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BOUNDEDNESS, LIMITS, AND STABILITY OF SOLUTIONS OF A PERTURBATION OF A NONHOMOGENEOUS RENEWAL EQUATION ON A SEMIAXIS

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Abstract. We consider a time-nonhomogeneous perturbation of the classical renewal equation with continuous time on the semiaxis that can be reduced to the Volterra integral equation with a nonnegative bounded (or with a substochastic) kernel. We assume that this kernel is approximated by a convolution kernel for large time intervals and that the latter is generated by a substochastic distribution. We find necessary and sufficient conditions for the existence of the limit of a solution of the perturbed equation under the assumption that the corresponding perturbation is small in a certain sense. We also obtain estimates for the deviation of the solutions of the perturbed equations from those of the initial equations.

Several applications are described.

1. Introduction

The risk processes in a nonhomogeneous environment are studied in [2]–[9].

We consider a time-nonhomogeneous generalization of the classical renewal equation on the semiaxis that can be reduced to the Volterra integral equation with a nonnegative bounded (or with a substochastic) kernel. We assume that the latter kernel is approximated in variation by a convolution kernel for large time intervals and that the convolution kernel is generated by a substochastic distribution on the positive semiaxis. This setting is motivated by the problem of the asymptotic behavior of the ruin function for risk processes with a varying intensity of premiums, which are considered, for example, in [7].

An analogous problem is considered in the author's paper [9] for the renewal equation on the whole axis. The restriction of the domain to the nonnegative semiaxis allows us to obtain more precise results.

The proofs in the first part of this paper use some ideas of the unpublished author's dissertation [10] and of the paper by Schmidli [5].

We have made certain assumptions on the corresponding perturbation and on the minimal solution of the perturbed equation and proved the existence of the limit at infinity for an arbitrary solution of the perturbed equation.

2. Perturbed renewal equation

Consider the semiaxis $\mathbb{R}_+ = [0, \infty)$. Let $\mathfrak{B}_+ = \mathfrak{B}(\mathbb{R}_+)$ be the Borel σ-algebra and let $\delta_a(B) = 1_{a \in B}$ be the Dirac measure.

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We introduce the following classes of functions:

$$B_0 \equiv \left\{ x\colon \mathbb{R}_+ \to \mathbb{R}, x \text{ is a Borel function such that } \sup_{s\le t} |x(s)| < \infty \text{ for all } t \ge 0 \right\},$$

$$B_0^+ \equiv B_0 \cap \{x\colon \mathbb{R}_+ \to \mathbb{R}_+\},$$

$$L_1^0 \equiv \left\{ y \in B_0\colon \lim_{t\to\infty} y(t) = 0, \int_{[0,\infty)} |y(s)|\, ds < \infty \right\}.$$

Then L_1^0 is a Banach space with the norm

$$\|y\|_{01} = \|y\|_0 + \|y\|_1, \qquad \|y\|_0 = \sup_{s\ge 0} |y(s)|, \qquad \|y\|_1 = \int_{[0,\infty)} |y(s)|\, ds. \tag{1}$$

Let G be a substochastic measure on $\mathfrak{B}_+$. The integral equation

$$x_0(t) = y(t) + \int_{[0,t]} x_0(t-s)\, G(ds), \qquad t \ge 0, \tag{2}$$

is called the *generalized renewal equation* on $\mathbb{R}_+$ generated by the measure G with an unknown function x_0 of the class B_0.

Note that, for a given function $y \in B_0$, equation (2) has a unique solution $x_0 \in B_0$ represented in the form of *convolution*

$$x_0(t) = y * U(t) \equiv \int_{[0,t]} y_0(t-s)\, U(ds) \tag{3}$$

(see [11]–[14]), where the σ-finite *renewal measure* U is defined by

$$U(B) = \sum_{n\ge 0} G^{*n}(B). \tag{4}$$

Recall (see [15]) that a measure G is of *absolute continuous type* if the convolution G^{*m} has an absolutely continuous component for some $m \ge 1$, that is, if

$$G^{*m}(B) \ge \nu(B)$$

for all Borel sets B and some nonnegative nonzero and absolutely continuous measure ν.

The classical renewal theorem for a distribution of the absolutely continuous type. *Let G be a probability distribution of absolutely continuous type. Then, for all functions $y \in L_1^0$, equation* (2) *has a unique solution x_0 in the class B_0. Moreover, the following limit:*

$$\lim_{t\to\infty} x_0(t) = m_G^{-1} \int_{[0,\infty)} y(s)\, ds, \qquad m_G \equiv \int_{[0,\infty)} s\, G(ds) \tag{5}$$

exists. In particular, $\sup_{t\in\mathbb{R}} |x(t)| < \infty$.

This result is proved in [12]–[14, Theorem 8, Appendix 3], and [15].

