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OVERSHOOT FUNCTIONALS FOR ALMOST SEMICONTINUOUS PROCESSES DEFINED ON A MARKOV CHAIN

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Abstract. The distributions of overshoot functionals are considered in the paper for almost semicontinuous processes defined on a finite irreducible Markov chain.

1. Introduction

The distributions of extremal values and overshoot functionals for semicontinuous processes (that is, for those processes that cross either a positive or a negative barrier in a continuous way) defined on a Markov chain are considered by many authors (see, for example, [1]–[3]). The distributions of extremal values are considered in the paper [4] for almost semicontinuous processes (that is, for those processes that cross either a positive or a negative barrier by means of exponential jumps only). Under some assumptions, these processes can be viewed as surplus risk processes with random premiums in a Markov environment. The distributions of some overshoot functionals are studied in this paper for lower almost semicontinuous processes defined on a Markov chain.

2. Main part

Consider a two dimensional Markov process

$$Z(t) = \{\xi(t), x(t)\}, \qquad t \geq 0,$$

where $x(t)$ is a finite irreducible aperiodic Markov chain whose phase space is

$$E' = \{1, \ldots, m\}$$

and whose matrix of transient probabilities is given by

$$\mathbf{P}(t) = e^{t\mathbf{Q}}, \qquad t \geq 0, \qquad \mathbf{Q} = \mathbf{N}(\mathbf{P} - \mathbf{I}).$$

Here

$$\mathbf{N} = \|\delta_{kr}\nu_k\|_{k,r=1}^{m},$$

ν_k are the parameters of the exponential random variables ζ_k (meaning the sojourn times of $x(t)$ at states k), $\mathbf{P} = \|p_{kr}\|$ is the matrix of transient probabilities of the embedded chain, and $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_m)$ is the stationary distribution. The process $\xi(t)$ is homogeneous with conditionally independent increments given that the values of $x(t)$ are fixed (see [1]).

The evolution of the process $Z(t)$ is described by the matrix characteristic function

$$\boldsymbol{\Phi}_t(\alpha) = \left\| \mathsf{E}\left[e^{\imath\alpha(\xi(t+u)-\xi(u))}, x(t+u) = r \ / \ x(u) = k \right] \right\|, \qquad u \geq 0,$$

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which can be represented in the following form:

$$\mathbf{\Phi}_t(\alpha) = \mathbf{E}\, e^{\imath\alpha\xi(t)} = e^{t\mathbf{\Psi}(\alpha)}, \qquad \mathbf{\Psi}(0) = \mathbf{Q}.$$

In what follows we study the processes with the cumulant

$$\mathbf{\Psi}(\alpha) = \int_0^\infty \left(e^{\imath\alpha x} - 1\right) d\mathbf{K}_0(x) + \mathbf{\Lambda}\mathbf{F}_0(0)\left(\mathbf{C}\left(\mathbf{C} + \imath\alpha\mathbf{I}\right)^{-1} - \mathbf{I}\right) + \mathbf{Q}, \tag{1}$$

where $d\mathbf{K}_0(x) = \mathbf{N}\, d\mathbf{F}(x) + \mathbf{\Pi}(dx)$,

$$\mathbf{F}(x) = \left\| \mathsf{P}\left\{\chi_{kr} < x; x(\zeta_1) = r \;/\; x(0) = k\right\}\right\|,$$

χ_{kr} are the heights of jumps of $\xi(t)$ at the moments when $x(t)$ jumps from the state k to the state r,

$$\mathbf{\Pi}(dx) = \mathbf{\Lambda} d\mathbf{F}_0(x), \qquad \mathbf{F}_0(x) = \left\|\delta_{kr} F_k^0(x)\right\|,$$

$F_k^0(x)$ are the distribution functions of the heights of jumps of $\xi(t)$ given $x(t) = k$, $\mathbf{\Lambda} = \|\delta_{kr}\lambda_k\|$, and where λ_k are the parameters of the exponential random variables ζ_k' (meaning the time between two successive jumps of $\xi(t)$ given $x(t) = k$). Let $\mathbf{C} = \|\delta_{kr} c_k\|$, where c_k are the parameters of the exponential negative jumps of $\xi(t)$ given $x(t) = k$. A process $Z(t)$ with a cumulant of this kind is called a lower almost semicontinuous process (this definition is introduced in [4]).

