A NEW METHOD OF DETERMINING THE DIFFERENTIAL PARAMETERS AND INVARIANTS OF QUADRATIC DIFFERENTIAL QUANTICS*

BY

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In the following article I propose to exhibit in a preliminary way a symbolic method, in close analogy with the symbolism used in the algebraic theory of invariants, for the construction and investigation of invariants of quadratic differential quantics. The method proves to be fully as successful as in algebra, the chief advantage lying in the fact that after the establishment of the fundamental principles of the method further reference to the formulas of transformation becomes unnecessary.†

§1. *Introduction.*

If in the binary quadratic differential quantic

$$A = \sum_{i,k=1}^{2} a_{ik} dx_i dx_k, \qquad (a_{ki} = a_{ik}), \tag{1}$$

where the coefficients a_{ik} are functions of $x_1 x_2$, we introduce new variables $y_1 y_2$ by the equations:

$$x_i = x_i(y_1, y_2) \qquad (i = 1, 2), \tag{2}$$

then A goes into the new expression:

$$A = A' = \sum a'_{ik} dy_i dy_k. \tag{3}$$

Let now $u, v, \cdots$ be arbitrary functions of $x_1 x_2$, and denote by $u', v', \cdots$ the same functions after the substitution (2) has been made. Let further Ω be a function of the coefficients a_{ik} and their derivatives and of $u, v, \cdots$ and their derivatives, and let Ω' be the expression into which Ω is transformed, if the old quantities a_{ik}, $u, v, \cdots$ and their derivatives with respect to $x_1 x_2$ are replaced

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† A similar remark applies to the method followed by Mr. HESSENBERG in his paper, *Über die Invarianten linearer und quadratischer binärer Differentialformen und ihre Anwendung auf die Deformation der Flächen,* Acta Mathematica, vol. 23, p. 121, 1900.

by the new quantities a'_{ik}, u', v', $\cdots$ and their derivatives with respect to $y_1 y_2$, where the a'_{ik} are defined by the equation (3). If Ω satisfies the equation:

$$\Omega' = \Omega, \tag{4}$$

Ω is called an *invariant of* the quadratic differential quantic A.

In this sense every arbitrary function $u, v, \cdots$ of $x_1 x_2$ is an invariant of A on account of the equations: $u' = u$, $v' = v$, $\cdots$.

Invariants Ω which actually contain at least one arbitrary function are generally called *differential parameters*, while those which involve no arbitrary function are called *differential invariants*.

The most important differential parameters are the following:*

$$\Delta_1 u = \sum_{r,s} A_{rs} \frac{\partial u}{\partial x_r} \frac{\partial u}{\partial x_s}, \tag{5}$$

$$\nabla(u, v) = \sum_{r,s} A_{rs} \frac{\partial u}{\partial x_r} \frac{\partial v}{\partial x_s}, \tag{6}$$

$$\Delta_2 u = \sum_{r,s} A_{rs} \frac{\partial^2 u}{\partial x_r \partial x_s} - \sum_{r,s,i,k} A_{rs} A_{ik} \begin{bmatrix} rs \\ k \end{bmatrix} \frac{\partial u}{\partial x_k}. \tag{7}$$

Here A_{ik} denotes the minor of the element a_{ik} in the determinant

$$D = |a_{ik}| \tag{8}$$

divided by the determinant D (which is always supposed to be different from zero). Further:

$$\begin{bmatrix} rs \\ k \end{bmatrix} = \tfrac{1}{2} \left(\frac{\partial a_{rk}}{\partial x_s} + \frac{\partial a_{sk}}{\partial x_r} - \frac{\partial a_{rs}}{\partial x_k} \right) \tag{9}$$

is the so-called *triple index symbol* introduced by CHRISTOFFEL.†

The transformed expression A' (3) being obtained from (1) by linear substitution of the differentials, the connection between the a'_{ik} and the a_{ik} will be the same as in the case of the transformation of an algebraic quantic with coefficients a_{ik} and the quantities $\partial x_i / \partial y_k$ as coefficients of the linear substitution.

Hence we have

$$D' = r^2 D, \tag{10}$$

where

$$r = \left| \frac{\partial x_i}{\partial y_k} \right| \qquad (i, k = 1, 2). \tag{11}$$

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* Cf. BIANCHI, *Vorlesungen über Differentialgeometrie*, Leipzig, 1899, pp. 41 and 47.

