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NORMAL LIMITING DISTRIBUTION OF THE NORMALIZED NUMBER OF EXTRANEOUS SOLUTIONS OF A COMPATIBLE SYSTEM OF NONLINEAR RANDOM EQUATIONS OVER THE FIELD GF(2)

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Abstract. Conditions are presented under which the distribution of the properly normalized number of extraneous solutions of a system of compatible random equations over the field GF(2) tends to a standard normal distribution.

1. Setting of the problem

Consider the system of equations

$$\sum_{k=1}^{g_q(n)} \sum_{1\le j_1<\dots<j_k\le n} a^{(q)}_{j_1\dots j_k} x_{j_1}\dots x_{j_k} = b_q, \qquad q=1,2,\dots,N, \tag{1}$$

over the field GF(2) that contains only two elements. We study this system under the condition

(A): the coefficients $a^{(q)}_{j_1\dots j_k}$, $1\le j_1<\dots<j_k\le n$, $k=1,2,\dots,g_q(n)$, $q=1,2,\dots,N$, are independent random variables, $\mathsf{P}\{a^{(q)}_{j_1\dots j_k}=1\}=1-\mathsf{P}\{a^{(q)}_{j_1\dots j_k}=0\}=p_{qk}$.

The elements b_q, $q=1,2,\dots,N$, on the right-hand side of (1) represent the result after the substitution of a fixed n-dimensional $(0,1)$-vector $\overline{x}^{(0)}$ on the left-hand side of system (1); the function $g_q(n)$ is nonrandom, $g_q(n)\in\{2,3,\dots,n\}$, $q=1,2,\dots,N$.

Denote by $M\left(\overline{x}^{(0)},f(n)\right)$ the family of all n-dimensional $(0,1)$-vectors $\overline{x}$ that do not coincide with $\overline{x}^{(0)}$ and have $|\overline{x}|$ nonzero coordinates such that $|\overline{x}|\ge f(n)$, $f(n)\in\{0,1,2,\dots,n\}$.

The number of solutions of system (1) that belong to the set $M\left(\overline{x}^{(0)},f(n)\right)$ is denoted by ν_n. These solutions are called *extraneous*. We are interested in finding conditions under which the appropriately normalized random variable ν_n possesses a limiting (as $n\to\infty$) normal distribution and $f(n)\ge 2$.

The case $f(n)=0$ and $\rho(n)\to\infty$, $n\to\infty$, is considered in [2]. Here and in what follows $\rho(n)$ means the number of nonzero coordinates of the vector $\overline{x}^{(0)}$, $\rho(n)=\left|\overline{x}^{(0)}\right|$.

2. Statement of the result

Theorem 1. *Let condition* (A) *hold. Assume that, as* $n\to\infty$,

$$2^{n-N}=[\lambda], \tag{2}$$

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where

$$\lambda = \frac{1}{\upsilon(1+\alpha+\omega)} \log_2 \frac{n}{\beta f(n) \log_2 n}, \tag{3}$$

$\upsilon = \upsilon(n)$, $\upsilon \geq 2$, $\alpha = \alpha(n)$, $\omega = \omega(n)$, $\beta = \beta(n)$, $\beta \geq c_0 > 2\ln 2$, $c_0 = \text{const}$, $f(n) \geq 2$, *and where the symbol* $[\cdot]$ *stands for the integer part of a real number,*

$$\lambda \to \infty, \tag{4}$$

$$\omega\sqrt{\lambda} \to \infty. \tag{5}$$

We further assume that, for an arbitrary q, $q = 1, 2, \ldots, N$, *there exists a nonempty set* T_q *such that*

$$T_q \subseteq \{2, \ldots, g_q(n)\} \cap \{2, \ldots, f(n)\}, \qquad T_q \neq \varnothing, \tag{6}$$

$$\frac{1}{2} - \delta_{qt} \leq p_{qt} \leq \frac{1}{2} + \delta_{qt}, \qquad \delta_{qt} = \delta_{qt}(n), \qquad t \in T_q,\ q = 1, 2, \ldots, N, \tag{7}$$

$$(2 + (1+\alpha+\omega)\ln 2)\lambda - \frac{\ln\lambda}{2} + \ln\left(\sum_{q=1}^{N} \prod_{t \in T_q} 2\delta_{qt}\right) \to -\infty, \tag{8}$$

$$(\alpha - 1)\lambda u(\alpha) \geq c_1, \tag{9}$$

where $c_1 = \text{const}$, $u(\alpha) = (\alpha-1)^{-1}(\alpha\ln\alpha - \alpha - 1)$ *if* $2 > \ln\alpha > 0$, *and* $u(\alpha) = \ln\alpha - 1$ *if* $\ln\alpha \geq 2$.

Then the distribution function of the random variable $\frac{\nu_n - \lambda}{\sqrt{\lambda}}$ *converges as* $n \to \infty$ *to the standard normal distribution function.*

3. Auxiliary results

Let W denote the family of all nonempty subsets of Ω. Denote the number of elements of Ω by $|\Omega| = r$, $1 \leq r < \infty$. Consider two subsets W_Δ and I_s in W:

$$W_\Delta \subseteq W, \qquad W_\Delta = \{\omega_1, \ldots, \omega_\Delta\}, \qquad |W_\Delta| = \Delta, \qquad \Delta \geq 1, \qquad \omega_i \neq \omega_j$$

for $i \neq j$, $i, j \in \{1, \ldots, \Delta\}$;

$$I_s \subseteq W, \qquad I_s = \{m_1, \ldots, m_s\}, \qquad |I_s| = s, \qquad s \geq 0, \qquad m_i \neq m_j$$

for $i \neq j$, $i, j \in \{1, \ldots, s\}$.

Proposition 1 ([3])**.** *Let*

$$|m_i \cap \omega_j| \equiv 0 \pmod 2, \qquad i = 1, \ldots, s,\ j = 1, \ldots, \Delta; \tag{10}$$

$$\Delta \in \left[2^{k-1}, 2^k - 1\right], \qquad 1 \leq k \leq r.$$

Then

$$s \leq 2^{r-k} - 1.$$

Proposition 2 ([3])**.** *Let* $\Omega = \{1, \ldots, r\}$, $3 \leq r < \infty$. *Let conditions* (10) *and*

$$\Delta = 2^k - 1, \qquad s = 2^{r-k} - 1, \qquad 1 \leq k \leq r - 2;$$

$$|\omega_j| \geq 3, \qquad j = 1, \ldots, \Delta, \tag{11}$$

hold for subsets W_Δ *and* I_s. *Then there exists a number* α, $\alpha \in \{1, \ldots, \Delta\}$, *such that*

$$|\omega_\alpha \cap m_{i_\nu}| = 2, \qquad \nu = 1, 2, 3, \qquad |\omega_\alpha \cap (a \cup b)| = 3,$$

for some $m_{i_\nu} \in I_s$, $\nu = 1, 2, 3$, *where* $a \neq b$, $a, b \in \{m_{i_\nu} : \nu = 1, 2, 3\}$.

