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WAVE EQUATION WITH A STABLE NOISE

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Abstract. The three-dimensional wave equation is studied in the paper. The right hand side of the equation has a symmetric α-stable distribution. Two cases are considered, namely the cases where the perturbation is a (1) "white noise" and (2) "colored noise". It is proved for both cases that a candidate for a solution (a function represented by the Kirchhoff formula) is a generalized solution.

1. Introduction

Stochastic partial differential equations play a leading role in modeling the behavior of complex systems where a randomness is involved. The number of studies devoted to such equations increases constantly. The majority of publications deal with the case where the randomness in the equation has a Gaussian distribution. In particular, the wave equation with a Gaussian random noise is studied in the papers [3, 4, 8, 10, 14]. The one-dimensional wave equation with a general random measure is considered in [2, 6].

In this paper, we continue the studies initiated in [9], where the wave equation is considered with a stable measure in the plane. The object of our studies is the wave equation in the space

$$\begin{cases} \left(\dfrac{\partial^2}{\partial t^2} - a^2\Delta\right)U(x,t) = f(x,t), \quad x \in \mathbb{R}^3,\ t > 0, \\ U(x,0) = 0, \\ \dfrac{\partial U}{\partial t}(x,0) = 0, \end{cases} \tag{1}$$

where the right hand side is a random perturbation of the form $f(x,t) = \sigma(x,t)\cdot \dot{Z}(x)$. Here $\sigma(x,t)$ is a non-random bounded function, while $\dot{Z}(x)$ is a random noise with a symmetric α-stable distribution. We consider two completely different cases, namely, the cases where the noise is (1) "white" (that is, it is a derivative in a certain sense of an independently scattered stable measure) and (2) "colored" generated by a field with dependent increments (more precisely, it is generated by a real harmonizable anisotropic fractional stable field). For both cases, the function represented by the Kirchhoff formula is considered as a potential solution of the equation. It is proved that this function is well defined and is a generalized solution of the equation. Since the theory of integration with respect to a stable colored noise is not developed yet, we provide the definition of the integral with respect to such a noise.

The paper is organized as follows. Auxiliary results concerning stable random variables and their distributions are given in Section 2. This section contains a brief description

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of decompositions into LePage series. Sections 3 and 4 contain statements and proofs of the main results of the paper for the white and colored noise, respectively.

2. Stable random variables

We consider symmetric α-stable ($S\alpha S$) random variables with $\alpha \in (0,2)$. More details concerning such random variables can be found in [12].

A random variable ξ whose characteristic function is given by

$$\mathsf{E}\left[e^{i\lambda\xi}\right] = e^{-\lambda^\alpha \|\xi\|_\alpha^\alpha}$$

is called an $S\alpha S$ random variable with the scale parameter $\|\xi\|_\alpha$.

A special role in the construction of processes and fields with stable distributions is played by an independently scattered $S\alpha S$ measure. In the current paper, we restrict our consideration to the case of $S\alpha S$ measures in $\mathbb{R}^3$ which are defined as functions $M\colon B_f(\mathbb{R}^3) \times \Omega \to \mathbb{R}$, where $B_f(\mathbb{R}^3)$ is a family of Borel subsets of finite Lebesgue measure such that

(1) for every Borel set $A \in B_f(\mathbb{R}^3)$, $M(A)$ is an $S\alpha S$ random variable with the scale parameter $\lambda(A)$; here $\lambda(A)$ denotes the Lebesgue measure of the set A;
(2) the random variables $M(A_1), \dots, M(A_n)$ are independent for all disjoint sets $A_1, \dots, A_n \in B_f(\mathbb{R}^3)$;
(3) the series $\sum_{n=1}^\infty M(A_n)$ converges almost surely, and moreover

$$M\left(\bigcup_{n=1}^\infty A_n\right) = \sum_{n=1}^\infty M(A_n)$$

almost surely for all sequences of disjoint sets $A_1, \dots, A_n, \dots \in B_f(\mathbb{R}^3)$ such that $\bigcup_{n=1}^\infty A_n \in B_f(\mathbb{R}^3)$.

The integral

$$I(f) = \iiint_{\mathbb{R}^3} f(x,t)\, M(dx)$$

is defined for a function $f(x,t) \in L^\alpha(\mathbb{R}^3)$ as the limit in probability of the integrals of simple functions with bounded supports. The following isometry property holds for such an integral:

$$\|I(f)\|_\alpha^\alpha = \iiint_{\mathbb{R}^3} |f(x,t)|^\alpha \, dx.$$

A powerful tool for an analysis of stable random variables is the LePage representation, which we describe briefly for the measure M. Let φ be an arbitrary continuous positive distribution density in $\mathbb{R}^3$ and let independent families $\{\Gamma_k, k \ge 1\}$, $\{\xi_k, k \ge 1\}$, and $\{g_k, k \ge 1\}$ be such that

- $\{\Gamma_k, k \ge 1\}$ is the sequence of moments of arrivals of a Poisson process with unit intensity;
- $\{\xi_k, k \ge 1\}$ are independent random vectors with density φ;
- $\{g_k, k \ge 1\}$ are independent centered normal random variables with $\mathsf{E}[|g_k|^\alpha] = 1$.

Then M has the same distribution as

$$M'(dx) = C_\alpha \sum_{k\ge 1} \Gamma_k^{-1/\alpha} \varphi(\xi_k)^{-1/\alpha} \delta_{\xi_k}(dx) g_k, \tag{2}$$

where

$$C_\alpha = \left(\frac{\Gamma(2-\alpha)\cos\frac{\pi\alpha}{2}}{1-\alpha}\right)^{1/\alpha}.$$

The series on the right hand side of (2) converges almost surely. Equality (2) is understood in the following sense: for all $f_1, f_2, \dots, f_n \in L^\alpha(\mathbb{R}^3)$, the distribution of the vector $\big(I(f_1), I(f_2), \dots, I(f_n)\big)$ coincides with that of $\big(I'(f_1), I'(f_2), \dots, I'(f_n)\big)$, where

$$I'(f) = C_\alpha \sum_{k\ge 1} \Gamma_k^{-1/\alpha} \varphi(\xi_k)^{-1/\alpha} f(\xi_k) g_k. \tag{3}$$

