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THE ASYMPTOTIC BEHAVIOR OF THE TOTAL NUMBER OF PARTICLES IN A CRITICAL BRANCHING PROCESS WITH IMMIGRATION

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ABSTRACT. A sequence of branching processes with immigration is considered in the case where the mean number of descendents of a particle tends to unity. The rate of growth and asymptotic behavior of the total number of particles in the population are found.

1. INTRODUCTION

Assume that $\{\xi_{k,j}^{(n)}, k, j \in \mathbb{N}\}$ and $\{\varepsilon_k^{(n)},\ k \in \mathbb{N}\}$ are two independent families of independent nonnegative integer valued and identically distributed random variables for all $n \in \mathbb{N}$. Let $\{X_k^{(n)}, k = 0, 1, \dots\}$, $n \in \mathbb{N}$, be a sequence of branching processes with immigration defined by the following recurrence relations:

$$(1) \qquad X_0^{(n)} = 0, \qquad X_k^{(n)} = \sum_{j=1}^{X_{k-1}^{(n)}} \xi_{k,j}^{(n)} + \varepsilon_k^{(n)}, \qquad k, n \in \mathbb{N}.$$

If the random variables $\xi_{k,j}^{(n)}$ and $\varepsilon_k^{(n)}$, $n \geq 1$, are treated as the number of descendents of a jth particle in a $(k-1)$th generation of a certain population of particles and the number of particles immigrating into the k^{th} generation, respectively, then $X_k^{(n)}$ is the total number of particles in the kth generation of the population.

A sequence of branching processes with immigration (1) is called almost critical if $\mathsf{E}\xi_{1,1}^{(n)} \to 1$ as $n \to \infty$. Consider random stepwise processes Z_n, $n \in \mathbb{N}$, defined by

$$Z_n(t) = \sum_{k=1}^{[nt]} X_k^{(n)}, \qquad t \geq 0, \quad n \in \mathbb{N}.$$

It is clear that the trajectories of the processes Z_n, $n \in \mathbb{N}$, belong to the Skorokhod space $D[0, \infty)$. The variable $Z_n(t)$ is the total number of particles (counted up to the moment $[nt]$) in the branching process with immigration $X_k^{(n)}$, $k \geq 0$.

Pakes [1] studies the rate of growth and asymptotic behavior (as $n \to \infty$) of the fluctuation of the total number of descendents of a particle in a branching Galton–Watson process up to the moment n under the assumption that the process does not vanish. Karpenko and Nagaev [2] investigate the limit behavior of the conditional distribution of the total number of descendents of a particle in the Galton–Watson process under the condition that the process vanishes at the moment n and in the case where the

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expectation of the number of particles generated by a single particle tends to unity as $n \to \infty$. A number of papers, say [3, 4, 8, 9], are devoted to functional limit theorems for a sequence of branching processes with immigration. On the other hand, the asymptotic behavior of the total number of particles Z_n has been studied quite a bit.

The aim of this paper is to study the asymptotic behavior of the process Z_n as well as that of its deviation from the mean value, $Z_n - \mathsf{E}Z_n$, for the case where the mean value of the number of descendants of a single particle tends to unity from the left with a rate being slower than n^{-1} as $n \to \infty$.

2. Mainstream

We assume that

$$m_n = \mathsf{E}\xi_{1,1}^{(n)}, \qquad \sigma_n^2 = \operatorname{Var}\xi_{1,1}^{(n)}, \qquad \lambda_n = \mathsf{E}\varepsilon_1^{(n)}, \qquad b_n^2 = \operatorname{Var}\varepsilon_1^{(n)}$$

are finite for all $n \in \mathbb{N}$. Throughout below d_n, $n \in \mathbb{N}$, denotes a certain sequence of positive numbers such that $d_n \to \infty$ and $n^{-1}d_n \to 0$ as $n \to \infty$; $W(t)$, $t \geq 0$, is a standard Wiener process in the space $D[0,\infty)$; $I(A)$ is the indicator of an event A; and the symbol $\xrightarrow{\mathsf{P}}$ denotes the convergence in probability of random variables.