Remark 1. It is proved in [3] that condition (11) does not hold for $k = r - 1$, $s = 1$.

Let $\overline{x}^{(1)}, \dots, \overline{x}^{(r)}$ be pairwise different n-dimensional $(0,1)$-vectors that do not coincide with $\overline{x}^{(0)}$, $\overline{x}^{(\nu)} = \left(\overline{x}_1^{(\nu)}, \dots, \overline{x}_n^{(\nu)}\right)$, $\nu = 0, \dots, r$, $1 \le r < \infty$.

Given a parameter $t \in \{1, \dots, n\}$, a family $\{\alpha_1, \dots, \alpha_t\}$, $1 \le \alpha_1 < \dots < \alpha_t \le n$, another parameter $\nu \in \{1, \dots, r\}$, and another family $\{u_1, \dots, u_\nu\}$, $1 \le u_1 < \dots < u_\nu \le r$, consider the set

$$m(\alpha_1, \dots, \alpha_t; u_1, \dots, u_\nu) = \left\{ x_{\alpha_1}^{(u_z)} \dots x_{\alpha_t}^{(u_z)} \oplus x_{\alpha_1}^{(0)} \dots x_{\alpha_t}^{(0)} \colon z = 1, \dots, \nu \right\}.$$

We say that a family $m(\alpha_1, \dots, \alpha_t; u_1, \dots, u_\nu)$ possesses property E if

$$x_{\alpha_1}^{(u_z)} \dots x_{\alpha_t}^{(u_z)} \oplus x_{\alpha_1}^{(0)} \dots x_{\alpha_t}^{(0)} = 1, \qquad z = 1, \dots, \nu,$$

and $x_{\alpha_1}^{(\upsilon)} \dots x_{\alpha_t}^{(\upsilon)} \oplus x_{\alpha_1}^{(0)} \dots x_{\alpha_t}^{(0)} = 0$ for an arbitrary υ, $\upsilon \in \{1, \dots, r\} \setminus \{u_1, \dots, u_\nu\}$, where the symbol $\oplus$ denotes the addition in the field GF(2). Denote by $\gamma_t^{\{u_1, \dots, u_\nu\}}$ the number of all elements of the set $\{m(\alpha_1, \dots, \alpha_t; u_1, \dots, u_\nu) \colon 1 \le \alpha_1 < \dots < \alpha_t \le n\}$ that possess property E.

Given $1 \le u_1 < \dots < u_\nu \le r$, $\nu = 1, \dots, r$, $t = 1, \dots, n$, let

$$\Gamma_{t,r}^{\{u_1, \dots, u_\nu\}} = \sum_{\psi=1}^{\nu} 2^{-1} \left(1 - (-1)^\psi\right) \sum_{W_1} \sum_{l=0}^{r-\nu} \sum_{W_2} \gamma_t^{\{\sigma_1, \dots, \sigma_\psi, \mu_1, \dots, \mu_l\}}, \tag{12}$$

where

$$W_1 = \{1 \le \sigma_1 < \dots < \sigma_\psi \le r, \sigma_z \in \{u_1, \dots, u_\nu\}, z = 1, \dots, \psi\}$$

and

$$W_2 = \{1 \le \mu_1 < \dots < \mu_l \le r, \mu_z \notin \{u_1, \dots, u_\nu\}, z = 1, \dots, l\}.$$

Here the symbol $\sum_{W_1}$ ($\sum_{W_2}$) denotes the sum over all families $\{\sigma_1, \dots, \sigma_\psi\}$ ($\{\mu_1, \dots, \mu_l\}$) that belong to the set W_1 (W_2).

We introduce the following notation: $i_{\{u_1 \dots u_\nu\}}$ $\left(j_{\{u_1 \dots u_\nu\}}\right)$ is the number of units (zeros) placed at those positions of all vectors $\overline{x}^{(u_1)}, \dots, \overline{x}^{(u_\nu)}$ where zeros (units) are placed in the vectors $\overline{x}^{(u_{\nu+1})}, \dots, \overline{x}^{(u_r)}, \overline{x}^{(0)}$. Let $F_{u_1, \dots, u_\nu}$ ($\Phi_{u_1, \dots, u_\nu}$) be the number of units (zeros) placed at the same positions in all vectors $\overline{x}^{(u_1)}, \dots, \overline{x}^{(u_\nu)}$, where zeros (units) are placed in the vector $\overline{x}^{(0)}$.

Using the above notation,

$$F_{u_1, \dots, u_\nu} = \sum_{l=0}^{r-\nu} \sum_{\substack{1 \le \mu_1 < \dots < \mu_l \le r \\ \mu_\cdot \notin \{u_1, \dots, u_\nu\}}} i_{\{u_1 \dots u_\nu \mu_1 \dots \mu_l\}}, \tag{13}$$

$$\Phi_{u_1, \dots, u_\nu} = \rho(n) - \sum_{q=1}^{\nu} \sum_{\substack{v_1 < \dots < v_q \\ v_\cdot \in \{u_1, \dots, u_\nu\}}} \sum_{l=0}^{r-\nu} \sum_{\substack{1 \le \mu_1 < \dots < \mu_l \le r \\ \mu_\cdot \notin \{u_1, \dots, u_\nu\}}} j_{\{v_1 \dots v_q \mu_1 \dots \mu_l\}}, \tag{14}$$

$$\gamma_t^{\{m_1 \cap m_2\}} \ge C_{i_{m_1} + i_{m_2}}^t - C_{i_{m_1}}^t - C_{i_{m_2}}^t, \tag{15}$$

where $m_i \subseteq \{1, 2, \dots, r\}$, $i = 1, 2$, $m_1 \ne m_2$, $m_1 \cap m_2 \ne \varnothing$.