Without loss of generality we assume in what follows that M is given by equality (2) and that the integral

$$I(f) = \iiint_{\mathbb{R}^3} f(x,t)\, M(dx)$$

is defined for a function $f(x,t) \in L^\alpha(\mathbb{R}^3)$ by equality (3). For simplicity, we also assume that

$$(\Omega, \mathcal{F}, \mathsf{P}) = (\Omega_\Gamma \otimes \Omega_\xi \otimes \Omega_g, \mathcal{F}_\Gamma \otimes \mathcal{F}_\xi \otimes \mathcal{F}_g, \mathsf{P}_\Gamma \otimes \mathsf{P}_\xi \otimes \mathsf{P}_g)$$

for all $\omega = (\omega_\Gamma, \omega_\xi, \omega_g)$, where $\Gamma_k(\omega) = \Gamma_k(\omega_\Gamma)$, $\xi_k(\omega) = \xi_k(\omega_\xi)$, and $g_k(\omega) = g_k(\omega_g)$ for all $k \ge 1$.

3. Wave equation with a stable white noise

Consider equation (1),

$$\begin{cases} \left(\dfrac{\partial^2}{\partial t^2} - a^2 \Delta\right) U(x,t) = \sigma(x,t) \dot{M}(x), \quad x \in \mathbb{R}^3,\ t > 0, \\ U(x,0) = 0, \\ \dfrac{\partial}{\partial t} U(x,0) = 0, \end{cases} \tag{4}$$

where the right hand side of the equation is the product of a non-random function $\sigma\colon \mathbb{R}^3 \times \mathbb{R}^+ \to \mathbb{R}$ and an "$S\alpha S$ white noise" $\dot{M}(x)$ being, in a certain sense, the Radon–Nikodym density of the independently scattered $S\alpha S$ measure M in $\mathbb{R}^3$. The function represented by the Kirchhoff formula, namely,

$$U(x,t) = \frac{1}{4\pi a} \iiint_{y\colon |x-y|<at} \frac{\sigma\left(y, t - \frac{|x-y|}{a}\right)}{|x-y|}\, M(dy_1, dy_2, dy_3), \tag{5}$$

is considered as a candidate for a solution of equation (4). We assume throughout that the function σ is bounded.

Theorem 1. *The integral in* (5) *is well defined for all* $(x,t) \in \mathbb{R}^3 \times \mathbb{R}^+$.

Proof. The integral in (5) is well defined if

$$\iiint_{y:|x-y|<at} \left| \frac{\sigma\left(y, t - \frac{|x-y|}{a}\right)}{|x-y|} \right|^\alpha dy_1\, dy_2\, dy_3 < \infty.$$

So, we check the condition:

$$\iiint_{y:|x-y|<at} \left| \frac{\sigma\left(y, t - \frac{|x-y|}{a}\right)}{|x-y|} \right|^\alpha dy_1\, dy_2\, dy_3 \le \iiint_{y:|x-y|<at} \frac{C^\alpha}{|x-y|^\alpha}\, dy_1\, dy_2\, dy_3$$
$$= 4\pi C^\alpha \frac{(at)^{3-\alpha}}{3-\alpha} < \infty,$$

and this is what had to be proved. □

Theorem 2. *The series*

$$U(x,t) = C_\alpha \sum_{k\geq 1} \Gamma_k^{-1/\alpha} \varphi(\xi_k)^{-1/\alpha} \frac{\sigma\left(\xi_k, t - \frac{|x-\xi_k|}{a}\right)}{|x-\xi_k|} g_k I_{\{|x-\xi_k|<at\}}$$

converges almost surely for all $(x,t) \in \mathbb{R}^3 \times \mathbb{R}^+$.

Proof. Put

$$u_k(x,t) = C_\alpha \Gamma_k^{-1/\alpha} \varphi(\xi_k)^{-1/\alpha} \frac{\sigma\left(\xi_k, t - \frac{|x-\xi_k|}{a}\right)}{|x-\xi_k|} g_k I_{\{|x-\xi_k|<at\}}.$$

If $\omega_\xi \in \Omega_\xi$ and $\omega_\Gamma \in \Omega_\Gamma$ are fixed, then u_k are independent and centered random variables. According to Kolmogorov's theorem, the statement of Theorem 2 follows if

$$\mathsf{E}_g\left[|U(x,t)|^2\right] = \sum_{k=1}^{\infty} \mathsf{E}_g\left[|u_k(x,t)|^2\right] < \infty$$

for almost all $\omega_\xi \in \Omega_\xi$ and $\omega_\Gamma \in \Omega_\Gamma$. To check the latter condition, we estimate

$$\begin{aligned}
\mathsf{E}_{\xi,g}\left[|U(x,t)|^2\right] &\leq \mathsf{E}_{\xi,g}\left[C_\alpha^2 \sum_{k\geq 1} \Gamma_k^{-2/\alpha} \varphi(\xi_k)^{-2/\alpha} \frac{\left|\sigma\left(\xi_k, t - \frac{|x-\xi_k|}{a}\right)\right|^2}{|x-\xi_k|^2} g_k^2 I_{\{|x-\xi_k|<at\}}\right] \\
&\leq \mathsf{E}_\xi\left[C \sum_{k\geq 1} \Gamma_k^{-2/\alpha} \varphi(\xi_k)^{-2/\alpha} \frac{1}{|x-\xi_k|^2} I_{\{|x-\xi_k|<at\}}\right] \\
&= C \sum_{k\geq 1} \Gamma_k^{-2/\alpha} \iiint_{y:\,|x-y|<at} \frac{\varphi(y)^{1-2/\alpha} dy}{|x-y|^2} \\
&\leq C \left(\inf_{y:\,|x-y|\leq at} \varphi(y)\right)^{1-2/\alpha} \sum_{k\geq 1} \Gamma_k^{-2/\alpha} \iiint_{y:\,|x-y|<at} \frac{dy}{|x-y|^2} \\
&\leq C_{x,t} \sum_{k\geq 1} \Gamma_k^{-2/\alpha}.
\end{aligned}$$

The strong law of large numbers implies that $\Gamma_k \sim k$ as $k \to +\infty$ almost surely with respect to P_Γ. Then $\mathsf{E}_{\xi,g}\left[|U(x,t)|^2\right] < \infty$ almost surely with respect to P_Γ. Hence

$$\mathsf{E}_g\left[|U(x,t)|^2\right] < \infty$$

almost surely with respect to $\mathsf{P}_\xi \otimes \mathsf{P}_\Gamma$. This completes the proof. □

Now we show that the function $U(x,t)$ satisfies equation (4) in the generalized sense.

