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ESTIMATES FOR THE PROBABILITY THAT A SYSTEM OF RANDOM EQUATIONS IS SOLVABLE IN A GIVEN SET OF VECTORS OVER THE FIELD GF(3)

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Abstract. Let P_n be the probability that a second order system of nonlinear random equations over the field $\mathbf{GF}(3)$ has a solution in a given set of vectors, where n is the number of unknowns in the system. A necessary and sufficient condition is found for $P_n \to 0$ as $n \to \infty$. Some rates of convergence to zero are found and some applications are described.

1. Setting of the problem. Statement of main results

Let

$$\sum\nolimits_{3}\limits_{1\le j_1<j_2\le n} a^{(\mu)}_{j_1 j_2} x_{j_1} x_{j_2} = \mathbf{0}, \qquad \mu \in J, \tag{1}$$

be a system of nonlinear random equations of the second order considered over the field $\mathbf{GF}(3)$, where $\sum_3$ denotes the summation in the field $\mathbf{GF}(3)$ and where $J = \{1, \dots, T\}$ and $T = T(n)$. Recall that $\mathbf{GF}(3)$ contains only three elements.

We assume that system (1) satisfies the following condition:

(A) the coefficients $a^{(\mu)}_{j_1 j_2}$, $1 \le j_1 < j_2 \le n$, $\mu \in J$, are independent random variables assuming values in the field $\mathbf{GF}(3)$ according to the distribution

$$\mathsf{P}\{a^{(\mu)}_{j_1 j_2} = \mathbf{1}\} = \mathsf{P}\{\mathbf{a^{(\mu)}_{j_1 j_2} = 2}\} = \mathbf{p}_\mu, \qquad \mathsf{P}\{\mathbf{a^{(\mu)}_{j_1 j_2} = 0}\} = \mathbf{1 - 2p}_\mu.$$

Let V_n be the family of all n-dimensional vectors $\bar{x}$, $\bar{x} = (x_1, x_2, \dots, x_n)$ whose coordinates belong to the field $\mathbf{GF}(3)$, and let $V'_n = V_n \setminus \{\bar{x}\colon |\bar{x}| \le 1\}$, where $|\bar{x}|$ denotes the number of nonzero coordinates of the vector $\bar{x}$.

Let $\bar{x}^{(1)}$ and $\bar{x}^{(2)}$ be two arbitrary vectors, where $\bar{x}^{(q)} \in V_n$, $\bar{x}^{(q)} = \left(x_1^{(q)}, \dots, x_n^{(q)}\right)$, $q = 1, 2$. Let $i_{c_1 c_2}$, $c_1, c_2 \in \mathbf{GF}(3)$, denote the number of pairs (c_1, c_2) among n possible pairs $\left(x_j{}^{(1)}, x_j{}^{(2)}\right)$, $1 \le j \le n$.

Let $i = i_{\mathbf{01}} + i_{\mathbf{02}}$ and $l = i_{\mathbf{10}} + i_{\mathbf{20}}$.

By M_n, we denote the maximal subset of the set V'_n (with respect to the inclusion) with the property that arbitrary vectors $\bar{x}^{(1)}, \bar{x}^{(2)} \in V'_n$ belong to M_n if and only if

$$i + l \ge 1. \tag{2}$$

For example, if $n = 3$, then

$$M_3 = \{(\mathbf{1}, \mathbf{1}, \mathbf{0}), (\mathbf{1}, \mathbf{0}, \mathbf{2}), (\mathbf{0}, \mathbf{2}, \mathbf{2}), (\mathbf{1}, \mathbf{2}, \mathbf{1})\}.$$

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Let θ_n be a random variable equal to the number of solutions of system (1) that belong to the set M_n.

In what follows we assume that the probability p_μ varies in such a way that

$$\frac{c\ln n}{n} \le p_\mu \le \frac{1}{2} - \frac{c\ln n}{n}, \tag{3}$$

where $\ln 3/\ln 2 < a_1 \le c = c(n) \le a_2 < \infty$ and where $\{a_r\colon r = 1, 2, \dots\}$ is a sequence of bounded positive constants.

Theorem 1.1. *Assume that conditions* (A) *and* (3) *hold. Then*

$$\mathsf{P}\{\theta_n > 0\} = o(1), \qquad n \to \infty, \tag{4}$$

if and only if

$$T = n\frac{\ln 2}{\ln 3} + A_n, \tag{5}$$

where $A_n \to \infty$ as $n \to \infty$.

Remark 1.1. The existence of solutions belonging to a given set of vectors for a system of equations is considered in [2] for different right hand sides. In [1], special solutions of a homogeneous system of linear random equations over a finite field are studied and the study of special solutions for the random linear inclusion is considered.

Theorem 1.2. *Let conditions* (A), (3), *and* (5) *hold. Assume that the parameters ε_1, $\varepsilon_1 \in (0,1)$, and c vary in such a way that*

$$0 < \gamma_1 \le \varepsilon_1 c \le \gamma_0 < \frac{4}{3}\left(1 - \frac{\ln 3}{c\ln 2}\right),$$

where γ_0 and γ_1 are fixed numbers.

Then there exist a real number $\varepsilon_2 \in (0,1)$ and natural number $n_0 = n_0(\varepsilon_1, \varepsilon_2, c)$ such that $\mathsf{P}\{\theta_n > 0\} \le Z_1$ for $n \ge n_0$, where

$$\begin{aligned} Z_1 = &\sum_{t=2}^{\left[\sqrt{\frac{\varepsilon_1 n}{\ln n}}\right]} \frac{1}{t!}\left(\frac{1}{n^{c\frac{\ln 2}{\ln 3}\left(1-\frac{\ln 3}{c\ln 2}-\frac{3}{4}\gamma_0\right)}}\right)^t \left(\frac{1}{n^{c\frac{A_n}{n}\left(1-\frac{3}{4}\gamma_0+\frac{3}{4}c\sqrt{\frac{\varepsilon_1\ln n}{n}}+\frac{3}{4}c\frac{\ln 2}{\ln 3}\frac{\sqrt{\varepsilon_1 n\ln n}}{A_n}\right)}}\right)^t \\ &+ 2^{n\sigma(\varepsilon_2)}\left(\frac{1}{3} + \frac{2}{3e^{\frac{3}{2}\gamma_1\left(1+\sqrt{\frac{\ln n}{\varepsilon_1 n}}\right)}}\right)^{n\frac{\ln 2}{\ln 3}+A_n} \\ &+ \left(\frac{\exp\left\{\frac{2}{n^{\frac{3}{2}nc\varepsilon_2^2\left(1+\frac{1}{\varepsilon_2 n}\right)}}\right\}}{3}\right)^{A_n} \exp\left\{\frac{2\ln 2}{n^{\frac{3}{2}nc\varepsilon_2^2\left(1+\frac{1}{\varepsilon_2 n}\right)-1}\ln 3}\right\} \end{aligned}$$

and where $\sigma(\varepsilon_2) = -\varepsilon_2\log_2\varepsilon_2 - (1-\varepsilon_2)\log_2(1-\varepsilon_2)$.

Theorem 1.3. *Let conditions* (A), (3), *and* (5) *hold. Assume that the parameters ε_1 and c vary in such a way that*

$$0 < \beta_1 \le \varepsilon_1 c \le \beta_0 < \frac{4}{3}\left(1 - \frac{(1+\alpha)\ln 3}{c\ln 2}\right),$$

where α, β_0, and β_1 are fixed numbers such that $\alpha > 0$ and

$$\alpha + \frac{3}{4}\beta_0 < 1 - \frac{\ln 3}{a_1\ln 2}.$$

Then, given an arbitrary fixed real number ε_2, $\varepsilon_2 \in (0,1)$, *there exists a positive integer number* n_0, $n_0 = n_0(\varepsilon_1, \varepsilon_2, c)$, *such that* $\mathsf{P}\{\theta_n > 0\} \le Z_2$ *for* $n \ge n_0$, *where*

$$Z_2 = \frac{e}{n^{2\alpha}} + 2^{n\sigma(\varepsilon_2)}\left(\frac{1}{3} + \frac{2}{3e^{\frac{3}{2}\beta_1}}\right)^{n^{\frac{\ln 2}{\ln 3}}} + \left(\frac{\exp\left\{\frac{2}{n^{\frac{3}{2}nc\varepsilon_2^2}}\right\}}{3}\right)^{A_n} \exp\left\{\frac{2\ln 2}{n^{\frac{3}{2}nc\varepsilon_2^2-1}\ln 3}\right\}.$$

Remark 1.2. The upper bound Z_1 (Z_2) approaches 0 as $n \to \infty$ under the assumptions of Theorem 1.2 (Theorem 1.3).

