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ON A REDUNDANT SYSTEM WITH RENEWALS

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Abstract. A two units redundant system with renewals of general type is considered. Stationary probabilities are found and the average duration of busy periods are established for the stationary regime.

1. Introduction

Redundancy is one of the main methods for increasing the reliability of systems [1]. In the current paper, we consider the simplest type of redundancy, namely the duplication, where each operating unit is associated with a single standby unit, which replaces the operating unit in case the latter fails. The failed unit is subject for renewal and, after the repair, has the characteristics being equivalent to the initial ones.

B. Epstein and T. Hosford [2] were the first to consider two units redundant systems. It is assumed in [2] that the distribution of the trouble-free time and that of the renewal time are exponential (this assumption does not always hold in practice). A special place in this topic is occupied by the papers [3] and [4] by B. V. Gnedenko. In contrast to [2], the papers [3] and [4] deal with general distributions. Moreover, new methods for analysis of systems are proposed in [3] and [4] that allow one to find solutions to such types of problems in the general case.

A number of papers are devoted to studying the reliability of redundant systems with renewal. These are papers, to mention a few, by I. M. Kovalenko [5,6], V. S. Korolyuk [7, 8], and A. D. Solovyev [10,11].

When studying a redundant system with renewal, a common problem is to find the distribution (or, at least, some of its characteristics) of the trouble-free time until the first failure. It is also clear that stationary characteristics of the system are also of primary interest. In the Markov case, this problem is reduced in a standard way to a question concerning a birth-death process (see, for example, [1] or [7]).

The problem becomes nontrivial if the distribution of the trouble-free time or renewal time is not exponential. In this case, the problem is close to the classical problems of the queueing theory. Some of the problems of this kind are solved by T. P. Maryanovich [12], I. M. Kovalenko [5] and A. D. Solovyev [10] for a general redundant system with the so-called unbounded renewal. A general approach based on semi-Markov processes, is proposed by V. S. Korolyuk [7,9].

Throughout the paper, $\zeta(t)$ denotes the total number of defective units in the system at a moment t,

$$p_k(t) = \mathsf{P}(\zeta(t) = k), \qquad k = 0, 1, 2.$$

We assume that $\zeta(0) = 0$ almost surely and say that the system is in a state k at a moment t if $\zeta(t) = k$.

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Let $\alpha_k(t)$ be the time spent by the system in a state k, $k = 0, 1, 2$, during the interval $(0, t)$.

It is clear that

$$\alpha_0(t) + \alpha_1(t) + \alpha_2(t) = t.$$

The aim of this paper is to prove that, for a redundant system with renewal, the limits

$$\lim_{t\to\infty} \frac{\alpha_k(t)}{t} = p_k \tag{1}$$

exist almost surely for $k = 0, 1, 2$, under rather mild assumptions and to find a simple formula for stationary probabilities p_k and for some other characteristics of the reliability. In doing so, we use the methods developed in [3, 4, 13].

Remark 1. In the theory of queueing systems, the stationary probabilities are usually defined as follows:

$$\lim_{t\to\infty} p_k(t) = p_k^*. \tag{2}$$

Denote by $I(A)$ the indicator of an event A. Then

$$\alpha_k(t) = \int_0^t I(\zeta(s) = k)\, ds.$$

Thus

$$\mathsf{E}\, \frac{\alpha_k(t)}{t} = \frac{1}{t} \int_0^t p_k(s)\, ds.$$

This implies that if limits (1) and (2) exist, then $p_k = p_k^*$ for all k. Clearly, this observation holds for queueing systems of a rather general form.

We suppose that the system is functioning properly if at least one of its units works properly. Then $K_\Gamma = p_0 + p_1$ is called the availability coefficient (see [1, p. 110]).

Throughout the paper a busy period is understood as a continuous period when the system is functioning properly; every busy period is followed by an idle period when both units fail.

Depending on the state of redundant units until they replace the initial unit, there are several types of redundancy. We recall briefly the main types of redundancy that usually are studied in the queueing theory.

(i) *Completely inactive standby redundancy.* The redundant unit is not initially hooked up and, by hypothesis, it cannot fail until it is put in place of the primary unit.

(ii) *Active redundancy.* The redundant unit is subject to the same rules as the basic unit.

(iii) *Partially energized standby redundancy.* The redundant unit is in a partially energized state up to the instant it is put in place of the primary unit. During the period it is in standby, it can fail but the probability of this is less than the same probability for the basic unit.

Each of the above types of redundancy of a standby system can be accompanied by the following types of renewal:

a) *Bounded renewal*: there is a unique redundant unit in the system.

b) *Unbounded renewal*: there are at least two redundant units in the system.

In what follows we use the following notation: type $(A/n/r)$ means a redundant system with renewal where n is the number of redundant units, r is the number of repairing units, $A \in \{i, ii, iii\}$, i is a completely inactive standby redundant reserve, ii is a partially energized standby redundant reserve, and iii is an active redundant reserve.

For a redundant system $n \equiv 2$. Therefore there are six types of redundant systems, namely $(i/2/1), \dots, (iii/2/2)$. For example, the type (ii/2/2) characterizes a redundant system with an active redundant reserve and unbounded renewal.

2. Completely inactive standby redundant reserve

We start with the case of bounded renewal, that is, we consider a redundant system of the type $(i/2/1)$. The functioning of the system can be described as follows. The primary unit works during a time τ; then it is either sent for repairing (if a repairing unit is idle), or takes a position in the queue for repairing. Simultaneously a redundant unit is put in place of the primary unit if the first works properly. After the repairing of the primary unit is completed, it is sent to reserve or is put in place of the primary unit if the place is free.