Theorem 3. *Let* $\theta(x,t) \in C_{fin}^\infty(\mathbb{R}^3 \times \mathbb{R}^+)$ *be an arbitrary function. Then*

$$\begin{aligned}
&\int_0^\infty \iiint_{\mathbb{R}^3} U(x,t)\left(\frac{\partial^2}{\partial t^2}\theta(x,t) - a^2\Delta\theta(x,t)\right) dx\, dt \\
&\qquad = \int_0^\infty \iiint_{\mathbb{R}^3} \theta(x,t)\sigma(x,t)\, M(dx)\, dt
\end{aligned} \tag{6}$$

with probability one.

Remark 1. The exceptional event of probability zero where equality (6) fails may depend on θ in Theorem 3.

Proof. We introduce the following notation:

$$\psi(x,t) = \frac{\partial^2}{\partial t^2}\theta(x,t) - a^2\Delta\theta(x,t),$$

$$\begin{aligned}
K(y) &= \int_0^\infty \iiint_{x:|x-y|<at} \frac{\varphi(y)^{-1/\alpha}\sigma\left(y, t-\frac{|x-y|}{a}\right)\psi(x,t)}{|x-y|}\,dx\,dt,\\
L(\psi) &= \int_0^\infty \iiint_{\mathbb{R}^3} U(x,t)\psi(x,t)\,dx\,dt\\
&= C_\alpha \int_0^\infty \iiint_{\mathbb{R}^3} \sum_{k=1}^\infty \Gamma_k^{-1/\alpha}\varphi(\xi_k)^{-1/\alpha}\frac{\sigma\left(\xi_k, t-\frac{|x-\xi_k|}{a}\right)}{|x-\xi_k|} g_k\psi(x,t) I_{\{|x-\xi_k|<at\}}\,dx\,dt\\
&= C_\alpha \sum_{k=1}^\infty \Gamma_k^{-1/\alpha} K(\xi_k) g_k,\\
R(\theta) &= \int_0^\infty \iiint_{\mathbb{R}^3} \theta(x,t)\sigma(x,t)\,M(dx)\,dt\\
&= C_\alpha \int_0^\infty \iiint_{\mathbb{R}^3} \sum_{k=1}^\infty \Gamma_k^{-1/\alpha}\varphi(\xi_k)^{-1/\alpha}\sigma(\xi_k,t)\theta(\xi_k,t) g_k\,dx\,dt.
\end{aligned}$$

Then equality (6) can be rewritten as $L(\psi) = R(\theta)$.

Assume that $\operatorname{supp}\theta \subset B(0,R)\times[0,R]$. Then $L(\psi) = 0$ for $|y| > R + at$, while

$$\begin{aligned}
|K(y)| &\le \left| \int_0^\infty \iiint_{x:\ |x-y|<at} \varphi(y)^{-1/\alpha}\,\frac{\sigma\left(y, t-\frac{|x-y|}{a}\right)}{|x-y|}\,\psi(x,t)\,dx\,dt \right|\\
&\le \int_0^R \iiint_{x:\ |x|<R} |\varphi(y)|^{-1/\alpha}\,\frac{\left|\sigma\left(y, t-\frac{|x-y|}{a}\right)\right|}{|x-y|}\,|\psi(x,t)|\,dx\,dt\\
&\le C\left(\inf_{x:\ |x|\le R+at}\varphi(x)\right)^{-1/\alpha} \sup_{\substack{x\in\mathbb{R}^3\\ t\ge 0}} |\psi(x,t)| \int_0^R \iiint_{x:\ |x|\le R} \frac{dx\,dt}{|x-y|} \le C_{R,\psi}
\end{aligned}$$

for $|y| \le R + at$. The equality

$$\mathsf{E}_g\left[L(\psi)^2\right] = C_\alpha^2 \sum_{k\ge 1} \Gamma_k^{-2/\alpha} K^2(\xi_k)$$

holds for all fixed $\omega_\xi \in \Omega_\xi$ and $\omega_\Gamma \in \Omega_\Gamma$. The above bound on $K(y)$ proved above implies that $\mathsf{E}_g\left[L(\psi)^2\right] < \infty$ for almost all $\omega_\xi \in \Omega_\xi$ and $\omega_\Gamma \in \Omega_\Gamma$. According to the strong law of large numbers $\Gamma_k \sim k$, $k \to +\infty$, almost surely with respect to P_Γ. Thus the series for $L(\psi)$ converges almost surely by Kolmogorov's theorem for an arbitrary function $\psi(x,t) \in C_{fin}^\infty(\mathbb{R}^3\times\mathbb{R}^+)$. The same is true for the series $R(\theta)$. It remains to check that the corresponding partial sums are equal, that is, to prove the equality

$$\begin{aligned}
&C_\alpha \sum_{k=1}^N \Gamma_k^{-1/\alpha}\varphi(\xi_k)^{-1/\alpha} g_k \int_0^\infty \iiint_{\mathbb{R}^3} \frac{\sigma\left(\xi_k, t-\frac{|x-\xi_k|}{a}\right)}{|x-\xi_k|}\,\psi(x,t) I_{\{|x-\xi_k|<at\}}\,dx\,dt\\
&\qquad = C_\alpha \sum_{k=1}^N \Gamma_k^{-1/\alpha}\varphi(\xi_k)^{-1/\alpha} g_k \int_0^\infty \theta(\xi_k,t)\sigma(\xi_k,t)\,dt.
\end{aligned} \tag{7}$$

Note that

$$\iiint_{\mathbb{R}^3}\int_{\frac{|x-y|}{a}}^{\infty}\frac{\sigma\left(y,t-\frac{|x-y|}{a}\right)}{|x-y|}\psi(x,t)\,dt\,dx=\int_0^{\infty}\iiint_{\mathbb{R}^3}\frac{\sigma(y,s)}{|x-y|}\psi\left(x,s+\frac{|x-y|}{a}\right)dx\,ds.$$