Theorem 1 provides some information on the rate of growth of the process Z_n.

Theorem 1. *Assume that*

(1) $m_n = 1 + \alpha d_n^{-1} + o\left(d_n^{-1}\right)$ *as* $n \to \infty$ *for some fixed* $\alpha < 0$*;*
(2) *the limit* $\lim_{n\to\infty} \sigma_n^2 = \sigma^2 \geq 0$ *exists and is finite;*
(3) *the limits* $\lim_{n\to\infty} = \lambda \geq 0$ *and* $\lim_{n\to\infty} b_n^2 = b^2 \geq 0$ *exist and are finite.*

Then

$$\frac{Z_n}{nd_n} \to Z \quad \text{as } n \to \infty$$

in the space $D[0,\infty)$ *equipped with the Skorokhod* J*-topology, where the limit process* Z *is defined by*

$$Z(t) = |\alpha|^{-1}\lambda t, \qquad t \geq 0.$$

Theorem 2 describes the asymptotic behavior of the deviation of the process Z_n from its mean value.

Theorem 2. *Assume that*

(1) $m_n = 1 + \alpha d_n^{-1} + o\left(d_n^{-1}\right)$ *as* $n \to \infty$ *for some fixed* $\alpha < 0$*;*
(2) *the limit* $\lim_{n\to\infty} d_n\sigma_n^2 = \sigma^2 \geq 0$ *exists and is finite;*
(3) *the limits* $\lim_{n\to\infty} \lambda_n = \lambda \geq 0$ *and* $\lim_{n\to\infty} b_n^2 = b^2 \geq 0$ *exist and are finite;*
(4) *for all* $\varepsilon > 0$*,*

$$d_n\mathsf{E}\big(\xi_{1,1}^{(n)} - m_n\big)^2 I\big(\big|\xi_{1,1}^{(n)} - m_n\big| > \varepsilon\sqrt{n}\big) \to 0, \qquad n \to \infty;$$

(5) *for all* $\varepsilon > 0$*,*

$$\mathsf{E}\big(\varepsilon_1^{(n)} - \lambda_n\big)^2 I\big(\big|\varepsilon_1^{(n)} - \lambda_n\big| > \varepsilon\sqrt{n}\big) \to 0, \qquad n \to \infty.$$

Then

$$\left(d_n\sqrt{n}\right)^{-1}(Z_n - \mathsf{E}Z_n) \to Y \quad \text{as } n \to \infty$$

in the space $D[0,\infty)$ *equipped with the Skorokhod* J*-topology, where the limit process* Y *is defined by*

$$Y(t) = |\alpha|^{-1}\left(|\alpha|^{-1}\lambda\sigma^2 + b^2\right)^{1/2} W(t), \qquad t \geq 0.$$

Remark 1. The asymptotic behavior of the process Z_n and that of its deviation are easy to extract from the results of papers [3, 4, 8] and Theorem 5.1 of [5] in the case where $m_n = 1 + \alpha n^{-1} + o\left(n^{-1}\right)$.

Remark 2. Theorems 1 and 2 show that the rate of convergence of m_n to unity influences essentially the rate of growth and asymptotic behavior of the deviation of the process Z_n.

Remark 3. In general, the conditions $b_n^2 \to b^2 > 0$ and $d_n\sigma^2 \to \sigma^2 > 0$ do not imply assumptions (4) and (5) of Theorem 2, respectively. For example, let $\varepsilon_k^{(n)}$ assume values 0, 1, and n with probabilities n^{-2}, $1-2n^{-2}$, and n^{-2}, respectively. Then $\lambda_n = 1+n^{-1}+o\left(n^{-1}\right)$ and $b_n^2 = 1-2n^{-1}+o\left(n^{-1}\right)$ as $n\to\infty$. For every $\varepsilon>0$,

$$\mathsf{E}\left(\varepsilon_1^{(n)}-\lambda_n\right)^2 I\left(\left|\varepsilon_1^{(n)}-\lambda_n\right| > \varepsilon\sqrt{n}\right) \approx \frac{(n-1)^2}{n^2} \to 1 \quad \text{as } n\to\infty.$$

Thus $b_n^2 \to 1$ as $n\to\infty$ in this case, but assumption 5 does not hold.

