{x\in X}\Big[\int_0^\infty \rho[\tau,\alpha,V_0]\frac{\partial}{\partial\tau}\varphi(\tau,x,V_0)\,d\tau + \int_0^\infty r_x(\tau)\rho[\tau,x,V_0]P\varphi(0,x,V_0)\\
&\qquad\qquad - \int_0^\infty r_x(\tau)\rho[\tau,x,V_0]\varphi(\tau,x,V_0)\Big]\\
&\qquad \times \sum_{y\in Y}\Big[\int_0^\infty \rho[\tau,y,V_1]\frac{\partial}{\partial\tau}\varphi(\tau,y,V_1)\,d\tau + \int_0^\infty r_y(\tau)\rho[\tau,y,V_1]P\varphi(0,y,V_1)\\
&\qquad\qquad - \int_0^\infty r_y(\tau)\rho[\tau,y,V_1]\varphi(\tau,y,V_1)\Big]\\
&\quad= \sum_{\alpha\in G}\Big[\int_{V_0}^{V_1}\Big\{\varphi(\infty,\alpha,v)\rho(\infty,\alpha,v) - \varphi(0,\alpha,v)\rho(0,\alpha,v)\\
&\qquad\qquad - \int_0^\infty \varphi(\tau,\alpha,v)\frac{\partial}{\partial\tau}\rho(\tau,\alpha,v)\Big\}\,dv\\
&\qquad + \int_{V_0}^{V_1}\int_0^\infty r_\alpha(\tau)\rho(\tau,\alpha,v)\,d\tau P\varphi(0,\alpha,v)\,dv\\
&\qquad - \int_0^\infty\int_{V_0}^{V_1} \varphi(\infty,\alpha,v)r_\alpha(\tau)\rho(\tau,\alpha,v)\,dv\,d\tau
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^\infty \big\{\varphi(\tau,\alpha,V_1)C(\alpha,V_1)\rho(\tau,\alpha,V_1-)\\
&\qquad\qquad -\varphi(\tau,\alpha,V_0)C(\alpha,V_0)\rho(\tau,\alpha,V_0+)\big\}\,d\tau\\
&\qquad\qquad\qquad +\int_0^\infty\int_{V_0}^{V_1}\varphi(\tau,\alpha,v)C(\alpha,v)\frac{\partial}{\partial v}\rho(\tau,\alpha,v)\,dv\,d\tau\Big]\\
&+\sum_{x\in X}\Big[\rho[\infty,x,V_0]\varphi(\infty,x,V_0)-\rho[0,x,V_0]\varphi(0,x,V_0)\\
&\qquad -\int_0^\infty \varphi(\tau,x,V_0)\frac{\partial}{\partial\tau}\rho[\tau,x,V_0]\,d\tau+\int_0^\infty r_x(\tau)\rho[\tau,x,V_0]P\varphi(0,x,V_0)\,d\tau\\
&\qquad\qquad -\int_0^\infty \varphi(\tau,x,V_0)r_x(\tau)\rho[\tau,x,V_0]\,d\tau\Big]\\
&+\sum_{y\in Y}\Big[\rho[\infty,y,V_1]\varphi(\infty,y,V_1)-\rho[0,y,V_1]\varphi(0,y,V_1)\\
&\qquad -\int_0^\infty \varphi(\tau,y,V_1)\frac{\partial}{\partial\tau}\rho[\tau,y,V_1]\,d\tau+\int_0^\infty r_y(\tau)\rho[\tau,y,V_1]P\varphi(0,y,V_1)\,d\tau\\
&\qquad\qquad -\int_0^\infty \varphi(\tau,y,V_1)r_y(\tau)\rho[\tau,y,V_1]\,d\tau\Big]\\
&=0.
\end{aligned}
$$