Denote by θ_s an exponential random variable whose parameter is $s > 0$ (that is, $\mathsf{P}\{\theta_s > t\} = e^{-st}$ for $t \geq 0$) and assume that θ_s is independent of $Z(t)$. Then the characteristic function of $\xi(\theta_s)$ can be written as follows:

$$\mathbf{\Phi}(s,\alpha) = \mathbf{E}\, e^{\imath\alpha\xi(\theta_s)} = s\int_0^\infty e^{-st}\mathbf{\Phi}_t(\alpha)\, dt = s\left(s\mathbf{I} - \mathbf{\Psi}(\alpha)\right)^{-1}. \tag{2}$$

We introduce the main functionals of interest:

$$\xi^\pm(t) = \sup_{0\le u\le t}(\inf)\xi(u), \qquad \xi^\pm = \sup_{0\le u\le \infty}(\inf)\xi(u);$$

$$\overline{\xi}(t) = \xi(t) - \xi^+(t), \qquad \check{\xi}(t) = \xi^-(t) - \xi(t),$$

$$\left.\begin{array}{ll} \tau^+(x) = \inf\{t\colon \xi(t) > x\}, & \gamma^+(x) = \xi(\tau^+(x)) - x, \\ \gamma_+(x) = x - \xi(\tau^+(x) - 0), & \gamma_x^+ = \gamma^+(x) + \gamma_+(x), \end{array}\right. \quad x \geq 0.$$

The distributions of the functionals $\xi^\pm(\theta_s)$, $\overline{\xi}(\theta_s)$, and $\check{\xi}(\theta_s)$ are obtained in [4]. The aim of this paper is to obtain explicitly the joint moment generating functions of overshoot functionals for lower almost semicontinuous processes and moment generating functions of the random vectors $\{\tau^+(x), \gamma^+(x)\}$, $\{\tau^+(x), \gamma_+(x)\}$, and $\{\tau^+(x), \gamma_x^+\}$.

Put

$$\mathbf{V}(s,x,u,v,\mu) = \mathbf{E}\left[e^{-s\tau^+(x) - u\gamma^+(x) - v\gamma_+(x) - \mu\gamma_x^+},\, \tau^+(x) < \infty\right],$$

$$\mathbf{W}(x,u,v,\mu) = \int_x^\infty e^{(u-v)x - (u+\mu)z}\, d\mathbf{K}_0(z), \qquad \overline{\mathbf{K}}_0(x) = \mathbf{W}(x,0,0,0),$$

$$\mathbf{P}_s = s\int_0^\infty e^{-st}\,\mathbf{P}(t)\, dt = s\left(s\mathbf{I} - \mathbf{Q}\right)^{-1},$$

$$\mathbf{P}_+(s,x) = \mathbf{P}\left\{\xi^+(\theta_s) < x\right\}, \quad x > 0, \qquad \mathbf{P}^-(s,x) = \mathbf{P}\left\{\overline{\xi}(\theta_s) < x\right\}, \quad x < 0,$$

$$\widetilde{\mathbf{P}}^0(s) = \mathbf{P}\left\{\xi(\theta_s) = 0\right\}, \qquad \mathbf{p}_\pm(s) = \mathbf{P}\left\{\xi^\pm(\theta_s) = 0\right\}, \qquad \check{\mathbf{R}}_-(s) = \mathbf{P}_s^{-1}\check{\mathbf{p}}_-(s),$$

$$\check{\mathbf{p}}_-(s) = \mathbf{P}\left\{\overline{\xi}(\theta_s) = 0\right\}, \qquad \check{\mathbf{q}}_-(s) = \mathbf{P}_s - \check{\mathbf{p}}_-(s), \qquad \check{\mathbf{R}}_c(s) = \check{\mathbf{R}}_-(s)\mathbf{C},$$

$$\mathbf{G}_+(s,x,u,v,\mu) = \int_{-\infty}^0 d\,\mathbf{P}^-(s,y)\mathbf{W}(x-y,u,v,\mu).$$

Lemma 1. *Consider a process $Z(t)$ with the cumulant of the form* (1)*. Then*

$$s\mathbf{V}(s,x,u,v,\mu)=\int_0^x d\,\mathbf{P}_+(s,y)\,\mathbf{P}_s^{-1}\,\mathbf{G}_+(s,x-y,u,v,\mu),\qquad x>0,\tag{3}$$

where

$$\begin{aligned}\mathbf{G}_+&(s,x,u,v,\mu)\\ &=\check{\mathbf{p}}_-(s)\int_x^\infty e^{(u-v)x-(u+\mu)z}\,d\mathbf{K}_0(z)\\ &\quad-\check{\mathbf{p}}_-(s)\mathbf{C}\left(\mathbf{P}_s^{-1}\,\check{\mathbf{p}}_-(s)\mathbf{C}-(u-v)\mathbf{I}\right)^{-1}e^{-(v+\mu)x}\\ &\quad\times\int_0^\infty\Big[(u+\mu)e^{-(u+\mu)z}\\ &\qquad-\left(\mathbf{P}_s^{-1}\,\check{\mathbf{p}}_-(s)\mathbf{C}+(\mu+v)\mathbf{I}\right)e^{-\left(\mathbf{P}_s^{-1}\,\check{\mathbf{p}}_-(s)\mathbf{C}+(v+\mu)\mathbf{I}\right)z}\Big]\\ &\qquad\qquad\times\mathbf{P}_s^{-1}\,\check{\mathbf{q}}_-(s)\overline{\mathbf{K}}_0(x+z)\,dz\end{aligned}\tag{4}$$

for $u-v\notin\sigma(\mathbf{P}_s^{-1}\,\check{\mathbf{p}}_-(s)\mathbf{C})$. Here the symbol $\sigma(A)$ stands for the spectrum of a matrix A.