To prove equality (7), it suffices to check that the corresponding terms are equal, that is, to prove that

$$\iiint_{\mathbb{R}^3}\frac{\psi\left(x,s+\frac{|x-y|}{a}\right)}{|x-y|}\,dx=\theta(y,s) \tag{8}$$

for all $s>0$ and $y\in\mathbb{R}^3$. Since $\operatorname{supp}\theta\subset B(0,R)\times[0,R]$, we get $\theta(x,s)=0$ for all $s\ge R$ and $x\in\mathbb{R}^3$. In particular, equality (8) is obvious for $s\ge R$. To prove equality (8) for $s\in(0,R)$, put

$$\tilde\theta(x,u)=\theta(x,R-u),\qquad u\le R.$$

Then

$$\frac{\partial^2}{\partial t^2}\theta(x,R-u)=\frac{\partial^2}{\partial t^2}\tilde\theta(x,u),\qquad \Delta\theta(x,R-u)=\Delta\tilde\theta(x,u);$$
$$\tilde\theta(x,0)=0;\qquad \frac{\partial}{\partial t}\tilde\theta(x,0)=0.$$

Consider the following boundary problem:

$$\begin{cases}\left(\dfrac{\partial^2}{\partial t^2}-a^2\Delta\right)V(x,t)=\frac{\partial^2}{\partial t^2}\tilde\theta(x,t)-a^2\Delta\tilde\theta(x,t), & x\in\mathbb{R}^3,\ t\in(0,R],\\ V(x,0)=0,\\ \dfrac{\partial V}{\partial t}(x,0)=0.\end{cases}$$

Obviously, the function $\tilde\theta(x,t)$ satisfies this equation. On the other hand, a solution of this equation is of the form

$$\tilde\theta(y,t)=\frac{1}{4\pi a}\iiint_{x:\ |x-y|<at}\frac{\frac{\partial^2}{\partial t^2}\tilde\theta\left(x,t-\frac{|x-y|}{a}\right)-a^2\Delta\tilde\theta\left(x,t-\frac{|x-y|}{a}\right)}{|x-y|}\,dx$$

by the Poisson–Parseval formula. Substituting $t=R-s$, one obtains

$$\begin{aligned}\theta(y,s)&=\frac{1}{4\pi a}\iiint_{x:|x-y|<a(R-s)}\frac{\frac{\partial^2}{\partial t^2}\theta\left(x,s+\frac{|x-y|}{a}\right)-a^2\Delta\theta\left(x,s+\frac{|x-y|}{a}\right)}{|x-y|}\,dx\\&=\frac{1}{4\pi a}\iiint_{\mathbb{R}^3}\frac{\psi\left(x,s+\frac{|x-y|}{a}\right)}{|x-y|}\,dx.\end{aligned}$$

The latter equality holds, since $s+\frac{|x-y|}{a}\ge R$ for $|x-y|\ge a(R-s)$, whence $\psi(x,t)=0$. Therefore equality (8) is proved. □

4. Wave equation with a stable colored noise

Our next aim is to consider equation (4) with the "colored" random noise. In contrast to the "white" noise, the values of the colored noise are not independent. To be more precise, consider a random noise generated by a real harmonizable anisotropic fractional stable field with Hurst parameter H,

$$Z^H(x)=\operatorname{Re}\iiint_{\mathbb{R}^3}\prod_{l=1}^{3}\frac{e^{ix_ly_l}-1}{|y_l|^{H+1/\alpha}}\,M(dy).$$

For some technical reasons, we restrict the consideration to the case of $\alpha \in (1,2)$ and $H \in (1/3, 1)$.

Properties of such a random field are studied in [7]. Its multifractional analogues are studied in the papers [5, 13]. In particular, it is proved in [5, 13] that this field has a version whose realizations satisfy the Hölder condition with index $\beta \in (0, H)$. The LePage decomposition for this field is given by

$$Z^H(x) = C_\alpha \operatorname{Re} \sum_{k=1}^{\infty} \Gamma_k^{-1/\alpha} \prod_{l=1}^{3} \frac{e^{ix_l \xi_{k,l}} - 1}{|\xi_{k,l}|^{H+1/\alpha}} \varphi(\xi_k)^{-1/\alpha} g_k,$$

where

$$\varphi(x) = \prod_{l=1}^{3} \frac{K}{|x_l| \left(\left|\log|x_l|\right| + 1\right)^{1+\eta}}.$$

Here η is a fixed positive constant, and

$$K = \left(\int_{-\infty}^{+\infty} |x|^{-1} \left(\left|\log|x|\right| + 1\right)^{-1-\eta} dx \right)^{-1}$$

is a normalizing constant. The LePage decomposition is one of the main tools when studying properties of realizations of the field Z^H such as continuity or the Hölder condition mentioned above.

One cannot find the theory of integration with respect to the field Z^H in the literature. For this reason, we introduce the corresponding integral here in the way described below. Denote by

$$\psi(x) = \frac{1}{(2\pi)^{3/2}} e^{-|x|^2/2}, \qquad x \in \mathbb{R}^3,$$

the density of the standard normal distribution in $\mathbb{R}^3$. For every $\varepsilon > 0$, let

$$Z^{H,\varepsilon}(x) = \left(Z^H * \psi_\varepsilon\right) x = \iiint_{\mathbb{R}^3} \psi_\varepsilon(x - z) Z^H(z)\, dz$$

be the convolution, where $\psi_\varepsilon(x) = \frac{1}{\varepsilon^3} \psi\left(\frac{x}{\varepsilon}\right)$. Let

$$\begin{aligned} X^\varepsilon(z, y) &= \frac{\partial^3}{\partial x_1 \partial x_2 \partial x_3} \psi_\varepsilon(y - z) Z^H(z) \\ &= \frac{\prod_{l=1}^3 (y_l - z_l)}{(2\pi)^{3/2} \varepsilon^9} e^{-\frac{|y-z|^2}{2\varepsilon^2}} \operatorname{Re} \iiint_{\mathbb{R}^3} \prod_{l=1}^{3} \frac{e^{iz_l t_l} - 1}{|t_l|^{H+1/\alpha}} M(dt) \end{aligned}$$

and put

$$Y^\varepsilon(y) = \iiint_{\mathbb{R}^3} X^\varepsilon(z, y)\, dz.$$

We are going to prove that one can interchange the order of integration in the latter integral. According to Theorem 4.1 of [11], this follows if

