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ASYMPTOTICS FOR TIME-CHANGED DIFFUSIONS

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Abstract. We consider time-changed diffusions driven by generators with discontinuous coefficients. The PDE's connections are investigated, and in particular some results on the asymptotic analysis according to the behaviour of the coefficients are presented.

1. Introduction

We consider the balls $\Omega_l \subset \Omega_\ell \subset \Omega_r$ centered at the same point where Ω_q is a ball with radius $q = l, \ell, r$, where $r = \ell + \varepsilon$ and $\varepsilon > 0$. Let us introduce two independent Brownian motions B^1, B^2 and the process (see the generator (4.9) below)

$$B_t^{(\alpha,\lambda)} := \begin{cases} B_t^1, & \text{on } \Omega_\ell \setminus \overline{\Omega_l}, \\ B_{\lambda t}^2, & \text{on } \Sigma_\varepsilon = \Omega_r \setminus \overline{\Omega_\ell}, \end{cases}$$

with skew condition on $\partial\Omega_\ell$:

$$\forall\, x \in \partial\Omega_\ell, \qquad \mathbb{P}_x(B_t^{(\alpha,\lambda)} \in \Omega_\ell \setminus \overline{\Omega_l}) = 1-\alpha \quad \text{and} \quad \mathbb{P}_x\big(B_t^{(\alpha,\lambda)} \in \Sigma_\varepsilon\big) = \alpha.$$

Moreover, $B^{(\alpha,\lambda)}$ is killed on $\partial\Omega_l$ and $\partial\Omega_r$. Since we have different variances depending on $\lambda > 0$, we refer to the process $B^{(\alpha,\lambda)}$ as a modified process. Obviously, $\alpha \in (0,1)$ is the skewness parameter and $B^{(\alpha,\lambda)}$ is called the skew process. We write $\alpha = \alpha_\varepsilon$, $\lambda = \lambda_\varepsilon$ by underling the dependence from ε ($\alpha_\varepsilon \to 0$ and $\lambda_\varepsilon \to 0$ as $\varepsilon \to 0$) and consider the collapsing domain $\Sigma_\varepsilon \subset \mathbb{R}^2$ (that is, with vanishing thickness ε).

Our aim is to study a killed diffusion on $\Omega_r \setminus \overline{\Omega_l}$ with skew condition on $\partial\Omega_\ell$ and different behaviour in $\Sigma_\varepsilon = \Omega_r \setminus \overline{\Omega_\ell}$ and $\Omega_\ell \setminus \overline{\Omega_l}$ under the assumption

$$\lim_{\varepsilon\to 0} \frac{\alpha\varepsilon}{\lambda} = 0. \tag{1.1}$$

For the classical case $\alpha = \lambda$ which has been extensively investigated in literature (see for example [2,3] and the references therein), the condition (1.1) becomes trivial.

Our new result is given by the asymptotic analysis obtained under (1.1) with $\alpha \neq \lambda$.

We first consider time-changed Brownian motions on the line. After that we pass to Brownian motion on balls and the associated skew-product representations (R, Θ). In particular, we study the time-changed Bessel process, and the winding number Θ can be neglected by exploiting the isotropy of the Brownian diffusions. Thus, we focus on the radial parts R which are driven by Markov generators with representation given in terms of scale functions and speed measures. The eigenfunction expansions of such generators of these Markov processes and their associated semigroups can be explicitly computed

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as solutions to the corresponding Sturm–Liouville problems; see for example [9, 14, 15]. In such works, the authors considered the approach based on scale functions and speed measures as in the present paper. However, the results obtained in the above-mentioned works concern diffusions on bounded domains with no collapsing subsets.

We obtain the following result.

Theorem 1.1. *Let X_t be a reflecting Brownian motion on $\overline{\Omega_\ell} \setminus \overline{\Omega_l}$ with boundary local time*

$$L_t^{\partial\Omega_\ell}(X) := \int_0^t \mathbf{1}_{\partial\Omega_\ell}(X_s)\, ds.$$

Under (1.1)*, we have that*

$$\begin{aligned}\lim_{\varepsilon\to 0} \mathbb{E}_x &\left[\int_0^\infty f\left(B_t^{(\alpha,\lambda)}\right) M_t^\varepsilon\, dt\right] \\ &= \mathbb{E}_x\left[\int_0^{\tau_l} f(X_t) \exp\left(-\left(\lim_{\varepsilon\to 0}\frac{\alpha}{(1-\alpha)\varepsilon}\right) L_t^{\partial\Omega_\ell}(X)\right) dt\right]\end{aligned}$$

where $M_t^\varepsilon := \mathbf{1}_{(t<\tau_\varepsilon)}$ with

$$\tau_\varepsilon := \inf\left\{s>0\colon B_s^{(\alpha,\lambda)} \notin \Omega_r \setminus \overline{\Omega_l}\right\} \quad \textit{and} \quad \tau_l = \inf\{s>0\colon X_s \in \partial\Omega_l\}.$$

An interesting connection is related to a conjecture of Feller. An elastic Brownian motion on $[0,\infty)$ with condition $\alpha u(0) = (1-\alpha)u'(0)$, $\alpha \in (0,1)$, is identical in law to a reflecting Brownian motion B^+ killed according to the conditional law

$$\mathbb{P}(T>t \mid B^+) = \exp\left(-\frac{\alpha}{1-\alpha} L_t^{\partial\Omega_\ell}\right).$$

We notice that the special cases $\alpha = 1$ or $\alpha = 0$ correspond to Dirichlet or Neumann conditions. For example, for $\alpha = 1$, the process $B^{(1,\lambda)}$ moves on $\Omega_r \setminus \Omega_\ell$ collapsing to $\partial\Omega_\ell$ as the thickness $\varepsilon \to 0$. Thus, we consider the only starting point $x \in \partial\Omega_\ell$. On the other hand, X_t moves on $\overline{\Omega_\ell}$. If X_t is forced to start at $x \in \partial\Omega_\ell$, the local time is positive and the right-hand side of (1.1) equals 0, that is, we have the Dirichlet condition on $\partial\Omega_\ell$. The Neumann boundary condition follows immediately by considering $B^{(0,\lambda)} \in \overline{\Omega_\ell} \setminus \overline{\Omega_l}$ and the right-hand side of (1.1) for $\alpha = 0$.

The plan of the work is the following. In Section 2 we give some preliminaries about the characterization of one-dimensional diffusions by means of scale functions and speed measures. In Section 3 we study the one-dimensional Brownian motion by exploiting the technique based on the corresponding scale function and speed measure. We provide some explicit results about the mean exit time. In Section 4 we study the two-dimensional Brownian motion by skew-product representation. and therefore we apply the technique introduced in Section 2 by considering the associated Bessel process. We provide some results on the asymptotic behaviour of the mean exit time. In Section 5 we give the proof of Theorem 1.1. Finally, in the last section we mention some possible extensions of our results to irregular domains and delayed diffusions.

2. Preliminaries

A one-dimensional stochastic process Y_t, $t \geq 0$, can be characterized by means of the scale function $S(\cdot)$ and the speed measure $M(\cdot)$ (see for example [11]). For a one-dimensional diffusion with generator

$$A = \frac{\rho(x)}{2}\frac{d}{dx}\left(a(x)\frac{d}{dx}\right) + b(x)\frac{d}{dx}, \tag{2.1}$$

we introduce

$$\mathfrak{s}(x) = S'(x) = \frac{1}{a(x)} \exp\left(-2\int^x \frac{b(\eta)}{\rho(\eta)a(\eta)}\,d\eta\right) \tag{2.2}$$

and the speed density

$$\mathfrak{m}(x) = (\rho(x)a(x)\mathfrak{s}(x))^{-1} \tag{2.3}$$

such that

$$\mathbb{P}_x(Y_t \in \cdot) = \int_{\cdot} K(t,x,y)\,\mathfrak{m}(dy)$$

and $\partial_t K(t,x,\cdot) = AK(t,x,\cdot)$. The corresponding integrals are therefore written as

$$S(x) = \int^x \mathfrak{s}(\eta)\,d\eta$$

for the scale function S and

$$M(x) = \int^x \mathfrak{m}(\eta)\,d\eta$$

for the speed measure M. We also write

$$M(l,x] = M(x) - M(l) \quad \text{and} \quad S(l,x] = S(x) - S(l)$$

in order to maintain the notation in [11]. Moreover, we notice that, from (2.2),

$$AS = \frac{\rho(x)}{2}\frac{d}{dx}\left(a(x)\frac{dS}{dx}\right) + b(x)\frac{dS}{dx} = \frac{\rho(x)}{2}\frac{d}{dx}\left(a(x)\mathfrak{s}(x)\right) + b(x)\mathfrak{s}(x) = 0.$$

Let B_t, $t \geq 0$, be a Brownian motion on the real line and denote by

$$P_t f(x) = \mathbb{E}_x f(B_t) = \mathbb{E} f(x + B_t) = \int_{\mathbb{R}} f(y)K(t,x,y)\,dy$$

its semigroup (with $K(t,x,y) = K(t,y-x)$, the heat kernel). Let B^α_t, $t \geq 0$, be a skew Brownian motion on the real line with $\alpha \in (0,1)$. A deep discussion about skew Brownian motion can be found in [13] and the references therein. The reader can also consult the interesting papers [10,18,19]. The skew Brownian motion is the solution to $dB^\alpha_t = \sigma\,dB_t + (2\alpha-1)dL^{\{\ell\}}_t$, where B is a standard Brownian motion and the symmetric local time (at $\ell \in \mathbb{R}$) of B^α is given by

$$L^{\{\ell\}}_t = \frac{1}{2}\left(L^{\ell^-}_t + L^{\ell^+}_t\right) \tag{2.4}$$