Let $(F(t,B), t \ge 0, B \in \mathfrak{B}_+)$ be a bounded nonnegative kernel.

Definition. The generalized Volterra integral equation

$$x(t) = y(t) + \int_0^t x(t-s)\, F(t, ds), \qquad t \in \mathbb{R}, \tag{6}$$

is called the nonhomogeneous perturbation of equation (2) for an unknown function $x \in B_0$ and a given function $y \in L_1^0$.

The kernel F is approximated by the measure G in variation

$$\Delta(t,B) \equiv F(t,B) - G(B) \to 0, \qquad t \to \infty, \tag{7}$$

uniformly in $B \in \mathfrak{B}_+$, that is,

$$\delta(t) \equiv |\Delta|\,(t,[0,t]) = \sup_{|x|\le 1}\left|\int_0^t x(s)\,\Delta(t,ds)\right| \to 0, \qquad t\to\infty. \tag{8}$$

Remark 1. Multiplying both sides of equation (6) by $\exp(\alpha t)$, we find that the functions and the kernel

$$x_\alpha(t) = e^{\alpha t}x(t), \qquad y_\alpha(t) = e^{\alpha t}y(t), \qquad F_\alpha(t,ds) = e^{\alpha s}\,F(t,ds) \tag{9}$$

satisfy an equation similar to (6), namely

$$x_\alpha(t) = y_\alpha(t) + \int_0^t x_\alpha(t-s)\,F_\alpha(t,ds), \qquad t\in\mathbb{R}. \tag{10}$$

We define the following linear bounded operators acting on the class B_0 and related to the kernels F and H:

$$\begin{aligned} F[x](t) &= \int_0^t F(t,ds)x(t-s), \qquad t\ge 0,\\ F*H[x](t) &= \int_0^t F*H(t,ds)x(t-s), \qquad t\ge 0,\\ F*H(t,B) &= \int_0^t F(t,ds)H(t-s, B\cap[s,t]-s),\\ F^n[x](t) &= \int_0^t F^{*n}(t,ds)x(t-s), \end{aligned} \tag{11}$$

$$\begin{aligned} F^{*n}(t,B) = \int_0^t F(t,dt_1)\int_0^{t-t_1} F(t-t_1,dt_2)\\ \dots\int_{t_1+\dots+t_{n-1}}^t F(t-t_1-\dots-t_{n-1}, dt_n-t_1-\dots-t_{n-1})1_{t_n\in B}. \end{aligned} \tag{12}$$

The following theorem generalizes a result obtained in [10].

Theorem 1. *Let G be a substochastic measure of absolutely continuous type and let its perturbation* (8) *be such that $\delta\in L_1^0$. Assume that $y\in L_1^0$. If a solution $x\in B_0$ of equation* (6) *is bounded, then there exists a finite limit* $\lim_{t\to\infty} x(t)$.

To study solutions of equation (6) we use the following notions.

Definition. (a) The sum of the series

$$x_0(t) = \sum_{n\ge 0} F^n[y](t), \qquad t\ge 0, \tag{13}$$

is called the minimal solution of equation (6) for a function $y\in B_0^+$ if partial sums of the series converge pointwise at every point of the set $\mathbb{R}_+\cup\{\infty\}$.

(b) If the difference

$$\sum_{n\ge 0} F^n[y^+](t) - \sum_{n\ge 0} F^n[y^-](t)$$

is well defined, then it is called the minimal solution of equation (6) for a given function $y\in B_0$.

The well-known properties of minimal solutions are listed in the following result (see [17, Chapter 1]).

Lemma 1. *Let $(F(t,B), t \geq 0, B \in \mathfrak{B}_+)$ be a bounded nonnegative kernel.*

(1) *The minimal solution x_0 given by* (13) *for a given function $y \in B_0^+$ is a solution of equation* (6) *in the class of Borel nonnegative functions assuming values in $\mathbb{R}_+ \cup \{\infty\}$.*
(2) *Let $x \in B_0^+$ be an arbitrary solution of equation* (6) *for a given function $y \in B_0^+$. Then $x(t) \geq x_0(t)$, $t \geq 0$.*
(3) *A solution $x \in B_0^+$ of equation* (6) *is minimal if and only if*

$$\lim_{t\to\infty} F^n[x](t) = 0, \qquad t \geq 0.$$

(4) *A minimal solution $x_0 \in B_0^+$ of equation* (6) *for a given function $y \in B_0^+$ is bounded if and only if*

$$w(t) \geq F[w](t) + y(t), \qquad t \geq 0, \tag{14}$$

for some bounded function $w \in B_0^+$. In this case, $w \geq x_0$.

Remark 2. The inequality $w \geq x_0$ follows from condition (14) with $w = c$. The converse is not true in general.