3. Proofs

Proof of Theorem 1. It is easy to see that

$$\mathsf{E}X_k^{(n)} = \frac{1-m_n^k}{1-m_n}\,\lambda_n, \qquad k=0,1,2,\dots . \tag{2}$$

Put $G_n(t) = (nd_n)^{-1} Z_n(t)$. The assumptions of Theorem 1 imply that

$$\mathsf{E}G_n(t) \to |\alpha|^{-1}\lambda t \quad \text{as } n\to\infty. \tag{3}$$

We are going to estimate $\operatorname{Var} G_n(t)$. Equality (2.13) of the paper [4] yields

$$\operatorname{Var} G_n(t) = (nd_n)^{-2}\left(U_n(t)b_n^2 + V_n(t)\lambda_n\sigma_n^2\right),$$

where

$$U_n(t) = \sum_{k=1}^{[nt]+1} \frac{1-m_n^{2(k-1)}}{1-m_n^2}\left(2\,\frac{1-m_n^{[nt]-k+2}}{1-m_n}-1\right),$$

$$V_n(t) = \sum_{k=1}^{[nt]+1} \frac{\left(1-m_n^{k-1}\right)\left(1-m_n^{k-2}\right)}{\left(1-m_n\right)\left(1-m_n^2\right)}\left(2\,\frac{1-m_n^{[nt]-k+2}}{1-m_n}-1\right).$$

It is easy to see that

$$U_n(t) \le \frac{2(1+nt)}{(1-m_n)^2}, \qquad V_n(t) \le \frac{2(1+nt)}{(1-m_n)^3}.$$

Then

$$\operatorname{Var} G_n(t) \le 2\alpha^{-2}\left(\frac{b_n^2}{n}+\frac{\lambda_n\sigma_n^2 d_n}{\alpha n}\right)t \to 0 \quad \text{as } n\to\infty \tag{4}$$

for all $t\ge 0$. Applying the Chebyshev inequality we conclude from here that $G_n(t) \xrightarrow{\mathsf{P}} Z(t)$ as $n\to\infty$ for all $t\ge 0$ in view of relation (3). Since the limit process Z is continuous and nonrandom, it remains to show that the sequence of processes $\{G_n(t), t\ge 0\}$, $n\in\mathbb{N}$, is dense (see Theorem 15.1 of [5]). Indeed, relations (3) and (4) imply that

$$\begin{aligned}\mathsf{E}\big(G_n(t)-G_n(s)\big)^2 &\le 3\left(\operatorname{Var} G_n(s)+\operatorname{Var} G_n(t)+\big(\mathsf{E}G_n(t)-\mathsf{E}G_n(s)\big)^2\right)\\ &\le 4\alpha^{-2}\lambda^2(t-s)^2\end{aligned}$$

for all $t,s\ge 0$ and all sufficiently large n. Hence the sequence of processes $\{G_n(t), t\ge 0\}$, $n\in\mathbb{N}$, is dense by the density criteria 15.5 and 12.3 of [5]. □

Now we are going to prove the following three auxiliary results and then use them to derive the statement of Theorem 2. Put

$$M_k^{(n)} = \sum_{j=1}^{X_{k-1}^{(n)}} \left(\xi_{k,j}^{(n)} - m_n\right) + \varepsilon_k^{(n)} - \lambda_k, \qquad k = 1, 2, \dots .$$

Denote by $F_k^{(n)}$ the σ-algebra generated by random variables $\{X_0^{(n)}, X_1^{(n)}, \dots, X_k^{(n)}\}$. It is clear that $\{M_k^{(n)}, k \geq 0\}$ is a martingale-difference with respect to the flow of σ-algebras $F_k^{(n)}$, $k \geq 0$.