(the time B^α spends at ℓ up to time t), where

$$L^{\ell^-}_t = 2(1-\alpha)L^{\{\ell\}}_t, \qquad L^{\ell^+}_t = 2\alpha L^{\{\ell\}}_t. \tag{2.5}$$

We refer to $L^{\ell^-}_t$ and $L^{\ell^+}_t$, respectively, as left and right local time. To be precise,

$$L^{\ell^-}_t = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^t \mathbf{1}_{(\ell-\varepsilon,\ell]}(B^\alpha_s)\,ds \quad \text{and} \quad L^{\ell^+}_t = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^t \mathbf{1}_{[\ell,\ell+\varepsilon)}(B^\alpha_s)\,ds.$$

Thus, $L^{\{\ell\}}_t = \lim_{\varepsilon\to 0}(2\varepsilon)^{-1}\int_0^t \mathbf{1}_{(\ell-\varepsilon,\ell+\varepsilon)}(B^\alpha_s)\,ds$. Let $u(x,t) = \mathbb{E}_x[f(B^\alpha_t)]$ (see for example [1,21] for the explicit representation of the kernel K^α for B^α). Then, u is the solution to

$$\partial_t u(x,t) = \frac{\sigma^2}{2}\partial_{xx}u(x,t), \qquad x \in \mathbb{R},\ t > 0, \tag{2.6}$$

subject to

$$u(\ell^-) = u(\ell^+), \tag{2.7}$$

$$(1-\alpha)u'(\ell^-) = \alpha u'(\ell^+) \tag{2.8}$$

with initial datum f. The infinitesimal generator A of B^α is given by

$$Au = \frac{\sigma^2}{2a(x)}\frac{d}{dx}\left(a(x)\frac{d}{dx}u\right), \tag{2.9}$$

where, for $\alpha \in (0,1)$,

$$a(x) = (1-\alpha)\mathbf{1}_{(-\infty,\ell]}(x) + \alpha\mathbf{1}_{(\ell,+\infty)}(x) = c_2\exp\left(-c_1\mathbf{1}_{(-\infty,\ell]}(x)\right) \tag{2.10}$$

and $c_1 = \ln(\alpha/(1-\alpha))$, $c_2 = \alpha$. We evidently have that

$$\mathbb{P}_x(B_t^\alpha \le \ell) = 1-\alpha \quad \text{and} \quad \mathbb{P}_x(B_t^\alpha > \ell) = \alpha.$$

3. The modified skew Brownian motion on the line

3.1. The process $B_t^{(\alpha,\lambda)}$. We focus on the process driven by the generator (2.1) with $b(\cdot) = 0$ and coefficients

$$a(x) = (1-\alpha)\mathbf{1}_{(0,\ell]}(x) + \alpha\mathbf{1}_{(\ell,\infty)}(x), \qquad \alpha \in (0,1) \tag{3.1}$$

and

$$\rho(x)a(x) = \sigma^2(x) = \mathbf{1}_{(0,\ell]}(x) + \lambda\mathbf{1}_{(\ell,\infty)}(x), \qquad \lambda > 0. \tag{3.2}$$

Let us consider the interval (l,r), where $r = \ell+\varepsilon$ with $\varepsilon \ge 0$ and $0 < l < \ell < r$. We denote by $B_t^{(\alpha,\lambda)}$, $t > 0$, the corresponding one-dimensional diffusion in (l,r). The process $B^{(\alpha,\lambda)}$ moves like a skew Brownian motion with reflecting barrier at ℓ and different variances inside (l,r) (it respectively moves like the standard Brownian motion B on (l,ℓ) and the Brownian motion $\sqrt{\lambda}B$ on (ℓ,r)). The probability law u of $B^{(\alpha,\lambda)}$ is the solution to

$$\partial_t u = Au = \begin{cases} \frac{1}{2a(x)}\partial_x(a(x)\partial_x u) & \text{in } (l,\ell), \\ \frac{\lambda}{2a(x)}\partial_x(a(x)\partial_x u) & \text{in } (\ell,r) \end{cases} \tag{3.3}$$

subject to

$$u(l) = 0, \tag{3.4}$$

$$u(r) = 0, \tag{3.5}$$

$$u(\ell^-) = u(\ell^+), \tag{3.6}$$

$$(1-\alpha)u'(\ell^-) = \alpha u'(\ell^+). \tag{3.7}$$

Remark 3.1. Notice that, if $\lambda = \alpha/(1-\alpha)$, then

$$\rho(x) = \frac{\sigma^2(x)}{a(x)} = \frac{1}{1-\alpha},$$

and therefore

$$Au = \frac{1}{2}\frac{d}{dx}\left(\sigma^2(x)\frac{d}{dx}u\right).$$

3.2. Scale function and speed measure. We define the scale function and the speed measure for the process $B_t^{(\alpha,\lambda)}$, $t > 0$, previously introduced. For the sake of simplicity we consider $\ell = 1$. From

$$\mathfrak{s}(x) = \frac{1}{a(x)} = \frac{1}{1-\alpha}\mathbf{1}_{(0,1]}(x) + \frac{1}{\alpha}\mathbf{1}_{(1,\infty)}(x) \tag{3.8}$$

we get the scale function

$$S(x) = \frac{x}{1-\alpha}\mathbf{1}_{(0,1]}(x) + \left(\frac{1}{1-\alpha} + \frac{x-1}{\alpha}\right)\mathbf{1}_{(1,\infty)}(x) \tag{3.9}$$

and recall that

$$S(\eta_1, \eta_2] = S(\eta_2) - S(\eta_1). \tag{3.10}$$

From the speed density

$$\mathfrak{m}(x) = \big(\sigma^2(x)\mathfrak{s}(x)\big)^{-1} = (1-\alpha)\mathbf{1}_{(0,1]}(x) + \frac{\alpha}{\lambda}\mathbf{1}_{(1,\infty)}(x) \tag{3.11}$$

we get the speed measure

$$M(x) = (1-\alpha)x\mathbf{1}_{(0,1]}(x) + \Big((1-\alpha) + \frac{\alpha}{\lambda}(x-1)\Big)\,\mathbf{1}_{(1,\infty)}(x) \tag{3.12}$$

with

$$M(\eta_1, \eta_2] = M(\eta_2) - M(\eta_1). \tag{3.13}$$

Notice that $S(\cdot)$ and $M(\cdot)$ are continuous functions. Moreover, since $dM(x) = \mathfrak{m}(x)\,dx$ and $dS(x) = \mathfrak{s}(x)\,dx$, the operator (2.1) can be written as

$$A = \frac{\sigma^2(x)\mathfrak{s}(x)}{2}\frac{d}{dx}\frac{1}{\mathfrak{s}(x)}\frac{d}{dx} = \frac{1}{2}\frac{d}{dM}\frac{d}{dS} \tag{3.14}$$

(see for example [20, Theorem 3.12]) and $(A, D(A))$, where

$$D(A) = \{g, Ag \in C_b\colon g \text{ satisfies } (3.4), (3.5), (3.6), (3.7)\}$$

is the infinitesimal generator of $B_t^{(\alpha,\lambda)}$, $t > 0$. That is a skew diffusion on (l, r) with no drift (that is, $b = 0$ in (2.1)), with transmission condition at $\ell = 1$ and variances 1 and λ respectively on $(l, 1)$ and $(1, r)$. We recall that $r = 1 + \varepsilon$.

3.3. **Exit time from an interval.** Let $\tau_y = \inf\{s \geq 0\colon B_s^{(\alpha,\lambda)} = y\}$ be the first time the process $B^{(\alpha,\lambda)}$ hits y. We also define

$$\tau_{(l,r)} := \tau_l \wedge \tau_r.$$

The solution to the Cauchy problem (3.3) with initial datum f can be written as follows:

$$P_t^{(\alpha,\lambda)} f(x) = \mathbb{E}_x\left[f\left(B_t^{(\alpha,\lambda)}\right)\mathbf{1}_{(t<\tau_{(l,r)})}\right]. \tag{3.15}$$

We use the fact that the probability function φ_ε defined as

$$\varphi_\varepsilon(x) = \mathbb{P}\left(\tau_r < \tau_l \mid B_0^{(\alpha,\lambda)} = x \in (l,r)\right) = \mathbb{P}_x(\tau_r < \tau_l) \tag{3.16}$$

solves

$$A\varphi_\varepsilon = 0, \tag{3.17}$$

$$\varphi_\varepsilon(l) = 0, \tag{3.18}$$

$$\varphi_\varepsilon(r) = 1, \tag{3.19}$$

$$\varphi_\varepsilon(1^-) = \varphi_\varepsilon(1^+), \tag{3.20}$$

$$(1-\alpha)\varphi_\varepsilon'(1^-) = \alpha\varphi_\varepsilon'(1^+), \tag{3.21}$$

and the problem to find $v_\varepsilon \in D(A)$ such that $Av_\varepsilon = -f$ can be solved by considering the function φ_ε, that is, ([11, page 197])

$$v_\varepsilon(x) = 2\big(1 - \varphi_\varepsilon(x)\big)\Sigma(l, x] + 2\varphi_\varepsilon(x)\Sigma(x, r], \tag{3.22}$$

where

$$\Sigma(l, x] = \int_l^x S(l, \eta] f(\eta)\mathfrak{m}(\eta)\,d\eta \quad \text{and} \quad \Sigma(x, r] = \int_x^r S(\eta, r] f(\eta)\mathfrak{m}(\eta)\,d\eta. \tag{3.23}$$