2. The first two factorial moments of the random variable θ_n

Lemma 2.1. *If condition* (A) *holds, then*

$$\mathsf{E}\,\theta_n = 3^{-T}\sum_{t=2}^{n}\binom{n}{t}Q_t, \tag{6}$$

where

$$Q_t = \prod_{\mu=1}^{T}\left(1 + 2(1-3p_\mu)^{\binom{t}{2}}\right). \tag{7}$$

Proof. Let the symbol $\xi(\bar{x})$ stand for the indicator of the random event that the vector $\bar{x}$, $\bar{x} \in M_n$, is a solution of system (1). Condition (A) implies that

$$\mathsf{E}\,\theta_n = \sum_{\bar{x}:\bar{x}\in M_n}\mathsf{E}\,\xi(\bar{x}) = \sum_{\bar{x}:\bar{x}\in M_n}\prod_{\mu=1}^{T}\mathsf{P}\left(\sum\nolimits_{3\ 1\le j_1<j_2\le n} a_{j_1j_2}^{(\mu)}x_{j_1}x_{j_2} = \mathbf{0}\right). \tag{8}$$

Denote by t the number of nonzero coordinates of an arbitrary fixed vector $\bar{x} \in M_n$. We will need the following relation:

$$\mathsf{P}\{\xi = a\} = \frac{1}{3} - \frac{1}{3}(1-3p^*)^k, \qquad a \in \mathbf{GF}(3),\ a \ne \mathbf{0}, \tag{9}$$

where $\xi = \xi_1 +_3 \cdots +_3 \xi_k$ (see [2]). Here $\xi_1, \dots, \xi_k$, $1 \le k < \infty$, are independent identically distributed random variables such that $\mathsf{P}\{\xi_s = a\} = p^*$, $a \in \mathbf{GF}(3)$, $a \ne \mathbf{0}$, and $\mathsf{P}\{\xi_s = \mathbf{0}\} = 1 - 2p^*$, $s = 1, \dots, k$. The symbol $+_3$ denotes the summation in the field $\mathbf{GF}(3)$.

Using relation (9), we obtain

$$\prod_{\mu=1}^{T}\mathsf{P}\left(\sum\nolimits_{3\ 1\le j_1<j_2\le n} a_{j_1j_2}^{(\mu)}x_{j_1}x_{j_2} = \mathbf{0}\right) = 3^{-T}Q_t. \tag{10}$$

The total number of vectors of the set M_n that have t nonzero coordinates is equal to the binomial coefficient $\binom{n}{t}$. Thus, with the help of relation (10), equality (8) can be rewritten in the form of (6). □

Let $I = \{i_{\mathbf{01}}, i_{\mathbf{02}}, i_{\mathbf{10}}, i_{\mathbf{20}}, i_{\mathbf{11}}, i_{\mathbf{22}}, i_{\mathbf{12}}, i_{\mathbf{21}}\}$.

Lemma 2.2. *If condition (A) holds, then*

$$\mathsf{E}\,\theta_n^{[2]} = 9^{-T}\sum_{t=3}^{n}\binom{n}{t}\sum_{i+l+h=t}\frac{t!}{h!\,i!\,l!}Q_t^*, \tag{11}$$

where

$$Q_t^* = \prod_{\mu=1}^{T}\left(1 + 2\left(\sum_{r=1}^{4}(1-3p_\mu)^{\Gamma^{(r)}}\right)\right). \tag{12}$$

The summation $\sum$ above is considered with respect to all indices i, l, and h such that $i+l+h=t$, $t-i\geq 2$, $t-l\geq 2$, and $i+l\geq 1$; the parameters $\Gamma^{(r)}$, $r=1,\dots,4$, are defined by the equalities

$$\Gamma^{(1)}=\binom{l}{2}+\binom{i}{2}+(i+l)(t-l-i), \tag{13}$$

$$\Gamma^{(2)}=\binom{t-l}{2}, \tag{14}$$

$$\Gamma^{(3)}=\binom{t-i}{2}, \tag{15}$$

$$\Gamma^{(4)}=\binom{l}{2}+\binom{i}{2}+\binom{t-l-i}{2}+(i+l)(t-l-i), \tag{16}$$

respectively.

Proof. Condition (A) together with the equality $\mathsf{E}\,\theta_n^{[2]}=\mathsf{E}\,\theta_n(\theta_n-1)$ and representation $\theta_n=\sum_{\bar{x}:\bar{x}\in M_n}\xi\left(\bar{x}\right)$ implies that

$$\mathsf{E}\,\theta_n^{[2]}=\sum\nolimits_1\mathsf{E}\,\xi(\bar{x}^{(1)})\xi(\bar{x}^{(2)}), \tag{17}$$

where the summation $\sum_1$ is considered with respect to all pairs of vectors $\left(\bar{x}^{(1)},\bar{x}^{(2)}\right)$ such that $\bar{x}^{(q)}\in M_n$, $q=1,2$, and $\bar{x}^{(1)}\neq\bar{x}^{(2)}$. With the help of equality (17) we find that

$$\begin{aligned}\mathsf{E}\,\theta_n^{[2]}&=\sum\nolimits_1\prod_{\mu=1}^{T}\mathsf{P}\left\{\bigcup\left\{A^{(\mu)}\left(\bar{x}^{(k)}\right)=y_k,A^{(\mu)}\left(\bar{x}^{(1)},\bar{x}^{(2)}\right)=y_{12},k=1,2\right\}\right\}\\&=\sum\nolimits_1\prod_{\mu=1}^{T}\sum\nolimits_2\mathsf{P}\left\{A^{(\mu)}\left(\bar{x}^{(1)},\bar{x}^{(2)}\right)=y_{12}\right\}\prod_{k=1,2}\mathsf{P}\left\{A^{(\mu)}\left(\bar{x}^{(k)}\right)=y_k\right\},\end{aligned} \tag{18}$$

where the symbol $\bigcup$ ($\sum_2$) means the union (summation) corresponding to all solutions of the following system of the two equations $y_1+_3y_{12}=\mathbf{0}$ and $y_2+_3y_{12}=\mathbf{0}$ over the field $\mathbf{GF}(3)$, where

$$A^{(\mu)}\left(\bar{x}^{(1)},\bar{x}^{(2)}\right)=\sum\nolimits_{\omega\in E^{(12)}}^{3}a_\omega^{(\mu)},A^{(\mu)}\left(\bar{x}^{(q)}\right)=\sum\nolimits_{\omega\in E^{(q)}}^{3}a_\omega^{(\mu)},\qquad q=1,2,$$

for $\mu\in J$, and where

$$E^{(12)}=\left\{(j_1,j_2),1\leq j_1<j_2\leq n\colon x_{j_1}^{(q)}x_{j_2}^{(q)}\neq\mathbf{0},q=1,2\right\},$$
$$E^{(q)}=\left\{(j_1,j_2),1\leq j_1<j_2\leq n\colon x_{j_1}^{(q)}x_{j_2}^{(q)}\neq\mathbf{0},x_{j_1}^{(q^*)}x_{j_2}^{(q^*)}=\mathbf{0}\right\},$$

$q\in\{1,2\}$, $q^*\in\{1,2\}$, $q^*\neq q$.

Let $\gamma^{(1)}$, $\gamma^{(2)}$, and $\gamma^{(3)}$ be the number of elements of the sets $E^{(1)}$, $E^{(2)}$, and $E^{(12)}$, respectively.