It is proved in [3] that

$$\Gamma_{t,r}^{\{u_1\dots u_\nu\}} = \sum_{\psi=1}^{\nu} (-2)^{\psi-1} \sum_{\substack{\sigma_1<\dots<\sigma_\psi \\ \sigma_\cdot\in\{u_1,\dots,u_\nu\}}} C^t_{F_{\sigma_1,\dots,\sigma_\psi}+\Phi_{\sigma_1,\dots,\sigma_\psi}} + \frac{1-(-1)^\nu}{2}\left(C^t_{\rho(n)} + \sum_{\psi=1}^{\nu}(-2)^\psi \sum_{\substack{\sigma_1<\dots<\sigma_\psi \\ \sigma_\cdot\in\{u_1,\dots,u_\nu\}}} C^t_{\Phi_{\sigma_1,\dots,\sigma_\psi}}\right), \tag{16}$$

where the number $\Gamma_{t,r}^{\{u_1\dots u_\nu\}}$, $\overline{x}^{(0)}$, $1 \le u_1 < \dots < u_\nu \le r$, $\nu \in \{1,\dots,r\}$, $t = 1,2,\dots,g_q(n)$, $q = 1,2,\dots,N$, is defined by relation (12).

Let $M(\nu_n)_r$ denote the r^{th} factorial moment of the random variable ν_n, $r \ge 1$.

Proposition 3. *Assume that condition* (A) *holds. Then*

$$M\left(\nu_n\right)_r = 2^{-rN} S(n,r;Q) \tag{17}$$

for all $1 \le r < \infty$ *and* $0 \le f(n) \le n$, *where*

$$S(n,r;Q) = \sum_{s=0}^{n-\rho(n)} \sum (n-\rho(n))! \left((n-\rho(n)-s)! \prod_{i\in I} i!\right)^{-1} \times \sum_{\substack{s'=0 \\ s'+s\ge 1}}^{\rho(n)} \sum{}' (\rho(n))! \left((\rho(n)-s')! \prod_{j\in J} j!\right)^{-1} Q, \tag{18}$$

$$Q = \prod_{q=1}^{N}\left(1 + \sum_{\nu=1}^{r} \sum_{1\le u_1<\dots<u_\nu\le r} \prod_{t=1}^{g_q(n)} (1-2p_{qt})^{\Gamma_{t,r}^{\{u_1,\dots,u_\nu\}}}\right).$$

The symbol $\sum$ ($\sum'$) *corresponds to the summation over all* $i \in I$ ($j \in J$) *where*

$$I = \left\{i_{\{u_1,\dots,u_\nu\}} : 1 \le u_1 < \dots < u_\nu \le r, \nu = 1,\dots,r\right\}$$
$$\left(J = \left\{j_{\{u_1,\dots,u_\nu\}} : 1 \le u_1 < \dots < u_\nu \le r, \nu = 1,\dots,r\right\}\right)$$

are such that

$$\sum_{i\in I} i = s \qquad \left(\sum_{j\in J} j = s'\right).$$

The numbers i, $i \in I$, *and* j, $j \in J$, *in equality* (18) *are such that*

$$\sum_{i\in I_{\{u\}}, j\in J_{\{u\}}} (i+j) \ge 1, \qquad u = 1,\dots,r, \tag{19}$$

$$\sum_{i\in I_{\{u\}}} i + \rho(n) - \sum_{j\in J_{\{u\}}} j \ge f(n), \qquad u = 1,\dots,r, \tag{20}$$

$$\sum_{i\in I_{\{u_1,u_2\}}, j\in J_{\{u_1,u_2\}}} (i+j) \ge 1, \qquad 1 \le u_1 < u_2 \le r. \tag{21}$$

Moreover,

$$\Gamma_{t,r}^{\{u_1,\dots,u_\nu\}} \ge \sum_{(i,j)\in T} \left(C_i^t + C_j^t\right) \tag{22}$$

for all $1 \le u_1 < \cdots < u_\nu \le r$, $\nu \in \{1, \dots, r\}$, *and* $t \in \{1, \dots, n\}$, *where* $T = I_{\{u_1,\dots,u_\nu\}} \times J_{\{u_1,\dots,u_\nu\}}$. *Here*

$$I_{\{u_r,\dots,u_\nu\}} = \left\{ i_{\{\sigma_1,\dots,\sigma_\psi,\mu_1,\dots,\mu_l\}} \colon A(\psi, l, r) \right\},$$
$$J_{\{u_r,\dots,u_\nu\}} = \left\{ j_{\{\sigma_1,\dots,\sigma_\psi,\mu_1,\dots,\mu_l\}} \colon A(\psi, l, r) \right\},$$

where $A(\psi, l, r)$ *stands for the following collection of restrictions*: $1 \le \sigma_1 < \cdots < \sigma_\psi \le r$, $\sigma_z \in \{u_1, \dots, u_\nu\}$, $z = 1, \dots, \psi$, $\psi = 1, \dots, \nu$, $\psi \equiv 1 \pmod 2$, $1 \le \mu_1 < \cdots < \mu_l \le r$, $\mu_1, \dots, \mu_l \notin \{u_1, \dots, u_\nu\}$, $l = 0, \dots, r - \nu$.

The proof of equality (17) is similar to that of equality (2.21) in [3] where the case $f(n) = 0$ is considered. The bound (22) is equivalent to the bound (2.12) proved in [3].

Proposition 4. *Assume that condition* (A) *holds. If* $0 \le f(n) \le n$, *then the expectation of the random variable* ν_n *is equal to*

$$\mathsf{E}\,\nu_n = \sum_{i=0}^{n-\rho(n)} C_{n-\rho(n)}^i \sum_{\substack{j=0 \\ i+j \ge 1}}^{\rho(n)} C_{\rho(n)}^j Q^*, \tag{23}$$

where

$$Q^* = \prod_{q=1}^{N} \left(\frac{1}{2} + \frac{1}{2} \prod_{t=1}^{g_q(n)} (1 - 2p_{qt})^{\Gamma_t(i,j)} \right), \tag{24}$$

$$\Gamma_t(i,j) = C_{i+\rho(n)-j}^t + C_{\rho(n)}^t - 2C_{\rho(n)-j}^t, \tag{25}$$
$$t = 1, 2, \dots, g_q(n), \qquad q = 1, 2, \dots, N,$$
$$i + \rho(n) - j \ge f(n).$$

Relation (23) follows from (17) with $r = 1$.

Proposition 5. *Assume that conditions* (19) *and* (20) *hold. Then*

$$\Gamma_{t,r}^{\{u\}} \ge C_{f(n)-1}^{t-1}, \qquad u = 1, 2, \dots, r, \tag{26}$$

for $t \in \{1, 2, \dots, f(n)\}$.