Proof. Since we assume the almost semicontinuity, relation (3) follows from [1, Corollary 3.4]. According to [4] (see Remark 1 therein) the distribution of $\overline{\xi}(\theta_s)$ is given by

$$\mathbf{P}^-(s,x)=\mathbf{P}\left\{\overline{\xi}(\theta_s)<x\right\}=e^{\check{\mathbf{p}}_-(s)\mathbf{C}\,\mathbf{P}_s^{-1}\,x}\check{\mathbf{q}}_-(s),\qquad x<0.$$

Then

$$\begin{aligned}\mathbf{G}_+(s,x,u,v,\mu)&=\int_{-\infty}^0 d\,\mathbf{P}^-(s,y)\mathbf{W}(x-y,u,v,\mu)\\ &=\check{\mathbf{p}}_-(s)\mathbf{W}(x,u,v,\mu)\\ &\quad+\check{\mathbf{p}}_-(s)\mathbf{C}\int_x^\infty e^{\mathbf{P}_s^{-1}\,\check{\mathbf{p}}_-(s)\mathbf{C}(x-y)}\,\mathbf{P}_s^{-1}\,\check{\mathbf{q}}_-(s)\mathbf{W}(y,u,v,\mu)\,dy.\end{aligned}\tag{5}$$

By the definition of the function $\mathbf{W}(y,u,v,\mu)$,

$$\begin{aligned}&\int_x^\infty e^{\mathbf{P}_s^{-1}\,\check{\mathbf{p}}_-(s)\mathbf{C}(x-y)}\,\mathbf{P}_s^{-1}\,\check{\mathbf{q}}_-(s)\mathbf{W}(y,u,v,\mu)\,dy\\ &\quad=-\left(\mathbf{P}_s^{-1}\,\check{\mathbf{p}}_-(s)\mathbf{C}-(u-v)\mathbf{I}\right)^{-1}e^{-(v+\mu)x}\\ &\qquad\times\Bigg[\left(\mathbf{P}_s^{-1}\,\check{\mathbf{p}}_-(s)\mathbf{C}+(\mu+v)\mathbf{I}\right)\int_0^\infty e^{-\left(\mathbf{P}_s^{-1}\,\check{\mathbf{p}}_-(s)\mathbf{C}+(v+\mu)\mathbf{I}\right)z}\,\mathbf{P}_s^{-1}\,\check{\mathbf{q}}_-(s)\overline{\mathbf{K}}_0(x+z)\,dz\\ &\qquad\qquad-(u+\mu)\int_0^\infty e^{-(u+\mu)z}\,\mathbf{P}_s^{-1}\,\check{\mathbf{q}}_-(s)\overline{\mathbf{K}}_0(x+z)\,dz\Bigg].\end{aligned}$$

Now equality (4) follows from (5). □

Note that

$$\lim_{x\to-\infty}\mathbf{P}^-(s,x)=\mathbf{P}\left\{\overline{\xi}(\theta_s)<-\infty\right\}=0.$$

Thus the representation

$$\begin{aligned}\mathbf{P}^-(s,x)&=e^{\check{\mathbf{p}}_-(s)\mathbf{C}\,\mathbf{P}_s^{-1}\,x}\mathbf{q}^-(s)=\mathbf{P}_s\mathbf{P}_s^{-1}e^{\check{\mathbf{p}}_-(s)\mathbf{C}\mathbf{P}_s^{-1}}\mathbf{P}_s\,\mathbf{P}_s^{-1}\,\check{\mathbf{q}}_-(s)\\ &=\mathbf{P}_s e^{\check{\mathbf{R}}_c(s)x}\left(\mathbf{I}-\check{\mathbf{R}}_-(s)\right)\end{aligned}$$

implies that the spectrum $\sigma\left(\check{\mathbf{R}}_c(s)\right)$ of the matrix $\check{\mathbf{R}}_c(s)$ consists of positive elements.

Let

$$\gamma_1(x)=\gamma^+(x),\qquad\gamma_2(x)=\gamma_+(x),\qquad\gamma_3(x)=\gamma_x^+.$$

Substituting $v=\mu=0$, $u\notin\sigma(\check{\mathbf{R}}_c(s))$, $u=\mu=0$, and $v=u=0$ we derive from equality (3) that

$$\mathbf{E}\left[e^{-s\tau^+(x)-u\gamma_i(x)},\tau^+(x)<\infty\right]=s^{-1}\int_0^x d\mathbf{P}_+(s,y)\,\mathbf{P}_s^{-1}\,\mathbf{G}_i(s,x-y,u),\qquad i=1,2,3,$$