The following property implies the boundedness of a solution of equation (6).

Definition. A kernel $F(t,B)$ is called (a) regular if there are some constants $a > 0$ and $T \geq 0$ such that

$$\inf_{t\geq T} F(t,[a,\infty)) > 0, \tag{15}$$

and (b) completely regular if $T = 0$ in condition (15).

Theorem 2. *Let $F(t,B)$ be a regular substochastic kernel and let $x \in B_0$ be a solution of equation* (6).

(a) *This solution is bounded if and only if the minimal solution $x_0 \in B_0$ of equation* (6) *with $y \in B_0$ is bounded.*
(b) *If a kernel $F(t,B)$ is completely regular, then the solution x coincides with the minimal solution, that is, $x = x_0$.*

The stability implies the regularity.

Lemma 2. *If a kernel $F(t,B)$ satisfies condition* (7) *and if a measure G is of absolutely continuous type, then the kernel F is regular.*

Theorems 1 and 2 imply the criterion for the existence of the limit for a solution of equation (6).

Theorem 3. *Let G be a measure of absolutely continuous type. Assume that the perturbation* (8) *of a substochastic kernel $F(t,B)$ is such that $\delta \in L_1^0$. Let $x \in B_0$ be a solution of equation* (6) *with a function $y \in L_1^0$.*

(a) *The finite limit $\lim_{t\to\infty} x(t)$ exists if and only if the minimal solution* (13) *of this equation is bounded.*
(b) *If condition* (a) *holds, then the above limit is equal to*

$$\lim_{t\to\infty} x(t) = m_G^{-1} \int_{[0,\infty)} (y(s) + \Delta[x](s))\, ds. \tag{16}$$

Remark 3. Equation (6) may have more than one bounded solution in case the kernel F is not completely regular even if all the assumptions of Theorem 3 are satisfied.

Example 1. Suppose all the assumptions of Theorem 3 hold. We also assume that

$$F(t,B) \le G(B)$$

for $t \ge 0$ and $B \in \mathfrak{B}_+$ (that is, $F(t,ds) = \rho(t,s)\,G(ds)$ with $\rho(t,s) \in [0,1]$) and that the probability measure G is of absolutely continuous type. Then the minimal solution of equation (6) with a function $y \in L_1^0$ is bounded and thus the limit $\lim_{t\to\infty} x(t)$ exists.

The stability of solutions of equation (2) can be expressed via certain numerical coefficients.

Theorem 4. *Let a measure G be of absolutely continuous type. Let x and x_0 be solutions in the class B_0 of equations* (6) *and* (2)*, respectively, with a function $y \in L_1^0$. If the perturbation $\delta \in L_1^0$ of the kernel G given by* (8) *is such that*

$$\|\delta\|_G \equiv \sup_{t\ge 0} \delta * U(t) < 1 \tag{17}$$

and if the renewal measure is given by (4)*, then the solution x is unique in the class B_0, coincides with the minimal solution, and satisfies the following inequalities:*

$$\sup_{t\ge 0} |x(t)| \le \left(1 - \|\delta\|_G\right)^{-1} \sup_{t\ge 0} |x_0(t)|\,, \tag{18}$$

$$\sup_{t\ge 0} |x(t) - x_0(t)| \le \|\delta\|_G \sup_{t\ge 0} |x(t)|\,. \tag{19}$$

Moreover, the condition $G(\mathbb{R}_+) = 1$ implies that

$$\left|\lim_{t\to\infty} x(t) - \lim_{t\to\infty} x_0(t)\right| \le m_G^{-1} \int_{\mathbb{R}_+} \delta(s)\,ds \sup_{t\ge 0} |x(t)|\,. \tag{20}$$

Remark 4. The renewal theorem implies that $\|\delta\|_G < \infty$ for $\delta \in L_1^0$ (see [13]–[15]). Moreover, it is proved in [11, 12] that

$$\|\delta\|_G \le C(G)\|\delta\|_{01}, \tag{21}$$

where the norm $\|\delta\|_{01}$ is defined by (1) and the constant $C(G)$ is completely determined by the measure G. The numerical estimation for $C(G)$ is given in [10] for various classes of measures G.