$$\iiint_{\mathbb{R}^3} \left| X^\varepsilon(z, y) \right| dz < \infty$$

almost surely. In turn, the latter condition is equivalent to

$$\iiint_{\mathbb{R}^3} \frac{\prod_{l=1}^3 |y_l - z_l|}{(2\pi)^{3/2} \varepsilon^9} e^{-\frac{|y-z|^2}{2\varepsilon^2}} \left(\iiint_{\mathbb{R}^3} \left| \operatorname{Re} \prod_{l=1}^{3} \frac{e^{iz_l t_l} - 1}{|t_l|^{H+1/\alpha}} \right|^\alpha dt \right)^{1/\alpha} dz < \infty$$

by Theorem 3.3 of [11]. Applying the inequality $|e^{iz_l t_l} - 1| \le 2 \wedge |z_l t_l|$ and making the change $u_l = z_l t_l$, $l = 1, 2, 3$, the latter integral is estimated from above by

$$\iiint_{\mathbb{R}^3} \frac{\prod_{l=1}^3 |y_l - z_l||z_l|^{H+1/\alpha-1}}{(2\pi)^{3/2}\varepsilon^9} e^{-\frac{|y-z|^2}{2\varepsilon^2}} \left(\iiint_{\mathbb{R}^3} \prod_{l=1}^3 \frac{2^\alpha \wedge |u_l|^\alpha}{|u_l|^{\alpha H+1}} du \right)^{1/\alpha} dz < \infty.$$

Therefore

$$\begin{aligned} Y^\varepsilon(y) = \frac{1}{(2\pi)^{3/2}\varepsilon^9} \iiint_{\mathbb{R}^3} \prod_{l=1}^3 \frac{1}{|t_l|^{H+1/\alpha}} \\ \times \operatorname{Re} \iiint_{\mathbb{R}^3} \left(\prod_{j=1}^3 \left(e^{iz_l t_l} - 1\right)(y_j - z_j) \right) e^{-\frac{|y-z|^2}{2\varepsilon^2}} dz\, M(dt). \end{aligned} \tag{9}$$

Following similar reasoning we prove that

$$Z^{H,\varepsilon}(x) = \int_0^{x_1} \int_0^{x_2} \int_0^{x_3} Y^\varepsilon(y)\, dy_3\, dy_2\, dy_1$$

for an arbitrary $x \in \mathbb{R}^3$, whence we conclude that the field $Z^{H,\varepsilon}$ is, at least in the weak sense, differentiable. Moreover

$$\frac{\partial^3}{\partial x_1 \partial x_2 \partial x_3} Z^{H,\varepsilon}(x) = Y^\varepsilon(x).$$

Thus it is natural to define the integral with respect to the field Z^H for a function $f\colon \mathbb{R}^3 \to \mathbb{C}$ as the limit in probability

$$\iiint_{\mathbb{R}^3} f(x)\, Z^H(dx) = \lim_{\varepsilon \to 0+} \iiint_{\mathbb{R}^3} f(x) \frac{\partial^3}{\partial x_1 \partial x_2 \partial x_3} Z^{H,\varepsilon}(x)\, dx \tag{10}$$

if it exists.

Consider the wave equation with an α-stable colored perturbation

$$\begin{cases} \left(\dfrac{\partial^2}{\partial t^2} - a^2 \Delta\right) U(x,t) = \dot{Z}^H(x), \quad x \in \mathbb{R}^3,\ t > 0, \\ U(x,0) = 0, \\ \dfrac{\partial U}{\partial t}(x,0) = 0. \end{cases} \tag{11}$$

As in the preceding section, a candidate for a solution of equation (11) is defined by the Kirchhoff formula

$$U(x,t) = \frac{1}{4\pi a} \iiint_{y\colon |x-y| < at} \frac{1}{|x-y|} Z^H(dy). \tag{12}$$

The integral on the right hand side of equality (12) is understood as the limit in probability (10).

The further proof is split into several steps. First we introduce the function $U'(x,t)$ as the result of the formal differentiation of LePage's series for the field $Z^H(x)$. Write a formal equality

$$\frac{\partial^3}{\partial x_1 \partial x_2 \partial x_3} Z^H(x) = C_\alpha \operatorname{Re} \left(-i \sum_{k=1}^\infty \Gamma_k^{-1/\alpha} \varphi(\xi_k)^{-1/\alpha} g_k \prod_{l=1}^3 \frac{\operatorname{sign} \xi_{k,l} e^{ix_l \xi_{k,l}}}{|\xi_{k,l}|^{H+1/\alpha-1}} \right)$$

and put for a moment

$$U'(x,t) = C_\alpha \sum_{k=1}^{\infty} \Gamma_k^{-1/\alpha} \varphi(\xi_k)^{-1/\alpha} \prod_{l=1}^{3} \frac{\operatorname{sign} \xi_{k,l}}{|\xi_{k,l}|^{H+1/\alpha-1}} g_k \times \operatorname{Im}\left(e^{i(x,\xi_k)} \iiint_{y:\,|x-y|<at} \frac{e^{i(y-x,\xi_k)}}{|x-y|}\, dy \right). \tag{13}$$

Then

$$\begin{aligned}
\iiint_{y:\,|x-y|<at} \frac{e^{i(y-x,\xi_k)}}{|x-y|}\, dy &= \iiint_{|z|<at} \frac{e^{i(z,\xi_k)}}{|z|}\, dz = \left|\xi_k \to |\xi_k|\, \overrightarrow{e_3}\right| \\
&= \iiint_{|z|<at} \frac{e^{iz_3|\xi_k|}}{|z|}\, dz = \int_0^{2\pi}\int_0^{\pi}\int_0^{at} \rho^2 \sin\theta \frac{e^{i\rho\cos\theta|\xi_k|}}{\rho}\, d\rho\, d\theta\, d\nu \\
&= 2\pi \int_0^{\pi} d\theta \int_0^{at} \rho \sin\theta e^{i\rho\cos\theta|\xi_k|}\, d\rho\, d\theta = -\frac{2\pi}{i\,|\xi_k|} \int_0^{at} e^{i\rho\cos\theta|\xi_k|}\Big|_0^{\pi}\, d\rho \\
&= \frac{4\pi}{|\xi_k|} \int_0^{at} \frac{e^{i\rho|\xi_k|} - e^{-i\rho|\xi_k|}}{2i}\, d\rho = \frac{4\pi}{|\xi_k|} \int_0^{at} \sin\rho\,|\xi_k|\, d\rho = \frac{4\pi}{|\xi_k|^2} \left(-\cos\rho\,|\xi_k|\right)\Big|_0^{at} \\
&= \frac{4\pi}{|\xi_k|^2} \left(1 - \cos at\,|\xi_k|\right).
\end{aligned}$$