Put

$$\Gamma^{(1)}=\gamma^{(1)}+\gamma^{(2)},\qquad\Gamma^{(2)}=\gamma^{(2)}+\gamma^{(3)}, \tag{19}$$

$$\Gamma^{(3)}=\gamma^{(1)}+\gamma^{(3)},\qquad\Gamma^{(4)}=\gamma^{(1)}+\gamma^{(2)}+\gamma^{(3)}. \tag{20}$$

Considering condition (A) and using equality (9), relation (18) can be rewritten as follows:

$$\mathsf{E}\,\theta_n^{[2]}=9^{-T}\sum\nolimits_1\prod_{\mu=1}^{T}\left(1+2\left(\sum_{r=1}^{4}(1-3p_\mu)^{\Gamma^{(r)}}\right)\right). \tag{21}$$

For arbitrary vectors $\bar{x}^{(1)}, \bar{x}^{(2)} \in M_n$, denote by t the total number of pairs (c_1, c_2), $(c_3, \mathbf{0})$, $(\mathbf{0}, c_4)$ among n possible pairs $\left(x_j{}^{(1)}, x_j{}^{(2)}\right)$, $1 \le j \le n$, with the property that $c_1c_2c_3c_4 \ne \mathbf{0}$ for all $c_1, c_2, c_3, c_4 \in \mathbf{GF}(3)$. Then $t = i + l + h$ and the total number of pairs of vectors $\left(\bar{x}^{(1)}, \bar{x}^{(2)}\right)$ for which equality (21) holds is found from the following equation:

$$\sum_{i+l+h=t} \frac{n!}{h!\, i!\, l!(n-t)!} = \binom{n}{t} \sum_{i+l+h=t} \frac{t!}{h!\, i!\, l!}.$$

The summation $\sum_1$ on the right hand side of (21) means the summation over all pairs of vectors $\left(\bar{x}^{(1)}, \bar{x}^{(2)}\right)$ such that $\bar{x}^{(1)} \ne \bar{x}^{(2)}$, $\bar{x}^{(q)} \in M_n$, $q = 1, 2$, and is equivalent to the summation with respect to all parameters i, l, and h written on the right hand side of (11). Note that the inequalities $t - i \ge 3$, $t - l \ge 3$, and $i + l \ge 1$ imply that $\left|\bar{x}^{(1)}\right| \ge 1$, $\left|\bar{x}^{(2)}\right| \ge 1$, and $\bar{x}^{(1)} \ne \bar{x}^{(2)}$, respectively.

Next we check equality (13). First, we find some explicit expressions for the parameters $\gamma^{(1)}$ and $\gamma^{(2)}$.

Our current goal is to show that

$$\gamma^{(1)} = \left|E^{(1)}\right| = \binom{l}{2} + l\,(t - l - i)\,. \tag{22}$$

Indeed, we represent $\gamma^{(1)}$ as a sum of two terms, namely

$$\gamma^{(1)} = \left|E_1^{(1)}\right| + \left|E_2^{(1)}\right|, \tag{23}$$

where

$$E_1^{(1)} = \left\{(j_1, j_2), 1 \le j_1 < j_2 \le n \colon x_{j_1}^{(1)}, x_{j_2}^{(1)} \ne \mathbf{0}; x_{j_1}^{(2)} = x_{j_2}^{(2)} = \mathbf{0}\right\},$$
$$E_2^{(1)} = \left\{(j_1, j_2), 1 \le j_1 < j_2 \le n \colon x_{j_1}^{(1)}, x_{j_2}^{(1)} \ne \mathbf{0}; x_{j}^{(2)} = \mathbf{0}, x_{j^*}^{(2)} \ne \mathbf{0}\right\},$$

$j \in \{j_1, j_2\}$, $j^* \in \{j_1, j_2\}$, $j \ne j^*$.

Since the sum $i_{\mathbf{10}} + i_{\mathbf{20}}$ means the total number of nonzero coordinates of the vector $\bar{x}^{(1)}$ corresponding to the nonzero coordinates of the vector $\bar{x}^{(2)}$ and since $i_{\mathbf{11}} + i_{\mathbf{22}} + i_{\mathbf{12}} + i_{\mathbf{21}}$ means the number of the nonzero coordinates in the vector $\bar{x}^{(1)}$ corresponding to the nonzero coordinates of the vector $\bar{x}^{(2)}$, we find

$$\left|E_1^{(1)}\right| = \binom{l}{2}, \tag{24}$$

$$\left|E_2^{(1)}\right| = l\,(t - l - i)\,. \tag{25}$$

Taking into account equalities (23)–(25) we obtain (22).

Similarly we have

$$\gamma^{(2)} = \left|E^{(2)}\right| = \binom{i}{2} + i\,(t - l - i)\,. \tag{26}$$

Thus (19), (22), and (26) imply relation (13). Finally, the definition of the set $E^{(12)}$ proves that

$$\gamma^{(3)} = \left|E^{(12)}\right| = \binom{t - i - l}{2}. \tag{27}$$

Now (19), (26), and (27) imply equality (14). Then we derive (15) from (20), (23), and (27). Using (20), (23), (26), and (27) we get equality (16). $\square$

Remark 2.1. We see from the proof of Lemma 2.2 that the number $i+l+h$ (see equality (11)) is equal to the sum of elements of the set I. In particular, $h = i_{\mathbf{11}}+i_{\mathbf{22}}+i_{\mathbf{12}}+i_{\mathbf{21}}$ and $l = i_{\mathbf{10}} + i_{\mathbf{20}}$.

3. Auxiliary results

Lemma 3.1. *If condition* (A) *holds and*

$$p_\mu \le \frac{1}{2} - \upsilon, \tag{28}$$

where $0 < \upsilon \le \frac{1}{2}$ *and* $\mu \in J$, *then*

$$\mathsf{E}\,\theta_n > 0 \tag{29}$$

for an arbitrary $n \ge 2$.

Proof. In view of (6) and (7), relation (29) follows if

$$Q_t > 0, \qquad n \ge 2. \tag{30}$$

To prove inequality (30) we represent the product Q_t defined by equality (7) as follows:

$$Q_t = \prod_{r=1}^{3} Q_{t;r}, \tag{31}$$

where $Q_{t;r}$ means the product of all factors on the right hand side of equality (7) for which the parameter μ belongs to the set W_r, $r = 1, 2, 3$. Here

$$W_1 = \left\{\mu, 1 \le \mu \le T \colon p_\mu \le \frac{1}{3}\right\},$$

$$W_2 = \left\{\mu, \quad 1 \le \mu \le T \colon \frac{1}{3} < p_\mu \le \frac{1}{2} - \upsilon, \binom{t}{2} \text{ is even, } t \ge 2\right\},$$

$$W_3 = \left\{\mu, \quad 1 \le \mu \le T \colon \frac{1}{3} < p_\mu \le \frac{1}{2} - \upsilon, \binom{t}{2} \text{ is odd, } t \ge 2\right\}.$$

Denote by η_r the number of elements of the set W_r, that is, $\eta_r = |W_r|$, $r = 1, 2, 3$. Then

$$\sum_{r=1}^{3} \eta_r = T. \tag{32}$$

The definition of the products $Q_{t;1}$ and $Q_{t;2}$ implies that

$$Q_{t;1} \ge 1, \qquad Q_{t;2} \ge 1. \tag{33}$$

Considering condition (28), we find

$$Q_{t;3} \ge (6\upsilon)^{\eta_3}. \tag{34}$$

It follows from (31)–(34) that $Q_t \ge (6\upsilon)^{\eta_3}$, whence we obtain inequality (30) and hence (29) is proved. ∎

Lemma 3.2. *Assume that conditions* (A) *and* (3) *hold. If*

$$T \le n\frac{\ln 2}{\ln 3} + m_0 \tag{35}$$

as $n \to \infty$, *where* m_0 *is a constant, then*

$$Q_t \ge a_3, \qquad n \to \infty, \tag{36}$$

for an arbitrary $t \in F$, *where*

$$F = \left[\left[\frac{n}{2}\right] - \left[\frac{n}{\ln n}\right]; n\right].$$

Proof. Taking into account representation (31) we see that relation (36) follows if, for $t \in F$ and as $n \to \infty$, there exists a constant a_4 such that

$$Q_{t;r} \geq a_4, \qquad r = 1, 2, 3. \tag{37}$$

Similarly to the proof of inequality (33) we obtain $Q_{t;1} \geq 1$ and $Q_{t;2} \geq 1$ for $t \in F$, $\mu \in W_1$, and $\mu \in W_2$ if $n > 1$.