Remark 5. If the function $\delta(t)$ has a bounded variation, then one can get a simple estimate for the norm on the left hand side of (17), namely

$$\sup_{t\ge 0} \delta * U(t) \le m_G^{-2} \left(m_G \int_0^\infty \delta(s)\,ds + (\delta(0) + \operatorname{var}\delta) \int_{\mathbb{R}_+} (t^2/2)\,G(dt) \right). \tag{22}$$

Example 2 (The condition $\delta \in L_1^0$ is essential)**.** Let $\alpha \ge 0$ and $\varepsilon \in [0,1)$. We consider the following kernel, measure, and function:

$$F(t,ds) = e^{-s}\left(1 - \varepsilon(1+t-s)^{-\alpha-1}\right) ds, \qquad G(ds) = e^{-s}\,ds, \qquad y(t) = e^{-t}, \qquad t \ge 0.$$

Then the solution of (2) equals $x_0(t) = 1$ and the perturbation function (8) is equivalent to $\delta(t) \approx \varepsilon(1+t)^{-\alpha-1}$, $t \to \infty$. The perturbation function belongs to the space L_1^0 if and only if $\alpha > 0$. Reducing equation (6) to the differential form allows us to evaluate $x(t) = (1+t)^{-\varepsilon}$ for $\alpha = 0$, and $x(t) = \exp(-\varepsilon(1-(1+t)^{-\alpha})/\alpha)$ for $\alpha > 0$. Thus $x(t) \to x_0(t)$ as $\varepsilon \to 0$ for all t. On the other hand, the convergence $x(\infty) \to x_0(\infty)$ as $\varepsilon \to 0$ holds if and only if $\alpha > 0$, that is, if $\delta \in L_1^0$.

Example 3 (The condition $\|\delta\|_G < 1$ is essential)**.** Consider the kernel

$$F(t, ds) = e^{-s} 1_{s<t} + \delta_t(ds)\varepsilon e^{-t}, \qquad G(ds) = e^{-s}\, ds,$$

where $\varepsilon \in (0, 1]$. Then the perturbation function (8) equals $\delta(t) = \varepsilon e^{-t}$ and

$$\|\delta\|_G = \sup_{t\geq 0} \left(\delta(t) + \int_0^t \delta(s)\, ds \right) = \varepsilon.$$

If $\varepsilon < 1$, then the unique solution of equation (6) (which is reduced to the linear differential equation in this case) exists and is of the form $x(t) = x_0(t) + \varepsilon x_0(t)/(1-\varepsilon)$. On the other hand, if $\varepsilon = 1$, then the uniqueness condition fails, since solutions of equation (6) are invariant with respect to the addition of an arbitrary constant.

Corollary 1. *Let all the assumptions of Theorem 4 hold. If $G(\mathbb{R}_+) = 1$, $m_G < \infty$, $\lim_{t\to\infty} x_0(t) \neq 0$, and*

$$0 < m_G^{-1} \int_{\mathbb{R}_+} \delta(s)\, ds \sup_{t\geq 0} |x_0(t)| \leq \frac{1}{2}\big(1 - \|\delta\|_G\big) \left| \lim_{t\to\infty} x_0(t) \right| \tag{23}$$

for δ defined in (8)*, then the limit* $\lim_{t\to\infty} x(t)$ *is nonzero.*

Remark 6. If $x \in B_0^+$, then

$$\delta_+(t) \equiv \sup_{0\leq x\leq 1} \left| \int_0^t x(s)\, \Delta(t, ds) \right| \tag{24}$$

can be used in definition (8) instead of the total variation of the function $\delta(t)$.

3. An application to the risk process

Below we describe an application of the nonhomogeneous renewal theorem to a certain model of the risk process. This model is based on the renewal process modelling the sequence of independent payments, while the ruin is not related to the income and to the balance between premiums and payments. Rather, the ruin is modelled by the ruin probabilities at sequential payment moments. The latter moments depend on the remaining capital evaluated at those moments.

Consider an insurance company with an initial capital $t > 0$ and independent premiums $(\xi_k, k \geq 1)$ that have a common distribution G of absolutely continuous type. Put $S_n = \xi_1 + \cdots + \xi_n$, $S_0 = 0$.

Assume that the company ruins with probability $\pi(t - S_{k-1})$ before the kth payment is made. This event depends on the current capital, and if it does not occur, then the company continues functioning on the market.

The event that the company ruins is assumed to be independent of the sequence of payments given the remaining capital at the corresponding moment.

Denote by $\nu(t)$ the total number of payments up to the ruin. Then

$$\mathsf{P}\big(\nu(t) = n \mid S_0, \ldots, S_n\big) = \left[\prod_{k=0}^{n-1} \big(1 - \pi(t - S_k)\big) \right] \pi(t - S_n), \qquad n \geq 0.$$

We define the distribution function of the remaining capital b at the moment of ultimate ruin:

$$x(t) = \mathsf{P}(t - b \leq S_{\nu(t)} < t) = \mathsf{P}(0 < t - S_{\nu(t)} \leq b).$$

This function is a solution of a nonhomogeneous renewal equation

$$x(t) = \pi(t) 1_{t\leq b} + (1 - \pi(t)) \int_0^t x(t-s)\, G(ds). \tag{25}$$

The function $x*G(t) = \mathsf{P}(t-S_{\nu(t)+1} \in (0,b])$ must substitute $x(t)$ if the ruin may happen only *after* the payment is made.