The probabilistic representation is given by

$$v_\varepsilon(x) = \mathbb{E}_x\left[\int_0^{\tau_{(l,r)}} f\left(B_t^{(\alpha,\lambda)}\right)\, dt\right].$$

We recall that $\alpha = \alpha_\varepsilon$ and $\lambda = \lambda_\varepsilon$. For the sake of simplicity we consider $f \equiv 1$, and the mean exit time

$$v_\varepsilon(x) = \mathbb{E}\left[\tau_{(l,r)} \;\middle|\; B_0^{(\alpha,\lambda)} = x \in (l,r)\right] = \mathbb{E}_x\left[\tau_{(l,r)}\right] \tag{3.24}$$

solves

$$Av_\varepsilon = -1, \qquad v_\varepsilon \in D(A). \tag{3.25}$$

We now obtain the explicit representation of φ_ε and therefore of v_ε. For the probability function (3.16), we get that ([11]) for $0 \le l < 1$ and $r > 1$ (that is the case here, in our analysis),

(3.26)

$$\varphi_\varepsilon(x) = \frac{S(l,x]}{S(l,r]} = \frac{\alpha x \mathbf{1}_{[0,1)}(x) + \big(\alpha + (1-\alpha)(x-1)\big)\mathbf{1}_{(1,1+\varepsilon]}(x) - \alpha l}{(1-\alpha)(r-1) - \alpha(l-1)}, \quad x \in (l,r)$$

$$= \begin{cases} \frac{\alpha x - \alpha l}{(1-\alpha)(r-1)-\alpha(l-1)}, & x \in (l,1],\\ \frac{(1-\alpha)(x-1)-\alpha(l-1)}{(1-\alpha)(r-1)-\alpha(l-1)}, & x \in (1,r). \end{cases}$$

Proposition 3.1. *We have that*

$$v_\varepsilon(x) = \begin{cases} 2\left(\frac{I_1}{I_2} - \frac{1}{\alpha}x\right)(\alpha x - \alpha l) + x^2 + l^2 - 2lx, & x \in (l,1],\\ 2\left(\frac{I_1}{I_2} - \frac{1}{\alpha} - \frac{(x-1)}{\lambda(1-\alpha)}\right)((1-\alpha)(x-1) - \alpha(l-1)) & \\ \quad +1 + l^2 + 2\frac{\alpha}{1-\alpha}\frac{(x-1)}{\lambda} - 2l - \frac{\alpha}{1-\alpha}\frac{2l}{\lambda}(x-1) + \frac{(x-1)^2}{\lambda}, & x \in (1,r), \end{cases} \tag{3.27}$$

where

$$I_1 = \frac{1-\alpha}{\alpha}(r-1) + \frac{(r-1)^2}{2\lambda} + \frac{(1-l^2)}{2}, \qquad I_2 = (1-\alpha)(r-1) - \alpha(l-1).$$

Proof. Consider (3.23) with $f = 1$. Then

$$\Sigma(x,r] = \int_x^r S(\eta, r]\mathfrak{m}(\eta)\, d\eta = S(r)M(x,r] - \int_x^r S(\eta)\mathfrak{m}(\eta)\, d\eta$$

and

$$\Sigma(l,x] = \int_l^x S(l,\eta]\mathfrak{m}(\eta)\, d\eta = \int_l^x S(\eta)\mathfrak{m}(\eta)\, d\eta - S(l)M(l,x],$$

where the last integral can be explicitly written by considering that

$$\begin{aligned}\int_l^x & S(\eta)\mathfrak{m}(\eta)\, d\eta\\ &= \int_l^x S(\eta)\mathfrak{m}(\eta)\, d\eta \mathbf{1}_{(0,1]}(x) + \left(\int_l^1 S(\eta)\mathfrak{m}(\eta)\, d\eta + \int_1^x S(\eta)\mathfrak{m}(\eta)\, d\eta\right)\mathbf{1}_{(1,\infty)}(x)\\ &= \frac{(x^2 - l^2)}{2}\mathbf{1}_{(0,1]}(x) + \left(\frac{(1-l^2)}{2} + \frac{\alpha}{1-\alpha}\frac{(x-1)}{\lambda} + \frac{(x-1)^2}{2\lambda}\right)\mathbf{1}_{(1,\infty)}(x)\end{aligned}$$

and

$$S(l)M(l,x] = -l^2 + lx\mathbf{1}_{(0,1]}(x) + \left(l + \frac{\alpha}{1-\alpha}\frac{l}{\lambda}(x-1)\right)\mathbf{1}_{(1,\infty)}(x).$$

Therefore, we have that

$$\Sigma(l,x] = \left(\frac{x^2+l^2}{2} - lx\right)\mathbf{1}_{(0,1]}(x)$$
$$+\left(\frac{(1+l^2)}{2} + \frac{\alpha}{1-\alpha}\frac{(x-1)}{\lambda} - l - \frac{\alpha}{1-\alpha}\frac{l}{\lambda}(x-1) + \frac{(x-1)^2}{2\lambda}\right)\mathbf{1}_{(1,\infty)}(x).$$

Rewrite (3.22) as follows:

$$v_\varepsilon(x) = 2\varphi_\varepsilon(x)\Big(\Sigma(x,r] - \Sigma(l,x]\Big) + 2\Sigma(l,x], \tag{3.28}$$

where

$$\begin{aligned}
&\Sigma(x,r] - \Sigma(l,x] \\
&\quad = S(r)M(x,r] - \int_l^r S(\eta)\mathfrak{m}(\eta)d\eta + S(l)M(l,x] \\
&\quad = S(r)\left(M(r)-M(x)\right) - \frac{(1-l^2)}{2} - \frac{\alpha}{1-\alpha}\frac{(r-1)}{\lambda} - \frac{(r-1)^2}{2\lambda} + S(l)\left(M(x)-M(l)\right) \\
&\quad = S(r)M(r) - S(l)M(l) - \left(S(r)-S(l)\right)M(x) - \frac{(1-l^2)}{2} - \frac{\alpha}{1-\alpha}\frac{(r-1)}{\lambda} - \frac{(r-1)^2}{2\lambda} \\
&\quad = I_1 - \frac{I_2}{\alpha(1-\alpha)}M(x)
\end{aligned}$$

with $I_2 = (1-\alpha)(r-1) - \alpha(l-1)$, $S(l)M(l) = l^2$ and

$$\begin{aligned}
I_1 &= S(r)M(r) - l^2 - \frac{(1-l^2)}{2} - \frac{\alpha}{1-\alpha}\frac{(r-1)}{\lambda} - \frac{(r-1)^2}{2\lambda} \\
&= \frac{1-\alpha}{\alpha}(r-1) + \frac{(r-1)^2}{2\lambda} + \frac{(1-l^2)}{2}.
\end{aligned}$$

By collecting all pieces together, we obtain

$$\begin{aligned}
v_\varepsilon(x) &= \begin{cases} 2\varphi_\varepsilon(x)\left(I_1 - \frac{I_2}{\alpha}x\right) + x^2 + l^2 - 2lx, & x\in(l,1], \\ 2\varphi_\varepsilon(x)\Big(I_1 - \frac{I_2}{\alpha(1-\alpha)}\left((1-\alpha)+\frac{\alpha}{\lambda}(x-1)\right)\Big) & \\ \quad +1+l^2+2\frac{\alpha}{1-\alpha}\frac{(x-1)}{\lambda} - 2l - \frac{\alpha}{1-\alpha}\frac{2l}{\lambda}(x-1) + \frac{(x-1)^2}{\lambda}, & x\in(1,r), \end{cases} \\
&= \begin{cases} 2\left(\frac{I_1}{I_2} - \frac{1}{\alpha}x\right)(\alpha x - \alpha l) + x^2 + l^2 - 2lx, & x\in(l,1], \\ 2\left(\frac{I_1}{I_2} - \frac{1}{\alpha} - \frac{1}{1-\alpha}\frac{(x-1)}{\lambda}\right)\left((1-\alpha)(x-1) - \alpha(l-1)\right) & \\ \quad +1+l^2+2\frac{\alpha}{1-\alpha}\frac{(x-1)}{\lambda} - 2l - \frac{\alpha}{1-\alpha}\frac{2l}{\lambda}(x-1) + \frac{(x-1)^2}{\lambda}, & x\in(1,r), \end{cases}
\end{aligned}$$

which is the claim. □

Remark 3.2. For $l=0$ (and $r=1+\varepsilon$), formula (3.27) becomes

$$v_\varepsilon(x) = \begin{cases} 2\alpha\frac{I_1}{I_2}x - x^2, & x\in(0,1], \\ 2(1-\alpha)\frac{I_1}{I_2}(x-1) - 2\frac{1-\alpha}{\alpha}(x-1) - \frac{(x-1)^2}{\lambda} & \\ \quad +2\alpha\frac{I_1}{I_2} - 1, & x\in(1,1+\varepsilon), \end{cases} \tag{3.29}$$

where

$$I_1 = \frac{1-\alpha}{\alpha}\varepsilon + \frac{\varepsilon^2}{2\lambda} + \frac{1}{2}, \qquad I_2 = (1-\alpha)\varepsilon + \alpha.$$

We get that $v_\varepsilon \in D(A)$ satisfies $Av_\varepsilon = -1$.