Now we check representation (37) for $r = 3$. Indeed, taking into account (3) and the inclusion $\mu \in W_3$, we prove that, for $t \in F$,

$$(1 - 3p_\mu)^{\binom{t}{2}} \geq -2^{-\frac{n^2}{8}(1+o(1))} \exp\left\{-\frac{3}{4}cn\,(1 + o(1)) \ln n\right\} \tag{38}$$

as $n \to \infty$, where $c > \ln 3 / \ln 2$.

Using (7), (35), and (38) we get

$$Q_{t;3} \geq a_5 \left(1 - a_6 2^{-\frac{n^2}{8}(1+o(1))} \exp\left\{-\frac{3}{4}cn\,(1 + o(1)) \ln n\right\}\right)^{n\frac{\ln 2}{\ln 3}}, \qquad n \to \infty.$$

This implies inequality (37) for $r = 1, 2, 3$. Now relations (31) and (37) prove (36). □

Lemma 3.3. *Let b and c be fixed integer numbers such that $0 < b < c$ and let ψ_n be a sequence of integer numbers such that $\psi_n / n \to 0$ as $n \to \infty$. Then*

$$\binom{n}{\left[\frac{b}{c}n\right] - \psi_n} < \frac{c^n \exp\left\{-\frac{\psi_n^2}{n}\left(\frac{c^2}{2b(c-b)} + O\left(\frac{\psi_n}{n}\right)\right)\right\}}{b^{\frac{b}{c}n - \psi_n} (c-b)^{\frac{c-b}{c}n + \psi_n}}, \qquad n \to \infty. \tag{39}$$

Proof. Relation (39) follows from the Stirling formula [4]. □

Lemma 3.4. *Let condition* (A) *hold and let $t \geq 4$ be an arbitrary number. Then, among the four parameters $\Gamma^{(l_0)}$, $l_0 = 1, \ldots, 4$, defined by relations* (13)–(16), *there are at least three parameters $\Gamma^{(l_1)}$, $\Gamma^{(l_2)}$, $\Gamma^{(l_3)}$, $l_1, l_2, l_3 \in \{1, 2, 3, 4\}$, $l_1 \neq l_2$, $l_2 \neq l_3$, $l_1 \neq l_3$, such that*

$$\Gamma^{(l_r)} \geq \frac{t}{2} - 1, \qquad r = 1, 2, 3.$$

Moreover, among these three parameters, there exists at least one parameter

$$\Gamma^{(l^*)}, \qquad l^* \in \{l_1, l_2, l_3\},$$

such that

$$\Gamma^{(l^*)} \geq \binom{\frac{t}{2}}{2}.$$

Proof. Let $i \geq \frac{t}{2}$. Then, applying relations (13)–(16), we prove that there are at least three parameters $\Gamma^{(l_1)}$, $\Gamma^{(l_2)}$, $\Gamma^{(l_3)}$, $l_1, l_2, l_3 \in \{1, 2, 3, 4\}$, $l_1 \neq l_2$, $l_2 \neq l_3$, $l_1 \neq l_3$, such that $\Gamma^{(l_r)} \geq \binom{\frac{t}{2}}{2}$, $t \geq 4$, $r = 1, 2, 3$.

Now let

$$i < \frac{t}{2}. \tag{40}$$

Consider separately all possible cases.

1) If inequality (40) holds and $l \geq \frac{t}{2}$, then

$$\Gamma^{(1)} \geq \binom{l}{2} \geq \binom{\frac{t}{2}}{2}, \qquad \Gamma^{(3)} \geq \binom{l}{2} \geq \binom{\frac{t}{2}}{2}, \qquad \Gamma^{(4)} \geq \Gamma^{(1)} \geq \binom{\frac{t}{2}}{2}$$

by (13), (15), and (16);

2) if inequality (40) holds and $h \geq \frac{t}{2}$, where $t = i + l + h$ and $h = i_{\mathbf{11}} + i_{\mathbf{22}} + i_{\mathbf{12}} + i_{\mathbf{21}}$, then we use relations (14), (15), and (16) and obtain the bounds

$$\Gamma^{(2)} \geq \binom{h}{2} \geq \binom{\frac{t}{2}}{2}, \qquad \Gamma^{(3)} \geq \binom{h}{2} \geq \binom{\frac{t}{2}}{2}, \qquad \Gamma^{(4)} \geq \binom{t-l-i}{2} = \binom{h}{2} \geq \binom{\frac{t}{2}}{2};$$

3) if inequality (40) holds and $l + h \geq \frac{t}{2}$, $l \geq 1$, and $h \geq 1$ (the cases where $l \geq \frac{t}{2}$ and $h = 0$ or $l = 0$ and $h \geq \frac{t}{2}$ are considered in cases 1) and 2) above), then $lh \geq \frac{t}{2} - 1$.

Indeed, let $l + h = \beta$, where $\beta \geq \frac{t}{2}$. Then $lh = l(\beta - l) \geq \beta - 1 \geq \frac{t}{2} - 1$. The inequality $l(\beta - l) \geq \beta - 1$ holds, since the function $f(x) = x(\beta - x)$ increases in the interval $[1; \beta/2]$, decreases in the interval $[\beta/2; \beta - 1]$, and attains its minimal value at $x = 1$ or at $x = \beta - 1$ (without loss of generality we assume that $1 \leq x \leq \beta - 1$).

Therefore (13), (15), (16) and the inequality $lh \geq \frac{t}{2} - 1$, $l + h \geq \frac{t}{2}$ imply that

$$\Gamma^{(1)} \geq (i+l)(t-l-i) = (i+l)h \geq lh \geq \frac{t}{2} - 1,$$

$$\Gamma^{(3)} = \binom{h+l}{2} \geq \binom{\frac{t}{2}}{2}, \qquad \Gamma^{(4)} \geq \Gamma^{(1)} \geq \frac{t}{2} - 1. \qquad \square$$

Let $p_{\min} = \min\limits_{1 \leq \mu \leq T} p_\mu$. In what follows the symbol ε_q stands for a positive fixed number whose precise value is specified for each appearance of q, $q \geq 1$.

Consider the sums

$$D_z = 3^{-T} \sum_{t \in \mathbb{R}_z} \binom{n}{t} Q_t,$$

where $z = 1, 2, 3$,

$$R_1 = \left[2; \left[\sqrt{\frac{\varepsilon_1 n}{\ln n}}\right]\right], \qquad R_2 = \left[\left[\sqrt{\frac{\varepsilon_1 n}{\ln n}}\right] + 1; [\varepsilon_2 n]\right], \qquad R_3 = [[\varepsilon_2 n] + 1; n].$$

Lemma 3.5. *Let conditions* (A), (3)*, and* (5) *hold. Assume that the parameters* ε_1 *and* c *are varying in such a way that*

$$\varepsilon_1 c \leq \gamma_0 < \frac{4}{3}\left(1 - \frac{\ln 3}{c \ln 2}\right).$$

Then

$$D_1 \leq \sum_{t=2}^{\left[\sqrt{\frac{\varepsilon_1 n}{\ln n}}\right]} \frac{1}{t!} \left(\frac{1}{n^{c\frac{\ln 2}{\ln 3}\left(1 - \frac{\ln 3}{c \ln 2} - \frac{3}{4}\gamma_0\right)}}\right)^t \tag{41}$$
$$\times \left(\frac{1}{n^{c\frac{A_n}{n}\left(1 - \frac{3}{4}\gamma_0 + \frac{3}{4}c\sqrt{\frac{\varepsilon_1 \ln n}{n}} + \frac{3}{4}c\frac{\ln 2}{\ln 3}\frac{\sqrt{\varepsilon_1 n \ln n}}{A_n}\right)}}\right)^t.$$

Proof. Taking into account (3) and (7) we obtain

$$Q_t \leq 3^T \left(1 - 2p_{\min}\binom{t}{2} + 3\left(p_{\min}\binom{t}{2}\right)^2\right)^T \tag{42}$$

for $t \in \left[2; \left[\sqrt{\frac{\varepsilon_1 n}{\ln n}}\right]\right]$. It follows from (42) that

$$D_1 \leq \sum_{t=2}^{\left[\sqrt{\frac{\varepsilon_1 n}{\ln n}}\right]} \frac{n^t}{t!} \exp\left\{-Ttp_{\min}\left(1 - \frac{3}{4}\frac{\varepsilon_1 n}{\ln n}\left(1 - \sqrt{\frac{\ln n}{\varepsilon_1 n}}\right)p_{\min}\right)\right\} \tag{43}$$

for all $t \in \left[2; \left[\sqrt{\frac{\varepsilon_1 n}{\ln n}}\right]\right]$.