Let τ be the failure-free time of the primary unit, η be the renewal time (repairing time), τ and η are independent random variables. Assume that τ_n, $n \ge 1$, are independent identically distributed random variables with the distribution function $F(x) = \mathsf{P}(\tau < x)$, $F(0+) = 0$. Let η_n, $n \ge 1$, be independent identically distributed random variables, $G(x) = \mathsf{P}(\eta < x)$, $G(0+) = 0$.

Denote by W'_k the k^{th} busy period. Let W''_k be the k^{th} idle period of the system, ν_k be the number of repairing of the primary unit by a redundant unit during the k^{th} busy period $+1$,

$$q = \mathsf{P}(\tau < \eta) = \int_0^\infty F(x)\, dG(x),$$

$$d = \mathsf{E}\min(\tau, \eta) = \int_0^\infty \mathsf{P}(\min(\tau, \eta) > x)\, dx = \int_0^\infty (1 - F(x))(1 - G(x))\, dx.$$

Note that the random variables ν_k have the geometric distribution with parameter q.

Theorem 1. *If a redundant system of the type* $(i/2/1)$ *is such that*

$$\mathsf{E}\,\tau = a < \infty, \qquad \mathsf{E}\,\eta = b < \infty, \tag{3}$$

then the limits (1) *exist and*

$$p_0 = \frac{a-d}{a+b-d}, \qquad p_1 = \frac{d}{a+b-d}, \qquad p_2 = \frac{b-d}{a+b-d}. \tag{4}$$

If $k \ge 2$, *then*

$$\mathsf{E}\,W'_k = \frac{a}{q}, \qquad \mathsf{E}\,W''_k = \frac{b-d}{q}.$$

If $\operatorname{Var}\tau = \sigma^2 < \infty$, $\operatorname{Var}\eta < \infty$, *then, for* $k \ge 2$,

$$\operatorname{Var}(W'_k - a\nu_k) = \frac{\sigma^2}{q},$$

$$\operatorname{Var} W''_k = \frac{1}{q}\int_0^\infty \int_t^\infty (x-t)^2\, dG(x)\, dF(t) - \frac{(b-d)^2}{q^2}.$$

Remark 2. It is clear that formula (4) implies the expression for the stationary coefficient of reliability of the system

$$K_\Gamma = p_0 + p_1 = \frac{a}{a+b-d}.$$

Below we find the stationary probabilities of states for all main types of redundant systems. It is clear that, similarly to the above reasoning, one can determine the stationary coefficient of reliability.

Remark 3. The following upper bound for the variance of the k^{th} busy period holds under the assumptions of Theorem 1,

$$\operatorname{Var} W_k' \leq \frac{\sigma^2}{q} + \frac{2a\sigma(1-q)^{1/2}}{q^{3/2}} + \frac{a^2(1-q)}{q^2}$$

for $k \geq 2$. Unfortunately, the method of the current paper does not allow us to evaluate the exact value of $\operatorname{Var} W_k'$.

An auxiliary result for regenerative processes plays an important role in the rest of the paper. Some related results on the existence of limits for a regenerative process can be found in [14, 15].

Definition 2.1. An ordered pair $\mathcal{L} = (T, \xi(t))$ is called a period of length T if T is a nonnegative random variable such that

$$\mathsf{P}(T = 0) < 1, \qquad \mathsf{P}(T < \infty) = 1,$$

and $\xi(t)$ is a stochastic process defined on $[0, T)$. In general, the random variable T and the stochastic process $\xi(t)$ are dependent.

Assume that $\mathcal{L}_i = (T_i, \xi_i(t))$ is an infinite sequence of independent periods being equivalent to $\mathcal{L}$. Introduce a stochastic process $X(t)$, $t \geq 0$, by

$$X(t) = \xi_i(t - S_{i-1}) \quad \text{for } t \in (S_{i-1}, S_i],$$

where $S_i = T_1 + \ldots + T_i$, $i \geq 1$, $S_0 = 0$.

Then $X(t)$ is called a *regenerative process.* The points S_i are called *regeneration moments*, while the interval (S_{i-1}, S_i) is called the i^{th} *regeneration period.*

Lemma 1. *Let $\xi(t)$ be a stochastic process that assumes a finite or an infinite number of values $0, 1, 2, \ldots, N$. Assume that there exist regeneration moments for the process $\xi(t)$, namely $S_0 = 0, S_1, S_2, \ldots$. Then $T_i = S_i - S_{i-1}$, $i = 1, 2, \ldots$, are independent identically distributed random variables. If additionally*

$$\mathsf{E}\, T_1 = u < \infty,$$

then the limit

$$\lim_{t\to\infty} \frac{\alpha_k(t)}{t} = \frac{\mathsf{E}\, \alpha_k(T_1)}{u} = \frac{1}{u} \int_0^\infty \mathsf{P}(\xi(t) = k, T_1 > t)\, dt \tag{5}$$

exists almost surely for all k, where $\alpha_k(t)$ is the sojourn time in the state k during the interval $(0, t)$ for the process $\xi(t)$.

Proof of Lemma 1. Let k be a certain fixed number. Then

$$\alpha_k(S_n) = \sum_{i=1}^n \alpha_k(S_{i-1}, S_i) = \sum_{i=1}^n \alpha_{ki},$$

$$S_n = \sum_{i=1}^n T_i,$$

where α_{ki} are independent identically distributed random variables,

$$\alpha_k(s, t) = \alpha_k(t) - \alpha_k(s)$$

is the sojourn time in the state k during the interval (s, t).

Then, by the strong law of large numbers,

$$\begin{aligned}\frac{\alpha_k(S_n)}{S_n} &= \frac{\frac{1}{n}\sum_{i=1}^n \alpha_{ki}}{\frac{1}{n}\sum_{i=1}^n T_i} \to \frac{\mathsf{E}\,\alpha_k(T_1)}{\mathsf{E}\,T_1} = \frac{1}{u}\,\mathsf{E}\int_0^\infty I(\xi(t)=k, T_1>t)\,dt \\ &= \frac{1}{u}\int_0^\infty \mathsf{P}(\xi(t)=k, T_1>t)\,dt\end{aligned} \tag{6}$$

almost surely as $n\to\infty$.