Related to (25), the renewal kernel in equation (6) equals

$$F(t,ds) = (1-\pi(t))\, G(ds) 1_{s<t}.$$

Assume that the limit $\theta \equiv \lim_{t\to\infty} \pi(t)$ exists and moreover $\sup_{t\geq 0} \pi(t) < 1$. Then the above kernel is regular and satisfies condition (7) with the measure $(1-\theta)G$.

We also assume the Cramér condition: there exists $\delta > 0$ such that

$$\widehat{G}(\delta) \equiv \mathsf{E} \exp(\delta \xi_1) < \infty. \tag{26}$$

(a) First we consider the case of $\theta \in (0,1)$. For simplicity we assume that $(1-\theta)\widehat{G}(\delta) \in (1,\infty)$ for some $\delta > 0$. If $\pi(t) - \theta \in L_1^0$, then Theorem 3 implies that, for every $b > 0$, the limit

$$\begin{aligned} \lim_{t\to\infty} \exp(\alpha t) x(t) = m_\alpha^{-1} \int_0^b \exp(\alpha t) \pi(t)\, dt \\ + m_\alpha^{-1} \int_0^\infty \exp(\alpha t)(\theta - \pi(t))\, dt \int_0^t x(t-s)\, G(ds) \end{aligned} \tag{27}$$

exists, where $m_\alpha = (1-\theta)\, \mathsf{E}\, \xi_1 \exp(\alpha \xi_1)$ and where the constant $\alpha > 0$ is a unique solution of the Lundberg equation

$$(1-\theta)\widehat{G}(\alpha) = 1. \tag{28}$$

Note that the second term in (27) containing the unknown x is significantly smaller than the first term in at least two cases, namely if $b \to \infty$ and $\theta - \pi(t) \to 0$ sufficiently quickly or if the norm $\|\theta - \pi(t)\|_{01}$ is sufficiently small.

(b) Let $\theta = \lim_{t\to\infty} \pi(t) = 0$. If $\pi \in L_1^0$, then

$$x(t) = \int_0^t U(ds) \pi(t-s) \big(1_{t-s\leq b} - x * G(t-s)\big) \tag{29}$$

and the limit

$$\lim_{t\to\infty} x(t) = m_G^{-1} \int_0^\infty \pi(s) \big(1_{s\leq b} - x(s)\big)\, ds \tag{30}$$

exists where $U = \sum_{n\geq 0} G^{*n}$ is the renewal measure.

If the norms $\|\pi\|_{01}$ in (1) are sufficiently small, then we obtain the first approximation as $\|\pi\|_{01} \to 0$:

$$\begin{aligned} x(t) &= \int_{t-b}^t U(ds) \pi(t-s)\, ds + o(\|\pi\|_{01}), \\ \lim_{t\to\infty} x(t) &= m_G^{-1} \int_0^b \pi(s)\, ds + o(\|\pi\|_{01}). \end{aligned} \tag{31}$$

Note that the probability that the capital is exhausted according to the schedule is equal to

$$q(t) = \mathsf{P}(S_{\nu(t)} \geq t),$$

which is a solution of the equation

$$q(t) = (1-\pi(t)) \left(1 - G(t) + \int_0^t q(t-s)\, G(ds)\right), \tag{32}$$

whence we derive the approximations

$$q(t) = 1 - U * \pi(t) + o(\|\pi\|_{01}),$$

$$\lim_{t\to\infty} q(t) = 1 - m_G^{-1} \int_0^\infty \pi(s)\, ds + o(\|\pi\|_{01}). \tag{33}$$

4. Proofs

The equalities (11) and (12) are obtained by iterating $F[H[x]]$ and $F[F[\dots F[x]\dots]$.

Recall that the kernel of the perturbation Δ and the perturbation function δ are defined by (7) and (8), respectively.

First we note that Remark 1 follows from the multiplication property of the exponential function.

Proof of Theorem 1. Let $x \in B_0$ be a solution of equation (6). We rewrite this equation in the form of the generalized renewal equation

$$x = y + \Delta[x] + G * x = y_1 + G * x, \tag{34}$$

where $y_1 = y + \Delta[x] \in L_1^0$, since $y \in L_1^0$ and the solution x is bounded. The latter property follows from

$$|\Delta[x](t)| = \left| \int_0^t \Delta(t, ds) x(t-s) \right| \le \delta(t) \sup_{s\ge 0} |x(s)| \in L_1^0. \tag{35}$$

Assume that G is a probability distribution, that is, $G(\mathbb{R}_+) = 1$. Then the desired result follows from the above renewal theorem for a distribution of absolutely continuous type.