Taking into account equality (2), we get the differential equations

$$
C(\alpha,v)\frac{\partial}{\partial v}\rho(\tau,\alpha,v)+r_\alpha(\tau)\rho(\tau,\alpha,v)+\frac{\partial}{\partial\tau}\rho(\tau,\alpha,v)=0, \tag{3}
$$

integral equations

$$
\sum_{\beta\in G}\int_0^\infty r_\beta(\tau)\rho(\tau,\beta,v)d\tau p_{\beta\alpha}=\rho(0,\beta,v),\qquad \alpha\in G, \tag{4}
$$

and the boundary conditions $\rho(\infty,\alpha,v)=0$, $\alpha\in G$, for the densities as well as the differential equations

$$
\begin{aligned}
&\frac{d}{d\tau}\rho[\tau,x,V_0]+r_x(\tau)\rho[\tau,x,V_0]-a_x\rho(\tau,x,V_0+)=0,\\
&\frac{d}{d\tau}\rho[\tau,y,V_1]+r_y(\tau)\rho[\tau,y,V_1]-b_y\rho(\tau,y,V_1-)=0
\end{aligned} \tag{5}
$$

for the atoms of the distribution where

$$
\rho(\tau,x,V_0+)=\lim_{v\downarrow V_0}\rho(\tau,x,v),\qquad \rho(\tau,y,V_1)=\lim_{v\uparrow V_1}\rho(\tau,y,v).
$$

The atoms of the measure satisfy the integral equations

$$
\begin{aligned}
&\sum_{x\in X}\int_0^\infty r_x(\tau)\rho[\tau,x,V_0]\,d\tau p_{xz}=\rho[0,z,V_0],\qquad z\in X,\\
&\sum_{y\in Y}\int_0^\infty r_y(\tau)\rho[\tau,y,V_1]\,d\tau p_{yz}=\rho[0,z,V_1],\qquad z\in Y,\\
&\sum_{x\in X}\int_0^\infty r_x(\tau)\rho[\tau,x,V_0]\,d\tau p_{xy}=b_y\int_0^\infty \rho(\tau,y,V_0+),\qquad y\in Y,\\
&\sum_{y\in Y}\int_0^\infty r_y(\tau)\rho[\tau,y,V_1]\,d\tau p_{yx}=a_x\int_0^\infty \rho(\tau,x,V_1-)\,d\tau,\qquad x\in X,
\end{aligned} \tag{6}
$$

and the boundary conditions

$$\rho[\infty, x, V_0] = \rho[\infty, y, V_1] = 0, \qquad x \in X,\ y \in Y. \tag{7}$$

Solving equations (3) and taking into account that

$$\exp\left\{-\int_0^\tau r_\alpha(s)\,ds\right\} = 1 - F_\alpha(\tau),$$

we get

$$\begin{aligned} \rho(\tau, x, v) &= h_x(v + a_x\tau)(1 - F_x(\tau)), \qquad x \in X,\\ \rho(\tau, y, v) &= h_y(v - b_y\tau)(1 - F_y(\tau)), \qquad y \in Y. \end{aligned} \tag{8}$$

Considering equation (4), we seek for a function $h_\alpha(\cdot)$, $\alpha \in G$, of the form

$$h_\alpha(x) = c_\alpha e^{sx}, \qquad c_\alpha > 0,\ \alpha \in G. \tag{9}$$

Then equation (4) implies the system of equations

$$\sum_{\beta\in G} c_\beta \int_0^\infty f_\beta(\tau) e^{sk_\beta\tau}\,d\tau\, p_{\beta\alpha} = c_\alpha, \qquad \alpha \in G, \tag{10}$$

where

$$k_\beta = \begin{cases} a_\beta, & \beta \in X,\\ -b_\beta, & \beta \in Y. \end{cases}$$

The Laplace transform of the function $f_\beta(\tau)$ is given by

$$\widehat{f}_\beta(sk_\beta) = \int_0^\infty f_\beta(\tau) e^{sk_\beta\tau}\,d\tau.$$

It is obvious that the system (10) with respect to unknowns c_α, $\alpha \in G$, has a solution if the determinant of the system is zero, that is,

$$\begin{vmatrix} -1 & \widehat{f}_{x_1}(a_1 s)p_{x_1x_2} & \dots & \widehat{f}_{x_1}(a_1 s)p_{x_1y_m}\\ \widehat{f}_{x_2}(a_2 s)p_{x_2x_1} & -1 & \dots & \widehat{f}_{x_2}(a_2 s)p_{x_2y_m}\\ - & - & - & -\\ \widehat{f}_{y_m}(-b_m s)p_{y_mx_1} & \widehat{f}_{y_m}(-b_m s)p_{y_mx_2} & \dots & -1 \end{vmatrix} = 0. \tag{11}$$