If $S_{n-1} < t \le S_n$, then

$$\frac{\alpha_k(S_{n-1})}{S_n} \le \frac{\alpha_k(t)}{t} \le \frac{\alpha_k(S_n)}{S_{n-1}} \tag{7}$$

almost surely. According to relation (6),

$$\frac{\alpha_k(S_n)}{S_{n-1}} = \frac{\alpha_k(S_n)}{S_n}\,\frac{S_n/n}{S_{n-1}/(n-1)}\,\frac{n}{n-1} \to \frac{\mathsf{E}\,\alpha_k(T_1)}{\mathsf{E}\,T_1} \tag{8}$$

almost surely. Similarly,

$$\frac{\alpha_k(S_{n-1})}{S_n} \to \frac{\mathsf{E}\,\alpha_k(T_1)}{\mathsf{E}\,T_1} \tag{9}$$

almost surely. Combining (7)–(9), we obtain (5). □

Proof of Theorem 1. We treat only the case of $0<q<1$. Nevertheless, relations (4) remain true for the degenerate cases $q=0$ and $q=1$, as well.

Note that $\zeta(t)$ is a regenerative process. We define the regeneration moments S_i for the process $\zeta(t)$. Let S_0 be the moments when the primary unit fails for the first time. For $i=1,2,\dots$, let S_i be the i^{th} moment when repair is finished for a unit and it is put in place of the primary unit, while the repair starts for the other unit. In fact, S_i are the transition moments from the state 2 to the state 1.

Consider the random events $A_n=\{\tau_n<\eta_n\}$ and put $\varepsilon_n = I(A_n)$, $n=1,2,\dots$. Then the random variable ε_n assumes the value 1 if the random event A_n occurs, otherwise ε_n assumes the value 0. It is clear that

$$\mathsf{P}(\varepsilon_n=1)=q,\qquad \mathsf{P}(\varepsilon_n=0)=1-q,\qquad q=\mathsf{P}(\tau<\eta).$$

Next we introduce the random variable ν:

$$\nu=\min(n\ge 1\colon \varepsilon_n=1).$$

It is well known (see [16]) that ν has the geometric distribution

$$\mathsf{P}(\nu=n)=q(1-q)^{n-1},\qquad n\ge 1,$$

and

$$\mathsf{E}\,\nu=\frac{1}{q},\qquad \operatorname{Var}\nu=\frac{1-q}{q^2}. \tag{10}$$

Figure 1 demonstrates the first regeneration period (S_0,S_1).

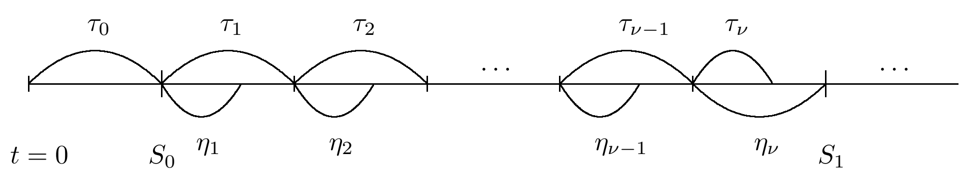

Figure 1

It is seen from Figure 1 that, during the regeneration period (S_0,S_1), the number of transitions of the types $1\to 0$ and $0\to 1$ are the same and equal $\nu-1$, while there is only one transition of each type $1 \to 2$ or $2 \to 1$. Similarly, we conclude from Figure 1 that the sojourn times in the states k during the interval (S_0, S_1) are given by

$$\begin{aligned}\overline{\alpha}_1 &= \sum_{i=1}^{\nu} \min(\tau_i, \eta_i),\\ \overline{\alpha}_2 &= T_1 - \sum_{i=1}^{\nu} \tau_i,\\ \overline{\alpha}_0 &= T_1 - \overline{\alpha}_1 - \overline{\alpha}_2,\end{aligned} \tag{11}$$

almost surely, where $\overline{\alpha}_k = \alpha_k(S_1) - \alpha_k(S_0)$, $k = 0, 1, 2$, $T_1 = S_1 - S_0$.

The length of the first regeneration period can be found from the relation

$$T_1 = S_1 - S_0 = \sum_{i=1}^{\nu} \max(\tau_i, \eta_i). \tag{12}$$

Since ν is the first moment when the event $\{\varepsilon_n = 1\}$ occurs, it is clear that ν is a Markov moment with respect to the flow of σ-algebras $(\mathfrak{F}_n = \sigma\{\tau_i, \eta_i, i = 1, 2, \dots, n\})$. Then the Wald identity (see [17]) and (10)–(12) imply

$$\begin{aligned}\mathsf{E}\, T_1 &= \mathsf{E}\,\nu\, \mathsf{E} \max(\tau, \eta) = \frac{1}{q}\big(\mathsf{E}\,\tau + \mathsf{E}\,\eta - \mathsf{E}\min(\tau, \eta)\big) = \frac{a + b - d}{q},\\ \mathsf{E}\,\overline{\alpha}_1 &= \mathsf{E}\,\nu\, \mathsf{E}\min(\tau, \eta) = \frac{d}{q},\\ \mathsf{E}\,\overline{\alpha}_2 &= \mathsf{E}\, T_1 - \mathsf{E}\,\nu\, \mathsf{E}\,\tau = \frac{a + b - d}{q} - \frac{a}{q} = \frac{b - d}{q},\\ \mathsf{E}\,\overline{\alpha}_0 &= \mathsf{E}\, T_1 - \mathsf{E}\,\overline{\alpha}_1 - \mathsf{E}\,\overline{\alpha}_2 = \frac{a - d}{q}.\end{aligned} \tag{13}$$