The latter formula is used to define the function U' by putting

$$U'(x,t) = \frac{C_\alpha}{a} \sum_{k=1}^{\infty} \Gamma_k^{-1/\alpha} \varphi(\xi_k)^{-1/\alpha} \prod_{l=1}^{3} \frac{\operatorname{sign} \xi_{k,l}}{|\xi_{k,l}|^{H+1/\alpha-1}} \sin(x,\xi_k) \frac{1-\cos at\,|\xi_k|}{|\xi_k|^2} g_k. \tag{14}$$

Proposition 1. *The series on the right hand side of equality* (14) *converges almost surely.*

Proof. Since the random variables g_k are independent,

$$\begin{aligned}
\mathsf{E}_g\left[U'(x,t)^2\right] &= \frac{C_\alpha^2}{a^2} \sum_{k=1}^{\infty} \Gamma_k^{-2/\alpha} \varphi(\xi_k)^{-2/\alpha} \prod_{l=1}^{3} \frac{1}{|\xi_{k,l}|^{2H+2/\alpha-2}} \sin^2(x,\xi_k) \frac{(1-\cos at\,|\xi_k|)^2}{|\xi_k|^4} \\
&=: \frac{C_\alpha^2}{a^2} \sum_{k=1}^{\infty} \Gamma_k^{-2/\alpha} g(t,x,\xi_k).
\end{aligned}$$

Then we estimate the expectation,

$$\begin{aligned}
\mathsf{E}_\xi\left[g(t,x,\xi_k)\right] &= \iiint_{\mathbb{R}^3} \varphi(y)^{1-2/\alpha} \frac{\sin^2(x,y)(1-\cos at\,|y|)^2}{|y|^4} \prod_{l=1}^{3} \frac{dy_l}{|y_l|^{2H+2/\alpha-2}} \\
&= K^{3-6/\alpha} \iiint_{\mathbb{R}^3} \frac{\sin^2(x,y)(1-\cos at\,|y|)^2}{|y|^4} \prod_{l=1}^{3} \frac{dy_l}{|y_l|^{2H-1}\left(\left|\log|y_l|\right|+1\right)^{1+\eta-2/\alpha-2\eta/\alpha}} \\
&\le K^{3-6/\alpha} \iiint_{\mathbb{R}^3} \frac{\left(1\wedge |x|^2\,|y|^2\right)\left(4\wedge a^4t^4\,|y|^4\right)}{|y|^4} \\
&\qquad \times \prod_{l=1}^{3} \frac{dy_l}{|y_l|^{2H-1}\left(\left|\log|y_l|\right|+1\right)^{1+\eta-2/\alpha-2\eta/\alpha}}.
\end{aligned}$$

Now we make the spherical change of variables

$$y_1 = \rho\sin\theta\cos\nu, \qquad y_2 = \rho\sin\theta\sin\nu, \qquad y_3 = \rho\cos\theta, \\ \rho > 0, \qquad \theta \in [0,\pi], \qquad \nu \in [0,2\pi]. \tag{15}$$

Using the inequality

$$\left(\left|\log|z|\right| + 1\right)^d \le C_\varepsilon\left(|z|^{-\varepsilon} \vee |z|^{\varepsilon}\right), \tag{16}$$

where $d = 2/\alpha + 2\eta/\alpha - 1 - \eta$, we prove the bound

$$\left(\left|\log|\rho\sin\theta\cos\nu|\right| + 1\right)^d \le C_\varepsilon\left(\rho^\varepsilon \vee \rho^{-\varepsilon}\right)|\sin\theta|^{-\varepsilon}|\cos\nu|^{-\varepsilon}. \tag{17}$$

Other logarithms are estimated similarly. Thus

$$\begin{aligned}\mathsf{E}_\xi\left[g(t,x,\xi_k)\right] \le C_\varepsilon \int_0^\infty\int_0^{2\pi}\int_0^\pi & \frac{\left(\rho^\varepsilon \vee \rho^{-\varepsilon}\right)^3\left(1\wedge|x|^2\rho^2\right)\left(4\wedge a^4t^4\rho^4\right)}{\rho^{6H-1}} \\ & \times|\sin\theta|^{3-4H-2\varepsilon}|\cos\theta|^{1-2H-\varepsilon}|\sin\nu|^{1-2H-\varepsilon}|\cos\nu|^{1-2H-\varepsilon}\,d\theta\,d\nu\,d\rho,\end{aligned}$$

whence

$$\mathsf{E}_\xi\left[g(t,x,\xi_k)\right] \le C_\varepsilon \int_0^\infty \frac{\left(\rho^\varepsilon \vee \rho^{-\varepsilon}\right)^3\left(1\wedge|x|^2\rho^2\right)\left(4\wedge a^4t^4\rho^4\right)}{\rho^{6H-1}}\,d\rho$$

for $\varepsilon < 2 - 2H$. The latter integral converges at zero, since $7 - 6H - 3\varepsilon > -1$. It converges at infinity for $1 - 6H + 3\varepsilon < -1$, that is, for $\varepsilon < 2(H - 1/3)$. Setting $\varepsilon = (1-H)\wedge(H-1/3)$, we obtain $\mathsf{E}_\xi\left[g(t,x,\xi_k)\right] < \infty$, whence

$$\mathsf{E}_{g,\xi}\left[U'(x,t)^2\right] = \frac{C_\alpha^2}{a^2}\sum_{k=1}^\infty \Gamma_k^{-2/\alpha}\mathsf{E}_\xi\left[g(t,x,\xi_k)\right] < \infty$$

almost surely with respect to P_Γ. As above, this together with the Kolmogorov theorem implies the almost sure convergence of the series for $U'(x,t)$ in (14). □

Put

$$U^\varepsilon(x,t) = \frac{1}{4\pi a}\iiint_{\mathbb{R}^3} \frac{\partial^3}{\partial x_1\partial x_2\partial x_3} Z^{H,\varepsilon}(y)\,\frac{dy}{|x-y|}.$$

Theorem 4. *The convergence in probability*

$$U^\varepsilon(x,t) \xrightarrow{\mathsf{P}} U'(x,t), \qquad \varepsilon \to 0+, \tag{18}$$

holds for all $t > 0$ and $x \in \mathbb{R}^3$.