The case where the balance condition holds. It is easy to check that $s = 0$ is a solution of (11). If the solution is unique, then equation (10) becomes of the form

$$\sum_{\beta\in G} c_\beta p_{\beta\alpha} = c_\alpha, \qquad \alpha \in G.$$

This implies that $c_\alpha = \sigma\rho_\alpha$ where ρ_α, $\alpha \in G$, is the stationary distribution of the embedded Markov chain $\kappa_l, l \in \mathbf{N}$, and σ is the normalizing factor defined from the condition $\int_Z \rho(z)\,dz = 1$. Then $h_\alpha(x) = \sigma\rho_\alpha$ and (9) yields

$$\rho(\tau, x, v) = \sigma\rho_x(1 - F_x(\tau)), \quad x \in X, \qquad \rho(\tau, y, v) = \sigma\rho_y(1 - F_y(\tau)), \quad y \in Y. \tag{12}$$

Solving equation (5) and taking into account (12), we get

$$\begin{aligned} \rho[\tau, x, V_0] &= \sigma\rho_x(1 - F_x(\tau))\left(a_x\tau + \frac{\rho[0, x, V_0]}{\sigma\rho_x}\right),\\ \rho[\tau, y, V_1] &= \sigma\rho_y(1 - F_y(\tau))\left(b_y\tau + \frac{\rho[0, y, V_1]}{\sigma\rho_y}\right). \end{aligned} \tag{13}$$

Substituting these results in (7), we obtain

$$\begin{aligned}
&\sigma \sum_{x\in X} \rho_x a_x m_x p_{xz} + \sum_{x\in X} \rho[0,x,V_0]p_{xz} = \rho[0,z,V_0], \qquad z\in X,\\
&\sigma \sum_{y\in Y} \rho_y b_y m_y p_{yz} + \sum_{y\in Y} \rho[0,y,V_1]p_{yz} = \rho[0,z,V_1], \qquad z\in Y,\\
&\sigma \sum_{y\in Y} \rho_y b_y m_y p_{yx} + \sum_{y\in Y} \rho[0,y,V_1]p_{yx} = \sigma\rho_x a_x m_x, \qquad x\in X,\\
&\sigma \sum_{x\in X} \rho_x a_x m_x p_{xy} + \sum_{x\in X} \rho[0,x,V_0]p_{xy} = \sigma\rho_y b_y m_y, \qquad y\in Y.
\end{aligned} \tag{14}$$

Put $P_X = \{p_{xz}, x, z\in X\}$, $P_Y = \{p_{xz}, x, z \in Y\}$, and

$$G_X = (I-P_X)^{-1} = \{g_{xz}, x, z\in X\}, \qquad G_Y = (I-P_Y)^{-1} = \{g_{yz}, y, z\in Y\}.$$

Since the Markov chain κ_l is ergodic, the matrices G_X and G_Y exist and are potentials [6]. Solving the first two equations of (14), we have

$$\begin{aligned}
\rho[0,z,V_0] &= \sigma \sum_{x\in X} a_x m_x \rho_x \sum_{k\in X} p_{xk} g_{kz}, \qquad z\in X,\\
\rho[0,z,V_1] &= \sigma \sum_{y\in Y} b_y m_y \rho_y \sum_{k\in Y} p_{yk} g_{kz}, \qquad z\in Y.
\end{aligned}$$

Substituting these relations in equality (14), we obtain the condition for the existence of the stationary distribution of the process (the so-called balance condition):

C2. We have

$$\begin{aligned}
\sum_{x\in X} \rho_x a_x m_x \sum_{z\in X} g_{xz} p_{zy} &= \rho_y b_y m_y, \qquad y\in Y,\\
\sum_{y\in Y} \rho_y a_y m_y \sum_{z\in Y} g_{yz} p_{zx} &= \rho_x b_x m_x, \qquad x\in X.
\end{aligned}$$

Therefore we proved the following result.