$$\begin{aligned}\mathbf{G}_1(s,x,u)&=\check{\mathbf{p}}_-(s)\int_x^\infty e^{u(x-z)}\,d\mathbf{K}_0(z)\\&\quad-\check{\mathbf{p}}_-(s)\mathbf{C}\left(\check{\mathbf{R}}_c(s)-u\mathbf{I}\right)^{-1}\\&\quad\times\int_0^\infty\left[ue^{-uz}-\check{\mathbf{R}}_c(s)e^{-\check{\mathbf{R}}_c(s)z}\right]\mathbf{P}_s^{-1}\,\check{\mathbf{q}}_-(s)\overline{\mathbf{K}}_0(x+z)\,dz,\end{aligned}$$

$$\mathbf{G}_2(s,x,v)=\check{\mathbf{p}}_-(s)e^{-vx}\overline{\mathbf{K}}_0(x)+\check{\mathbf{p}}_-(s)\mathbf{C}e^{-vx}\int_0^\infty e^{-\left(\check{\mathbf{R}}_c(s)+v\mathbf{I}\right)z}\,\mathbf{P}_s^{-1}\,\check{\mathbf{q}}_-(s)\overline{\mathbf{K}}_0(x+z)\,dz,$$

$$\begin{aligned}\mathbf{G}_3(s,x,\mu)&=\check{\mathbf{p}}_-(s)\int_x^\infty e^{-\mu z}\,d\mathbf{K}_0(z)\\&\quad-e^{-\mu x}\int_0^\infty\left[\mu e^{-\mu z}-\left(\check{\mathbf{R}}_c(s)+\mu\mathbf{I}\right)e^{-\left(\check{\mathbf{R}}_c(s)+\mu\mathbf{I}\right)z}\right]\mathbf{P}_s^{-1}\,\check{\mathbf{q}}_-(s)\overline{\mathbf{K}}_0(x+z)\,dz.\end{aligned}$$

Inverting with respect to u we obtain

$$\mathbf{E}\left[e^{-s\tau^+(x)},\gamma_i(x)\in dz,\tau^+(x)<\infty\right]=s^{-1}\int_0^x d\mathbf{P}_+(s,y)\check{\mathbf{R}}_-(s)\,d_z\mathbf{g}_i^*(s,x-y,z),$$

$$d_z\mathbf{g}_i^*(s,x,z)=d_z\mathbf{w}_i^*(x,z)+\mathbf{C}\int_x^\infty e^{\check{\mathbf{R}}_c(s)(x-y)}\left(\mathbf{I}-\check{\mathbf{R}}_-(s)\right)d_z\mathbf{w}_i^*(y,z)\,dy,$$

$$d_z\mathbf{w}_1^*(x,z)=d_z\mathbf{K}_0(x+z),\qquad d_z\mathbf{w}_2^*(x,z)=d_zI\left\{z>x\right\}\overline{\mathbf{K}}_0(x),$$

$$d_z\mathbf{w}_3^*(x,z)=I\{z\geq x\}\,d\mathbf{K}_0(z).$$

The case of $x=0$ is treated in the following result.

Theorem 1. *Consider a process $Z(t)$ with the cumulant of the form (1). If $z>0$, then*

$$\begin{aligned}&\mathbf{E}\left[e^{-s\tau^+(0)},\gamma^+(0)>z,\tau^+(0)<\infty\right]\\&\quad=s^{-1}\widetilde{\mathbf{P}}^0(s)\left(\overline{\mathbf{K}}_0(z)+\mathbf{C}\int_z^\infty e^{(z-y)\check{\mathbf{R}}_c(s)}\,\mathbf{P}_s^{-1}\,\check{\mathbf{q}}_-(s)\overline{\mathbf{K}}_0(y)\,dy\right),\end{aligned}$$

$$\mathbf{E}\left[e^{-s\tau^+(0)},\gamma_+(0)>z,\tau^+(0)<\infty\right]=s^{-1}\widetilde{\mathbf{P}}^0(s)\mathbf{C}\int_z^\infty e^{-y\check{\mathbf{R}}_c(s)}\,\mathbf{P}_s^{-1}\,\check{\mathbf{q}}_-(s)\overline{\mathbf{K}}_0(y)\,dy,$$

$$\begin{aligned}&\mathbf{E}\left[e^{-s\tau^+(0)},\gamma_0^+>z,\tau^+(0)<\infty\right]\\&\quad=s^{-1}\widetilde{\mathbf{P}}^0(s)\left(\overline{\mathbf{K}}_0(z)+\mathbf{C}\left(\check{\mathbf{R}}_c(s)\right)^{-1}\int_z^\infty\left(\mathbf{I}-e^{-y\check{\mathbf{R}}_c(s)}\right)\mathbf{P}_s^{-1}\,\check{\mathbf{q}}_-(s)\,d\mathbf{K}_0(y)\right).\end{aligned}\tag{6}$$

Proof. Relation (3) implies that

$$\mathbf{V}(s,x,u,v,\mu)=\overline{\mathbf{P}}_+(s,x)\,\mathbf{P}_s^{-1}+s^{-1}\int_0^x d\,\mathbf{P}_+(s,y)\,\mathbf{P}_s^{-1}\,\overline{\mathbf{G}}_+(s,x-y,u,v,\mu),\quad x>0,\tag{7}$$

where

$$\overline{\mathbf{G}}_+(s,x,u,v,\mu)=\mathbf{G}_+(s,x,u,v,\mu)-\mathbf{G}_+(s,x,0,0,0).$$