Proof. The inner integral on the right hand side of the expression for $\frac{\partial^3}{\partial x_1\partial x_2\partial x_3}Z^{H,\varepsilon}$ is transformed as follows:

$$\begin{aligned}\frac{1}{\sqrt{2\pi\varepsilon^3}}\int_{\mathbb{R}}\left(e^{iz_it_i} - 1\right)(y_j - z_j)\,e^{-\frac{|y_j-z_j|^2}{2\varepsilon^2}}dz_j &= \frac{1}{\sqrt{2\pi\varepsilon^3}}\int_{\mathbb{R}} e^{iz_it_i}\,(y_j - z_j)\,e^{-\frac{|y_j-z_j|^2}{2\varepsilon^2}}dz_j \\ &= e^{iy_jt_j}\int_{\mathbb{R}} e^{i(z_j-y_j)t_j}(y_j - z_j)e^{-\frac{|y_j-z_j|^2}{2\varepsilon^2}}\frac{dz_j}{\sqrt{2\pi\varepsilon^3}} \\ &= e^{iy_jt_j}\int_{\mathbb{R}} e^{ixt_j}xe^{-\frac{|x|^2}{2\varepsilon^2}}\frac{dx}{\sqrt{2\pi\varepsilon^3}} = e^{iy_jt_j}\widehat{\psi'_\varepsilon}(t_j) \\ &= e^{iy_jt_j}(-it_j)\widehat{\psi_\varepsilon}(t_j) = -it_je^{iy_jt_j}e^{-\varepsilon^2t_j^2/2}.\end{aligned}$$

Then

$$\frac{\partial^3}{\partial x_1\partial x_2\partial x_3}Z^{H,\varepsilon}(y) = i\iiint_{\mathbb{R}^3} e^{i(z,y)}\prod_{j=1}^3\frac{\operatorname{sign} z_j e^{-\varepsilon^2z_j^2/2}}{|z_j|^{H+1/\alpha-1}}\,M(dz).$$

Following the same reasoning as led us to (9) we obtain

$$U^\varepsilon(x,t) = \frac{1}{4\pi a}\,\mathrm{Re}\left(i\iiint_{y:|x-y|<at}\frac{1}{|x-y|}\right.$$
$$\left.\times\iiint_{\mathbb{R}^3} e^{i(z,y)}\prod_{j=1}^{3}\frac{\operatorname{sign} z_j}{|z_j|^{H+1/\alpha-1}}\, e^{-\varepsilon^2|z|^2/2}\, M(dz)\, dy\right)$$
$$= \frac{1}{4\pi a}\,\mathrm{Re}\left(i\iiint_{\mathbb{R}^3}\prod_{j=1}^{3}\frac{\operatorname{sign} z_j}{|z_j|^{H+1/\alpha-1}}\, e^{-\varepsilon^2|z|^2/2}\iiint_{y:|x-y|<at}\frac{e^{i(z,x)}}{|x-y|}\,dy\, M(dz)\right)$$
$$= \frac{1}{a}\,\mathrm{Re}\left(i\iiint_{\mathbb{R}^3}\prod_{j=1}^{3}\frac{\operatorname{sign} z_j}{|z_j|^{H+1/\alpha-1}}\, e^{-\varepsilon^2|z|^2/2}\,\frac{e^{i(z,x)}}{|z|^2}\,(1-\cos at\,|z|)\, M(dz)\right)$$
$$= \frac{1}{a}\iiint_{\mathbb{R}^3}\prod_{j=1}^{3}\frac{\operatorname{sign} z_j}{|z_j|^{H+1/\alpha-1}}\, e^{-\varepsilon^2|z|^2/2}\sin(z,x)\,\frac{1-\cos at\,|z|}{|z|^2}\, M(dz).$$

Let

$$U_N(x,t) = \frac{C_\alpha}{a}\sum_{k=1}^{N}\varphi(\xi_k)^{-1/\alpha}\prod_{j=1}^{3}\frac{\operatorname{sign}\xi_{k,j}}{|\xi_{k,j}|^{H+1/\alpha-1}}\sin(x,\xi_k)\,\frac{1-\cos at\,|\xi_k|}{|\xi_k|^2}\, g_k,$$
$$U_N^\varepsilon(x,t) = \frac{C_\alpha}{a}\sum_{k=1}^{N}\Gamma_k^{-1/\alpha}\varphi(\xi_k)^{-1/\alpha}\prod_{j=1}^{3}\frac{\operatorname{sign}\xi_{k,j}}{|\xi_{k,j}|^{H+1/\alpha-1}}\, e^{\frac{-\varepsilon^2|\xi_k|^2}{2}}\sin(\xi_k,x)\frac{1-\cos at\,|\xi_k|}{|\xi_k|^2}\, g_k$$

be the partial sums of the LePage series. For a given $\delta>0$,

$$\begin{aligned}\mathsf{P}\left(|U'(x,t)-U^\varepsilon(x,t)|>\delta\right) \le{}& \mathsf{P}\left(|U_N(x,t)-U_N^\varepsilon(x,t)|>\frac{\delta}{3}\right)\\ &+\mathsf{P}\left(|U'(x,t)-U_N(x,t)|>\frac{\delta}{3}\right)\\ &+\mathsf{P}\left(|U^\varepsilon(x,t)-U_N^\varepsilon(x,t)|>\frac{\delta}{3}\right).\end{aligned}\tag{19}$$

It is easy to see that

$$U_N^\varepsilon(x,t)\to U_N(x,t),\qquad \varepsilon\to 0+,$$

almost surely for all $t\ge 0$, $x\in\mathbb{R}^3$, and $N\ge 1$. Then

$$\begin{aligned}&\varlimsup_{\varepsilon\to 0+}\mathsf{P}\left(|U'(x,t)-U^\varepsilon(x,t)|>\delta\right)\\ &\quad\le \mathsf{P}\left(|U'(x,t)-U_N(x,t)|>\frac{\delta}{3}\right)+\varlimsup_{\varepsilon\to 0+}\mathsf{P}\left(|U^\varepsilon(x,t)-U_N^\varepsilon(x,t)|>\frac{\delta}{3}\right).\end{aligned}$$