Remark 3.3. Under (1.1), v_ε given in (3.29) converges pointwise in $(0,1)$ to

$$v(x) = Cx - x^2,$$

where

$$C = \begin{cases} 2, & \text{if } \alpha/\varepsilon \to 0, \\ \frac{2+G}{1+G}, & \text{if } \alpha/\varepsilon \to G \in (0,\infty), \\ 1, & \text{if } \alpha/\varepsilon \to \infty. \end{cases}$$

We can immediately verify that v satisfies Neumann, Robin, and Dirichlet boundary condition at $\ell = 1$ depending on the asymptotic behaviour of α/ε. Indeed, we have that

$$\lim_{\varepsilon \to 0} 2\alpha \frac{I_1}{I_2} = 2 \lim_{\varepsilon \to \infty} \frac{1 + \frac{\varepsilon}{2\lambda}\frac{\alpha}{1-\alpha} + \frac{1}{2}\frac{\alpha}{(1-\alpha)\varepsilon}}{1 + \frac{\alpha}{(1-\alpha)\varepsilon}} = \begin{cases} 2, & \alpha/\varepsilon \to 0, \\ \frac{2+G}{1+G}, & \alpha/\varepsilon \to G \in (0,\infty), \\ 1, & \alpha/\varepsilon \to \infty. \end{cases}$$

4. The modified skew Brownian motion on the plane

4.1. The process $R_t^{(\alpha,\lambda)}$. Let $\Theta(t)$, $t \geq 0$, be a Brownian motion on

$$\mathbb{S}_r^1 = \left\{ x \in \mathbb{R}^2 \colon |x| = r \right\},$$

with the radius r, and let $R(t)$, $t \geq 0$, be the 2-dimensional Bessel process. Then, the planar Brownian motion $\mathbf{B}_t$, $t \geq 0$, can be written by considering the skew-product representation (R, Θ), where (see [12, p. 269])

$$R(t),\ t > 0 \quad \text{and} \quad \Theta\left(\int_0^t (R(s))^{-2}\, ds \right),\ t > 0. \tag{4.1}$$

Assume that $\mathbf{B}$ is starting at $x \in \Omega_\ell$, the disc of radius $\ell > 0$. Let us consider the additive functional

$$\mathfrak{f}(t) = \text{meas}\{0 \leq s \leq t \colon \mathbf{B}_s \in \Omega_\ell\}, \qquad t > 0, \tag{4.2}$$

which is the time the Brownian motion spends on the disc up to time t. The Bessel process running with the new clock $\mathfrak{f}^{-1}$, that is, $R(\mathfrak{f}^{-1}(t))$, $t > 0$, is the Bessel process reflecting at ℓ. The Brownian motion in the disc $\Omega_\ell \subset \mathbb{R}^2$ reflecting at $\partial\Omega_\ell$ is identical in law to the skew representation above running with the clock $\mathfrak{f}^{-1}$. The spherical part $\Theta\big(\int_0^{\mathfrak{f}^{-1}(t)} (R(s))^{-2}\, d\mathfrak{f}\big)$ is identical in law ([12, p. 272]) to the spherical part in (4.1) with no time-change, that is, the standard Brownian angle sampled on the support of $\mathfrak{f}$ up to time $\mathfrak{f}^{-1}$. Therefore, due to the isotropic nature of the motion, the Brownian motion in the disc Ω_ℓ of radius $\ell > 0$ and reflecting at ℓ can be studied by considering only its time-changed radial part, that is, $R(\mathfrak{f}^{-1}(t))$, $t > 0$.

The Bessel process is a diffusion with state space $(0,\infty)$ associated with the infinitesimal generator

$$\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx}. \tag{4.3}$$

The generator above can be written as

$$\frac{1}{x}\frac{d}{dx}\left(x \frac{d}{dx} \right), \tag{4.4}$$

and we denote by $R(t) = |\mathbf{B}_t|$, $t > 0$, the Bessel process which is the solution to

$$dR(t) = \frac{1}{2}\frac{dt}{R(t)} + dB(t) \tag{4.5}$$

(where $B(t)$ is an independent one-dimensional Brownian motion). If $R(t)$, $t > 0$, starts from 0, then it leaves 0 immediately, never to return. If the process starts from $x > 0$, then it never hits 0. If we consider the additive functionals (4.2) and

$$\bar{\mathfrak{f}}(t) = \text{meas}\{0 \le s \le t \colon \mathbf{B}_s \notin \Omega_\ell\}, \qquad t > 0, \tag{4.6}$$

we can construct a skew motion by exploiting the same reasoning as above. In particular, $\mathbf{B}_{\bar{\mathfrak{f}}^{-1}}$, that is, $\mathbf{B}$ time-changed by the inverse $\bar{\mathfrak{f}}^{-1}$, is a Brownian motion moving out the disc and reflecting on the boundary $\partial\Omega_\ell$ (from outside), whereas $\mathbf{B}_{\mathfrak{f}^{-1}}$ is a Brownian motion moving on the disc and reflecting on the boundary $\partial\Omega_\ell$ (from inside). By taking the α portion of trajectories out of the disc and the $(1-\alpha)$ portion of trajectories in the disc, we construct a process which is identical in law to the skew Brownian motion on the plane with transmission condition on $\partial\Omega_\ell$. Since the skew-product allows us to consider only the time change of radial part R (the spherical part is identical in law to the time-changed spherical part), we can study the skew Bessel process instead of the skew Brownian motion. The Skew Bessel process has also been considered for example in [4, 8].

We formalize the problem and clarify the aspects mentioned above. Let Ω_ℓ be the disc of radius ℓ and let Ω_r be the disc or radius $r = \ell + \varepsilon$, so that $\Omega_\ell \subset \Omega_r$. Moreover, we denote by $\Sigma_\varepsilon = \Omega_r \setminus \overline{\Omega_\ell}$ the ε-neighbourhood of $\partial\Omega_\ell$. Let $\Omega_l \subset \Omega_\ell$ be a disc of radius $0 < l < \ell$. Our aim is to study the Brownian motion on Ω_r with Dirichlet conditions on $\partial\Omega_l$ and $\partial\Omega_r$, and the transmission condition on $\partial\Omega_\ell$. Thus, we study a killed skew Brownian motion. Moreover, we require that the process exhibits different variances; we have a standard Brownian motion in $\Omega_\ell \setminus \overline{\Omega_l}$ and a Brownian motion with variance λt in Σ_ε. The transmission condition is written as follows:

$$u|_{\partial\Omega_\ell^-} = u|_{\partial\Omega_\ell^+} \qquad \text{continuity on the boundary,} \tag{4.7}$$

$$(1-\alpha)\partial_\nu u|_{\partial\Omega_\ell^-} = \alpha\partial_\nu u|_{\partial\Omega_\ell^+} \qquad \text{partial reflection,} \tag{4.8}$$

where $\partial_\nu u$ is the normal derivative of u. The infinitesimal generator $(A, D(A))$ is therefore given by

$$Au = \begin{cases} \frac{1}{2}\Delta u & \text{on } \Omega_\ell \setminus \overline{\Omega_l}, \\ \frac{\lambda}{2}\Delta u & \text{on } \Omega_r \setminus \overline{\Omega_\ell}, \end{cases} \tag{4.9}$$

with

$$D(A) = \{g, Ag \in C_b \colon g|_{\partial\Omega_l} = g|_{\partial\Omega_r} = 0,\ f \text{ satisfies (4.7), (4.8)}\}. \tag{4.10}$$

As in the previous section we assume that $\alpha = \alpha_\varepsilon$, $\lambda = \lambda_\varepsilon$ are parameters depending on ε. We are interested in the asymptotic analysis (as $\varepsilon \to 0$) for the solution

$$u \in D(A) \quad \text{s.t. } Au = -f \tag{4.11}$$

on the collapsing domain Ω_r under condition (1.1). Denote by $\tau_{l,r}$ the first exit time from $\Omega_r \setminus \overline{\Omega_l}$ of the planar Brownian motion $\mathbf{B}$ started at $x \in \Omega_r \setminus \overline{\Omega_l}$. Killing the process by $\tau_{l,r}$ we get a motion on $\Omega_r \setminus \overline{\Omega_l}$. Since we can find a time change such that $\mathbf{B}_{\mathfrak{f}^{-1}}$ is a skew Brownian motion with reflecting barrier on $\partial\Omega_\ell$, we consider the time-changed skew-product representation. The skew-product representation $(R(\mathfrak{f}^{-1}), \Theta(\mathfrak{f}^{-1}))$ of $\mathbf{B}_{\mathfrak{f}^{-1}}$ can be replaced, for our purposes, by $(R(\mathfrak{f}^{-1}), \Theta(t))$, and we are allowed to study only the time-changed Bessel process. With (4.4) in mind, the generator of the Bessel process we are interested in can be rewritten as follows:

$$Au = \frac{\sigma^2(x)}{2a(x)}\frac{d}{dx}\left(a(x)\frac{d}{dx}u\right) = \begin{cases} \frac{1}{2}\left(\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx}\right)u & \text{on } (l, \ell), \\ \frac{\lambda}{2}\left(\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx}\right)u & \text{on } (\ell, r), \end{cases} \tag{4.12}$$

as will be discussed below.