Proof. Using equalities (13), (14), and (16) we obtain

$$\Gamma_{t,r}^{\{u\}} = C_{F_u+\Phi_u}^t + C_{\rho(n)}^t - 2C_{\Phi_u}^t, \qquad u = 1, 2, \dots, r, \tag{27}$$

where $F_u = \sum_{i \in I_{\{u\}}} i$, $\Phi_u = \rho(n) - \Phi_u^*$, $\Phi_u^* = \sum_{j \in J_{\{u\}}} j$. In view of inequality (19) we get $F_u + \Phi_u^* \ge 1$. Thus inequality (26) follows from a bound for $\Gamma_{t,r}^{\{u\}}$ in the following two cases:

1) $\Phi_u^* \ge 0$, $F_u \ge 1$,
2) $\Phi_u^* \ge 1$, $F_u = 0$.

Let $\Phi_u^* \ge 0$. Then $F_u \ge 1$ and (27) implies

$$\Gamma_{t,r}^{\{u\}} \ge C_{F_u+\Phi_u}^t - C_{\Phi_u}^t \ge C_{F_u+\Phi_u-1}^{t-1} \ge C_{f(n)-1}^{t-1},$$

where the latter inequality follows from condition (20).

If $F_u = 0$ and $\Phi_u^* \ge 1$, then $\Gamma_{t,r}^{\{u\}} \ge C_{\rho(n)}^t - C_{\Phi_u}^t \ge C_{\Phi_u}^{t-1} \ge C_{f(n)-1}^{t-1}$. Proposition 5 is proved. □

Proposition 6. *Assume that conditions* (20) *and* (21) *hold. Then*

$$\Gamma_{t,r}^{\{u_1,u_2\}} \ge C_{f(n)-1}^{t-1}, \qquad 1 \le u_1 < u_2 \le r, \tag{28}$$

for all $t \in \{1, 2, \dots, f(n)\}$.

Proof. Using equalities (13), (14), and (16) we obtain

$$\Gamma_{t,r}^{\{u_1,u_2\}} = C^t_{F_{u_1}+\Phi_{u_1}} + C^t_{F_{u_2}+\Phi_{u_2}} - 2C^t_{F_{u_1,u_2}+\Phi_{u_1,u_2}}, \qquad 1 \le u_1 < u_2 \le r, \tag{29}$$

where $F_{u_1,u_2} = \sum_{l=0}^{r-2} \sum_{\substack{1\le\mu_1<\dots<\mu_l\le r \\ \mu.\notin\{u_1,u_2\}}} i_{\{u_1u_2\mu_1\dots\mu_l\}}$,

$$\Phi_{u_1,u_2} = \rho(n) - \sum_{l=0}^{r-2} \sum_{\substack{1\le\mu_1<\dots<\mu_l\le r \\ \mu.\notin\{u_1,u_2\}}} \left(j_{\{u_1\mu_1\dots\mu_l\}} + j_{\{u_2\mu_1\dots\mu_l\}}\right)$$

$$- \sum_{l=0}^{r-2} \sum_{\substack{1\le\mu_1<\dots<\mu_l\le r \\ \mu.\notin\{u_1,u_2\}}} j_{\{u_1u_2\mu_1\dots\mu_l\}}, \qquad 1 \le u_1 < u_2 \le r.$$

Now we rewrite (29) as follows:

$$\Gamma_{t,r}^{\{u_1,u_2\}} = C^t_{F_{u_1}+\Phi_{u_1}} - C^t_{F_{u_1,u_2}+\Phi_{u_1,u_2}} + C^t_{F_{u_2}+\Phi_{u_2}} - C^t_{F_{u_1,u_2}+\Phi_{u_1,u_2}}, \qquad 1 \le u_1 < u_2 \le r.$$

Recalling the definition of the numbers F_u, Φ_u, $u \in \{u_1, u_2\}$, and F_{u_1,u_2}, Φ_{u_1,u_2} we get

$$\Gamma_{t,r}^{\{u_1,u_2\}} = C^t_{F_{u_1}+\Phi_{u_1}} - C^t_{F_{u_1}+\Phi_{u_1}-\psi_*} + C^t_{F_{u_2}+\Phi_{u_2}} - C^t_{F_{u_2}+\Phi_{u_2}-\psi^*} \tag{30}$$

for $1 \le u_1 < u_2 \le r$, where

$$\psi_* = \sum_{l=0}^{r-2} \sum_{\substack{1\le\mu_1<\dots<\mu_l\le r \\ \mu.\notin\{u_1,u_2\}}} \left(i_{\{u_1\mu_1\dots\mu_l\}} + j_{\{u_2\mu_1\dots\mu_l\}}\right),$$

$$\psi^* = \sum_{l=0}^{r-2} \sum_{\substack{1\le\mu_1<\dots<\mu_l\le r \\ \mu.\notin\{u_1,u_2\}}} \left(i_{\{u_2\mu_1\dots\mu_l\}} + j_{\{u_1\mu_1\dots\mu_l\}}\right).$$

The inequality $\psi_* + \psi^* \ge 1$ holds in view of (21). If $\psi_* \ge 0$ and $\psi^* \ge 1$, then

$$\Gamma_{t,r}^{\{u_1,u_2\}} \ge C^t_{F_{u_2}+\Phi_{u_2}} - C^t_{F_{u_2}+\Phi_{u_2}-1} = C^{t-1}_{F_{u_2}+\rho(n)-\Phi^*_{u_2}-1} \ge C^{t-1}_{f(n)-1}.$$

Here we used equality (30) and condition (20).

If $\psi_* \ge 1$ and $\psi^* = 0$, then

$$\Gamma_{t,r}^{\{u_1,u_2\}} \ge C^{t-1}_{F_{u_1}+\rho(n)-\Phi^*_{u_1}-1} \ge C^{t-1}_{f(n)-1}.$$

Proposition 6 is proved. □

Proposition 7 ([2])**.** *Let X and Y be random variables assuming nonnegative integer values. Denote $MX = \lambda^*$. Let the distributions of these random variables depend on the parameter n in such a way that*

$$M(Y)_r \le C(\lambda^*)^r \tag{31}$$

for all $r \le (\alpha+\gamma)\lambda^*$ and some constant C. If

$$\varliminf_{n\to\infty} (\alpha-1)\lambda^* u(\alpha) \ge c_1 \tag{32}$$

as $n \to \infty$ and $\lambda^* \to \infty$, where c_1, $u(\alpha)$, and α are defined in the statement of Theorem 1*, and if*

$$\gamma \ge 0, \tag{33}$$

$$\max_{1\le r\le(\alpha+\gamma)\lambda^*}\left|M(X)_r(M(Y)_r)^{-1}-1\right|\frac{e^{2\lambda^*}}{\sqrt{\lambda^*}}\to 0,\qquad n\to\infty, \tag{34}$$

then

$$\max_{0\le t\le\gamma\lambda^*}\left|\mathsf{P}\left\{X\ge t\right\}-\mathsf{P}\left\{Y\ge t\right\}\right|\to 0,\qquad n\to\infty.$$

Remark 2. Proposition 7 is proved in [4] for the case $\alpha=5$ and $\gamma=2$.