Proposition 1 implies that

$$\mathsf{P}\left(|U'(x,t)-U_N(x,t)|>\frac{\delta}{3}\right)\to 0,\qquad N\to\infty.$$

Therefore

$$\varlimsup_{\varepsilon\to 0+}\mathsf{P}\left(|U'(x,t)-U^\varepsilon(x,t)|>\delta\right)\le\varlimsup_{N\to\infty}\varlimsup_{\varepsilon\to 0+}\mathsf{P}\left(|U^\varepsilon(x,t)-U_N^\varepsilon(x,t)|>\frac{\delta}{3}\right).\tag{20}$$

Now

$$\varlimsup_{\varepsilon\to 0+}\mathsf{P}\left(|U^\varepsilon(x,t)-U_N^\varepsilon(x,t)|>\frac{\delta}{3}\right)\le\mathsf{E}_{\Gamma,\xi}\left[\sup_{\varepsilon>0}\mathsf{P}_g\left(|U^\varepsilon(x,t)-U_N^\varepsilon(x,t)|>\frac{\delta}{3}\right)\right].\tag{21}$$

For an arbitrary $\varepsilon > 0$,

$$\mathsf{P}_g\left(|U^\varepsilon(x,t) - U_N^\varepsilon(x,t)| > \frac{\delta}{3}\right) \le \frac{9}{\delta^2}\mathsf{E}_g\left[|U^\varepsilon(x,t) - U_N^\varepsilon(x,t)|^2\right] \tag{22}$$

by the Chebyshev inequality. Now we conclude that

$$\begin{aligned}
&\mathsf{E}_g\left[|U^\varepsilon(x,t) - U_N^\varepsilon(x,t)|^2\right] \\
&\quad \le C\sum_{k=N+1}^{\infty}\Gamma_k^{-2/\alpha}\varphi(\xi_k)^{-2/\alpha}\prod_{j=1}^{3}\frac{1}{|\xi_{k,j}|^{2H+2/\alpha-2}}\sin^2(\xi_k,x)\frac{(1-\cos at\,|\xi_k|)^2}{|\xi_k|^4} \\
&\quad =: C\sum_{k=N+1}^{\infty}\Gamma_k^{-2/\alpha}g(t,x,\xi_k)
\end{aligned}$$

by considering the LePage decomposition. Proposition 1 implies that the series

$$\sum_{k=1}^{\infty}\Gamma_k^{-2/\alpha}g(t,x,\xi_k)$$

converges almost surely with respect to $\mathsf{P}_\Gamma \otimes \mathsf{P}_\xi$. Taking into account inequality (22) we get

$$\sup_{\varepsilon>0}\mathsf{P}_g\left(|U^\varepsilon(x,t) - U_N^\varepsilon(x,t)| > \frac{\delta}{3}\right) \to 0$$

as $N \to \infty$ almost surely with respect to $\mathsf{P}_\Gamma \otimes \mathsf{P}_\xi$. Now we derive from bounds (20) and (21) that

$$\overline{\lim_{\varepsilon\to 0+}}\,\mathsf{P}\left(|U'(x,t) - U^\varepsilon(x,t)| > \delta\right) = 0$$

by applying the Lebesgue dominated convergence theorem. The proof of Theorem 4 is complete. ☐

Like the case of the wave equation with a white noise, the function $U(x,t)$ satisfies equation (11) in the generalized sense.

Theorem 5. *For an arbitrary function $\theta(x,t) \in C_{fin}^\infty(\mathbb{R}^3 \times \mathbb{R}^+)$,*

$$\begin{aligned}
&\int_0^\infty\iiint_{\mathbb{R}^3}U(x,t)\left(\frac{\partial^2}{\partial t^2}\theta(x,t) - a^2\Delta\theta(x,t)\right)dx\,dt \\
&\qquad = \int_0^\infty\iiint_{\mathbb{R}^3}\theta(x,t)\sigma(x,t)Z^H(dx)\,dt
\end{aligned} \tag{23}$$

with probability one.

Proof. Since the reasoning is completely analogous to what had been demonstrated above, we provide only an outlined proof. Put

$$\frac{\partial^2}{\partial t^2}\theta(x,t) - a^2\Delta\theta(x,t) = \psi(x,t).$$

Then equality (23) can be rewritten as

$$\int_0^\infty\iiint_{\mathbb{R}^3}U(x,t)\psi(x,t)\,dx\,dt = \int_0^\infty\iiint_{\mathbb{R}^3}\theta(x,t)\sigma(x,t)\,Z^H(dx)dt.$$

Note that

$$\int_0^\infty\iiint_{\mathbb{R}^3}\theta(x,t)\sigma(x,t)\,Z^H(dx)\,dt = \lim_{\varepsilon\to 0}\int_0^\infty\iiint_{\mathbb{R}^3}\theta(x,t)\sigma(x,t)\,Z^{H,\varepsilon}(dx)\,dt.$$

Since the field $Z^{H,\varepsilon}$ is smooth, the standard properties of partial differential equations imply that

$$\int_0^\infty \iiint_{\mathbb{R}^3} U^\varepsilon(x,t)\psi(x,t)\,dx\,dt = \int_0^\infty \iiint_{\mathbb{R}^3} \theta(x,t)\sigma(x,t)\,Z^{H,\varepsilon}(dx)\,dt.$$

Further, similarly to the proof of Theorem (2), one can show that

$$\int_0^\infty \iiint_{\mathbb{R}^3} U^\varepsilon(x,t)\psi(x,t)\,dx\,dt \xrightarrow{\mathsf{P}} \int_0^\infty \iiint_{\mathbb{R}^3} U(x,t)\psi(x,t)\,dx\,dt.$$

On the other hand,

$$\lim_{\varepsilon\to 0} \int_0^\infty \iiint_{\mathbb{R}^3} \theta(x,t)\sigma(x,t)\,Z^{H,\varepsilon}(dx)\,dt = \int_0^\infty \iiint_{\mathbb{R}^3} \theta(x,t)\sigma(x,t)\,Z^{H}(dx)\,dt$$

by the definition. □

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