If $G(\mathbb{R}_+) < 1$, then the corresponding renewal measure (4) is finite, since

$$U(\mathbb{R}_+) \le (1 - G(\mathbb{R}_+))^{-1}.$$

Thus (34) and (3) imply that the limit

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} \int_0^t y_1(t-s)\, U(ds) = \lim_{t\to\infty} y_1(t) U(\mathbb{R}_+) = 0$$

exists. □

Proof of Lemma 1. (1) By definition, $x_0 = \sum_{n\ge 0} F^n[y] \ge 0$. Thus the Lebesgue monotone convergence theorem for $y \in B_0^+$ implies that

$$x_0 = y + \sum_{n\ge 1} F[F^{n-1}[y]] = y + F\left[\sum_{n\ge 1} F^{n-1}[y]\right] = y + F[x_0].$$

(2) Let $y \in B_0^+$. For an arbitrary solution $x \in B_0^+$, we iterate equation (6) and obtain the equality

$$x = \sum_{0\le k<n} F^k[y] + F^n[x], \tag{36}$$

whence

$$x \ge \sum_{0\le k<n} F^k[y] \to \sum_{0\le k} F^k[y] = x_0, \qquad n \to \infty.$$

(3) Sufficiency. We deduce from equality (36) that

$$x = \lim_{n\to\infty} \sum_{0\le k<n} F^k[y] + \lim_{n\to\infty} F^n[x] = x_0 + 0,$$

where x_0 is the minimal solution of equation (6).

The necessity follows from the convergence of series (13):

$$F^n[x_0] = F^n\left[\sum_{k\geq 0} F^k[y]\right] = \sum_{k\geq n} F^k[y] \to 0, \qquad n\to\infty.$$

(4) To prove the sufficiency we iterate (14) and take into account the positivity of w:

$$w \geq \sum_{0\leq k<n} F^k[y] + F^n[w] \geq \sum_{0\leq k<n} F^k[y] \to x_0, \qquad n\to\infty.$$

Thus $w \geq x_0$ and the solution x_0 is bounded.

The necessity of inequality (14) follows if one considers $w = x_0$. □

Proof of Remark 2. A unique solution of the renewal equation $x = y + x * E$ with the measure $E(ds) = \exp(-s)\,ds$ is of the form $x = y + 1 * y$. The inequality $c \geq y + c * E$ is equivalent to $y(t) \leq c\exp(-t)$ and the latter inequality does not hold for

$$y(t) = (1+t)^{-2} \in L_1^0,$$

while $\sup x < \infty$. □

Lemma 3. *Suppose a substochastic kernel F and a function $r \in B_0^+$ satisfy the following inequality:*

$$r \leq F[r], \qquad t \geq 0. \tag{37}$$

(a) *If the kernel F is regular, then* $\sup r < \infty$.
(b) *If F is completely regular, then $r = 0$.*

Proof. Take the constants a and T as defined in (15), where $\alpha > 0$ is the value of the right hand side of (15). We define substochastic kernels $Q(t,B) = F(t, B\cap(a,\infty))$, $K = F - Q$, and a function $q(t) = Q(t,\mathbb{R}_+) \in [0,1]$. By definition, $q(t) \geq \alpha$ for $t \geq T$, and

$$K(t,\mathbb{R}_+) = F(t,\mathbb{R}_+) - q(t) \leq 1 - q(t).$$

Since $Q(t,[0,a]) = 0$, we deduce from (37) that, for $u \geq T$,

$$\begin{aligned} r(u) &\leq K[r](u) + \int_0^u Q(u,ds)r(u-s) \\ &= K[r](u) + \int_a^u Q(u,ds)r(u-s) \\ &\leq (1-q(u))\sup_{s\leq u} r(s) + q(u)\sup_{s\leq u-a} r(s) \\ &\leq (1-\alpha)\sup_{s\leq u} r(s) + \alpha\sup_{s\leq u-a} r(s). \end{aligned}$$

Let $M = \sup_{s<T} r(s)$ and $\rho(t) = \sup_{T\leq s\leq t} r(s)$. Passing to the supremum with respect to $u \in [T,t]$ and considering the equality $\sup_{0\leq s\leq t} r(s) = \max(M,\rho(t))$, we derive from the latter inequality that

$$\rho(t) \leq (1-\alpha)\max(M,\rho(t)) + \alpha\max(M,\rho(t-a)) \tag{38}$$

for $t \geq T$.

We prove the boundedness of r (and ρ) by contradiction. Assume that $\rho(t) \to \infty$ as $t \to \infty$. Then, for some $T_1 \geq T$ and for all $t \geq T_1$, inequality (38) reduces to $\rho(t) \leq (1-\alpha)\rho(t) + \alpha\rho(t-a)$. Thus $\rho(t) \leq \rho(t-a)$ for $t \geq T_1$ and $\sup\rho \leq \rho(T_1) < \infty$. This contradicts the assumption $\rho(t) \to \infty$, whence $\sup r = \sup\rho < \infty$ and statement (a) is proved.