Now conditions (3) and (5) together with inequality (43) imply bound (41). $\square$

Lemma 3.6. *Let conditions* (A), (3), *and* (5) *hold. Assume that the parameters* ε_1 *and* c *are varying in such a way that* $\varepsilon_1 c \geq \gamma_1 > 0$. *Then there exists a number* ε_2, $0 < \varepsilon_2 < 1$, *such that*

$$D_2 \leq 2^{n\sigma(\varepsilon_2)} \left(\frac{1}{3} + \frac{2}{3e^{\frac{3}{2}\gamma_1\left(1+\sqrt{\frac{\ln n}{\varepsilon_1 n}}\right)}} \right)^{n\frac{\ln 2}{\ln 3}+A_n}, \tag{44}$$

where $\sigma(\varepsilon_2) = -\varepsilon_2 \log_2 \varepsilon_2 - (1-\varepsilon_2)\log_2(1-\varepsilon_2)$.

Proof. For $t \in \left[\left[\sqrt{\frac{\varepsilon_1 n}{\ln n}}\right]+1, [\varepsilon_2 n]\right]$, we get

$$Q_t \leq \left(1 + 2\exp\left\{-3p_{\min}\binom{\left[\sqrt{\frac{\varepsilon_1 n}{\ln n}}\right]+1}{2}\right\}\right)^T. \tag{45}$$

Now relations (3), (5), and (45) yield

$$D_2 \leq \left(\frac{1}{3} + \frac{2}{3e^{\frac{3}{2}c\varepsilon_1\left(1+\sqrt{\frac{\ln n}{\varepsilon_1 n}}\right)}} \right)^{n\frac{\ln 2}{\ln 3}+A_n} \sum_{t=\left[\sqrt{\frac{\varepsilon_1 n}{\ln n}}\right]+1}^{[\varepsilon_2 n]} \binom{n}{t}. \tag{46}$$

The inequality

$$\sum_{t=0}^{[\varepsilon_2 n]} \binom{n}{t} \leq 2^{n\sigma(\varepsilon_2)}$$

implies bound (44) in view of inequality (46), where

$$\sigma(\varepsilon_2) = -\varepsilon_2 \log_2 \varepsilon_2 - (1-\varepsilon_2)\log_2(1-\varepsilon_2)$$

(see [3]). □

Lemma 3.7. *Let conditions* (A), (3), *and* (5) *hold. Then*

$$D_3 \leq \left(\frac{\exp\left\{\frac{2}{n^{\frac{3}{2}nc\varepsilon_2^2\left(1+\frac{1}{\varepsilon_2 n}\right)}}\right\}}{3} \right)^{A_n} \exp\left\{ \frac{2\ln 2}{n^{\frac{3}{2}nc\varepsilon_2^2\left(1+\frac{1}{\varepsilon_2 n}\right)-1}\ln 3} \right\} \tag{47}$$

for $\varepsilon_2 > 0$.

Proof. For $t \in [[\varepsilon_2 n]+1, n]$, we have

$$Q_t \leq \left(1 + 2\exp\left\{-3p_{\min}\binom{[\varepsilon_2 n]+1}{2}\right\}\right)^T. \tag{48}$$

Using (3) and (48), we obtain

$$D_3 \leq \frac{2^n}{3^T} \exp\left\{ \frac{2T}{\exp\left\{\frac{3}{2}c\varepsilon_2^2 n\,(\ln n)\left(1+\frac{1}{\varepsilon_2 n}\right)\right\}} \right\}. \tag{49}$$

Now we derive

$$D_3 \leq \frac{2^n}{3^{\frac{\ln 2}{\ln 3}n+A_n}} \exp\left\{ \frac{2n\frac{\ln 2}{\ln 3}+2A_n}{n^{\frac{3}{2}nc\varepsilon_2^2\left(1+\frac{1}{\varepsilon_2 n}\right)}} \right\}$$

from (5) and (49), whence inequality (47) follows. □

4. Proof of Theorem 1.1

Proof. Sufficiency. We show that (5) implies

$$\mathsf{E}\,\theta_n = o(1), \qquad n \to \infty. \tag{50}$$

Considering (6) and (7), the expectation $\mathsf{E}\,\theta_n$ can be written as follows:

$$\mathsf{E}\,\theta_n = \sum_{h=1}^{3} D_h, \tag{51}$$

where D_1, D_2, and D_3 are defined above.

Taking into account representation (51), relation (50) follows from

$$D_h = o(1), \qquad n \to \infty, \tag{52}$$

for $h = 1, 2, 3$. Using (41), (44), and (47), one easily checks relation (52) for $h = 1$, $h = 2$, and $h = 3$, respectively.

Now relation (50) follows from (51) and (52). Using (50) and Chebyshev's inequality, we prove (4).

Necessity. Let $\mathsf{P}\{\theta_n > 0\} \to 0$ as $n \to \infty$. We show that (5) holds. If equality (5) does not hold, then equality (35) holds. Our current goal is to show that there exists a positive constant C such that

$$\mathsf{P}\{\theta_n > 0\} \geq C > 0, \qquad n \to \infty. \tag{53}$$

In other words, relation (53) means that, with a positive probability, there exists a solution that belongs to the set M_n. First we prove the following upper bounds:

$$(\mathsf{E}\,\theta_n)^{-1} \leq a_5, \tag{54}$$

$$\mathsf{E}\,\theta_n^{[2]}\,(\mathsf{E}\,\theta_n)^{-2} \leq a_6 \tag{55}$$

and use them further in the inequality

$$\mathsf{P}\{\theta_n > 0\} \geq \left((\mathsf{E}\,\theta_n)^{-1} + \mathsf{E}\,\theta_n^{[2]}\,(\mathsf{E}\,\theta_n)^{-2}\right)^{-1} \tag{56}$$

(see [5]).

Then relations (6) and (29) together with Lemma 3.2 imply that

$$(\mathsf{E}\,\theta_n)^{-1} \leq 3^T 2^{-n} \delta_n, \tag{57}$$

where

$$\delta_n \leq a_3^{-1}\left(2^{-n}\sum_{t\in F}\binom{n}{t}\right)^{-1}, \qquad n \to \infty. \tag{58}$$

Lemma 3.3 with $b = 1$, $c = 2$, and $\psi_n = [n/\ln n]$ allows one to conclude that

$$2^{-n}\sum_{t\in F}\binom{n}{t} \to 1, \qquad n \to \infty,$$

which together with (35), (57), and (58) proves (54).

Similarly to the proof of (54), we make sure that

$$\left(3^T 2^{-n}\,\mathsf{E}\,\theta_n\right)^{-1} \leq a_7, \qquad n \to \infty. \tag{59}$$

Next, relation (59) implies that inequality (55) follows from

$$9^T 4^{-n}\,\mathsf{E}\,\theta_n^{[2]} \leq a_8, \qquad n \to \infty. \tag{60}$$

Considering (11), the left hand side of (60) can be rewritten as follows:

$$9^T 4^{-n}\,\mathsf{E}\,\theta_n^{[2]} = 4^{-n} S\left(n; Q_t^*\right), \tag{61}$$

where

$$S\left(n; Q_t^*\right)=\sum_{t=3}^{n}\binom{n}{t}\sum_{i+l+h=t}\frac{t!}{i!\,l!\,h!}Q_t^*. \tag{62}$$

We represent $S\left(n; Q_t^*\right)$ as a sum of two terms $S_1\left(n; Q_t^*\right)$ and $S_2\left(n; Q_t^*\right)$, namely

$$S\left(n; Q_t^*\right)=S_1\left(n; Q_t^*\right)+S_2\left(n; Q_t^*\right), \tag{63}$$

where $S_1\left(n; Q_t^*\right)$ differs from $S\left(n; Q_t^*\right)$ by the set of summation on the right hand side of (62) where the indices i, l, and h are such that

$$\Gamma^{(r)} \geq \binom{\varepsilon n}{2}, \tag{64}$$

where ε is a constant such that $0<\varepsilon<1$, and where $\Gamma^{(r)}$, $r=1,\ldots,4$, are defined by equalities (13)–(16). Here $S_2\left(n; Q_t^*\right)$ is the sum of the rest of the terms in $S\left(n; Q_t^*\right)$.