Since

$$\begin{aligned}\mathbf{V}_k(s,x,u) &= \mathbf{E}\left[e^{-s\tau^+(x)-u\gamma_k(x)}, \tau^+(x)<\infty\right]\\ &= \overline{\mathbf{P}}_+(s,x)\,\mathbf{P}_s^{-1} - u\int_0^\infty e^{-uz}\,\mathbf{E}\left[e^{-s\tau^+(x)}, \gamma_k(x)>z, \tau^+(x)<\infty\right]dz\end{aligned}$$

for $k=1,2,3$, equality (7) yields

$$\begin{aligned}&\int_0^\infty e^{-uz}\,\mathbf{E}\left[e^{-s\tau^+(x)}, \gamma_k(x)>z, \tau^+(x)<\infty\right]dz\\ &\qquad = -\frac{1}{su}\int_{-0}^x d\,\mathbf{P}_+(s,y)\,\mathbf{P}_s^{-1}\,\overline{\mathbf{G}}_k(s,x-y,u),\end{aligned} \tag{8}$$

where

$$\overline{\mathbf{G}}_k(s,x,u) = \check{\mathbf{p}}_-(s)\overline{\mathbf{W}}_k(x,u) + \int_{-\infty}^{0-} d\,\mathbf{P}^-(s,y)\overline{\mathbf{W}}_k(x-y,u), \tag{9}$$

$$\begin{aligned}\overline{\mathbf{W}}_1(x,u) &= \int_x^\infty \left(e^{u(x-z)}-\mathbf{I}\right)d\mathbf{K}_0(z), \qquad \overline{\mathbf{W}}_2(x,u) = \left(e^{-ux}-\mathbf{I}\right)\overline{\mathbf{K}}_0(x),\\ \overline{\mathbf{W}}_3(x,u) &= \int_x^\infty \left(e^{-uz}-\mathbf{I}\right)d\mathbf{K}_0(z).\end{aligned} \tag{10}$$

Passing to the limit as $x\to 0$ we obtain from equality (8) that

$$\int_0^\infty e^{-uz}\,\mathbf{E}\left[e^{-s\tau^+(0)}, \gamma_k(0)>z, \tau^+(0)<\infty\right]dz = -\frac{1}{su}\mathbf{p}_+(s)\,\mathbf{P}_s^{-1}\,\overline{\mathbf{G}}_k(s,0,u). \tag{11}$$

Considering relation (9) for $x=0$ and taking into account equality (10) we get

$$\begin{aligned}\overline{\mathbf{G}}_1(s,0,u) &= -u\check{\mathbf{p}}_-(s)\Bigg(\int_0^\infty e^{-uz}\overline{\mathbf{K}}_0(z)\,dz\\ &\qquad + \mathbf{C}\int_0^\infty e^{-uz}\int_z^\infty e^{(z-y)\check{\mathbf{R}}_c(s)}\,\mathbf{P}_s^{-1}\,\check{\mathbf{q}}_-(s)\overline{\mathbf{K}}_0(y)\,dy\,dz\Bigg),\\ \overline{\mathbf{G}}_2(s,0,u) &= -u\check{\mathbf{p}}_-(s)\mathbf{C}\int_0^\infty e^{-uz}\int_z^\infty e^{-y\check{\mathbf{R}}_c(s)}\,\mathbf{P}_s^{-1}\,\check{\mathbf{q}}_-(s)\overline{\mathbf{K}}_0(y)\,dy\,dz,\\ \overline{\mathbf{G}}_3(s,0,u) &= -u\check{\mathbf{p}}_-(s)\Bigg(\int_0^\infty e^{-uz}\overline{\mathbf{K}}_0(z)\,dz\\ &\qquad + \mathbf{C}\left(\check{\mathbf{R}}_c(s)\right)^{-1}\\ &\qquad \times\int_0^\infty e^{-uz}\int_z^\infty \left(\mathbf{I}-e^{-y\check{\mathbf{R}}_c(s)}\right)\mathbf{P}_s^{-1}\,\check{\mathbf{q}}_-(s)\,d\mathbf{K}_0(y)\,dz\Bigg).\end{aligned}$$

Substituting the above expressions for $\overline{\mathbf{G}}_k(s,0,u)$ into (11) and inverting with respect to u we prove (6). □

Consider some corollaries of Theorem 2.3. We also apply some of results of [2]; namely, we need an analog of the Pollachek–Khinchine formula and two-sided Lundberg inequality. Assume that $\chi_{kr}=0$, $k,r=1,\dots,m$. Almost semicontinuous processes satisfying these assumptions can be treated as surplus risk processes with random premiums in a Markov environment.