Remark 3. In what follows, all limits are considered as $n\to\infty$.

4. Proof of Theorem 1

We show that one can apply Proposition 7 under assumptions of Theorem 1. Let the random variable Y in Proposition 7 possess the Poisson distribution with parameter 2^m, $m=n-N$, while the distribution of the random variable X coincides with the distribution of the random variable ν_n, and let the parameter γ equal $\gamma=1+\omega$.

We are going to check condition (31). Using equality (23) we will provide a bound for the expectation of the random variable ν_n. Using (25) and Proposition 5 with $r=1$, where T_q satisfies condition (6), we prove that $\Gamma_t(i,j)\ge 1$ for all $t\in T_q$, $q=1,2,\dots,N$.

The latter result and relations (4), (7), and (8) allow us to obtain the following representation for the product Q^* defined in (24):

$$Q^*=2^{-N}\left(1+O\left(\sum_{q=1}^{N}\prod_{t\in T_q}2\delta_{qt}\right)\right).$$

Taking into account (23) we find

$$M\nu_n=2^{-N}\left(1+O\left(\sum_{q=1}^{N}\prod_{t\in T_q}2\delta_{qt}\right)\right)(2^n-\sigma_0), \tag{35}$$

where $\sigma_0=1+\sum_i C^i_{n-\rho(n)}\sum_j C^j_{\rho(n)}$, $i+j\ge 1$, $i+\rho(n)-j<f(n)$.

Since $f(n)=o\left(\frac{n}{\log_2 n}\right)$ in view of conditions (3) and (4),

$$\sigma_0\le 2^{2nH(p)}, \tag{36}$$

where $H(p)=-p\log_2 p-(1-p)\log_2(1-p)$, $p=\frac{f(n)}{n}$. For the proof of (36), we have used the inequality

$$\sum_{k=0}^{r}C_n^k\le 2^{nH\left(\frac{r}{n}\right)}, \tag{37}$$

where $1\le r\le\frac{n}{2}$ (see [5]).

Considering relations (2), (35), and (36) we obtain

$$M\nu_n=[\lambda]\left(1+O\left(2^{-n(1-2H(p))}\right)+O\left(\sum_{q=1}^{N}\prod_{t\in T_q}2\delta_{qt}\right)\right). \tag{38}$$

Taking into account the notation introduced in the statement of Proposition 7 we rewrite relation (38) as

$$\lambda^*=[\lambda](1+o(1)). \tag{39}$$

Conditions (2), (3), and (8) and equalities $M(Y)_r=2^{mr}$ together with (38) imply (31) for a constant $C>1$.

Now we are going to check condition (34). We rewrite equality (17) as follows:

$$M\left(\nu_n\right)_r = \frac{1}{2^{rN}} \sum_{\Delta=0}^{2^r-1} S^{(\Delta)}\left(n, r; Q\right). \tag{40}$$

The difference between $S^{(\Delta)}(n, r; Q)$ and $S(n, r; Q)$ can be explained as follows. All $i \in I$ and $j \in J$ used to create $S(n, r; Q)$ according to the rule (18) assume only those values for which there are exactly Δ collections

$$\begin{gathered} \omega_k = \left\{u_1^{(k)}, \ldots, u_{\xi_k}^{(k)}\right\}, \qquad 1 \le u_1^{(k)} < \cdots < u_{\xi_k}^{(k)} \le r, \\ \xi_k \in \{1, 2, \ldots, r\}, \qquad k = 1, 2, \ldots, \Delta, \end{gathered} \tag{41}$$

such that, for each of them, there exists a number $t^{(k)} \in \{2, \ldots, f(n)\}$ for which

$$\Gamma_{t^{(k)}, r}^{\omega_k} = 0, \tag{42}$$

while

$$\Gamma_{t,r}^{\{\theta_1, \ldots, \theta_q\}} \ge 1 \tag{43}$$

for any collection $\{\theta_1, \ldots, \theta_q\}$, $1 \le \theta_1 < \cdots < \theta_q \le r$, $q = 1, \ldots, r$, with

$$\{\theta_1, \ldots, \theta_q\} \ne \omega_k, \qquad k = 1, 2, \ldots, \Delta,$$

and for all $t \in \{2, \ldots, f(n)\}$.

Now we show that

$$\sup_{1 \le r \le (\alpha+\gamma)\lambda^*} \left| \frac{S^{(0)}\left(n, r; Q\right)}{2^{rN} M\left(Y\right)_r} - 1 \right| \frac{e^{2\lambda^*}}{\sqrt{\lambda^*}} \to 0. \tag{44}$$

First of all we mention that the equality $\Delta = 0$ may happen, indeed. If for all i, $i \in I$, and (or) all j, $j \in J$, at least one of the two inequalities $i \ge f(n)$ or $j \ge f(n)$ holds, then, in view of (22), bound (43) holds for all collections $\{\theta_1, \ldots, \theta_q\}$, $1 \le \theta_1 < \cdots < \theta_q \le r$, $q = 1, \ldots, r$, and $t \in \{2, \ldots, f(n)\}$.

In turn, it is possible that the equality $i = f(n)$ and (or) equality $j = f(n)$ hold for all $i \in I$, and (or) all $j \in J$, since

$$2^r f(n) \le \max\left(n - \rho(n), \rho(n)\right)$$

by (3) and (39). Thus the parameter Δ may attain the zero value.

Let $u = (2^r - 1) \sum_{q=1}^{N} \prod_{t \in T_q} 2\delta_{qt}$. Using inequalities (7) and (43) and the relation

$$u \to 0 \tag{45}$$

in the case $\Delta = 0$, the product Q can asymptotically be written as $Q = 1 + O\left(u\right)$, whence

$$S^{(0)}\left(n, r; Q\right) = \left(1 + O\left(u\right)\right)\left(2^{rn} - \sigma_1 - \sigma_2 - \sigma_3\right), \tag{46}$$

where

$$\sigma_1 = \sum_{\mu=1}^{r} \sum_{1 \le u_1 < \cdots < u_\mu \le r} S_{\langle u_1, \ldots, u_\mu \rangle}^{(0)}\left(n, r; 1\right). \tag{47}$$