Since the initial period $(0, S_0)$ does not influence the asymptotic behavior of the process $\zeta(t)$, one can apply Lemma 1. Indeed, by Lemma 1,

$$\lim_{t\to\infty} \frac{\alpha_k(t - S_0)}{t - S_0} = \frac{\mathsf{E}\,\overline{\alpha}_k}{\mathsf{E}\, T_1}$$

almost surely for $k = 0, 1, 2$. It is clear from Figure 1 that

$$\alpha_k(t - S_0) = \alpha_k(t) \quad \text{for } k = 1, 2$$

and for all $t > S_0$ and that

$$\alpha_0(t - S_0) + \tau_0 = \alpha_0(t).$$

The latter equalities together with (13) yield (4).

According to Figure 1, the length of the busy period during the first regeneration is equal to $\overline{\alpha}_0 + \overline{\alpha}_1$, while the length of the idle period is equal to $\overline{\alpha}_2$. Thus

$$\mathsf{E}\, W_k' = \mathsf{E}\sum_{i=1}^{\nu} \tau_i = \mathsf{E}\,\nu\, \mathsf{E}\,\tau = \frac{a}{q}$$

for $k \geq 2$. Similarly,

$$\operatorname{Var}(W_k' - a\nu_k) = \mathsf{E}\left|\sum_{i=1}^{\nu}(\tau_i - a)\right|^2 = \operatorname{Var}\tau\, \mathsf{E}\,\nu = \frac{\sigma^2}{q}$$

(see [17]).

To complete the proof note that

$$\mathsf{E}\,W_k'' = \mathsf{E}\,\overline{\alpha}_2 = \frac{b-d}{q},$$
$$\mathsf{E}\,|W_k''|^2 = \mathsf{E}\left(|\eta-\tau|^2/(\eta>\tau)\right). \qquad \square$$

The bound mentioned in Remark 3 follows directly from Theorem 1 and equalities (10).

Remark 4. If the random variable τ has the exponential distribution,

$$F(x) = 1-\exp(-\lambda x), \qquad x \geq 0,$$

then

$$d = \frac{1}{\lambda} - G^*(\lambda), \qquad q = \lambda d = 1-\lambda G^*(\lambda),$$

where $G^*(z) = \int_0^\infty \exp(-zx)G(x)\,dx$ is the Laplace transform of the function $G(x)$.

In addition, if $G(x) = 1-\exp(-\mu x)$, then the parameters are easy to evaluate:

$$a = \frac{1}{\lambda}, \qquad b = \frac{1}{\mu}, \qquad d = \frac{1}{\lambda+\mu}, \qquad q = \lambda d = \frac{\lambda}{\lambda+\mu}.$$

Hence relations (4) with $\rho = \frac{\lambda}{\mu}$ are rewritten as follows:

$$p_0 = \frac{1}{1+\rho+\rho^2}, \qquad p_1 = \frac{\rho}{1+\rho+\rho^2}, \qquad p_2 = \frac{\rho^2}{1+\rho+\rho^2}.$$

These equalities can, of course, be derived from known results for stationary probabilities for birth and death processes.

Now we turn to the case of an unbounded renewal and consider a redundant system of the type $(i/2/2)$.

Let $S_0 = 0, S_1, S_2, \dots$ be the moments when the redundant system returns to the state 0 (when both units are in order). More precisely, S_1 is the first moment of transition from the state 1 to the state 0. Analogously, S_k is the k^{th} moment of transition from the state 1 to the state 0. As above, we put $T_i = S_i - S_{i-1}$, $\alpha_{ki} = \alpha_k(S_i) - \alpha_k(S_{i-1})$, $\overline{\alpha}_k = \alpha_{k1}$.

If $\zeta(t)$ is a regenerative process with regeneration moments $S_0, S_1, S_2, \dots$, then, for all $k = 0, 1, 2$, (α_{ki}) is a sequence of independent identically distributed random variables being identically distributed with $\overline{\alpha}_k$.

Theorem 2. *Assume that assumption* (3) *holds for a redundant system of type* $(i/2/2)$. *If the random variable* τ *has the exponential distribution, that is, if* $F(x) = 1-\exp(-\lambda x)$, $x \geq 0$, *then the limits* (1) *exist and*

$$p_0 = \frac{2}{2+2\rho+\rho^2}, \qquad p_1 = \frac{2\rho}{2+2\rho+\rho^2}, \qquad p_2 = \frac{\rho^2}{2+2\rho+\rho^2}, \tag{14}$$

where $\rho = \lambda b$.

The moments $S_0, S_1, S_2, \dots$ *when the system returns to the state* 0 *are regeneration moments and the mean sojourn time in the states during a single regeneration period is given by*

$$\mathsf{E}\,\overline{\alpha}_0 = \frac{1}{\lambda}, \qquad \mathsf{E}\,\overline{\alpha}_1 = \frac{\rho}{\lambda}, \qquad \mathsf{E}\,\overline{\alpha}_2 = \frac{\rho^2}{2\lambda}. \tag{15}$$

Theorem 2 will be derived below from the results of the next section.

3. Active redundant reserve

First we consider the case of unbounded renewal. Hence we consider a redundant system of type $(ii/2/2)$.

Theorem 3. *If condition* (3) *holds for a redundant system of type* $(ii/2/2)$, *then the limits* (2) *exist and*

$$p_0^* = \frac{1}{(1+\rho)^2}, \qquad p_1^* = \frac{2\rho}{(1+\rho)^2}, \qquad p_2^* = \frac{\rho^2}{(1+\rho)^2}, \tag{16}$$

where $\rho = b/a$.