4.1.1. *Scale function and speed measure.* We consider the Bessel process started at 0 with no drift which is characterized by the functions

$$\mathfrak{s}(x) = x^{-1} \quad \text{and} \quad \mathfrak{m}(x) = x, \qquad x \in (0,\infty). \tag{4.13}$$

We notice that $\mathfrak{m}(\{0\}) = 0$ and $\{0\}$ is an entrance nonexit boundary. The scale function turns out to be

$$S(x) = \ln x. \tag{4.14}$$

Our scope here is to consider the skew Bessel process started at 0 and reflecting at $\ell > 0$ with different variances before and after ℓ. The coefficients are given by

$$\sigma^2(x) = \mathbf{1}_{(0,\ell]}(x) + \lambda \mathbf{1}_{(\ell,\infty)}(x), \qquad \lambda > 0, \tag{4.15}$$

and

$$a(x) = (1-\alpha)x\mathbf{1}_{(0,\ell]}(x) + \alpha x \mathbf{1}_{(\ell,\infty)}(x). \tag{4.16}$$

Thus, we obtain

$$\mathfrak{s}(x) = \frac{x^{-1}}{1-\alpha}\mathbf{1}_{(0,\ell]}(x) + \frac{x^{-1}}{\alpha}\mathbf{1}_{(\ell,\infty)}(x) \tag{4.17}$$

and

$$\mathfrak{m}(x) = \big(\sigma^2(x)\mathfrak{s}(x)\big)^{-1} = (1-\alpha)x\mathbf{1}_{(0,\ell]}(x) + \frac{\alpha}{\lambda}x\mathbf{1}_{(\ell,\infty)}(x) \tag{4.18}$$

with scale function

$$S(x) = \frac{\ln x}{1-\alpha}\mathbf{1}_{(0,\ell]}(x) + \left(\frac{\ln \ell}{1-\alpha} + \frac{\ln x/\ell}{\alpha}\right)\mathbf{1}_{(\ell,\infty)}(x) \tag{4.19}$$

and speed measure

$$M(x) = \frac{1-\alpha}{2}x^2\mathbf{1}_{(0,\ell]}(x) + \left(\frac{1-\alpha}{2}\ell^2 + \frac{\alpha}{2\lambda}(x^2-\ell^2)\right)\mathbf{1}_{(\ell,\infty)}(x). \tag{4.20}$$

The infinitesimal generator (2.1) of the modified skew Bessel process can therefore be rewritten as

$$L = \frac{1}{2}\frac{d}{dM}\frac{d}{dS}, \tag{4.21}$$

with S and M as above and ([20, Theorem VII.3.12]) $Lg(\ell) = Lg(\ell^+) = Lg(\ell^-)$ with

$$D(L) = \left\{ g, Lg \in C_b([0,\infty))\colon \frac{dg^+}{dS}(0^+) = 0,\ (1-\alpha)g'(\ell^-) = \alpha g'(\ell^+) \right\}, \tag{4.22}$$

where

$$\frac{dg^+}{dS}(x) = \lim_{h\downarrow 0}\frac{g(x+h)-g(x)}{S(x+h)-S(x)}$$

is the S-derivative of g. Thus, we have the Bessel process on the positive real line with skew reflection at ℓ. We denote by $R_t^{(\alpha,\lambda)}$, $t \geq 0$, the Bessel process on (l,r) with generator $(A, D(A))$, where A is given in (4.12) and

$$D(A) = \big\{g, Ag \in C([l,r])\colon \\ g \text{ satisfies } g(l) = g(r) = 0, g(\ell^-) = g(\ell^+),\ (1-\alpha)g'(\ell^-) = \alpha g'(\ell^+)\big\}.$$

4.2. **Exit time from a disc.** We study the exit time of $B^{(\alpha,\lambda)}$ by considering the exit time of $R^{(\alpha,\lambda)}$.

Let us first consider the probability

$$\varphi_\varepsilon(x) = \mathbb{P}\left(\tau_r < \tau_l \mid R_0^{(\alpha,\lambda)} = x \in (l,r)\right) = \mathbb{P}_x(\tau_r < \tau_l), \tag{4.23}$$

where $\tau_y = \inf\{s > 0\colon R_s^{(\alpha,\lambda)} = y\}$. For $0 < l < \ell < r < \infty$, we obtain that

$$\varphi_\varepsilon(x) = \frac{S(x)-S(l)}{S(r)-S(l)} = \frac{1}{S(r)-S(l)} \begin{cases} \frac{\ln x/l}{1-\alpha}, & x \in (l,\ell], \\ \frac{\ln \ell/l}{1-\alpha} + \frac{\ln x/\ell}{\alpha}, & x \in (\ell, r) \end{cases} \tag{4.24}$$

solves

$$\begin{gathered} A\varphi_\varepsilon = 0, \\ \varphi_\varepsilon(l) = 0, \\ \varphi_\varepsilon(r) = 1, \\ \varphi_\varepsilon(\ell^-) = \varphi_\varepsilon(\ell^+), \\ (1-\alpha)\varphi_\varepsilon'(\ell^-) = \alpha\varphi_\varepsilon'(\ell^+). \end{gathered}$$

Our aim is to find

$$v_\varepsilon \in D(A) \text{ s.t. } Av_\varepsilon = -1. \tag{4.25}$$

The probabilistic representation is given in terms of the mean exit time

$$v_\varepsilon(x) = \mathbb{E}\left[\tau_{(l,r)} \mid R_0^{(\alpha,\lambda)} = x \in (l,r)\right] = \mathbb{E}_x\left[\tau_{(l,r)}\right]. \tag{4.26}$$

Set

$$\Pi^*(r) = \left(\frac{1-\alpha}{2\alpha} - \frac{1}{2\lambda}\right)\ell^2 \ln\frac{r}{\ell} + \frac{\ell^2 - l^2}{4} + \frac{r^2-\ell^2}{4\lambda} \tag{4.27}$$

and

$$\Pi_3(r) = \frac{\ln \ell/l}{1-\alpha} + \frac{\ln r/\ell}{\alpha}.$$

We present the following explicit result.

Proposition 4.1. *The solution to* (4.25) *is written as*

$$v_\varepsilon(x) = \begin{cases} \frac{2\Pi^*(r)}{(1-\alpha)\Pi_3(r)}\ln x/l - \frac{x^2-l^2}{2}, & x \in (l,\ell], \\ \frac{2\Pi^*(r)}{(1-\alpha)\Pi_3(r)}\ln \ell/l - \frac{\ell^2-l^2}{2} \\ \quad + \left(\frac{2\Pi^*(r)}{\alpha\Pi_3(r)} + \frac{\ell^2}{\lambda} - \frac{1-\alpha}{\alpha}\ell^2\right)\ln x/\ell - \frac{x^2-\ell^2}{2\lambda}, & x \in (\ell, r). \end{cases} \tag{4.28}$$

Proof. We use the same technique as in the previous section, based on the scale function and speed measure (see [11]). From (3.23) with $f = 1$, we obtain

$$\Sigma^+ = \Sigma(x,r] = \int_x^r S(\eta,r]\mathfrak{m}(\eta)\,d\eta = S(r)M(x,r] - \int_x^r S(\eta)\mathfrak{m}(\eta)\,d\eta$$

and

$$\Sigma^- = \Sigma(l,x] = \int_l^x S(l,\eta]\mathfrak{m}(\eta)\,d\eta = \int_l^x S(\eta)\mathfrak{m}(\eta)\,d\eta - S(l)M(l,x].$$

The solution v_ε can be obtained by calculation as in (3.22) or (3.28). Set, for $x > \ell$,

$$\Pi_1(x) = \frac{1}{2}\left(\ell^2 + \frac{\alpha}{1-\alpha}\frac{x^2-\ell^2}{\lambda}\right)\ln\ell, \tag{4.29}$$

$$\Pi_2(x) = \frac{1}{2}\left(\frac{1-\alpha}{\alpha}\ell^2 + \frac{x^2-\ell^2}{\lambda}\right)\ln\frac{x}{\ell}, \tag{4.30}$$

$$\Pi_3(x) = \frac{\ln \ell/l}{1-\alpha} + \frac{\ln x/\ell}{\alpha}, \tag{4.31}$$

$$\Pi_4(x) = \frac{\ell^2}{2}\ln\ell - \frac{l^2}{2}\ln l - \frac{\ell^2-l^2}{4} + \frac{\alpha}{1-\alpha}\frac{x^2-\ell^2}{2\lambda}\ln\ell + \frac{x^2}{2\lambda}\ln\frac{x}{\ell} - \frac{x^2-\ell^2}{4\lambda}. \tag{4.32}$$