Theorem 1. *If conditions C1 and C2 hold, then the stationary distribution $\rho(\cdot)$ of the process $\xi(t)$ is characterized by the equalities*

$$\rho(\tau,x,v) = \sigma\rho_x(1-F_x(\tau)), \quad x\in X, \qquad \rho(\tau,y,v) = \sigma\rho_y(1-F_y(\tau)), \quad y\in Y,$$

for densities and

$$\begin{aligned}
\rho[\tau,x,V_0] &= \sigma\rho_x(1-F_x(\tau))\left(a_x\tau + \frac{\sum_{z\in X} a_z m_z \rho_z \sum_{k\in X} p_{zk} g_{kx}}{\rho_x}\right),\\
\rho[\tau,y,V_1] &= \sigma\rho_y(1-F_y(\tau))\left(b_y\tau + \frac{\sum_{z\in Y} b_z m_z \rho_z \sum_{k\in Y} p_{zk} g_{ky}}{\rho_y}\right)
\end{aligned}$$

for atoms.

The case where the balance does not hold. Now we consider the case where the solution of equation (11) is not unique. Denote by $\{s_0 = 0, s_1, \dots, s_l\}$, $l \geq 1$, the set of solutions of equation (11). Every s_i of this set corresponds to a solution $\{c^i_\alpha, \alpha\in G\}$ of equation (10) being unique up to a constant factor ($s_0 = 0$ corresponds to the solution $c\rho_\alpha$). Thus we seek densities of the form

$$
\begin{aligned}
\rho(\tau, x, v) &= \sum_{i=0}^{l} c_x^i e^{s_i(v+a_x\tau)}(1-F_x(\tau)), \qquad x\in X,\\
\rho(\tau, y, v) &= \sum_{i=0}^{l} c_y^i e^{s_i(v-b_y\tau)}(1-F_y(\tau)), \qquad y\in Y.
\end{aligned} \tag{15}
$$

Solving equation (5) and taking into account (15), we have

$$
\begin{aligned}
\rho[\tau, x, V_0] &= \sum_{i=1}^{l} \frac{c_x^i}{s_i}(1-F_x(\tau))e^{s_iV_0}\left(e^{a_xs_i\tau}-1\right) + c\rho_x a_x\tau(1-F_x(\tau))\\
&\quad + \rho[0,x,V_0](1-F_x(\tau)), \qquad x\in X,\\
\rho[\tau, y, V_1] &= \sum_{i=1}^{l} \frac{c_y^i}{s_i}(1-F_y(\tau))e^{s_iV_1}\left(1-e^{-b_ys_i\tau}\right) + c\rho_y b_y\tau(1-F_y(\tau))\\
&\quad + \rho[0,y,V_1](1-F_y(\tau)), \qquad y\in Y.
\end{aligned} \tag{16}
$$

Substituting (16) in the first two equations of (6), we get

$$
\begin{aligned}
\rho[0, z, V_0] &= \sum_{i=1}^{l} \frac{e^{s_iV_0}}{s_i}\sum_{x\in X}\left[c_x^i(\widehat{f}(s_ia_x)-1)\sum_{k\in X}p_{xk}g_{kz}\right]\\
&\quad + c\sum_{x\in X}\rho_x a_x m_x\sum_{k\in X}p_{xk}g_{kz}, \qquad z\in X,\\
\rho[0, z, V_1] &= \sum_{i=1}^{l} \frac{e^{s_iV_1}}{s_i}\sum_{y\in Y}\left[c_y^i(1-\widehat{f}(-s_ib_y))\sum_{k\in Y}p_{yk}g_{kz}\right]\\
&\quad + c\sum_{y\in Y}\rho_y b_y m_y\sum_{k\in Y}p_{yk}g_{kz}, \qquad z\in Y.
\end{aligned} \tag{17}
$$

To find $\{c_\alpha^i, \alpha\in G\}$, we combine these equalities and the last two equations in (6). This leads to the following condition.