Lemma 1. *The representation*

$$W_n(t) = [d_n (1 - m_n)]^{-1} \left(\widetilde{M}_n^{(1)}(t) - m_n \widetilde{M}_n^{(2)}(t)\right)$$

holds, where

$$W_n(t) = \left(d_n \sqrt{n}\right)^{-1} \left(Z_n(t) - \mathsf{E} Z_n(t)\right),$$

$$\widetilde{M}_n^{(1)}(t) = n^{-1/2} \sum_{j=1}^{[nt]} M_j^{(n)}, \qquad \widetilde{M}_n^{(2)}(t) = n^{-1/2} \sum_{j=1}^{[nt]} m_n^{[nt]-j} M_j^{(n)}.$$

Proof of Lemma 1. The variable $X_k^{(n)}$ is represented as

$$X_k^{(n)} = m_n X_{k-1}^{(n)} + \lambda_n + M_k^{(n)}, \qquad k = 1, 2, \dots,$$

in view of equality (1), whence $\mathsf{E} X_k^{(n)} = m_n \mathsf{E} X_{k-1}^{(n)} + \lambda_n$, $k = 1, 2, \dots$. Therefore the random variables $X_k^{(n)} - \mathsf{E} X_k^{(n)}$, $k \geq 0$, satisfy the recurrence equation

$$X_k^{(n)} - \mathsf{E} X_k^{(n)} = m_n \left(X_{k-1}^{(n)} - \mathsf{E} X_{k-1}^{(n)}\right) + M_k^{(n)}, \qquad k = 1, 2, \dots .$$

The solution of the latter recurrence equation is given by

$$X_k^{(n)} - \mathsf{E} X_k^{(n)} = \sum_{j=1}^{k} m_n^{k-j} M_j^{(n)}, \qquad k = 1, 2, \dots .$$

Summing up the latter equalities with respect to k from 1 to $[nt]$ and normalizing the result appropriately, we complete the proof of the lemma. □

Lemma 2. *If assumptions* (1)–(3) *of Theorem* 2 *hold, then*

$$\widetilde{M}_n^{(2)}(t) \to 0 \quad \textit{as } n \to \infty$$

in the Skorokhod space $D[0, \infty)$.

Proof of Lemma 2. It is easy to see that

$$\begin{aligned} & n^{-1} \sum_{j=1}^{[nt]} m_n^{2([nt]-1)} \mathsf{E} \left(\left(M_k^{(n)}\right)^2 / F_{j-1}^{(n)} \right) \\ & = \frac{\sigma_n^2}{n} \sum_{j=1}^{[nt]} m_n^{2([nt]-1)} X_{j-1}^{(n)} + \frac{b_n^2}{n} \cdot \frac{1 - m_n^{2[nt]}}{1 - m_n^2}. \end{aligned} \tag{5}$$

By assumptions (1)–(3) of Theorem 2 and by using relation (2) we get

$$\frac{b_n^2}{n} \cdot \frac{1 - m_n^{2[nt]}}{1 - m_n^2} \sim \frac{b^2}{2|\alpha|} \cdot \frac{d_n}{n} \to 0, \qquad \frac{\sigma_n^2}{n} \sum_{j=1}^{[nt]} m_n^{2([nt]-j)} \mathsf{E} X_{j-1}^{(n)} \sim \frac{\lambda \sigma^2}{2\alpha^2} \cdot \frac{d_n}{n} \to 0$$

as $n \to \infty$. This together with (5) implies that

$$n^{-1} \sum_{j=1}^{[nt]} m_n^{2([nt]-j)} \mathsf{E}\left(\left(M_j^{(n)} \right)^2 / F_{j-1}^{(n)} \right) \xrightarrow{\mathsf{P}} 0 \quad \text{as } n \to \infty.$$

Then

$$n^{-1} \sum_{j=1}^{[nt]} m_n^{2([nt]-j)} \mathsf{E}\left(\left(M_j^{(n)} \right)^2 I \left(m_n^{[nt]-j} \left| M_j^{(n)} \right| > \varepsilon \sqrt{n} \right) / F_{j-1}^{(n)} \right) \xrightarrow{\mathsf{P}} 0$$

as $n \to \infty$. Therefore all the assumptions of Theorem 7.1.11 of [6] hold, and we complete the proof of Lemma 2. □

Lemma 3. *Assume that all the assumptions of Theorem* 2 *hold. Then the weak convergence*

$$\widetilde{M}_n^{(1)} \to \left(|\alpha|^{-1} \lambda \sigma^2 + b^2 \right)^{1/2} W \quad \text{as } n \to \infty \tag{6}$$

holds in the Skorokhod space $D[0, \infty)$.