We can immediately verify that

$$S(r)M(r) = \Pi_1(r)+\Pi_2(r), \qquad S(r)-S(l) = \Pi_3(r), \qquad S(l)M(l) = \frac{l^2}{2}\ln l, \tag{4.33}$$

$$S(l)M(x) = \frac{x^2}{2}\ln l\,\mathbf{1}_{(0,\ell]}(x) + \left(\frac{\ell^2}{2}\ln l + \frac{\alpha}{1-\alpha}\frac{x^2-\ell^2}{2\lambda}\ln l\right)\mathbf{1}_{(\ell,\infty)}(x), \tag{4.34}$$

$$\int_l^r S(\eta)\mathfrak{m}(\eta)\,d\eta = \Pi_4(r). \tag{4.35}$$

Moreover, we have that

$$\begin{aligned}\Sigma(x,r]-\Sigma(l,x] &= S(r)M(x,r] - \int_l^r S(\eta)\mathfrak{m}(\eta)\,d\eta + S(l)M(l,x]\\ &= S(r)M(r) - S(r)M(x) - \Pi_4(r) + S(l)M(x) - S(l)M(l)\\ &= \Pi_1(r)+\Pi_2(r) - \Pi_3(r)M(x) - \Pi_4(r) - S(l)M(l)\end{aligned}$$

and

$$\begin{aligned}\Sigma(l,x] &= S(l)M(l) - S(l)M(x) + \begin{cases}\frac{x^2}{2}\ln x - \frac{l^2}{2}\ln l - \frac{x^2-l^2}{4}, & x\in(l,\ell],\\ \Pi_4(x), & x\in(\ell,r)\end{cases}\\ &= \begin{cases}\frac{x^2}{2}\ln x/l - \frac{x^2-l^2}{4}, & x\in(l,\ell],\\ \frac{x^2}{2\lambda}\ln x/\ell - \frac{x^2-\ell^2}{4\lambda} + \frac{\alpha}{1-\alpha}\frac{1}{2\lambda}(x^2-\ell^2)\ln\ell/l + \frac{\ell^2}{2}\ln\ell/l - \frac{\ell^2-l^2}{4}, & x\in(\ell,r).\end{cases}\end{aligned} \tag{4.36}$$

Thus, by considering (3.28) and φ_ε in (4.23), we get that

$$v_\varepsilon(x) = 2\varphi_\varepsilon(x)\big(\Sigma(x,r]-\Sigma(l,x]\big) + 2\Sigma(l,x] = v^*(x) + 2\Sigma(l,x] \tag{4.37}$$

where

$$v^*(x) = \begin{cases}\frac{2}{\Pi_3(r)}\left(\Pi_1(r)+\Pi_2(r)-\Pi_3(r)M(x)-\Pi_4(r)-\frac{l^2}{2}\ln l\right)\frac{\ln x/l}{1-\alpha}, & x\in(l,\ell],\\ \frac{2}{\Pi_3(r)}\left(\Pi_1(r)+\Pi_2(r)-\Pi_3(r)M(x)-\Pi_4(r)-\frac{l^2}{2}\ln l\right)\Pi_3(x), & x\in(\ell,r).\end{cases}$$

By setting

$$\begin{aligned}\Pi^*(r) &= \Pi_1(r)+\Pi_2(r)-\Pi_4(r)-\frac{l^2}{2}\ln l\\ &= \left(\frac{1-\alpha}{2\alpha}-\frac{1}{2\lambda}\right)\ell^2\ln\frac{r}{\ell} + \frac{\ell^2-l^2}{4} + \frac{r^2-\ell^2}{4\lambda},\end{aligned} \tag{4.38}$$

after straightforward manipulation we get the claim. □

4.3. **Asymptotics.** Let us introduce the Bessel diffusion $R^{(0)}$ on (l, ℓ) with infinitesimal generator $(\mathcal{A}, D(\mathcal{A}))$, where $\mathcal{A}$ is given in (4.3). Now consider $v \in D(\mathcal{A})$ such that $\mathcal{A}v = -f$ with $f \in C([l, \ell])$. We have that

$$v(x) = \mathbb{E}_x \left[\int_0^\infty f\left(R_t^{(0)}\right) M_t^{(0)} \, dt \right], \tag{4.39}$$

where the multiplicative functional $M_t^{(0)}$ uniquely characterizes the process and agrees with the following choice for the generator $\mathcal{A}$:

i) $\mathcal{A} = A_R,\ D(A_R) = \{g, A_R g \in C([l, \ell]) \colon g(l) = 0,\ g'(\ell) + Gg(\ell) = 0\}$,
ii) $\mathcal{A} = A_N,\ D(A_N) = \{g, A_N g \in C([l, \ell]) \colon g(l) = 0,\ g'(\ell) = 0\}$,
iii) $\mathcal{A} = A_D,\ D(A_D) = \{g, A_D g \in C([l, \ell]) \colon g(l) = 0,\ g(\ell) = 0\}$.

We provide the following result concerning (4.28) and (4.39) for $f = 1$.

Theorem 4.1. *Under* (1.1), $v_\varepsilon \to v \in D(\mathcal{A})$ *pointwise in* (l, ℓ). *In particular:*

i) $v_\varepsilon \to v \in D(A_R)$ *if* $\alpha/\varepsilon \to G \in (0, \infty)$;
ii) $v_\varepsilon \to v \in D(A_N)$ *if* $\alpha/\varepsilon \to \infty$;
iii) $v_\varepsilon \to v \in D(A_D)$ *if* $\alpha/\varepsilon \to 0$.

Proof. Let us write

$$\frac{2\Pi^*(r)}{(1-\alpha)\Pi_3(r)} = \frac{\left(\frac{1-\alpha}{\alpha} \ln \frac{r}{\ell} - \frac{1}{\lambda} \ln \frac{r}{\ell}\right) \ell^2 + \frac{\ell^2 - l^2}{2} + \frac{r^2 - \ell^2}{2\lambda}}{\ln \ell/l + \frac{1-\alpha}{\alpha} \ln r/\ell}.$$

Throughout, we denote by $a_\varepsilon \sim b_\varepsilon$ the fact that $a_\varepsilon / b_\varepsilon \to 1$ as $\varepsilon \to 0$. By considering that

$$\ln\left(1 + \frac{\varepsilon}{\ell}\right) \sim \frac{\varepsilon}{\ell}$$

we get that

$$\frac{2\Pi^*(\ell+\varepsilon)}{(1-\alpha)\Pi_3(\ell+\varepsilon)} \sim \frac{\left(\frac{1-\alpha}{\alpha}\varepsilon - \frac{\varepsilon}{\lambda}\right)\ell + \frac{\ell^2 - l^2}{2} + \frac{\ell\varepsilon}{\lambda} + \frac{\varepsilon^2}{2\lambda}}{\ln \ell/l + \frac{1-\alpha}{\alpha}\frac{\varepsilon}{\ell}} = \frac{\frac{1-\alpha}{\alpha}\varepsilon\ell + \frac{\ell^2 - l^2}{2} + \frac{\varepsilon^2}{2\lambda}}{\ln \ell/l + \frac{1-\alpha}{\alpha}\frac{\varepsilon}{\ell}},$$

which can be rewritten as

$$\frac{2\Pi^*(\ell+\varepsilon)}{(1-\alpha)\Pi_3(\ell+\varepsilon)} \sim \frac{\ell + \frac{\ell^2 - l^2}{2}\frac{\alpha}{(1-\alpha)\varepsilon} + \frac{\alpha\varepsilon}{2\lambda(1-\alpha)}}{\frac{\alpha}{(1-\alpha)\varepsilon}\ln \ell/l + \frac{1}{\ell}}. \tag{4.40}$$

Therefore, under (1.1), from (4.40) we have that:

i) if $\alpha/\varepsilon \to G > 0$, then

$$\frac{2\Pi^*(\ell+\varepsilon)}{(1-\alpha)\Pi_3(\ell+\varepsilon)} \to \frac{\frac{\ell}{G} + \frac{\ell^2 - l^2}{2}}{\frac{1}{\ell G} + \ln \ell/l}, \tag{4.41}$$

ii) if $\alpha/\varepsilon \to 0$, then

$$\frac{2\Pi^*(\ell+\varepsilon)}{(1-\alpha)\Pi_3(\ell+\varepsilon)} \to \ell^2, \tag{4.42}$$

iii) if $\alpha/\varepsilon \to \infty$, then

$$\frac{2\Pi^*(\ell+\varepsilon)}{(1-\alpha)\Pi_3(\ell+\varepsilon)} \to \frac{\ell^2 - l^2}{2\ln \ell/l}. \tag{4.43}$$

The solution (4.39) can therefore be written as

$$v(x) = C \ln x/l - \frac{(x^2 - l^2)}{2},$$

where the coefficient C is given by (4.41) or (4.42) or (4.43), depending on the asymptotic behaviour of α/ε. □

5. Proof of Theorem 1.1

Proof. We exploit the pointwise convergence of v_ε to v and the characterization by multiplicative functionals of the corresponding semigroups. For the Brownian motion on the line we consider the result in Remark 3.3, whereas for the Brownian motion on the plane we consider the result in Theorem 4.1. Then, the proof moves in both cases by following the same arguments. For this reason, we present the proof only for the planar case.