C3. Among solutions $\{c_\alpha^i, \alpha\in G\}$ of equations (10) that correspond to s_i, there are nonzero solutions $\{c_\alpha^{i_r}, \alpha\in G, r=1,\dots,d\}$, $r\le l$, such that

$$
\begin{aligned}
&\sum_{r=1}^{d}\sum_{x\in X}\left[c_x^{i_r}(\widehat{f}(s_{i_r}a_x)-1)\sum_{z\in X}g_{xz}p_{zy}\right] + c\sum_{x\in X}\rho_x a_x m_x\sum_{z\in X}g_{xz}p_{zy}\\
&\qquad = \sum_{r=1}^{d} c_x^{i_r}(1-\widehat{f}(-s_{i_r}b_y)) + c\rho_y b_y m_y, \qquad y\in Y,\\
&\sum_{r=1}^{d}\sum_{y\in Y}\left[c_y^{i_r}(1-\widehat{f}(-s_{i_r}b_y))\sum_{z\in Y}g_{yz}p_{zx}\right] + c\sum_{y\in Y}\rho_y a_y m_y\sum_{z\in Y}g_{yz}p_{zx}\\
&\qquad = \sum_{r=1}^{d} c_x^{i_r}(\widehat{f}(s_{i_r}a_x)-1) + c\rho_x a_x m_x, \qquad x\in X.
\end{aligned}
$$

Therefore we have proved the following result.

Theorem 2. *If conditions C1 and C3 hold, then the Markov process $\xi(t)$ has the stationary distribution $\rho(\cdot)$ whose density is given by*

$$\rho(\tau,x,v)=\sigma_1\sum_{r=1}^{d}c_x^{i_r}e^{s_{i_r}(v+a_x\tau)}(1-F_x(\tau)),\qquad x\in X,$$
$$\rho(\tau,y,v)=\sigma_1\sum_{r=1}^{d}c_y^{i_r}e^{s_{i_r}(v-b_y\tau)}(1-F_y(\tau)),\qquad y\in Y,$$

where $c^o_\alpha=c\rho_\alpha$, and whose atoms are given by

$$\rho[\tau,x,V_0]=\sigma_1\bigg[\sum_{i=1}^{l}\frac{c_x^i}{s_i}(1-F_x(\tau))e^{s_iV_0}(e^{a_xs_i\tau}-1)+c\rho_xa_x\tau(1-F_x(\tau))$$
$$+\rho[0,x,V_0](1-F_x(\tau))\bigg],\qquad x\in X,$$
$$\rho[\tau,y,V_1]=\sigma_1\bigg[\sum_{i=1}^{l}\frac{c_y^i}{s_i}(1-F_y(\tau))e^{s_iV_1}(1-e^{-b_ys_i\tau})+c\rho_yb_y\tau(1-F_y(\tau))$$
$$+\rho[0,y,V_1](1-F_y(\tau))\bigg],\qquad y\in Y,$$

where σ_1 is defined from the condition $\int_Z\rho(z)\,dz=1$.

Note also that the case of $c=0$ is not excluded.

Example. Consider Theorem 2 for the case of $n=m=1$. Then $X=\{x\}$, $Y=\{y\}$, the matrix of transient probabilities P of the embedded (to the process $\kappa(t)$) Markov chain κ_l is $P=\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$, and equation (11) is

$$\widehat{f}_x(sa)\widehat{f}_y(-sb)=1. \tag{18}$$

Assume that the balance condition does not hold. Let s_i satisfy (18). Then the corresponding $\{c_x^i,c_y^i\}$ satisfy equalities

$$c_x^i\widehat{f}_x(s_ia)=c_y^i,\qquad c_y^i\widehat{f}_y(-s_ib)=c_x^i,\qquad i=1,\dots,l,$$

and condition C3 becomes of the form

$$\sum_{i=1}^{l}c_x^i(\widehat{f}_x(sa)-1)+ca_xm_x\rho_x=\sum_{i=1}^{l}c_y^i(1-\widehat{f}_y(-sb))+ca_ym_y\rho_y.$$