The terms on the right-hand side of (47) are given by

$$\begin{aligned} S_{\langle u_1, \ldots, u_\mu \rangle}^{(0)}\left(n, r; 1\right) = \sum_{s=0}^{n-\rho(n)} \sum \left(n - \rho(n)\right)! \left(\left(n - \rho(n) - s\right)! \prod_{i \in I} i!\right)^{-1} \\ \times \sum_{\substack{s'=0 \\ s'+s \ge 1}}^{\rho(n)} {\sum}' \left(\rho(n)\right)! \left(\left(\rho(n) - s'\right)! \prod_{j \in J} j!\right)^{-1}. \end{aligned}$$

The symbols $\sum$ and $\sum'$ are defined in equality (18) with the following additional conditions:

$$\sum_{i\in I_{\{u\}}} i+\rho(n)-\sum_{j\in J_{\{u\}}} j<f(n), \qquad u\in\{u_1,u_2,\dots,u_\mu\},$$

$$\sum_{i\in I_{\{u\}}} i+\rho(n)-\sum_{j\in J_{\{u\}}} j\geq f(n), \qquad u\in\{1,2,\dots,r\}\setminus\{u_1,u_2,\dots,u_\mu\},$$

and

$$\Gamma_{t,r}^{\{\theta_1,\dots,\theta_\mu\}}\geq 1, \qquad 1\leq\theta_1<\dots<\theta_\mu\leq r, \qquad \mu=1,\dots,r; \tag{48}$$

$$\sigma_2=\sum_{q=1}^{2^r-1} S_q^{(0)}(n,r;1).$$

The difference between $S_q^{(0)}(n,r;1)$, $1\leq q\leq 2^r-1$, and $S(n,r;1)$ can be explained as follows: $i\in I$ and $j\in J$ on the right-hand side of equality (18) change in the case of $S_q^{(0)}(n,r;1)$ in such a way that exactly q expressions $\Gamma_{t,r}^{\{u_1,\dots,u_\nu\}}$ have the property that

$$\Gamma_{t,r}^{\{u_1,\dots,u_\nu\}}=0 \tag{49}$$

for at least one value of the parameter $t\in\{2,\dots,f(n)\}$, and the sum $\sum$ ($\sum'$) is considered over all $i\in I$ ($j\in J$) such that $\sum_{i\in I} i=s$ ($\sum_{j\in J} j=s'$) and, moreover, (19) and (21) hold;

$$\sigma_3=\sum_{s=0}^{n-\rho(n)}\sum\frac{(n-\rho(n))!}{\left(\prod_{i\in I} i!\right)(n-\rho(n)-s)!}\sum_{s'=0}^{\rho(n)}\sum'\frac{\rho(n)!}{\left(\prod_{j\in J} j!\right)(\rho(n)-s')!},$$

where the sum $\sum$ ($\sum'$) is considered over all $i\in I$ ($j\in J$) such that $\sum_{i\in I} i=s$ ($\sum_{j\in J} j=s'$) and at least one of the C_{r+1}^2 inequalities corresponding to relations (19) and (21) does not hold.

Now we estimate σ_1. First we obtain a bound for $S_{\langle u_1,\dots,u_\mu\rangle}^{(0)}(n,r;1)$. Note that

$$S_{\langle u_1,\dots,u_\mu\rangle}^{(0)}(n,r;1)\leq S_{\langle u_1\rangle}^{(0)}(n,r;1),$$

since the restriction

$$\sum_{i\in I_{\{u\}}} i+\rho(n)-\sum_{j\in J_{\{u\}}} j<f(n),$$

$u\in\{u_2,\dots,u_\mu\}$, is dropped on the right-hand side of the inequality, but the restriction $\sum_{i\in I_{\{u_1\}}} i+\rho(n)-\sum_{j\in J_{\{u_1\}}} j<f(n)$ is kept. In turn, the sum $S_{\langle u_1\rangle}^{(0)}(n,r;1)$ can be represented in the following way:

$$\begin{aligned} &S_{\langle u_1\rangle}^{(0)}(n,r;1)\\ &\quad=\sum_{s=0}^{n-\rho(n)} C_{n-\rho(n)}^{s}\sum_{s_1+s_2=s} C_s^{s_1}\left\{\sum\nolimits_1\frac{s_1!}{\prod_{i\in I_{\{u_1\}}} i!}\right\}\left\{\sum\nolimits_2\frac{s_2!}{\prod_{i\in I\setminus I_{\{u_1\}}} i!}\right\}\\ &\qquad\times\sum_{\substack{s'=0\\ s'+s\geq 1}}^{\rho(n)} C_{\rho(n)}^{s'}\sum_{s_1'+s_2'=s'} C_{s'}^{s_2'}\left\{\sum\nolimits_1'\frac{s_1'!}{\prod_{j\in J_{\{u_1\}}} j!}\right\}\left\{\sum\nolimits_2'\frac{s_2'!}{\prod_{j\in J\setminus J_{\{u_1\}}} j!}\right\}, \end{aligned} \tag{50}$$

where $\sum_1$ ($\sum_1$) is the sum considered over all $i\in I_{\{u_1\}}$ ($j\in J_{\{u_1\}}$) such that $\sum i=s_1$ ($\sum j=s_1'$) and where $\sum_2$ ($\sum_2$) is the sum considered over all $i\in I\setminus I_{\{u_1\}}$ ($j\in J\setminus J_{\{u_1\}}$) such that $\sum i=s_2$ ($\sum j=s_2'$).

Since $\left|I_{\{u_1\}}\right| = \left|J_{\{u_1\}}\right| = 2^{r-1}$ and $s_1 < f(n)$, $s_1' > \rho(n) - f(n)$, we take (37) and (50) into account and obtain

$$S^{(0)}_{\langle u_1\rangle}(n, r; 1) \le 2^{(r-1)(n+2f(n))+3nH(p)},$$

whence $\sigma_1 \le 2^{r+(r-1)(n+2f(n))+3nH(p)}$. The latter bound with $r \le (1+\alpha+\omega)\lambda^*$ implies that

$$\frac{\sigma_1}{2^{rn}} \frac{e^{2\lambda^*}}{\sqrt{\lambda^*}} \to 0. \tag{51}$$

Now we are going to show that

$$\frac{\sigma_2}{2^{rn}} \frac{e^{2\lambda^*}}{\sqrt{\lambda^*}} \to 0. \tag{52}$$

Denote by M_1 ($\widetilde{M_1}$) the collection of all i, $i \in I$ (j, $j \in J$), that do not belong to $I_{\{u_1,\dots,u_\nu\}}$ ($J_{\{u_1,\dots,u_\nu\}}$) for all collections $\{u_1, \dots, u_\nu\}$ that satisfy condition (49). Then

$$\sigma_2 \le 2^{r+(r-1)n+r2^{r+1}f(n)+2nH(\widetilde{p})},$$

where $\widetilde{p} = \frac{2^r f(n)}{n}$. Here we used Proposition 1 with $2^{z-1} \le q \le 2^z - 1$ which implies that $|M_1| \le 2^{r-z} - 1$ and $\left|\widetilde{M_1}\right| \le 2^{r-z} - 1$. Then we obtain for $r \le (1+\alpha+\omega)\lambda^*$ that

$$\frac{\sigma_2}{2^{rn}} \frac{e^{2\lambda^*}}{\sqrt{\lambda^*}} \le \frac{1}{\sqrt{\lambda^*}} \left(\frac{1+o(1)}{2}\right)^n,$$

whence (52) follows.