Relations (3), (12), (35), and (64) imply that

$$S_1\left(n; Q_t^*\right) \leq a_9 S_1\left(n; 1\right), \qquad n\to\infty. \tag{65}$$

The inequality $S_1\left(n; 1\right)\leq 4^n$ together with relation (65) yields

$$S_1\left(n; Q_t^*\right) \leq a_9 4^n \tag{66}$$

as $n\to\infty$.

Next we represent the sum $S_2\left(n; Q_t^*\right)$ as follows:

$$S_2\left(n; Q_t^*\right)=\sum_{k=1}^{4} S_{2;k}\left(n; Q_t^*\right), \tag{67}$$

where $S_{2;k}\left(n; Q_t^*\right)$ differs from $S_2\left(n; Q_t^*\right)$ by the set of summation on the right hand side of (62). Namely, the summation on the right hand side of (62) is considered with respect to all those elements of the set I such that there exist $l_1,\ldots,l_k\in\{1,2,3,4\}$ for which $\Gamma^{(l_s)}<\binom{\varepsilon n}{2}$ and $\Gamma^{(r)}\geq\binom{\varepsilon n}{2}$, where $r\in\{1,2,3,4\}\setminus\{l_1,\ldots,l_k\}$, $s=1,\ldots,k$, $k=1,\ldots,4$.

For each $k=1,\ldots,4$, we represent $S_{2;k}\left(n; Q_t^*\right)$ in the following form:

$$S_{2;k}\left(n; Q_t^*\right)=\sum_{1\leq t_1<\cdots<t_k\leq 4} S_{2;k;t_1,\ldots,t_k}\left(n; Q_t^*\right), \tag{68}$$

where $S_{2;k;t_1,\ldots,t_k}\left(n; Q_t^*\right)$ denotes the sum of all terms of $S_{2;k}\left(n; Q_t^*\right)$ for which

$$\Gamma^{(t_l)}<\binom{\varepsilon n}{2}, \quad l=1,\ldots,k, \qquad \Gamma^{(t')}\geq\binom{\varepsilon n}{2}, \quad t'\in\{1,2,3,4\}\setminus\{t_1,\ldots,t_k\}.$$

We show that, for all $k=1$,

$$S_{2;k}\left(n; Q_t^*\right)\leq a_{10}4^n\left(1+o(1)\right) \tag{69}$$

as $n\to\infty$.

Using (3), (12), and (35) and recalling the definition of the sum

$$S_{2;1;1}(n; Q_t^*)$$
$$(S_{2;1;4}(n; Q_t^*)),$$

we obtain

$$S_{2;1;1}\left(n; Q_t^*\right)\leq a_{11}2^n S_{2;1;1}\left(n; 1\right) \tag{70}$$

$$\left(S_{2;1;4}\left(n; Q_t^*\right)\leq a_{11}2^n S_{2;1;4}\left(n; 1\right)\right). \tag{71}$$

The inequality $\Gamma^{(1)} < \binom{\varepsilon n}{2}$ ($\Gamma^{(4)} < \binom{\varepsilon n}{2}$) and relation (13) ((16)) imply that all parameters i, l, and h involved in forming the sum $S(n; Q_t^*)$ (see (62)) do not exceed εn. Then the polynomial theorem implies that, for $k = 1$,

$$S_{2;k;1}(n; 1) \leq \exp\{\sigma_1(\varepsilon) n\} \tag{72}$$

$$(S_{2;k;4}(n; 1) \leq \exp\{\sigma_2(\varepsilon) n\}) \tag{73}$$

as $n \to \infty$, where $\sigma_r(\varepsilon_0) \to 0$ as $\varepsilon_0 \to 0$ for $r = 1, 2, \dots$.

Taking into account (70) and (72) ((71) and (73)), we prove the following bound:

$$S_{2;1;1}(n; Q_t^*) \leq a_{11} 2^n \exp\{\sigma_1(\varepsilon) n\} \tag{74}$$

$$(S_{2;1;4}(n; Q_t^*) \leq a_{11} 2^n \exp\{\sigma_2(\varepsilon) n\}). \tag{75}$$

Further, the inequalities $\Gamma^{(2)} < \binom{\varepsilon n}{2}$ and $t - l = i + h \geq 2$ (see Lemma 2.2) and relation (14) allow one to rewrite the sum $S_{2;1;2}(n; Q_t^*)$ as follows:

$$S_{2;1;2}(n; Q_t^*) = \sum_{l=1}^{3} S_{2;1;2}^{(l)}(n; Q_t^*), \tag{76}$$

where

$$S_{2;1;2}^{(l)}(n; Q_t^*) = \sum_{t=3}^{n} \binom{n}{t} \sum_{q \in \mathbb{R}_l} \binom{t}{q} \sum_{i+h=q} \frac{q!}{i!\, h!} Q_t^*, \qquad l = 1, 2, 3.$$

The closed intervals R_l, $l = 1, 2, 3$, with integer end points are given by

$$R_1 = \left[2; \left[\sqrt{\frac{\varepsilon' n}{\ln n}}\right]\right], \qquad R_2 = \left[\left[\sqrt{\frac{\varepsilon' n}{\ln n}}\right] + 1; [\varepsilon'' n]\right], \qquad R_3 = [[\varepsilon'' n] + 1; t],$$

where ε' and ε'' are fixed positive numbers such that $0 < \varepsilon', \varepsilon'' < 1$.

Taking into account (12), we have for $q \in \mathbb{R}_1$

$$Q_t^* \leq 3^T \left(1 - 2p_{\min}\binom{q}{2}\left(1 + O\left(p_{\min}\binom{q}{2}\right)\right)\right) + 2\exp\left\{-3p_{\min}\binom{\varepsilon n}{2}\right\}\Bigg)^T \tag{77}$$

as $n \to \infty$. Bound (77) together with relations (3) and (35) implies

$$Q_t^* \leq a_{12} 3^{\frac{\ln 2}{\ln 3} n} \left(1 - \frac{qc \ln n}{n}(1 + O(\varepsilon')) + 2n^{-\frac{3}{2}c\varepsilon^2 n(1+o(1))}\right)^{\frac{\ln 2}{\ln 3} n}$$

as $n \to \infty$.

Thus, if $q \in \mathbb{R}_1$, then

$$Q_t^* \leq a_{13} 2^n \exp\left\{\frac{2 \ln 2 \,(1 + o(1))}{n^{\frac{3}{2}c\varepsilon^2 n(1+o(1)) - 1} \ln 3}\right\} \left(\frac{1}{n^{c\frac{\ln 2}{\ln 3}(1+o(1)+O(\varepsilon'))}}\right)^q \tag{78}$$

as $n \to \infty$. The definition of the sum $S_{2;1;2}^{(1)}(n; Q_t^*)$ and relation (78) imply

$$S_{2;1;2}^{(1)}(n; Q_t^*) \leq a_{14} 4^n \sum_{q=0}^{\infty} \frac{1}{q!} \left(\frac{2}{n^{c\frac{\ln 2}{\ln 3}(1+O(\varepsilon'))-1}}\right)^q \tag{79}$$

as $n \to \infty$. Using equality (12), we get for $q \in \mathbb{R}_2$

$$Q_t^* \leq \left(1 + 2\exp\left\{-3p_{\min}\binom{\left[\sqrt{\frac{\varepsilon' n}{\ln n}}\right] + 1}{2}\right\} + 6\exp\left\{-3p_{\min}\binom{\varepsilon n}{2}\right\}\right)^T. \tag{80}$$

Now we apply (3), (35), and (80) to prove that

$$Q_t^* \le \left(1 + \frac{2}{e^{\frac{3}{2}c\varepsilon'(1+o(1))}} + \frac{6}{n^{\frac{3}{2}c\varepsilon^2 n(1+o(1))}}\right)^{\frac{\ln 2}{\ln 3}n+m_0} \tag{81}$$

as $n \to \infty$ for $q \in \mathbb{R}_2$.