Since the balance condition does not hold, the latter condition holds for $c=0$ only. Thus the density of the stationary distribution is given by

$$\rho(\tau,x,v)=\sigma_1\sum_{i=1}^{l}c_x^ie^{s_i(v+a\tau)}(1-F_x(\tau)),\qquad\rho(\tau,y,v)=\sigma_1\sum_{i=1}^{l}c_y^ie^{s_i(v-b\tau)}(1-F_x(\tau)),$$

while the atoms are given by

$$\rho[0,x,V_0]=\sigma_1\sum_{i=1}^{l}\frac{e^{s_iV_0}}{s_i}c_x^i\widehat{f}(s_ia),$$
$$\rho[0,y,V_1]=\sigma_1\sum_{i=1}^{l}\frac{e^{s_iV_1}}{s_i}c_y^i\widehat{f}(-s_ib).$$

In order to apply the above result, one needs to know the stationary distribution $\overline{\rho}(\cdot)$ of the process $\zeta(\kappa(t), v(t))$ [3, 4]. This distribution can be found from the distribution of $\rho(\cdot)$ by

$$\overline{\rho}(\cdot) = \int_0^\infty \rho(\tau, \cdot)\, d\tau.$$

Let $f_0(x) = \lambda e^{-\lambda x}$ and $f_1(x) = p^2 x e^{-px}$. Then

$$\widehat{f}_o(sa)\widehat{f}_1(-sb) = \frac{\lambda p^2}{(\lambda - as)(p + bs)^2} = 1. \tag{19}$$

The integral $\int_0^\infty \rho(\tau, \cdot)\, d\tau$ converges if $as_i < \lambda$ and $bs_i > -p$. These assumptions also imply that equation (19) has the unique solution

$$s_1 = \frac{b\lambda - 2ap + \sqrt{4abp\lambda + (b\lambda)^2}}{2ab}.$$

This result allows us to find the stationary distribution of $\rho(\cdot)$. The balance condition holds in this case if $2b\lambda = ap$.

Remark. We assume in Theorem 2 that the roots of equation (11) are real and the number of roots is finite. If we omit this assumption and assume instead that the stationary distribution is a convergent Fourier series, then the result may have more applications. However this case requires a deeper consideration that will be published elsewhere.

Bibliography

1. V. S. Korolyuk and V. V. Korolyuk, *Stochastic Models of Systems*, Lybid', Kyiv, 1993; English transl., Kluwer Academic Publishers, Dordrecht, 1999. MR1817881; MR1753470 (2002b:60169)
2. V. S. Korolyuk and A. V. Svishchuk, *Semi-Markov Random Evolutions*, Lybid', Kyiv, 1992; English transl., Kluwer Academic Publishers, Dordrecht, 1995. MR1286045 (95j:60143); MR1472977 (98e:60145)
3. A. A. Pogoruĭ and A. F. Turbin, *Estimation of stationary efficiency of a production line with two unreliable aggregates*, Kibernetika i Sistemnyi Analiz **6** (2002), 35–42; English transl. in Cybernetics and Systems Analysis **38** (2002), no. 6, 823–829.
4. A. F. Turbin and A. A. Pogoruĭ, *Computation of stationary efficiency indices for inventory control systems with feedback*, Intellectualization of Systems for Processing Informational Messages, Nats. Akad. Nauk Ukrainy, Inst. Mat., Kyiv, 1995, pp. 191–204. (Russian) MR1365762
5. I. I. Gikhman and A. V. Skorokhod, *The Theory of Stochastic Processes*, vol. II, Nauka, Moscow, 1973; English transl., Springer-Verlag, New York–Heidelberg, 1975. MR2058260 (2005a:60003)
6. V. S. Korolyuk and A. F. Turbin, *Mathematical Foundations of the State Lumping of Large Systems*, Kluwer Academic Publishers, Dordrecht, 1994. MR1281385 (95e:60071)
7. A. N. Korlat, V. N. Kuznetsov, M. M. Novikov, and A. F. Turbin, *Semi-Markov Repairable Systems and Queuing Systems*, Shtiintsa, Kishinev, 1991. (Russian) MR1187752 (94e:60075)

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