The relation

$$\frac{\sigma_3}{2^{rn}} \frac{e^{2\lambda^*}}{\sqrt{\lambda^*}} \to 0 \tag{53}$$

is valid, since $\sigma_3 \le \exp\left\{\left(C^2_{r+1} + rn - n\right)\ln 2\right\}$ and

$$\frac{\sigma_3}{2^{rn}} \frac{e^{2\lambda^*}}{\sqrt{\lambda^*}} \le \frac{1}{\sqrt{\lambda^*}} \left(\frac{1+o(1)}{2}\right)^n$$

for $r \le (1+\alpha+\omega)\lambda^*$.

Using condition (8), we prove for $r \le (1+\alpha+\omega)\lambda^*$ that

$$u \frac{e^{2\lambda^*}}{\sqrt{\lambda^*}} \to 0. \tag{54}$$

Taking into account (46) we represent the fraction $\frac{S^{(0)}(n,r;Q)}{2^{rn}}$ as

$$1 - 2^{-rn} \sum_{k=1}^{3} \sigma_k + O(u).$$

Thus relation (44) which we want to check can be rewritten as follows:

$$\sup_{1\le r\le(1+\alpha+\omega)\lambda^*} \left(2^{-rn} \sum_{k=1}^{3} \sigma_k + O(u)\right) \frac{e^{2\lambda^*}}{\sqrt{\lambda^*}} \to 0. \tag{55}$$

In turn, relation (55) follows immediately from (51)–(54).

In order to complete the proof of condition (34) and in view of (40) and (44) one needs to show that

$$\frac{1}{2^{rn}} \left(\sum_{\Delta=1}^{2^r-1} S^{(\Delta)}(n, r; Q)\right) \frac{e^{2\lambda^*}}{\sqrt{\lambda^*}} \to 0 \tag{56}$$

for $1 \le r \le (1+\alpha+\omega)\lambda^*$.

Having this observation in mind, we rewrite the sum $\sum_{\Delta=1}^{2^r-1} S^{(\Delta)}(n,r;Q)$ as

$$\sum_{\Delta=1}^{2^r-1} S^{(\Delta)}(n,r;Q) = S_1 + S_2, \tag{57}$$

where $S_1 = \sum_{z=2}^{r} \sum_{\Delta\in\mathbf{G}_{1z}} S^{(\Delta)}(n,r;Q)$,

$$\mathbf{G}_{1z} = \left\{\Delta\colon 2^{z-1} \le \Delta < 2^z - 1\right\}, \qquad z = 2,3,\dots,r,$$

$S_2 = \sum_{z=1}^{r} \sum_{\Delta\in\mathbf{G}_{2z}} S^{(\Delta)}(n,r;Q)$, and $\mathbf{G}_{2z} = \{\Delta\colon \Delta = 2^z - 1\}$, $z = 1,2,\dots,r$.

Using (43) and conditions (6) and (7) we obtain for $\Delta \in \mathbf{G}_{1z}$ that

$$|Q| \le 2^{zN} Q_1, \tag{58}$$

where $Q_1 = (1-2^{-z})^N \exp\left\{\frac{2^z-\Delta-1}{(2^z-1)(2^r-1)}u\right\}$.

Similarly we prove for $\Delta \in \mathbf{G}_{2z}$ that

$$|Q| \le 2^{zN} Q_2, \tag{59}$$

where $Q_2 = \exp\left\{\frac{2^r-\Delta-1}{2^z(2^r-1)}u\right\}$.

Denote by M_1 ($\widetilde{M}_1$) the family of all i, $i \in I$ (j, $j \in J$), that do not belong to I_{ω_k} (J_{ω_k}) for all sets ω_k, $k = 1,2,\dots,\Delta$, that satisfy (41) and (42).

Put $M_2 = I \setminus M_1$, $\widetilde{M}_2 = J \setminus \widetilde{M}_1$, and let z be the least integer number for which $\Delta \le 2^z - 1$, $1 \le z \le r$. Applying Proposition 1 we get

$$|M_1| \le 2^{r-z} - 1, \qquad \left|\widetilde{M}_1\right| \le 2^{r-z} - 1. \tag{60}$$

With the help of (22) and (42) we deduce that

$$0 \le i \le f(n) \qquad (0 \le j \le f(n)) \tag{61}$$

for all $i \in M_2$ ($j \in \widetilde{M}_2$).

Taking into account (58), (60), and (61) we derive the inequality

$$2^{-rn}\,|S_1|\,\frac{e^{2\lambda^*}}{\sqrt{\lambda^*}} \le \frac{1+O(u)}{\sqrt{\lambda^*}} \exp\left\{-2^{-r}n\left(1-\beta^{-1}2\ln 2 + o(1)\right)\right\}$$

for all $1 \le r \le (1+\alpha+\omega)\lambda^*$, whence

$$2^{-rn} S_1 \frac{e^{2\lambda^*}}{\sqrt{\lambda^*}} \to 0, \tag{62}$$

since $2^{-r}n \to \infty$ for r specified above and for $\beta \ge c_0 > 2\ln 2$. Relation (45) holds in view of condition (3).

Our next aim is to show that

$$2^{-rn} S_2 \frac{e^{2\lambda^*}}{\sqrt{\lambda^*}} \to 0 \tag{63}$$

for $1 \le r \le (1+\alpha+\omega)\lambda^*$.

Let

$$|M_1| < 2^{r-z} - 1, \qquad \left|\widetilde{M}_1\right| < 2^{r-z} - 1 \tag{64}$$

for $\Delta \in \mathbf{G}_{2z}$.

Using (59) and (61) we conclude that

$$2^{-rn}\,|S_2|\,\frac{e^{2\lambda^*}}{\sqrt{\lambda^*}} \le \lambda^{-3/2}\,(1+O(u)) \exp\left\{-2^{-r+1}n\left(1-\beta^{-1}\ln 2 + o(1)\right)\right\},$$

whence (63) follows.