Recalling the definition of the sum $S_{2;1;2}^{(2)}(n; Q_t^*)$ we deduce from the polynomial formula and bound (81) that

$$S_{2;1;2}^{(2)}(n; Q_t^*) \le a_{15} 2^n e^{\sigma_3(\varepsilon'')n} \left(1 + \frac{2}{e^{\frac{3}{2}c\varepsilon'(1+o(1))}} + \frac{6}{n^{\frac{3}{2}c\varepsilon^2 n(1+o(1))}}\right)^{\frac{\ln 2}{\ln 3}n} \tag{82}$$

as $n \to \infty$. For $q \in \mathbb{R}_3$, we take into account equality (12) and similarly to (82) find that

$$Q_t^* \le \left(1 + \frac{2}{n^{\frac{3}{2}c(\varepsilon'')^2 n(1+o(1))}} + \frac{6}{n^{\frac{3}{2}c\varepsilon^2 n(1+o(1))}}\right)^{\frac{\ln 2}{\ln 3}n+m_0} \tag{83}$$

as $n \to \infty$. Again using the polynomial formula together with bound (83) we obtain

$$S_{2;1;2}^{(3)}(n; Q_t^*) \le a_{16} 4^n \tag{84}$$

as $n \to \infty$. Combining (76), (79), (82), and (84) we get

$$S_{2;1;2}(n; Q_t^*) \le a_{17} 4^n (1 + o(1)) \tag{85}$$

as $n \to \infty$. Further, let $\Gamma^{(3)} < \binom{\varepsilon n}{2}$, $t - i \ge 2$. Then

$$S_{2;1;3}(n; Q_t^*) = \sum_{l=1}^{3} S_{2;1;3}^{(l)}(n; Q_t^*),$$

where

$$S_{2;1;3}^{(l)}(n; Q_t^*) = \sum_{t=3}^{n} \binom{n}{t} \sum_{q \in \mathbb{R}_l} \binom{t}{q} \sum_{l+h=q} \frac{q!}{l!\,h!} Q_t^*, \qquad l = 1, 2, 3.$$

This representation together with (15) allows one to prove similarly to (85) that

$$S_{2;1;3}(n; Q_t^*) \le a_{18} 4^n (1 + o(1)) \tag{86}$$

as $n \to \infty$. Now we derive inequality (69) with $k = 1$ from relations (68), (74), (75), (85), and (86).

Next we show that, for $k = 2$,

$$S_{2;k}(n; Q_t^*) \le a_{19} 5^{\frac{\ln 2}{\ln 3}n} e^{\sigma_4(\varepsilon)n} \tag{87}$$

as $n \to \infty$.

Indeed, relations (3), (12), (35), and (68) imply that, for $k = 2$,

$$S_{2;k}(n; Q_t^*) \le a_{20} 5^{\frac{\ln 2}{\ln 3}n} \left(\sum_{1 \le t_1 < t_2 \le 4} S_{2;k;t_1,t_2}(n; 1)\right) \tag{88}$$

as $n \to \infty$. Then the inequalities $\Gamma^{(t_1)} < \binom{\varepsilon n}{2}$ and $\Gamma^{(t_2)} < \binom{\varepsilon n}{2}$, where $1 \le t_1 < t_2 \le 4$, together with (13)–(16) yield that the parameters i, l, and h on the right hand side of (62) do not exceed εn. This, in turn, implies the bound

$$\max_{1 \le t_1 < t_2 \le 4} S_{2;2;t_1,t_2}(n; 1) \le a_{21} e^{\sigma_5(\varepsilon)n},$$

whence

$$\sum_{1\le t_1<t_2\le 4} S_{2;2;t_1,t_2}\left(n;1\right) \le a_{22}e^{\sigma_5(\varepsilon)n}. \tag{89}$$

Now inequality (87) with $k=2$ follows from (88) and (89).

Next we prove that

$$S_{2;k}\left(n;Q_t^*\right) \le a_{23}7^{\frac{\ln 2}{\ln 3}n}e^{\sigma_6(\varepsilon)n} \tag{90}$$

for $k=3$ as $n\to\infty$.

Indeed, relations (3), (12), (35), and (68) imply that

$$S_{2;k}\left(n;Q_t^*\right) \le a_{24}7^{\frac{\ln 2}{\ln 3}n}\left(\sum_{1\le t_1<t_2<t_3\le 4} S_{2;k;t_1,t_2,t_3}\left(n;1\right)\right) \tag{91}$$

for $k=3$ as $n\to\infty$.

The bound

$$\sum_{1\le t_1<t_2<t_3\le 4} S_{2;3;t_1,t_2,t_3}\left(n;1\right) \le a_{25}e^{\sigma_7(\varepsilon)n} \tag{92}$$

is proved analogously to (89). Then (91) and (92) prove inequality (90) for $k=3$.

Finally, we show that

$$S_{2;k}\left(n;Q_t^*\right) \le a_{26}4^n\left(1+o(1)\right) \tag{93}$$

for $k=4$ as $n\to\infty$. Note that the parameters i, l, and h on the right hand side of (62) are such that

$$\max\left(i,l,h\right) < \varepsilon n. \tag{94}$$

Inequality (94) follows from $\Gamma^{(r)} < \binom{\varepsilon n}{2}$, $r=1,\dots,4$, in view of (13)–(16).

In particular, inequality (94) allows one to represent $S_{2;4}\left(n;Q_t^*\right)$ in the following form:

$$S_{2;4}\left(n;Q_t^*\right) = \sum_{p=1}^{4} S_{2;4}^{(p)}\left(n;Q_t^*\right), \tag{95}$$

where

$$S_{2;4}^{(p)}\left(n;Q_t^*\right) = \sum_{t\in\mathbb{R}_p}\binom{n}{t}\sum_{i+l+h=t}\frac{t!}{i!\,l!\,h!}Q_t^*, \qquad p=1,\dots,4. \tag{96}$$

The closed intervals $\mathbb{R}_p$, $p=1,\dots,4$, whose end points are integers, are equal to

$$R_1=[3;7], \qquad R_2=\left[8;\left[\frac{n}{\ln^2 n}\right]\right], \qquad R_3=\left[\left[\frac{n}{\ln^2 n}\right]+1;\left[\frac{\delta n}{\ln n}\right]\right],$$
$$R_4=\left[\left[\frac{\delta n}{\ln n}\right]+1;[\varepsilon n]\right],$$

where δ is a constant such that $0<\delta<\frac{2}{3a_2}$.

We show that

$$S_{2;4}^{(1)}\left(n;Q_t^*\right) \le a_{27}4^n\left(1+o(1)\right) \tag{97}$$

as $n\to\infty$. If $t=3$, the inequalities $i+l\ge 1$, $t-i\ge 2$, and $t-l\ge 2$ (see Lemma 2.2) imply that $i\in\{0,1\}$, $l\in\{0,1\}$, and $i+l\ne 0$. Further, we consider separately all possible combinations of the parameters i and l.

1. If $i=1$ and $l=0$ or $i=0$ and $l=1$, then we derive from (12)—(16) that

$$\begin{aligned}&\binom{n}{3}\sum_{i+l+h=3}\frac{3!}{i!\,l!\,h!}\prod_{\mu=1}^{T}\left(1+2\left(\sum_{r=1}^{4}(1-3p_\mu)^{\Gamma^{(r)}}\right)\right)\\&\quad=\binom{n}{3}\prod_{\mu=1}^{T}\left(1+4\,(1-3p_\mu)^3+2\,(1-3p_\mu)^2+2\,(1-3p_\mu)\right).\end{aligned}\tag{98}$$

In view of (3), (35), and (98) we deduce that

$$\binom{n}{3}\sum_{i+l+h=3}\frac{3!}{i!\,l!\,h!}\prod_{\mu=1}^{T}\left(1+2\left(\sum_{r=1}^{4}(1-3p_\mu)^{\Gamma^{(r)}}\right)\right)\le\frac{a_{28}4^n}{n^{\frac{6c\ln 2}{\ln 3}(1+o(1))-3}}\tag{99}$$

as $n\to\infty$.