Proof of Lemma 3. Since the sequence $\left(M_k^{(n)}, F_k^{(n)} \right)$, $k \geq 1$, is a martingale-difference, Theorem 7.1.11 of [6] implies that we only need to show that

$$n^{-1} \sum_{j=1}^{[nt]} \mathsf{E}\left(\left(M_j^{(n)} \right)^2 / F_{j-1}^{(n)} \right) \xrightarrow{\mathsf{P}} \left(|\alpha|^{-1} \lambda \sigma^2 + b^2 \right) t \tag{7}$$

and that, for all $\varepsilon > 0$,

$$R_n(\varepsilon, t) = n^{-1} \sum_{j=1}^{[nt]} \mathsf{E}\left(\left(M_j^{(n)} \right)^2 I \left(\left| M_j^{(n)} \right| > \varepsilon \sqrt{n} \right) / F_{j-1}^{(n)} \right) \xrightarrow{\mathsf{P}} 0 \tag{8}$$

as $n \to \infty$. Relation (7) follows in view of

$$\mathsf{E}\left(\left(M_j^{(n)} \right)^2 / F_k^{(n)} \right) = \sigma_n^2 X_{k-1}^{(n)} + b_n^2,$$

Theorem 1, and assumptions (1)–(3) of Theorem 2.

Now we pass to the proof of relation (8). Put

$$N_{n,k}^{(1)} = \sum_{j=1}^{X_{k-1}^{(n)}} (\xi_{k,j} - m_n), \qquad N_{n,k}^{(2)} = \varepsilon_k^{(n)} - \lambda_n.$$

Note that

$$I(|X + Y| > 2\varepsilon) \leq I(|X| > \varepsilon) + I(|Y| > \varepsilon) \tag{9}$$

for all random variables X and Y and for every $\varepsilon > 0$. This together with the elementary inequality $(a + b)^2 \leq 2(a^2 + b^2)$ implies that

$$R_n(2\varepsilon, t) \leq 2 \sum_{i,j=1}^{2} R_{i,j}^{(n)}(\varepsilon, t)$$

with probability one, since $M_k^{(n)} = N_{n,k}^{(1)} + N_{n,k}^{(2)}$, where

$$R_{i,j}^{(n)}(\varepsilon, t) = n^{-1} \sum_{k=1}^{[nt]} \mathsf{E}\left(\left(N_{n,k}^{(i)} \right)^2 I \left(\left| N_{n,k}^{(j)} \right| > \varepsilon \sqrt{n} \right) / F_{k-1}^{(n)} \right), \qquad i, j = 1, 2.$$

Therefore relation (8) follows if

$$R_{i,j}^{(n)}(\varepsilon,t) \xrightarrow{\mathsf{P}} 0 \quad \text{as } n\to\infty \tag{10}$$

for $i,j=1,2$ and for all $t>0$, $\varepsilon>0$.

First we treat the case of $i=j=1$ in (10). We have $\left(N_{n,k}^{(1)}\right)^2 = J_k^{(n)} + L_k^{(n)}$, where

$$J_k^{(n)} = \sum_{j=1}^{X_{k-1}^{(n)}} \left(\xi_{k,j}^{(n)} - m_n\right)^2, \qquad L_k^{(n)} = 2\sum_{i=1}^{X_{k-1}^{(n)}} \sum_{j=i+1}^{X_{k-1}^{(n)}} \left(\xi_{k,i}^{(n)} - m_n\right)\left(\xi_{k,j}^{(n)} - m_n\right).$$

Now we introduce the random variables

$$S_{k,j}^{(n)} = N_{n,k}^{(1)} - \left(\xi_{k,j}^{(n)} - m_n\right), \qquad j=1,2,\dots,X_{k-1}^{(n)}.$$