Remark 5. It follows from the results of [10] for a system of type $(ii/2/2)$ that the mean duration of a busy period and that of the idle period for such a system in the stationary regime are given by

$$\lim_{k\to\infty} \mathsf{E}\, W_k' = \frac{b}{2}\left(\frac{(a+b)^2}{b^2} - 1\right) \quad \text{and} \quad \lim_{k\to\infty} \mathsf{E}\, W_k'' = \frac{b}{2},$$

respectively.

Remark 6. If the random variable τ has the exponential distribution, $F(x) = 1 - \exp(-\lambda x)$, then relations (14) and (16) with $\rho = \lambda b$ follow from [12] in the case of redundant systems of types $(i/2/2)$ and $(ii/2/2)$.

Theorem 4. *Let condition* (3) *hold for a redundant system of type* $(ii/2/2)$. *If the random variable* τ *has the exponential distribution,* $F(x) = 1 - \exp(-\lambda x)$, $x \ge 0$, *then the limits* (1) *and* (2) *exist and the stationary probabilities* $p_k = p_k^*$ *are given by equalities* (16) *with* $\rho = \lambda b$.

The moments $S_0, S_1, S_2, \dots$ *when the system returns to the state* 0 *are regeneration moments and the mean sojourn time in the states during a single regeneration period is given by*

$$\mathsf{E}\,\overline{\alpha}_0 = \frac{1}{2\lambda}, \qquad \mathsf{E}\,\overline{\alpha}_1 = \frac{\rho}{\lambda}, \qquad \mathsf{E}\,\overline{\alpha}_2 = \frac{\rho^2}{2\lambda}. \tag{17}$$

The proof of the above theorems relies on the following auxiliary result.

Let (ξ_i', ξ_i'') be an alternating renewal process (see [14, 15]). Note that (ξ_i', ξ_i'') is a renewal process with a finite renewal time in the language of the book [1]. Usually ξ_i' are treated as a failure-free time of a certain unit and ξ_i'' as its renewal time.

Let $\xi(t) = 1$ if the unit is functioning properly at the moment t, otherwise $\xi(t) = 0$. Then

$$\mathsf{E}\,\xi_i' = a, \qquad \mathsf{E}\,\xi_i'' = b, \qquad \rho = \frac{b}{a}, \qquad v = \frac{\rho}{1+\rho}.$$

Let a certain system contain n units and transitions between the states for an i^{th} unit described by an alternating renewal process $\xi_i(t)$, where $(\xi_i(t), i = 1, 2, \dots, n)$ are independent copies of $\xi(t)$. Further let $S(t) = \sum_{i=1}^n (1 - \xi_i(t))$ be the number of units in the system that are in the renewal mode at the moment t.

Lemma 2. *If assumption* (3) *holds for the random variables* (ξ_i', ξ_i''), *then*

$$\lim_{t\to\infty} \mathsf{P}(S(t) = k) = \frac{n!}{k!\,(n-k)!} v^k (1-v)^{n-k}$$

for all $k = 0, 1, 2, \dots, n$.

Proof of Lemma 2. Put $v(t) = \mathsf{P}(\xi(t) = 0)$. At a fixed moment of time t, the system is described by a scheme of independent Bernoulli trials. Therefore

$$\mathsf{P}(S(t) = k) = \frac{n!}{k!\,(n-k)!} v(t)^k (1 - v(t))^{n-k}. \tag{18}$$

It is well known (see [1]) that, the limit

$$\lim_{t\to\infty} v(t) = v = \frac{b}{a+b} \tag{19}$$

exists. Now Lemma 2 follows from (18) and (19). □

To prove Theorem 3 we note that a redundant system of type $(ii/2/2)$ is equivalent to a system consisting of $n = 2$ independent alternating renewal processes.

Indeed, the principles of functioning and basic characteristics of the system do not change if the primary unit is repaired all the time by the repairing unit labeled with 1, while the redundant unit is always repaired by the unit labeled with 2. Then the functioning of every such a pair (primary unit–repairing unit) is described by an alternating renewal process. Moreover, all these processes are independent and identically distributed.

Therefore Theorem 3 follows directly from Lemma 2.

It is clear that the observation mentioned above holds for a general redundant system with renewal of type $(ii/n/n)$. Namely, such a system is equivalent to a system consisting of n independent alternating renewal processes. Therefore the stationary probabilities for such a system can be obtained from Lemma 2.

Proof of Theorem 4. Denote by τ and τ' the trouble-free operating time for the primary and redundant units, respectively, and let (τ_i) and (τ_i') be their independent copies.

Let $S_0, S_1, S_2, \dots$ be sequential moments of hitting the state 0. In view of the memoryless property of the exponential distribution we conclude that the distribution of the residual trouble-free time for the main and redundant units at moment S_k is the same as that of (τ_0, τ_0') at moment $S_0 = 0$ and it does not depend on the prehistory. This, in fact, means that the period (S_k, S_{k+1}) is equivalent to the period (S_0, S_1) and is independent of the latter one. Hence $S_0, S_1, S_2, \dots$ are the generation moments of the system.