2. If $i=1$ and $l=1$, then similarly to the proof of (99) we obtain

$$\binom{n}{3}\sum_{i+l+h=3}\frac{3!}{i!\,l!\,h!}\prod_{\mu=1}^{T}\left(1+2\left(\sum_{r=1}^{4}(1-3p_\mu)^{\Gamma^{(r)}}\right)\right)\le\frac{a_{29}4^n}{n^{\frac{4c\ln 2}{\ln 3}(1+o(1))-3}}\tag{100}$$

as $n\to\infty$.

With the help of relations (99) and (100) we obtain that, for $t=3$,

$$\binom{n}{t}\sum_{i+l+h=t}\frac{t!}{i!\,l!\,h!}\prod_{\mu=1}^{T}\left(1+2\left(\sum_{r=1}^{4}(1-3p_\mu)^{\Gamma^{(r)}}\right)\right)\le\frac{a_{30}4^n}{n^{a_{31}(1+o(1))}}\tag{101}$$

as $n\to\infty$. Now one can easily check that bound (101) holds for $t=4,\dots,7$, whence we derive inequality (97).

Then we show that

$$S_{2;4}^{(2)}\,(n;Q_t^*)\le a_{32}4^n\sum_{t=0}^{\infty}\frac{1}{t!}\left(\frac{3}{n^{c\frac{\ln 2}{\ln 3}(1+o(1))-1}}\right)^t\tag{102}$$

as $n\to\infty$.

Put $t_{p,\min}=\min_{t\in\mathbb{R}_p}t$ and $t_{p,\max}=\max_{t\in\mathbb{R}_p}t$ for $p=2,3$.

Taking into account relation (12) and Lemma 3.4 we get for $t\in\mathbb{R}_p$, $p=2,3$, that

$$\begin{aligned}Q_t^*&\le\left(1+2\left(1+(1-3p_{\min})^{\frac{t}{4}\left(\frac{t}{2}-1\right)}+2\,(1-3p_{\min})^{\left(\frac{t}{2}-1\right)}\right)\right)^T\\&\le\left(9-3p_{\min}t\left(1-\frac{2}{t_{p,\min}}\right)H_t\right)^T,\end{aligned}\tag{103}$$

where

$$H_t=\frac{t_{p,\min}}{4}+2-3p_{\min}\left(\frac{t_{p,\max}}{2}-1\right)\left(\frac{1}{2}\left(\frac{t_{p,\min}}{4}\right)^2+1\right).$$

Using (3), (35), and (103) (for $p=2$), we get, for $t\in\mathbb{R}_2$,

$$Q_t^*\le a_{33}4^n\left(\frac{1}{n^{c\frac{\ln 2}{\ln 3}(1+o(1))}}\right)^t\tag{104}$$

as $n\to\infty$. The definition of $S_{2;4}^{(2)}\,(n;Q_t^*)$ and inequality (104) complete the proof of (102).

Next we prove that

$$S_{2;4}^{(3)}\,(n;Q_t^*)\le a_{34}4^n\left(\frac{3\ln^2 n}{n^{c\frac{\ln 2}{\ln 3}\left(1-\frac{3c\delta}{2}\right)(1+o(1))}}\right)^{\frac{n}{\ln^2 n}}\sum_{t=0}^{\infty}\left(\frac{3\ln^2 n}{n^{c\frac{\ln 2}{\ln 3}\left(1-\frac{3c\delta}{2}\right)(1+o(1))}}\right)^t\tag{105}$$

as $n \to \infty$. Taking into account (3), (35), and (103) (for $p = 3$), we obtain, for $t \in \mathbb{R}_3$,

$$Q_t^* \le a_{35} 4^n \left(\frac{1}{n^{c \frac{\ln 2}{\ln 3}\left(1 - \frac{3c\delta}{2}\right)(1+o(1))}} \right)^t \tag{106}$$

as $n \to \infty$. Thus bound (105) follows from (96) and (106).

Then we show that

$$S_{2;4}^{(4)}\left(n; Q_t^*\right) \le a_{36} 2^n \exp\left\{\sigma_8\left(\varepsilon\right) n\right\} \left(1 + \frac{2}{\exp\left\{\frac{3}{2} c\delta\left(1 + o(1)\right)\right\}}\right)^{\frac{\ln 2}{\ln 3} n} \tag{107}$$

as $n \to \infty$. Relations (3) and (35) together with Lemma 3.4 imply that

$$Q_t^* \le a_{37} 2^n \left(1 + \frac{2}{\exp\left\{\frac{3}{2} c\delta\left(1 + o(1)\right)\right\}}\right)^{\frac{\ln 2}{\ln 3} n} \tag{108}$$

as $n \to \infty$. Relation (94) proves that

$$S_{2;4}^{(4)}\left(n; 1\right) \le a_{38} n^{1/2} \exp\left\{\sigma_8\left(\varepsilon\right) n\right\} \tag{109}$$

in view of the polynomial formula. Considering (96), (109), and (108) we get inequality (107).

Combining (95), (97), (102), (105), and (107) we obtain (93).

For $S_2\left(n; Q_t^*\right)$, relations (67), (69), (87), (90), and (93) imply

$$S_2\left(n; Q_t^*\right) \le a_{39} 4^n \left(1 + o(1)\right) \tag{110}$$

as $n \to \infty$. Now $S\left(n; Q_t^*\right) \le a_{40} 4^n \left(1 + o(1)\right)$ as $n \to \infty$ by (63), (66), and (110). Thus bound (60) holds in view of relation (61). Inequalities (59) and (60) prove (55).

Summarizing, if (35) holds, then (54) and (55) hold as well. This together with (56) allows us to conclude that relation (53) holds, too. In turn, this contradicts the property that, with probability approaching zero as $n \to \infty$, there exists a solution of system (1) that belongs to the set M_n. □

5. Proof of Theorem 1.2

Proof. Theorem 1.2 follows from (41), (44), (47), and (51). □

Examples to Theorem 1.2

	1.	2.	3.	4.
ε_1	0.1	0.05	0.05	0.01
ε_2	0.02	0.01	0.01	0.01
c	5	10	10	100
γ_0	0.7	1	1	1
γ_1	0.5	0.4	0.4	1
n	500	1000	1000	10000
A_n	$\ln n$	$\ln n$	$\ln\ln n$	$\sqrt{\ln n}$
Z_1	1.1171×10^{-3}	5.1482×10^{-4}	1.1217×10^{-1}	3.5646×10^{-2}

Examples to Theorem 1.3

	1.	2.	3.	4.
ε_1	0.1	0.05	0.05	0.01
ε_2	0.02	0.01	0.01	0.01
c	5	10	10	100
β_0	0.7	1	1	1
β_1	0.5	0.4	0.4	1
n	500	1000	1000	10000
A_n	$\ln n$	$\ln n$	$\ln\ln n$	$\sqrt{\ln n}$
a_1	10	15	50	100
α	0.2	0.1	0.2	0.23
Z_2	2.2747×10^{-1}	6.8333×10^{-1}	1.1965×10^{-1}	3.9291×10^{-2}

6. Proof of Theorem 1.3

The assumptions of Theorem 1.3 imply that the terms D_1, D_2, and D_3 on the right hand side of relation (51) are such that $D_1 \le D_1'$, $D_2 \le D_2'$, and $D_3 \le D_3'$ (this can be proved similarly to the proof of inequalities (41), (44), and (47), respectively), where $D_1' = e/n^{2\alpha}$,

$$D_2' = 2^{n\sigma(\varepsilon_2)} \left(\frac{1}{3} + \frac{2}{3e^{\frac{3}{2}\beta_1}} \right)^{n^{\frac{\ln 2}{\ln 3}}}, \qquad D_3' = \left(\frac{\exp\left\{ \frac{2}{n^{\frac{3}{2}nc\varepsilon_2^2}} \right\}}{3} \right)^{A_n} \exp\left\{ \frac{2\ln 2}{n^{\frac{3}{2}nc\varepsilon_2^2 - 1} \ln 3} \right\}.$$

The above bounds prove Theorem 1.3.

7. Conclusion

A necessary and sufficient condition is found showing the probability of the random event that a second order system of nonlinear random equations over the field $\mathbf{GF}(3)$ has a solution that belongs to a given set of vectors (Theorem 1.1). The condition is given in terms of the number of equations and number of unknowns.

Under various assumptions concerning the parameter c, defined by equality (3), several bounds are found for the above probability (Theorems 1.2 and 1.3). Some examples are given for Theorems 1.2 and 1.3.

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