Using inequality (9) we get

$$\begin{aligned} &n^{-1}\sum_{k=1}^{[nt]} \mathsf{E}\left(J_k^{(n)} I\left(\left|N_{n,k}^{(1)}\right| > 2\varepsilon\sqrt{n}\right)/F_{k-1}^{(n)}\right) \\ &\quad\le n^{-1}\sum_{k=1}^{[nt]} \mathsf{E}\left(\sum_{j=1}^{X_{k-1}^{(n)}} \left(\xi_{k,j}^{(n)} - m_n\right)^2 I\left(\left|\xi_{k,j}^{(n)} - m_n\right| > \varepsilon\sqrt{n}\right)/F_{k-1}^{(n)}\right) \\ &\qquad + n^{-1}\sum_{k=1}^{[nt]} \mathsf{E}\left(\sum_{j=1}^{X_{k-1}^{(n)}} \left(\xi_{k,j}^{(n)} - m_n\right)^2 I\left(\left|S_{k,j}^{(n)}\right| > \varepsilon\sqrt{n}\right)/F_{k-1}^{(n)}\right) \\ &\quad= A_n + B_n. \end{aligned} \tag{11}$$

Since the random variables $\xi_{k,j}^{(n)}$, $k,j\in\mathbb{N}$, are independent and identically distributed, assumptions (3) in Theorem 2 and in Theorem 1 yield,

$$A_n \overset{\mathsf{P}}{\sim} |\alpha|^{-1}\lambda t d_n \mathsf{E}\left(\left(\xi_{1,1}^{(n)} - m_n\right)^2 I\left(\left|\xi_{1,1}^{(n)} - m_n\right| > \varepsilon\sqrt{n}\right)\right) \to 0 \quad \text{as } n\to\infty \tag{12}$$

for all $t>0$, where $\varphi \overset{\mathsf{P}}{\sim} \psi$ means that $\varphi\psi^{-1} \xrightarrow{\mathsf{P}} 1$ as $n\to\infty$.

Next we consider B_n. Since the random variables $\xi_{k,j}^{(n)} - m_n$ and $S_{k,j}^{(n)}$ are independent, we apply the Chebyshev inequality for conditional probabilities to make sure that

$$B_n \le \frac{4}{\varepsilon^2 n^2}\sigma_n^4 \sum_{k=1}^{[nt]} \left(X_k^{(n)}\right)^2 \tag{13}$$

with probability one. Applying Lemma 2.1 of [2] we obtain

$$\sigma_n^4 n^{-2}\sum_{k=1}^{[nt]} \mathsf{E}\left(X_k^{(n)}\right)^2 \le 2\alpha^{-2}\left(d_n\sigma_n^2\right)^2\left(|\alpha| b_n^2 + \lambda_n^2\right)[nt]n^{-2} \to 0 \tag{14}$$

as $n\to\infty$. This together with (13) and Markov's inequality implies that

$$B_n \xrightarrow{\mathsf{P}} 0 \quad \text{as } n\to\infty.$$

The latter relation together with (12) leads to

$$n^{-1}\sum_{k=1}^{[nt]} \mathsf{E}\left(J_k^{(n)} I\left(\left|N_{n,k}^{(1)}\right| > \varepsilon\sqrt{n}\right)/F_{k-1}^{(n)}\right) \xrightarrow{\mathsf{P}} 0 \quad \text{as } n\to\infty \tag{15}$$

in view of (11). Next, we apply the Cauchy–Bunyakovskiĭ and Chebyshev inequalities for conditional probabilities and conclude that

$$n^{-1}\sum_{k=1}^{[nt]}\mathsf{E}\left(\left|L_k^{(n)}\right| I\left(\left|N_{n,k}^{(1)}\right|>\varepsilon\sqrt{n}\right)/F_{k-1}^{(n)}\right)\le 2^{1/2}\varepsilon^{-1}n^{-3/2}\sigma_n^3\sum_{k=1}^{[nt]}\left(X_{k-1}^{(n)}\right)^2 \tag{16}$$

with probability one. Similarly to inequality (14),

$$(nd_n)^{-3/2}\sum_{k=1}^{[nt]}\mathsf{E}\left(X_{k-1}^{(n)}\right)^2\le 2\left(n^{-1}d_n\right)^{1/2}\alpha^{-2}\left(|\alpha|b_n^2+\lambda_n^2\right)t\to 0\quad\text{as } n\to\infty.$$

This together with (16) and (15) implies (10) for the case of $i=j=1$.