Let $T_1 = S_1 - S_0 = S_1$ be the duration of the first regeneration period. As above, denote by $\overline{\alpha}_k$, $k = 0, 1, 2$, the sojourn time in the state k during the first regeneration period. According to Remark 1, Lemma 1, and equalities (16) of Theorem 3 we have

$$\begin{aligned} \frac{\mathsf{E}\,\overline{\alpha}_0}{\mathsf{E}\,T_1} &= p_0 = p_0^* = \frac{1}{(1+\rho)^2}, \\ \frac{\mathsf{E}\,\overline{\alpha}_1}{\mathsf{E}\,T_1} &= p_1 = p_1^* = \frac{2\rho}{(1+\rho)^2}, \\ \frac{\mathsf{E}\,\overline{\alpha}_2}{\mathsf{E}\,T_1} &= p_2 = p_2^* = \frac{\rho^2}{(1+\rho)^2}. \end{aligned} \tag{20}$$

The mean sojourn time in the state 0 during the first regeneration period is easy to evaluate,

$$\mathsf{E}\,\overline{\alpha}_0 = \mathsf{E}\min(\tau_0, \tau_0') = \frac{1}{2\lambda}.$$

Combining the latter equality and relation (20) we obtain

$$\mathsf{E}\,T_1 = \frac{(1+\rho)^2}{2\lambda}, \qquad \mathsf{E}\,\overline{\alpha}_1 = \frac{\rho}{\lambda}, \qquad \mathsf{E}\,\overline{\alpha}_2 = \frac{\rho^2}{2\lambda}.$$

This yields relations (17). □

The following result holds for the case of bounded renewal.

Theorem 5. *If condition* (3) *holds for a redundant system of type* (*ii*/2/1) *and the random variable τ has the exponential distribution, $F(x) = 1 - \exp(-\lambda x)$, $x \geq 0$, then the stationary probabilities* (1) *exist and moreover*

$$p_0 = \frac{1-q}{1-q+2\rho}, \qquad p_1 = \frac{2q}{1-q+2\rho}, \qquad p_2 = \frac{2(\rho-q)}{1-q+2\rho}.$$

The mean duration of the k^{th} busy period and that of the idle period are given by

$$\mathsf{E}\, W_k' = \frac{1+q}{2\lambda q}, \qquad \mathsf{E}\, W_k'' = \frac{\rho - q}{\lambda q},$$

for $k \geq 2$, where $\rho = \lambda b$.

Proof of Theorem 5 is given in the next section.

4. Partially energized standby redundancy

We start with the case of unbounded renewal. Theretofore we consider a redundant system of type (*iii*/2/2). Assume that τ, the trouble-free operating time of the primary unit, and τ', the trouble-free operating time of the reserved unit, have the exponential distribution,

$$\mathsf{P}(\tau < x) = 1 - e^{-\lambda x}, \qquad \mathsf{P}(\tau' < x) = 1 - e^{-\lambda' x}, \qquad x \geq 0. \tag{21}$$

Theorem 6. *Assume that conditions* (3) *and* (21) *hold for a redundant system of type* (*iii*/2/2). *Then the limits* (1) *exist and*

$$\begin{aligned} p_0 &= \frac{2\lambda}{2\lambda + (\lambda+\lambda')(2\rho+\rho^2)}, \\ p_1 &= \frac{2(\lambda+\lambda')\rho}{2\lambda + (\lambda+\lambda')(2\rho+\rho^2)}, \\ p_2 &= \frac{(\lambda+\lambda')\rho^2}{2\lambda + (\lambda+\lambda')(2\rho+\rho^2)}. \end{aligned} \tag{22}$$

The moments of return to the state 0 *are regeneration moments and the mean sojourn time in the states on a single regeneration period is given by*

$$\mathsf{E}\,\overline{\alpha}_0 = \frac{1}{\lambda+\lambda'}, \qquad \mathsf{E}\,\overline{\alpha}_1 = \frac{\rho}{\lambda}, \qquad \mathsf{E}\,\overline{\alpha}_2 = \frac{\rho^2}{2\lambda}, \tag{23}$$

where $\rho = \lambda b$.

Proof of Theorem 6. The regeneration moments for a system of type (*iii*/2/2) are introduced in the same way as those in the preceding case, namely these are hitting moments for the state 0. Now we denote these moments by $S_0^* = 0, S_1^*, S_2^*, \dots$. Then we compare the first regeneration periods $(0, S_1^*)$ and $(0, S_1)$ for systems of types (*iii*/2/2) and (*ii*/2/2), respectively.

Recall that the distribution functions of the random variables τ and τ' are defined by equalities (21) for systems of type (*iii*/2/2). Moreover, the random variables τ and τ' have the same distribution $F(x) = 1 - e^{-\lambda x}$, $x \geq 0$, in the case of a system of type (*ii*/2/2).

It is clear that the first transition from the state 0 to the state 1 on $(0, S_1^*)$ occurs at the moment $t_0^* = \min(\tau_0^*, \tau_0^{*\prime})$, while that on $(0, S_1)$ happens at the moment $t_0 = \min(\tau_0, \tau_0')$.

The further behavior of the system (*iii*/2/2) on the interval (t_0^*, S_1^*) and that of the system (*ii*/2/2) on the interval (t_0, S_1) are equivalent in the probabilistic sense, since the difference between these systems is seen only if they stay at the state 0.

Let $\overline{\alpha}_k^*$ and $\overline{\alpha}_k$ be the sojourn time at the state k of systems $(iii/2/2)$ and $(ii/2/2)$, respectively, during the first regeneration periods $(0, S_1^*)$ and $(0, S_1)$, $k = 0, 1, 2$. Then $\overline{\alpha}_k^*$ and $\overline{\alpha}_k$, $k = 1, 2$, are identically distributed and

$$\mathsf{E}\,\overline{\alpha}_1^* = \mathsf{E}\,\overline{\alpha}_1 = \frac{\rho}{\lambda}, \qquad \mathsf{E}\,\overline{\alpha}_2^* = \mathsf{E}\,\overline{\alpha}_2 = \frac{\rho^2}{2\lambda}, \tag{24}$$

by equality (17).