From the previous Theorem 4.1 we have the pointwise convergence

$$\mathbb{E}_x \left[\int_0^\infty f\left(R_t^{(\alpha,\lambda)}\right) M_t^\varepsilon \, dt \right] \to \mathbb{E}_x \left[\int_0^\infty f\left(R_t^{(0)}\right) M_t^{(0)} \, dt \right]$$

for $f = -1$, where $M_t^{(0)}$ is determined by the boundary conditions on (l, ℓ). Since the multiplicative functional uniquely characterizes the semigroup ([5, Proposition 1.9]), we can consider a nonconstant f. Under the Robin boundary condition we have that

$$\begin{aligned}
\mathbb{E}_x \left[\int_0^\infty f\left(R_t^{(0)}\right) M_t^{(0)} \, dt \right] &= \mathbb{E}_x \left[\int_0^\infty f\left(R_t^{(0)}\right) \mathbf{1}_{(t<\zeta\wedge\tau_l)} \, dt \right] \\
&= \mathbb{E}_x \left[\int_0^\infty f\left(R_t^{(0)}\right) e^{-GL_t^{\{\ell\}}} \mathbf{1}_{(t<\tau_l)} \, dt \right],
\end{aligned}$$

where ζ is the (elastic) lifetime of $R^{(0)}$ such that $(t < \zeta) \equiv \left(L_t^{\{\ell\}}(R^{(0)}) < \gamma\right)$ and γ is the independent exponential r.v. with parameter $G \in (0, \infty)$. Observe that we obtain $\mathcal{A} = A_R$ in Theorem 4.1 under the hypothesis that (1.1) holds and $\alpha/\varepsilon \to G$, that is, trivially,

$$\mathbb{E}_x \left[\int_0^\infty f\left(R_t^{(0)}\right) M_t^{(0)} \, dt \right] = \mathbb{E}_x \left[\int_0^\infty f\left(R_t^{(0)}\right) e^{-\left(\lim_{\varepsilon\to 0} \frac{\alpha}{(1-\alpha)\varepsilon}\right) L_t^{\{\ell\}}} \mathbf{1}_{(t<\tau_l)} \, dt \right].$$

The skew-product representation allows us to consider only the radial part of the process $B^{(\alpha,\lambda)}$. Thus, we can transfer the result about $R^{(\alpha,\lambda)}$ to $B^{(\alpha,\lambda)}$ and we prove that Theorem 1.1 holds under the Robin boundary condition.

By following the same argument as before, we can easily prove that Theorem 1.1 holds under the Neumann boundary condition. Indeed, in this case we have that

$$\forall\, t \geq 0, \qquad \mathbb{P}_x \left(e^{-GL_t^{\{\ell\}}} = 1 \right) = 1,$$

and therefore, we get that

$$\mathbb{E}_x \left[\int_0^\infty f\left(R_t^{(0)}\right) \mathbf{1}_{(t<\tau_l)} \, dt \right] = \mathbb{E}_x \left[\int_0^\infty f\left(R_t^{(0)}\right) e^{-\left(\lim_{\varepsilon\to 0} \frac{\alpha}{(1-\alpha)\varepsilon}\right) L_t^{\{\ell\}}} \mathbf{1}_{(t<\tau_l)} \, dt \right]$$

with $\alpha/\varepsilon \to 0$.

Since $L^{\{\ell\}}_t$, $t \geq 0$, is a positive continuous additive functional (PCAF), the process $\frac{\alpha}{(1-\alpha)\varepsilon} L^{\{\ell\}}_t$, $t \geq 0$, is a PCAF. For $x \in (l, \ell)$,

$$\begin{aligned}
&\mathbb{E}_x \left[e^{-\frac{\alpha}{(1-\alpha)\varepsilon} L^{\{\ell\}}_t} \mathbf{1}_{(t<\tau_l)} \right] \\
&\quad = \mathbb{E}_x \left[e^{-\frac{\alpha}{(1-\alpha)\varepsilon} L^{\{\ell\}}_t} \mathbf{1}_{(t<\tau_l)}, (t < \tau_\ell) \cup (t \geq \tau_\ell) \right] \\
&\quad = \mathbb{E}_x \left[e^{-\frac{\alpha}{(1-\alpha)\varepsilon} L^{\{\ell\}}_t} \mathbf{1}_{(t<\tau_l)} \;\middle|\; t < \tau_\ell \right] \mathbb{P}_x(t < \tau_\ell) \\
&\qquad + \mathbb{E}_x \left[e^{-\frac{\alpha}{(1-\alpha)\varepsilon} L^{\{\ell\}}_t} \mathbf{1}_{(t<\tau_l)} \;\middle|\; t \geq \tau_\ell \right] \mathbb{P}_x(t \geq \tau_\ell) \\
&\quad = \mathbb{E}_x \left[\mathbf{1}_{(t<\tau_l)} \;\middle|\; t < \tau_\ell \right] \mathbb{P}_x(t < \tau_\ell) + \mathbb{E}_x \left[e^{-\frac{\alpha}{(1-\alpha)\varepsilon} L^{\{\ell\}}_t} \mathbf{1}_{(t<\tau_l)} \;\middle|\; L^{\{\ell\}}_t > 0 \right] \mathbb{P}_x(t \geq \tau_\ell) \\
&\quad = \mathbb{E}_x \left[\mathbf{1}_{(t<\tau_l)} \mathbf{1}_{(t<\tau_\ell)} \right] + \mathbb{E}_x \left[e^{-\frac{\alpha}{(1-\alpha)\varepsilon} L^{\{\ell\}}_t} \mathbf{1}_{(t<\tau_l)} \;\middle|\; L^{\{\ell\}}_t > 0 \right] \mathbb{P}_x(t \geq \tau_\ell) \\
&\quad = \mathbb{E}_x \left[\mathbf{1}_{(t<\tau_l \wedge \tau_\ell)} \right] + \mathbb{E}_x \left[e^{-\frac{\alpha}{(1-\alpha)\varepsilon} L^{\{\ell\}}_t} \mathbf{1}_{(t<\tau_l)} \;\middle|\; L^{\{\ell\}}_t > 0 \right] \mathbb{P}_x(t \geq \tau_\ell).
\end{aligned}$$

Then, we verify that

$$\begin{aligned}
\mathbb{E}_x \left[\int_0^\infty f\left(R^{(0)}_t\right) M^{(0)}_t \, dt \right] &= \mathbb{E}_x \left[\int_0^\infty f(R^{(0)}_t) \mathbf{1}_{(t<\tau_l \wedge \tau_\ell)} \, dt \right] \\
&= \mathbb{E}_x \left[\int_0^\infty f\left(R^{(0)}_t\right) e^{-\left(\lim_{\varepsilon \to 0} \frac{\alpha}{(1-\alpha)\varepsilon}\right) L^{\{\ell\}}_t} \mathbf{1}_{(t<\tau_l)} \, dt \right]
\end{aligned}$$

if $\alpha/\varepsilon \to \infty$. □

6. The modified skew Brownian motion on $\mathbb{R}^d$

Our results can be extended to diffusions in higher dimensions. We can follow different approaches depending on the regularity of the domains. For instance, in the case of d-dimensional balls we skip a detailed discussion about the Bessel process and emphasize only the fact that we can use the same argument as in the previous sections based on the speed measure and the scale function. Indeed, we can consider the skew-product representation and study the Bessel process with speed measure $\mathfrak{m}(x) = x^{d-1}$.

Since we are interested in irregular domains we also approach the problem via Dirichlet form theory. More precisely, the problem $Au = -f$, where A is given in (4.9) with domain (4.10), can be formulated by considering in $\Omega_{\ell+\varepsilon} \setminus \overline{\Omega_l}$ (here Ω_l denotes the d-dimensional ball)

$$-\frac{\sigma^2}{2a(x)} \nabla \left(a(x) \nabla u \right) = f, \tag{6.1}$$

where

$$a(x) = (1-\alpha) \mathbf{1}_{\overline{\Omega_\ell} \setminus \overline{\Omega_l}}(x) + \alpha \mathbf{1}_{\Omega_{\ell+\varepsilon} \setminus \overline{\Omega_\ell}}(x), \qquad \alpha \in (0,1) \tag{6.2}$$

and

$$\rho(x) a(x) = \sigma^2(x) = \mathbf{1}_{\overline{\Omega_\ell} \setminus \overline{\Omega_l}}(x) + \lambda \mathbf{1}_{\Omega_{\ell+\varepsilon} \setminus \overline{\Omega_\ell}}(x), \qquad \lambda > 0. \tag{6.3}$$

We consider the measure

$$dm = \frac{1}{\rho(x)} \, dx$$

(where we denote by dx the Lebesgue measure on $\mathbb{R}^d$) under the assumption that (1.1) holds true. This ensures that

$$\int_{\Omega_{\ell+\varepsilon} \setminus \overline{\Omega_l}} dm \to \int_{\Omega_\ell \setminus \overline{\Omega_l}} dx \quad \text{as } \varepsilon \to 0. \tag{6.4}$$