Next we consider the case of $i=1$, $j=2$. Since $N_{n,k}^{(1)}$ and $N_{n,k}^{(2)}$ are independent, the Chebyshev inequality for conditional probabilities and Theorem 1 imply that

$$R_{1,2}^{(n)}(\varepsilon,t)\le\frac{b_n^2\sigma_n^2}{\varepsilon^2n^2}\sum_{k=1}^{[nt]}X_{k-1}^{(n)}\overset{\mathsf{P}}{\sim}\frac{\lambda\sigma^2b^2}{\varepsilon^2|\alpha|}t\cdot\frac{1}{n}\to 0\quad\text{as } n\to\infty.$$

Similarly we have

$$R_{2,1}^{(n)}(\varepsilon,t)\xrightarrow{\mathsf{P}}0\quad\text{as } n\to\infty.$$

Relation (10) follows directly from assumption (4) in the case of $i=j=2$, since the random variables $\varepsilon_k^{(n)}$, $k\in\mathbb{N}$, are independent and identically distributed. The proof of Lemma 3 is completed. □

The proof of Theorem 2 follows directly from Lemmas 1–3 and Theorem 4.1 of [5].

4. Examples

Below are two examples of sequences of branching processes with immigration for which assumptions of Theorems 1 and 2 hold.

Example 1. Let $\xi_{1,1}^{(n)}$ have the Bernoulli distribution with success probability p_n such that $p_n=1+\alpha d_n^{-1}+o\left(d_n^{-1}\right)$ as $n\to\infty$, where $\alpha<0$ is a fixed number, the immigration process is governed by a Poisson law with parameter $\lambda_n\ge 0$, and there exists a finite nonnegative number λ such that $\lambda_n\to\lambda$ as $n\to\infty$. It is easy to check that all the assumptions of Theorems 1 and 2 hold. In this case,

$$Z(t)=|\alpha|^{-1}\lambda t,\qquad Y(t)=|\alpha|^{-1}(2\lambda)^{1/2}W(t).$$

If both $\xi_{1,1}^{(n)}$ and $\varepsilon_1^{(n)}$ have the Bernoulli distribution with the same success probability $p_n=1+\alpha d_n^{-1}$, $\alpha<0$, then

$$Z(t)=|\alpha|^{-1}t,\qquad Y(t)=|\alpha|^{-1}W(t).$$

Example 2. Let $\xi_{1,1}^{(n)}$ assume three values 0, 1, and 2 with probabilities $2d_n^{-1}$, $1-3d_n^{-1}$, and d_n^{-1}, respectively, and let the random variable $\varepsilon_1^{(n)}$ have the geometric distribution with success probability p_n, that is,

$$\mathsf{P}\left(\varepsilon_1^{(n)}=k\right)=p_n(1-p_n)^{k-1},\qquad k=1,2,\dots.$$

We have

$$m_n=1-d_n^{-1},\qquad\sigma_n^2=d_n^{-1}(3-d_n^{-1}),\qquad\lambda_n=p_n^{-1},\qquad b_n^2=(1-p_n)p_n^{-2}.$$

Let $p_n\to p>0$. It is clear that assumptions (1), (2), and (3) of Theorems 1 and 2 hold and moreover that

$$\alpha=-1,\qquad\sigma^2=3,\qquad\lambda=p^{-1},\qquad b^2=(1-p)p^{-2}.$$

Assumptions (4) and (5) of Theorem 2 can also be easily checked. In the case under consideration,

$$Z(t) = p^{-1}t, \qquad Y(t) = \left(p^{-1}\left(2 + p^{-1}\right)\right)^{1/2} W(t).$$

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