In statement (b) of the lemma, $T = 0$, whence $M = 0$ in equality (38). Considering the case of $t < a$, we have $\rho(t) \le (1-\alpha)\rho(t)$. Hence $\rho(t) = 0$ for $t < a$. As above, we conclude by induction that (38) implies $\rho(t) \le \rho(a-0) = 0$ if $t \ge a$. □

Proof of Theorem 2. (a) The function $r = x - x_0 \in B_0$ satisfies the equation $r = F[r]$ in view of linearity. Using convexity, we obtain $|r| \le F[|r|]$. Thus Lemma 3(a) implies that $\sup |r| < \infty$ and this completes the proof.

(b) Lemma 3(b) implies that $r = 0$. □

Proof of Lemma 2. Choose $a > 0$ such that $G([a,\infty)) > 0$. It follows from (7) that

$$F(t,[a,\infty)) \ge \frac{1}{2} G([a,\infty)) > 0$$

for all $t \ge T$ and some $T \ge 0$. □

Proof of Theorem 3. The kernel $F(t,B)$ is regular by Lemma 2. Since $x \in B_0$, the limit $\lim_{t\to\infty} x(t)$ exists and is finite if and only if the solution x is bounded (see Theorem 1). Therefore Theorem 2(a) completes the proof. The expression for the limit (16) follows from that of the corresponding renewal equation (34) and from the renewal theorem (5). □

Proof of Remark 3. Consider equation (6) with the following kernel and limit measure:

$$F(t,ds) = \delta_{t/2}(ds) 1_{t<1} + \exp(-s)\, ds 1_{t\ge 1}, \qquad G(ds) = \exp(-s)\, ds.$$

Consider the function $y(t) = t 1_{t<1}$. Then all assumptions of Theorem 3 hold. Straightforward calculations show that the minimal solution of equation (6) is

$$x_0(t) = 2t 1_{t<1} + 2e^{-1} 1_{t\ge 1},$$

while the bounded function $r(t) = 1_{t<1} + (1-e^{-1}) 1_{t\ge 1}$ is a solution of the equation $r = F[r]$. Thus the set of solutions of equation (6) contains the functions $x_0 + cr$ for all $c \in \mathbb{R}$. □

Proof of Example 1. The assumption $F(t,B) \le G(B)$ for $t \ge 0$ and $B \in \mathfrak{B}_+$ implies that $\sup \sum_{k\ge 0} F^k[y^\pm] \le \sup \sum_{k\ge 0} G^{*k} * y^\pm = \sup U * y^\pm < \infty$. □

Proof of Theorem 4. The solution $x \in B_0$ of equation (6) also satisfies equation (34). Moreover, $y_1 = y + \Delta[x] \in B_0$, since $L_1^0 \subset B_0$ and since the kernel Δ is bounded and transforms B_0 to B_0. Thus a unique solution in B_0 of equation (34) is of the form

$$x = U * y_1 = U * y + U * \Delta[x] = x_0 + U * \Delta[x] \tag{39}$$

with the renewal measure (4). By definition (8),

$$\begin{aligned} \sup_{t\le T} |U * \Delta[x](t)| &\le \sup_{t\le T} \int_0^t U(ds) \left| \int_0^{t-s} \Delta(t-s,du) x(t-s-u) \right| \\ &\le \sup_{t\le T} \int_0^t U(ds) \delta(t-s) \sup_{u\le T} |x(u)| \le \|\delta\|_G \sup_{u\le T} |x(u)| \end{aligned} \tag{40}$$

for all $T \ge 0$. Thus (39) implies that

$$\sup_{t\le T} |x(t)| \le \sup_{t\le T} |x_0(t)| + \|\delta\|_G \sup_{t\le T} |x(t)|,$$

whence we obtain (18) by passing to the limit as $T \to \infty$.

Representation (39) written in the form $x - x_0 = U * \Delta[x]$ proves inequality (19) upon applying bounds (40).