Now we check that if $\Delta = 2^z - 1$, $1 \le z \le r$, and $z \in \{r-1, r\}$ or $r \in \{1, 2\}$, then there exists k, $k \in \{1, 2, \dots, \Delta\}$, such that the parameter ξ_k defined by (41) does not exceed $\xi_k \le 2$. Indeed, if $z = r$ or $r \in \{1, 2\}$, then such a number k obviously exists. If $z = r - 1$, then Remark 1 implies that there exists an index k for which $\xi_k \le 2$.

The inequality $\Gamma_{t,r}^{\omega_k} \ge 1$ holds for $k \in \{1, 2, \dots, \Delta\}$ such that $\xi_k \le 2$ (in view of Propositions 5 and 6 and condition (6)) which contradicts (42). Therefore the expression $\Delta = 2^z - 1$ extends throughout below to all z, $1 \le z \le r-2$, $3 \le r < \infty$, and k for which $\xi_k \ge 3$.

Hence we may assume that

$$\xi_k \ge 3, \qquad k = 1, 2, \dots, \Delta, \quad \Delta = 2^z - 1, \quad 1 \le z \le r-2, \quad 3 \le r < \infty, \tag{65}$$

$$|M_1| = \left|\widetilde{M_1}\right| = 2^{r-z} - 1. \tag{66}$$

According to Proposition 2 (its assumptions hold in view of (65) and (66)), the set M_1 ($\widetilde{M_1}$) contains at least three elements $i_{m_\nu} \in M_1$ ($j_{\widetilde{m}_\nu} \in \widetilde{M_1}$), $\nu = 1, 2, 3$, for which the set ω_k ($\omega_{\widetilde{k}}$) occurred in relations (42) is such that

$$|\omega_\eta \cap m(\eta, \nu)| = 2, \qquad \nu = 1, 2, 3, \qquad |\omega_\eta \cap (a_\eta \cup b_\eta)| = 3, \qquad \eta \in \{k, \widetilde{k}\}, \tag{67}$$

for some $k \in \{1, 2, \dots, \Delta\}$ ($\tilde{k} \in \{1, 2, \dots, \Delta\}$) and arbitrary

$$a_\eta, b_\eta \in \{m(\eta, \nu) \colon \nu = 1, 2, 3\}, \qquad a_\eta \ne b_\eta,$$

where

$$m(\eta, \nu) = \begin{cases} m_\nu, & \text{if } \eta = k; \text{ or} \\ \widetilde{m}_\nu, & \text{if } \eta = \widetilde{k}, \end{cases} \qquad \nu = 1, 2, 3.$$

Then (67) implies that $|\omega_\eta \cap a_\eta \cap b_\eta| = 1$, $\eta \in \{k, \widetilde{k}\}$, whence

$$\Gamma_{t,r}^{\omega_\eta} \ge \gamma_t^{\{a_\eta \cap b_\eta\}} \ge t^{-1} l_* (l^* - 2^{-1}(l_* - 1)) C^{t-2}_{(l^*/2)+(3l_*/4)-(5/4)} \tag{68}$$

according to (12), (15), and the inequality $C_\alpha^t - C_{\alpha-\beta}^t \ge \beta C^{t-1}_{\alpha - \frac{1}{2}(1+\beta)}$ proved in [1], where α, β, and t are positive integer numbers such that $\alpha - \beta \ge t$ and where

$$(l_*, l^*) \in \{(i_*, i^*), (j_*, j^*)\}, \qquad i_* = \min\left\{i_{a_\eta}, i_{b_\eta}\right\}, \qquad i^* = \max\left\{i_{a_\eta}, i_{b_\eta}\right\},$$

and

$$j_* = \min\left\{j_{a_\eta}, j_{b_\eta}\right\}, \qquad j^* = \max\left\{j_{a_\eta}, j_{b_\eta}\right\}, \qquad \eta \in \{k, \widetilde{k}\}.$$

If $i_* \ge f(n)$ and $j_* \ge f(n)$, then relation (68) for $t \in \{2, \dots, f(n)\}$ implies $\Gamma_{t,r}^{\omega_\eta} \ge 1$ which contradicts (42). Hence (65) and (66) imply that the set M_1 ($\widetilde{M_1}$) contains at least one element $i \in M_1$ ($j \in \widetilde{M_1}$) such that

$$i < f(n) \qquad (j < f(n)). \tag{69}$$

This yields the following bound for S_2 if conditions (59), (65), (66), and (69) hold:

$$|S_2| \le (1 + O(u))\, 2^{r2^{r+1} f(n) + rn + 2nH(\widetilde{p})} \sum_{z=1}^{r-2} 2^{-zm} \left(1 - 2^{-r+z+1}\right)^n.$$

The latter inequality together with (45) leads to (63).

It remains to check relation (63) if condition (65) holds and

$$|M_1| = 2^{r-z} - 1, \qquad \left|\widetilde{M_1}\right| < 2^{r-z} - 1, \qquad |M_1| < 2^{r-z} - 1, \qquad \left|\widetilde{M_1}\right| = 2^{r-z} - 1.$$

This can be proved similarly to the above proof based on conditions (64), (65), and (66).

Relations (57), (62), and (63) prove (56). Combining (40), (44), and (56) we make sure that condition (34) in Proposition 7 holds. Inequality (33) holds, since $\gamma = 1 + \omega$ and $\omega > 0$ in view of (5).

Now we obtain (32) with the help of (4), (9), and (39). Thus all the assumptions of Proposition 7 are checked and therefore

$$\max_{0 \le t \le (1+\omega)\lambda^*} |\mathsf{P}\{\nu_n \ge t\} - \mathsf{P}\{Y \ge t\}| \to 0, \tag{70}$$

where $\lambda^* = M\nu_n$ and where Y is a Poisson random variable with parameter $[\lambda]$. Since the random variable $\frac{Y-[\lambda]}{\sqrt{[\lambda]}}$ converges as $\lambda \to \infty$ to the standard normal distribution, relation (70) is equivalent to

$$\max_{-\sqrt{\lambda(1+o(1))} \le l \le \omega\sqrt{\lambda(1+o(1))}} \left| \mathsf{P}\left\{ \frac{\nu_n - \lambda}{\sqrt{\lambda}} \ge l \right\} - \mathsf{P}\left\{ \frac{Y - [\lambda]}{\sqrt{[\lambda]}} \ge l \right\} \right| \to 0,$$

where $l = \frac{t-\lambda}{\sqrt{\lambda}}$. The theorem is proved.

5. Concluding remarks

We found sufficient conditions under which the appropriately normalized total number of extraneous solutions of a compatible system of nonlinear random equations with n unknowns over the field GF(2) converges as $n \to \infty$ to a normal distribution. The normalizing sequence is given explicitly.

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