Let ζ^* be the moment of the first jump of the process $\xi(t)$. Then

$$\zeta^*_{kr} \doteq \begin{cases}\zeta_k+\zeta^*_{jr}, & \zeta'_k>\zeta_k,\ x(\zeta_k)=j,\\ \zeta'_k, & \zeta'_k<\zeta_k\end{cases}$$

(see [1, p. 42]), where the indices kr mean that $x(\zeta^*) = r$, $x(0) = k$, $k, r = 1, \dots, m$. Taking into account the definition of the process $Z(t)$, the latter relations imply that

$$\begin{aligned}
\mathsf{E}\, e^{-s\zeta^*_{kr}} &= \mathsf{E}\left[e^{-s\zeta^*}, x(\zeta^*) = r \;/\; x(0) = k\right] \\
&= \mathsf{E}\left[e^{-s\zeta'_k}, \zeta'_k < \zeta_k\right] \delta_{kr} + \sum_{j=1}^{m} \mathsf{E}\left[e^{-s\zeta^*_{jr} + \zeta_k}, \zeta'_k > \zeta_k, x(\zeta_k) = j\right] \\
&= \int_0^\infty \lambda_k e^{-sy} e^{-\lambda_k y} e^{-\nu_k y}\, dy \delta_{kr} + \sum_{j=1}^{m} \int_0^\infty e^{-sy} \nu_k e^{-\nu_k y} e^{-\lambda_k y}\, \mathsf{E}\, e^{-s\zeta^*_{jr}} p_{kj}\, dy \\
&= \lambda_k \left(s + \lambda_k + \nu_k\right)^{-1} \delta_{kr} + \sum_{j=1}^{m} \nu_k (s + \nu_k + \lambda_k)^{-1} p_{kj}\, \mathsf{E}\, e^{-s\zeta^*_{jr}}
\end{aligned}$$

(see [1, p. 64]). The latter equality can be rewritten in the matrix form as follows:

$$\mathbf{E} e^{-s\zeta^*} = \mathbf{\Lambda} \left(s\mathbf{I} + \mathbf{\Lambda} + \mathbf{N}\right)^{-1} + \left(s\mathbf{I} + \mathbf{\Lambda} + \mathbf{N}\right)^{-1} \mathbf{N}\mathbf{P}\, \mathbf{E} e^{-s\zeta^*}.$$

This implies the following representation for the moment generating function of the first jump moment:

$$\mathbf{E} e^{-s\zeta^*} = \left(s\mathbf{I} + \mathbf{\Lambda} - \mathbf{Q}\right)^{-1} \mathbf{\Lambda}.$$

Since $\widetilde{\mathbf{P}}^0(s) = \left(\mathbf{I} - \mathbf{E}\, e^{-s\zeta^*}\right) \mathbf{P}_s$, we obtain

$$\lim_{s \to 0} s^{-1} \widetilde{\mathbf{P}}^0(s) = \left(\mathbf{\Lambda} - \mathbf{Q}\right)^{-1} = \left\| \mathsf{P}\left\{x(\zeta^*) = r \;/\; x(0) = k\right\} \right\| \mathbf{\Lambda}^{-1}.$$

Let

$$m_1^0 = \sum_{k=1}^{m} \pi_k \int_R x \lambda_k\, dF_k^0(x).$$

Corollary 1. *If $m_1^0 < 0$, then*

$$1 - \psi_i(u) = \mathsf{P}_i\left\{\xi^+ \le u\right\} = \mathbf{e}'_i \sum_{n=0}^{\infty} \mathbf{G}_+^{*n}(u) \left(\mathbf{I} - \|\mathbf{G}\|\right) \mathbf{e}, \tag{12}$$

where

$$\begin{aligned}
\mathbf{G}_+(y, \infty) &= \mathbf{P}\left\{\gamma^+(0) > y, \tau^+(0) < \infty\right\} \\
&= \left(\mathbf{\Lambda} - \mathbf{Q}\right)^{-1} \left(\mathbf{\Lambda} \overline{\mathbf{F}}_0(y) + \mathbf{C} \int_{-\infty}^{0} e^{\check{\mathbf{R}}_-(0)\mathbf{C}x} \left(\mathbf{I} - \check{\mathbf{R}}_-(0)\right) \mathbf{\Lambda} \overline{\mathbf{F}}_0(y - x)\, dx\right),
\end{aligned} \tag{13}$$

$$\|\mathbf{G}\| = \int_0^\infty \mathbf{G}_+(dx), \qquad \mathbf{e} = (1, \dots, 1)', \qquad \mathbf{e}'_i = \left(0, \dots, \overset{i}{1}, \dots, 0\right).$$

Proof. In fact, equality (12) is known (see the proof of Proposition 2.2 in [2]). Equality (13) follows from the first equality in (6) by passing to the limit as $s \to 0$. □