Under the condition $G(\mathbb{R}_+) = 1$, the limits

$$\lim_{t\to\infty} x(t) = m_G^{-1} \int_{\mathbb{R}_+} y_1(s)\, ds, \qquad \lim_{t\to\infty} x_0(t) = m_G^{-1} \int_{\mathbb{R}_+} y(s)\, ds$$

exist by the assumptions of the theorem and by the above *renewal theorem for a distribution of absolutely continuous type.* Thus

$$\begin{aligned}\lim_{t\to\infty} |x(t) - x_0(t)| &= \left| m_G^{-1} \int_{\mathbb{R}_+} (y_1(s) - y(s))\, ds \right| \\ &= m_G^{-1} \left| \int_{\mathbb{R}_+} \Delta[x](s)\, ds \right| \\ &\le m_G^{-1} \int_{\mathbb{R}_+} \delta(s)\, ds \sup_{t\ge 0} |x(t)|\,,\end{aligned}$$

whence (20) follows. □

Proof of Corollary 1. The inequalities (20), (18), and (23) imply that

$$\left| \lim_{t\to\infty} x(t) - \lim_{t\to\infty} x_0(t) \right| \le m_G^{-1} \int_{\mathbb{R}_+} \delta(s)\, ds (1 - \|\delta\|_G)^{-1} \sup_{t\ge 0} |x_0(t)| \le \frac{1}{2} \left| \lim_{t\to\infty} x_0(t) \right|,$$

whence the result follows. □

Proof of Remark 5. Inequality (22) follows from Daley's inequality [16] upon the substitution $\delta(t) = \delta(0) + \int_0^t d\delta(s)$, namely

$$\delta * U(t) = m_G^{-1} \int_0^t \delta(s)\, ds + \delta(0) V(t) + \int_0^t d\delta(s) V(t-s),$$

where $0 \le V(t) = U(t) - m_G^{-1} t \le m_G^{-2} \int_{\mathbb{R}_+} (t^2/2)\, G(dt)$. □

Proof of Remark 6. In contrast to the case of $x \in B_0$, the proof of inequality (35) for the case of $x \in B_0^+$ is based on the assumption that $0 \le x(t) \le 1$. □

Proof of the application to the claim process. To derive (25), we first evaluate

$$\begin{aligned}\mathsf{P}(0 < S_{\nu(t)} \le b \mid S_n, n \ge 1) &= \sum_{n\ge 0} \left[\prod_{k=0}^{n-1} (1 - \pi(t - S_k)) \right] \pi(t - S_n) 1_{0<t-S_n\le b} \\ &= \pi(t) 1_{0<t\le b} + (1 - \pi(t)) \sum_{n\ge 1} \left[\prod_{k=1}^{n-1} (1 - \pi(t - \xi_1 - S'_{k-1})) \right] \\ &\quad \times \pi\big(t - \xi_1 - S'_{n-1}\big) 1_{0<t-\xi_1-S'_{n-1}\le b} \\ &= \mathsf{P}\left(0 < S'_{\nu'(t-\xi_1)} \le b \;\middle|\; \xi_1, S'_n, n \ge 1\right) 1_{\xi_1<t},\end{aligned}$$

where the random variables $S'_k = \xi_2 + \cdots + \xi_{k+1}$ have the same distribution as the S_k and do not depend on ξ_1. Considering the mathematical expectation of both sides, we prove the desired result.

(a) For $\alpha \ge 0$, we introduce

$$x_\alpha(t) = \exp(\alpha t) x(t), \qquad \pi_\alpha(t) = \exp(\alpha t) \pi(t).$$

According to Remark 1, equation (25) can be rewritten as

$$x_\alpha(t) = \pi_\alpha(t) 1_{t\le b} + \int_0^t x_\alpha(t-s) F_\alpha(t, ds), \tag{41}$$

where $F_\alpha(t, ds) = (1 - \pi(t)) \exp(\alpha s)\, G(ds)$. If $\theta \in (0, 1)$, then the limit (as $t \to \infty$) distribution for $F_\alpha(t, ds)$ is given by

$$(1 - \theta) \exp(\alpha s)\, G(ds).$$

In view of (28), this is a probability distribution. As in the case of G, the limit distribution is of absolutely continuous type. Further, the perturbation kernel (7) is given by $\Delta(t, ds) = (\theta - \pi(t)) \exp(\alpha s)\, G(ds)$, while the perturbation function is

$$\delta(t) \le |\pi(t) - \theta| / (1 - \theta) \in L_1^0.$$

Applying equality (16) of Theorem 3(b) to equation (41), we prove (27).

(b) Representation (29) is a corollary of (3) for the renewal equation (41). Equality (30) follows from the renewal theorem (16).

From equality (29), inequality (21), and the boundedness of x, we obtain that

$$\sup |x| = O(\sup U * \pi) = O(\|\pi\|_{01}), \qquad \|\pi\|_{01} \to 0.$$

Thus (31) follows from equality (29). Equality (32) takes the form $1 - q = y_1 + (1 - q) * G$, where

$$\begin{aligned} y_1(t) &= -(1 - \pi(t))(1 - G(t)) + (1 - G(t)) + \pi(t) q * G(t) \\ &= \pi(t)(1 - G(t)) + \pi(t) q * G(t) = O(\|\pi\|_{01}), \qquad \|\pi\|_{01} \to 0. \end{aligned}$$

Thus representations (33) hold. □

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