It is clear that $\overline{\alpha}_0^* = t_0^*$ and

$$\mathsf{E}\,\overline{\alpha}_0^* = \mathsf{E}\min(\tau_0^*, \tau_0^{*\prime}) = \frac{1}{\lambda + \lambda'}. \tag{25}$$

Equalities (24) and (25) allow us to evaluate the mean duration of the regeneration interval for systems of type $(iii/2/2)$, namely

$$\mathsf{E}\,T_1^* = \mathsf{E}\,\overline{\alpha}_0^* + \mathsf{E}\,\overline{\alpha}_1^* + \mathsf{E}\,\overline{\alpha}_2^* = \frac{2\lambda + (\lambda + \lambda')(2\rho + \rho^2)}{2\lambda(\lambda + \lambda')}.$$

It remains to apply Lemma 1 and evaluate the stationary probabilities (22) with the help of the following formula:

$$p_k = \frac{\mathsf{E}\,\overline{\alpha}_k^*}{\mathsf{E}\,T_1^*}, \qquad k = 0, 1, 2. \qquad \square$$

Remark 7. It is clear that the case of completely inactive standby redundancy is a partial case of partially energized standby redundancy. Thus Theorem 2 is a corollary of Theorem 6 for $\lambda' = 0$.

Finally, we turn to the case of bounded renewal and consider the redundant system $(iii/2/1)$.

Theorem 7. *If conditions* (3) *and* (21) *hold for a redundant system of type* $(iii/2/1)$, *then the limits* (1) *exist and*

$$\begin{aligned} p_0 &= \frac{1 - q}{1 - q + b(\lambda + \lambda')}, \qquad p_1 = \frac{q(\lambda + \lambda')}{\lambda\big[1 - q + b(\lambda + \lambda')\big]}, \\ p_2 &= \frac{(\lambda b - q)(\lambda + \lambda')}{\lambda\big[1 - q + b(\lambda + \lambda')\big]}. \end{aligned} \tag{26}$$

The mean duration of the k^{th} *operating period and that of idle period are given by*

$$\mathsf{E}\,W_k' = \frac{\lambda + q\lambda'}{\lambda(\lambda + \lambda')q} \quad \text{and} \quad \mathsf{E}\,W_k'' = \frac{\lambda b - q}{\lambda q} \tag{27}$$

for $k \geq 2$, *respectively.*

Proof of Theorem 7. The regeneration periods of the system are introduced as in the proof of Theorem 1. Namely, S_0 is the first failure moment of one of the units, S_i is the i^{th} moment when the repair is finished for a unit, while the repair of the other one is started, $i = 1, 2, \ldots$.

Similarly to the proof of Theorem 1, we introduce the random events $A_n = \{\tau_n < \eta_n\}$ and random variables $\varepsilon_n = I(A_n)$ and $\nu = \min(n \geq 1 : \ \varepsilon_n = 1)$.

Let $\widetilde{\tau}_i = \min(\tau_i^*, \tau_i^{*\prime})$, where τ_i^* and $\tau_i^{*\prime}$, $i = 1, 2, \ldots, \nu - 1$, are the residual times of the primary and redundant units after the i^{th} renewal in the first regeneration period.

The first regeneration period of a system of type $(iii/2/1)$ is depicted below:

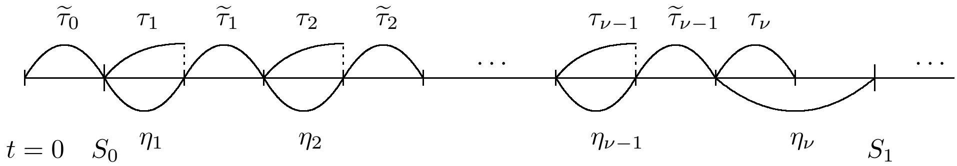

Figure 2

If $\overline{\alpha}_k$ is the sojourn time in the state k, $k=0,1,2$, in the interval (S_0,S_1), then we conclude from Figure 2 that

$$\overline{\alpha}_0=\sum_{i=1}^{\nu-1}\widetilde{\tau}_i,\quad \overline{\alpha}_1=\sum_{i=1}^{\nu}\min(\tau_i,\eta_i),$$
$$T_1=S_1-S_0=\sum_{i=1}^{\nu}\eta_i+\sum_{i=1}^{\nu-1}\widetilde{\tau}_i, \tag{28}$$
$$\overline{\alpha}_2=T_1-\overline{\alpha}_0-\overline{\alpha}_1.$$

We use the memory-less property of the exponential distribution once more to prove that the sequence $(\widetilde{\tau}_1,\widetilde{\tau}_2,\dots)$ does not depend on the sequence

$$\big((\tau_1,\eta_1),(\tau_2,\eta_2),\dots,(\tau_\nu,\eta_\nu)\big).$$

The sequence $(\widetilde{\tau}_1,\widetilde{\tau}_2,\dots)$ does not depend on ν, as well.

Now we pass to the mathematical expectations in equalities (28):

$$\mathsf{E}\,\overline{\alpha}_0=\mathsf{E}(\nu-1)\,\mathsf{E}\,\widetilde{\tau}_i=\left(\frac{1}{q}-1\right)\frac{1}{\lambda+\lambda'}=\frac{1-q}{q(\lambda+\lambda')},$$
$$\mathsf{E}\,\overline{\alpha}_1=\mathsf{E}\,\nu\,\mathsf{E}\min(\tau,\eta)=\frac{d}{q}=\frac{1}{\lambda},$$
$$\mathsf{E}\,T_1=\mathsf{E}\,\nu\,\mathsf{E}\,\eta+\mathsf{E}(\nu-1)\,\mathsf{E}\,\widetilde{\tau}_1=\frac{b}{q}+\frac{1-q}{q(\lambda+\lambda')}, \tag{29}$$
$$\mathsf{E}\,\overline{\alpha}_2=\mathsf{E}\,\nu\,\mathsf{E}\,\eta-\mathsf{E}\,\nu\,\mathsf{E}\min(\tau,\eta)=\frac{b}{q}-\frac{1}{\lambda}.$$

It remains to use relation (5) of Lemma 1 to obtain equalities (26). It is also clear that (27) follows from (29). □

Remark 8. Since the case of partially energized standby redundancy for $\lambda'=\lambda$ becomes the case of active redundancy, Theorem 5 is a simple corollary of Theorem 7.