Let $k(r)$ be the real eigenvalue whose real part is maximal among eigenvalues of the matrix $\mathbf{K}(r) = \mathbf{\Psi}(-\imath r)$ (that is, $k(r)$ is the Perron root of the matrix $\mathbf{K}$). Let $\gamma > 0$ be a solution of the equation $k(r) = 0$ and let $\boldsymbol{\nu} = (\nu_1, \dots, \nu_m)$ and $\mathbf{h} = (h_1, \dots, h_m)'$ be the left and right eigenvectors of the matrix $\mathbf{K}(\gamma)$ such that their coordinates are positive and $\boldsymbol{\nu}\mathbf{h} = 1$. Put

$$C_+ = \max_{j \in E'} \frac{1}{h_j} \sup_{x \ge 0} \frac{\overline{F_j^0}(x)}{\int_x^\infty e^{\gamma(y-x)} F_j^0(dy)}, \qquad C_- = \min_{j \in E'} \frac{1}{h_j} \inf_{x \ge 0} \frac{\overline{F_j^0}(x)}{\int_x^\infty e^{\gamma(y-x)} F_j^0(dy)}.$$

Corollary 2. *If $m_1^0 < 0$, then*

$$C_- h_i e^{-\gamma u} \le \psi_i(u) \le C_+ h_i e^{-\gamma u} \tag{14}$$

for all $i \in E'$ and $u \ge 0$.

Proof. The proof of the corollary is similar to that of Theorem 3.11 in [2]. □

Example. Let $Z(t) = \{\xi(t), x(t)\}$ be a process defined on the Markov chain $x(t)$ whose infinitesimal matrix is $\mathbf{Q} = \left(\begin{smallmatrix} -1 & 1 \\ 1 & -1 \end{smallmatrix}\right)$.

Assume that $\chi_{kr} = 0$, $k, r = 1, 2$. We also assume that the component $\xi(t)$ given $x(t) = i$ is represented as follows:

$$\xi_i(t) = S_i(t) - S_i'(t) = \sum_{k \le \nu_i'(t)} \eta_k^i - \sum_{k \le \nu_i(t)} \xi_k^i, \qquad i = 1, 2,$$

where $S_i(t)$ and $S_i'(t)$ are generalized Poisson processes with positive jumps $\xi_k^i > 0$ and $\eta_k^i > 0$ and parameters λ_i and $\lambda_i' = 1$, respectively. Furthermore let

$$\mathsf{P}\left\{\xi_k^i > x\right\} = e^{-c_i x}, \qquad \frac{\partial}{\partial x}\mathsf{P}\left\{\eta_k^i < x\right\} = {\delta_i}^2 x e^{-\delta_i x}, \qquad x \ge 0,\ i = 1, 2.$$

Consider an auxiliary process $Z_1(t) = \{\xi_1(t), x(t)\} = \{-\xi(t), x(t)\}$ whose cumulant is given by

$$\mathbf{\Psi}_1(-\imath r) = \begin{pmatrix} \frac{-3r^3 + (2c_1 - 6\delta_1)r^2 + \left(4c_1\delta_1 - 2\delta_1^2\right)r + c_1\delta_1}{(r - c_1)(r + \delta_1)^2} & 1 \\ 1 & \frac{-3r^3 + (2c_2 - 6\delta_2)r^2 + \left(4c_2\delta_2 - 2\delta_2^2\right)r + c_2\delta_2}{(r - c_2)(r + \delta_2)^2} \end{pmatrix}.$$

Then the stationary distribution is $\boldsymbol{\pi} = \left(\frac{1}{2}, \frac{1}{2}\right)$. If $c_1 = \frac{1}{3}$, $c_2 = \frac{1}{2}$, $\delta_1 = 2$, and $\delta_2 = 1$, then $m_1^0 = 1 > 0$. According to Theorem 3 of [4], ξ_1^- has a nondegenerate distribution. Consider the matrix

$$\mathbf{G}(r) := r\mathbf{\Psi}_1^{-1}(-\imath r)\left(\mathbf{C} - r\mathbf{I}\right)^{-1} = \frac{1}{D(r)}\begin{pmatrix} g_{11}(r) & g_{12}(r) \\ g_{21}(r) & g_{22}(r) \end{pmatrix},$$

where $D(r) = 48r^5 + 263r^4 + 387r^3 + 114r^2 - 51r - 8$, $g_{11}(r) = 3(r+2)^2(6r^3 + 10r^2 - 1)$, $g_{12}(r) = 2(r+1)^2(r+2)^2(3r-1)$, $g_{21}(r) = 3(r+1)^2(r+2)^2(2r-1)$, and

$$g_{22}(r) = 2(r+1)^2\left(9r^3 + 34r^2 + 16r - 4\right).$$

Since $D(r)$ has four negative roots,

$$-\rho_1 = -3.25672, \qquad -\rho_2 = -1.59682, \qquad -\rho_3 = -0.794382, \qquad -\rho_4 = -0.133485,$$

and one positive root $r_0 = 0.30224$, the entries of the matrix $\mathbf{G}(r)$ are such that

$$G_{ij}(r) = C_{ij}^0 + \frac{C_{ij}^1}{r + \rho_1} + \frac{C_{ij}^2}{r + \rho_2} + \frac{C_{ij}^3}{r + \rho_3} + \frac{C_{ij}^4}{r + \rho_4} + \frac{C_{ij}^5}{r - r_0}.$$