By multiplying by a test function $\psi \in H_0^1(\Omega_r \setminus \overline{\Omega_l})$ and integrating in dm we have

$$\begin{aligned} -\int_{\Omega_r \setminus \overline{\Omega_l}} \frac{\sigma^2}{2a(x)} \nabla\left(a(x)\nabla u\right)\psi\, dm &= \int_{\Omega_r \setminus \overline{\Omega_l}} f\psi\, dm \\ -\int_{\Omega_r \setminus \overline{\Omega_l}} \nabla\left(a(x)\nabla u\right)\psi\, dx &= 2\int_{\Omega_r \setminus \overline{\Omega_l}} f\psi\, dm \\ (1-\alpha)\int_{\Omega_\ell \setminus \overline{\Omega_l}} \nabla u \nabla \psi\, dx + \alpha \int_{\Omega_{\ell+\varepsilon} \setminus \overline{\Omega_\ell}} \nabla u \nabla\psi\, dx & \\ = 2(1-\alpha)\int_{\Omega_\ell \setminus \overline{\Omega_l}} f\psi\, dx + 2\frac{\alpha}{\lambda}\int_{\Omega_{\ell+\varepsilon} \setminus \overline{\Omega_\ell}} f\psi\, dx. & \end{aligned}$$

Let Ω^* be an open regular domain such that $\Omega^* \supset \overline{\Omega_r}$. We recall that $\alpha = \alpha_\varepsilon$ and $\lambda = \lambda_\varepsilon$. Now we consider the sequence of energy functionals in $L^2(\Omega^*)$:

$$F_\varepsilon[u] = \begin{cases} \int_{\Omega_\ell \setminus \overline{\Omega_l}} (1-\alpha)|\nabla u|^2\, dx & \\ \quad +\alpha \int_{\Omega_{\ell+\varepsilon} \setminus \overline{\Omega_\ell}} |\nabla u|^2\, dx, & \text{if } u|_{\Omega_r \setminus \overline{\Omega_l}} \in H_0^1(\Omega_r \setminus \overline{\Omega_l}), \\ +\infty, & \text{otherwise in } L^2(\Omega^*). \end{cases} \tag{6.5}$$

We study the Mosco convergence of (6.5). First we recall the notion of *M-convergence* of functionals, introduced in [16] (see also [17]).

Definition 6.1. A sequence of functionals $F_\varepsilon \colon H \to (-\infty, +\infty]$ is said to M-converge to a functional $F \colon H \to (-\infty, +\infty]$ in a Hilbert space H if

(a) for every $u \in H$ there exists u_ε converging strongly to u in H such that

$$\limsup F_\varepsilon[u_\varepsilon] \le F[u], \quad \text{as } \varepsilon \to 0; \tag{6.6}$$

(b) for every w_ε converging weakly to u in H

$$\liminf F_\varepsilon[w_\varepsilon] \ge F[u], \quad \text{as } \varepsilon \to 0. \tag{6.7}$$

In the following we recall the results concerning the asymptotic behaviour of the functional (6.5) according to different rates of α/ε (for the proof, see for example Theorem II.2 of [2] in the framework of Γ convergence). We recall the following results.

Theorem 6.1. *Under*

$$\frac{\alpha}{\varepsilon} \to G \quad \text{as } \varepsilon \to 0, \tag{6.8}$$

the sequence of functionals F_ε, defined in (6.5), M-converges in $L^2(\Omega^*)$ to the functional

$$F_G[u] = \begin{cases} \int_{\Omega_\ell \setminus \overline{\Omega_l}} |\nabla u|^2\, dx + G\int_{\partial\Omega_\ell} u^2\, ds & \text{if } u|_{\Omega_\ell \setminus \overline{\Omega_l}} \in H^1(\Omega_\ell \setminus \overline{\Omega_l}) \text{ s.t. } u = 0 \text{ on } \partial\Omega_l, \\ +\infty, & \text{otherwise in } L^2(\Omega^*). \end{cases} \tag{6.9}$$

Theorem 6.2. *Under*

$$\frac{\alpha}{\varepsilon} \to 0 \quad \text{as } \varepsilon \to 0, \tag{6.10}$$

the sequence of functionals F_ε, defined in (6.5), M-converges in $L^2(\Omega^*)$ to the functional

$$F_0[u] = \begin{cases} \int_{\Omega_\ell \setminus \overline{\Omega_l}} |\nabla u|^2\, dx, & \text{if } u|_{\Omega_\ell \setminus \overline{\Omega_l}} \in H^1(\Omega_\ell \setminus \overline{\Omega_l}) \text{ s.t. } u = 0 \text{ on } \partial\Omega_l, \\ +\infty, & \text{otherwise in } L^2(\Omega^*). \end{cases} \tag{6.11}$$

Theorem 6.3. *Under*

$$\frac{\alpha}{\varepsilon} \to \infty \quad \text{as } \varepsilon \to 0, \tag{6.12}$$

the sequence of functionals F_ε, defined in (6.5), *M-converges in $L^2(\Omega^*)$ to the functional*

$$F_\infty[u] = \begin{cases} \int_{\Omega_\ell \setminus \overline{\Omega_l}} |\nabla u|^2 \, dx, & \text{if } u|_{\Omega_\ell \setminus \overline{\Omega_l}} \in H_0^1(\Omega_\ell \setminus \overline{\Omega_l}), \\ +\infty, & \text{otherwise in } L^2(\Omega^*). \end{cases} \tag{6.13}$$

From the previous theorems we obtain the convergence of the solutions; see Theorems II.1, II.2, and III.3 of [2] (see also [3]). We note that following Lemma 6.1 plays a key role in this framework (see Lemma III.1 of [2]).

Lemma 6.1. *Let $a\colon \mathbb{R}^d \to \mathbb{R}$ be a continuous function and $u \in H^1(\mathbb{R}^d)$. Then*

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{\Omega_{\ell+\varepsilon} \setminus \overline{\Omega_\ell}} a(x)u(x)\, dx = \int_{\partial\Omega_\ell} a(s)u(s)\, ds.$$

Proof. We set

$$\omega_\varepsilon = \sup\{|a(s+t\nu) - a(s)|, s \in \partial\Omega_\ell, |t| \le \varepsilon\}$$

(ν is the outward normal vector) and we have $\lim_{\varepsilon\to 0} \omega_\varepsilon = 0$. As

$$\begin{aligned}
&\left| \frac{1}{\varepsilon} \int_{\Omega_{\ell+\varepsilon} \setminus \overline{\Omega_\ell}} a(x)u(x)\, dx - \int_{\partial\Omega_\ell} a(s)u(s)\, ds \right| \\
&\quad = \left| \frac{1}{\varepsilon} \int_{\partial\Omega_\ell} ds \int_0^\varepsilon a(s+t\nu)u(s+t\nu)\, dt - \int_{\partial\Omega_\ell} a(s)u(s)\, ds \right| \\
&\quad \le \frac{1}{\varepsilon} \int_{\partial\Omega_\ell} ds \int_0^\varepsilon |a(s+t\nu)u(s+t\nu) - a(s)u(s)|\, dt \\
&\quad = \frac{1}{\varepsilon} \int_{\partial\Omega_\ell} ds \int_0^\varepsilon |a(s+t\nu)u(s+t\nu) - a(s+t\nu)u(s) + a(s+t\nu)u(s) - a(s)u(s)|\, dt \\
&\quad \le \frac{1}{\varepsilon} \int_{\partial\Omega_\ell} ds \int_0^\varepsilon |a(s+t\nu)||u(s+t\nu) - u(s)|\, dt \\
&\qquad + \frac{1}{\varepsilon} \int_{\partial\Omega_\ell} ds \int_0^\varepsilon |a(s+t\nu) - a(s)||u(s)|\, dt \\
&\quad \le \frac{c}{\varepsilon} \int_{\partial\Omega_\ell} ds \int_0^\varepsilon \left(\int_0^t |\nabla u(s+\xi\nu)|\, d\xi \right) dt + \omega_\varepsilon \int_{\partial\Omega_\ell} |u(s)|\, ds \\
&\quad \le c \int_{\Omega_{\ell+\varepsilon} \setminus \overline{\Omega_\ell}} |\nabla u(x)|\, dx + \omega_\varepsilon \int_{\partial\Omega_\ell} |u(s)|\, ds,
\end{aligned}$$

passing to the limit as ε goes to 0 we achieve the result. □

We remark that, by Lemma 6.1 and by (1.1), we obtain that

$$\lim_{\varepsilon\to 0} \frac{\alpha}{\lambda} \int_{\Omega_{\ell+\varepsilon} \setminus \overline{\Omega_\ell}} f\psi\, dx = \lim_{\varepsilon\to 0} \frac{\alpha\varepsilon}{\lambda} \frac{1}{\varepsilon} \int_{\Omega_{\ell+\varepsilon} \setminus \overline{\Omega_\ell}} f\psi\, dx = 0.$$

Remark 6.1. From the previous calculation we obtain substantially different limit problems if $\alpha\varepsilon/\lambda \to F \neq 0$. We will investigate this aspect in a forthcoming paper.

Remark 6.2. Diffusions across fractal interfaces can be studied by using the approach based on the Dirichlet form, in particular, by extending Theorems 6.1, 6.2 and 6.3; see [7]. This framework has been considered in [6] in order to study the asymptotic behaviour of corresponding multiplicative functionals only in the case that $\alpha = \lambda$. Our aim is therefore to extend such results on irregular domains in the case that $\alpha \neq \lambda$.

Remark 6.3. The approaches considered in the present paper can also be extended to Brownian motions time-changed by an inverse to a stable subordinator. That is, we can consider a delayed Brownian motion, and it is our feeling that the asymptotic analysis could be deeply affected by the delay.

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