Remark 9. Denote by $\chi(t)$ the total number of changes of states in a redundant system during the interval $(0,t)$ and by $\chi_k(t)$ the number of those changes when the system hits the state k, $k=0,1,2$. It is clear that

$$\chi(t)=\chi_0(t)+\chi_1(t)+\chi_2(t).$$

Analyzing the proofs of Theorems 1, 5, and 7 we see that

$$\lim_{t\to\infty}\frac{\chi_k(t)}{\chi(t)}=p_k'\quad\text{almost surely}$$

for $k=0,1,2$. Moreover

$$p_0'=\frac{1-q}{2},\qquad p_1'=\frac{1}{2},\qquad p_2'=\frac{q}{2}. \tag{30}$$

Consider a redundant system of type $(i/2/1)$. Let $\zeta_k = \zeta(t_k + 0)$, $t_0 = 0, t_1, t_2, \dots$, be the moments when the system changes its state. Then the sequence (ζ_k) is a Markov chain with the transition matrix

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 1-q & 0 & q \\ 0 & 1 & 0 \end{pmatrix}.$$

The equations for the stationary probabilities of a Markov chain

$$p'_k = \sum_i p'_i p_{ik}, \qquad k = 0, 1, 2,$$

allow us to evaluate p'_0, p'_1, and p'_2. These probabilities can also be derived from equalities (30).

It is well known (see, for example, [18]) that the stationary probabilities of states coincide with the stationary probabilities of the embedded Markov chain in the case of a single channel queueing system with unbounded queue and Poissonian flow of customers. Comparing relations (4) and (30), it becomes clear that the stationary probabilities of a redundant system with renewal can essentially be different from the stationary probabilities of the embedded Markov chain even if the random variables τ_i and η_i have the exponential distributions.

Bibliography

[1] B. V. Gnedenko, Yu. K. Belyayev, and A. D. Solovyev, *Mathematical Methods of Reliability Theory*, "Nauka", Moscow, 1965; English transl. Scripta Technica, Inc. *Probability and Mathematical Statistics*, Translation edited by Richard E. Barlow, vol. 6, Academic Press, New York–London, 1969. MR0345234

[2] B. Epstein and T. Hosford, *Reliability of some two units redundant systems*, Proc. 6-th Nat. Symp. on Reliability and Quality Control, 1960, pp. 466–476.

[3] B. V. Gnedenko, *On a two-unit redundant system*, Izv. Akad. Nauk SSSR Tehn. Kibernet. **4** (1964), 3–12. (Russian) MR0191630

[4] B. V. Gnedenko, *Doubling with renewal*, Izv. Akad. Nauk SSSR Tehn. Kibernet. **5** (1964), 111–118. (Russian) MR0173541

[5] I. N. Kovalenko, *Studies in the Reliability Analysis of Complex Systems*, "Naukova Dumka", Kiev, 1975. (Russian) MR0451435

[6] I. N. Kovalenko and N. Yu. Kuznetsov, *Modelling highly reliable systems*, Reliability of Technical Systems (I. A. Ushakov, ed.), "Radio i svyaz", Moscow, 1985. (Russian)

[7] V. S. Korolyuk and A. F. Turbin, *Semi-Markov Processes and their Applications*, "Naukova Dumka", Kiev, 1976. (Russian) MR0420902

[8] V. S. Korolyuk and A. F. Turbin, *Markov Renewal Processes in Problems of the Reliability of Systems*, "Naukova Dumka", Kiev, 1982. (Russian) MR695006

[9] V. S. Korolyuk, *Stochastic Models of Systems*, "Naukova Dumka", Kiev, 1989. (Russian) MR1001967

[10] A. D. Solovyev, *Reliability of systems with renewal*, Cybernetics for Service to Communism, vol. 2, "Energiya", Moscow, 1964. (Russian)

[11] A. D. Solovyev, *Methods for evaluation of reliability of systems with renewal*, Reliability of Technical Systems (I. A. Ushakov, ed.), "Radio i svyaz", Moscow, 1985. (Russian)

[12] T. P. Maryanovich, *Some problems of reliability of systems with reserves*, Ukrain. Mat. Zh. **15** (1963), no. 2, 145–157. (Russian) MR0158758

[13] I. K. Matsak, *On a single-server queueing system with refusal*, Teor. Imovirnost. Matem. Statyst. **90** (2014), 136–142; English transl. in Theor. Probability and Math. Statist. **90** (2015), 153–160. MR3242027

[14] D. R. Cox, *Renewal Theory*, Methuen & Co. Ltd., London; John Wiley & Sons, Inc., New York, 1962. MR0153061

[15] Walter L. Smith, *Renewal theory and its ramifications*, J. Roy. Statist. Soc. Ser. **B 20** (1958), 243–302. MR0099090

[16] I. I. Gikhman, A. V. Skorokhod, and M. I. Yadrenko, *Theory of Probability and Mathematical Statistics*, "Vyshcha shkola", Kiev, 1988. (Russian)

[17] A. N. Kolmogorov, *Probability Theory and Mathematical Statistics. Selected Works*, "Nauka", Moscow, 1986. (Russian) MR861120
[18] V. V. Anisimov, O. K. Zakusylo, and V. S. Donchenko, *Elements of the Queueing Theory and Asymptotic Analysis of Systems*, "Vyshcha shkola", Kiev, 1987. (Russian)

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