The projection for functions of the form

$$\mathbf{G}(r) = \mathbf{C}_0 + \int_{-\infty}^{\infty} e^{rx}\mathbf{g}(x)\,dx$$

is defined by

$$[\mathbf{G}(r)]^- = \int_{-\infty}^{0} e^{rx}\mathbf{g}(x)\,dx.$$

Considering the projection we get

$$G_{ij}^-(r) = [G_{ij}(r)]^- = \frac{C_{ij}^1}{r + \rho_1} + \frac{C_{ij}^2}{r + \rho_2} + \frac{C_{ij}^3}{r + \rho_3} + \frac{C_{ij}^4}{r + \rho_4}.$$

Since

$$\check{\mathbf{R}}_+ = \left(\mathbf{G}^-(0) + (\mathbf{\Lambda} - \mathbf{Q})^{-1}\right)^{-1} \mathbf{P}_0 = \begin{pmatrix} 0.22 & 0.22 \\ 0.17 & 0.17 \end{pmatrix},$$

Theorem 3 of [4] implies that

$$\begin{aligned} \mathsf{E}_i\left[e^{r\xi_1^-}, \xi_1^- < 0\right] &= \mathbf{E}\left[e^{r\xi_1^-}, \xi_1^- < 0\right] \cdot \mathbf{e} = [\mathbf{G}(r)]^- \check{\mathbf{R}}_+ \cdot \mathbf{e} \\ &= \frac{A_i^1}{r+\rho_1} + \frac{A_i^2}{r+\rho_2} + \frac{A_i^3}{r+\rho_3} + \frac{A_i^4}{r+\rho_4}, \qquad i = 1, 2, \end{aligned}$$

where $\mathbf{e} = (1,1)'$. Inverting with respect to r we evaluate the distribution of ξ_1^- as follows:

$$\mathsf{P}_i\left\{\xi_1^- < x\right\} = \sum_{k \le 4} \frac{A_i^k}{\rho_k} e^{\rho_k x}, \qquad x < 0.$$

Therefore

$$\begin{aligned} \psi_1(u) &= \mathsf{P}_1\left\{\xi^+ > u\right\} = \mathsf{P}_1\left\{\xi_1^- < -u\right\} \\ &\approx -0.04e^{-3.26u} + 0.001e^{-1.6u} + 0.079e^{-0.79u} + 0.75e^{-0.13u}, \\ \psi_2(u) &= \mathsf{P}_2\left\{\xi^+ > u\right\} = \mathsf{P}_2\left\{\xi_1^- < -u\right\} \\ &\approx -0.01e^{-3.26u} - 0.016e^{-1.6u} + 0.004e^{-0.79u} + 0.85e^{-0.13u}. \end{aligned}$$

On the other hand, one can use bounds (14):

$$\begin{aligned} 0.665e^{-0.13u} &\le \psi_1(u) \le 0.935e^{-0.13u}, \\ 0.757e^{-0.13u} &\le \psi_2(u) \le 1.064e^{-0.13u}. \end{aligned}$$

Passing to the limit in (6) as $s \to 0$ we deduce that

$$\mathbf{P}\left\{\gamma^+(0) > z, \tau^+(0) < \infty\right\} \approx \begin{pmatrix} e^{-2z}(0.48 + 0.86z) & e^{-z}(0.31 + 0.22z) \\ e^{-2z}(0.21 + 0.34z) & e^{-z}(0.61 + 0.49z) \end{pmatrix},$$

$$\begin{aligned} \mathbf{P}\left\{\gamma_+(0) > z, \tau^+(0) < \infty\right\} \approx &\begin{pmatrix} 0.1e^{-2z}(1+z) & e^{-z}(0.2 + 0.1z) \\ 0.09e^{-2z}(1+z) & e^{-z}(0.18 + 0.09z) \end{pmatrix} \\ &+ \begin{pmatrix} e^{-2.3z}(0.0016 + 0.002z) & -e^{-1.3z}(0.02 + 0.013z) \\ -e^{-2.3z}(0.004 + 0.004z) & e^{-1.3z}(0.05 + 0.03z) \end{pmatrix}, \end{aligned}$$

$$\begin{aligned} \mathbf{P}\left\{\gamma_0^+ > z, \tau^+(0) < \infty\right\} \approx &\begin{pmatrix} e^{-2z}(0.49 + 0.97z + 0.2z^2) & e^{-z}(0.3(1+z) + 0.1z^2) \\ e^{-2z}(0.2 + 0.4z + 0.18z^2) & e^{-z}(0.7(1+z) + 0.09z^2) \end{pmatrix} \\ &+ \begin{pmatrix} -e^{-2.3z}(0.005 + 0.01z) & e^{-1.3z}(0.03 + 0.04z) \\ e^{-2.3z}(0.01 + 0.03z) & -e^{-1.3z}(0.075 + 0.1z) \end{pmatrix}. \end{aligned}$$

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