† *Über die Transformation der homogenen Differentialausdrücke zweiten Grades*, Crelle's Journal, vol. 70, p. 48, formula (4).

If now F is an invariant of A, then

$$\frac{\partial F'}{\partial y_1} dy_1 + \frac{\partial F'}{\partial y_2} dy_2 = \frac{\partial F}{\partial x_1} dx_1 + \frac{\partial F}{\partial x_2} dx_2 ;$$

and if Φ is also an invariant, we have:

$$\frac{\partial F'}{\partial y_1}\frac{\partial \Phi'}{\partial y_2} - \frac{\partial F'}{\partial y_2}\frac{\partial \Phi'}{\partial y_1} = r\left(\frac{\partial F}{\partial x_1}\frac{\partial \Phi}{\partial x_2} - \frac{\partial F}{\partial x_2}\frac{\partial \Phi}{\partial x_1}\right). \tag{12}$$

With respect to any functions U and V of $x_1 x_2$ let us introduce the following abbreviations:

$$\frac{\partial U}{\partial x_i} = U_i, \quad \frac{\partial^2 U}{\partial x_i \partial x_k} = U_{ik}, \quad \text{etc.}; \tag{13}$$

$$U_1 V_2 - U_2 V_1 = (U, \ V). \tag{14}$$

Further, with respect to a transformation from the variables $x_1 x_2$ to the variables $y_1 y_2$ we use the abbreviation:

$$\frac{1}{\sqrt{D}} = a. \tag{15}$$

Then from (12) and (10) we have in these notations:

$$a' (F, \ \Phi)' = a (F, \ \Phi),^* \tag{16}$$

and accordingly the fundamental theorem:

THEOREM I. *If F and Φ are invariants of A then $a(F, \ \Phi)$ is again an invariant of A.*

§2. *The Symbolic Method.*

Let now f be a (symbolic) function of $x_1 x_2$, and f_1 and f_2 its two derivatives: then on setting

$$f_i f_k = a_{ik}, \tag{17}$$

we may consider the expression $(f_1 dx_1 + f_2 dx_2)^2$ as symbolically equal to A. In this case we shall say for shortness that *f is a symbol of A*.

If $f, \phi, \cdots$ are several symbols of A, since $f' = f$, $\phi' = \phi, \cdots$, these symbols are in the first place invariants of A. Building up now according to theorem I other invariants out of these and of arbitrary functions $u, v, \cdots$ and forming products with these symbolic invariants as factors, we have at once by the same reasoning as in algebra an actual invariant of A before us, in case every

* [In subsequent papers it would be desirable to introduce for $a(F, \Phi)$ the notation $\{F, \Phi\}$.]

symbol $f, \phi, \cdots$ appears precisely twice and in such a way that only the connections $f_i f_k$, $\phi_i \phi_k$, $\cdots$ occur. (We shall presently see, however, that the last restriction can be partly removed.)

Thus the expression $a^2(f, u)^2$ satisfies the conditions just stated. Indeed we have

$$a^2(f, u)^2 = \frac{1}{D}(f_1^2 u_2^2 - 2f_1 f_2 u_1 u_2 + f_2^2 u_1^2) = \frac{1}{D}(a_{11}u_2^2 - 2a_{12}u_1u_2 + a_{22}u_1^2)$$

$$= A_{11}u_1^2 + 2A_{12}u_1u_2 + A_{22}u_2^2 = \sum A_{rs}\frac{\partial u}{\partial x_r}\frac{\partial u}{\partial x_s} = \Delta_1 u,$$

that is,

(18) $$a^2(f, u)^2 = \Delta_1 u.$$

Likewise we obtain:

(19) $$a^2(f, u)(f, v) = \nabla(u, v).$$

From (17) there follows:

$$\frac{\partial a_{ik}}{\partial x_l} = f_i f_{kl} + f_k f_{il},$$

and similarly:

$$\frac{\partial a_{kl}}{\partial x_i} = f_k f_{li} + f_l f_{ki},$$

$$\frac{\partial a_{li}}{\partial x_k} = f_l f_{ik} + f_i f_{lk}.$$

Hence we have the symbolic form:

(20) $$f_i f_{kl} = \tfrac{1}{2}\left(\frac{\partial a_{ik}}{\partial x_l} + \frac{\partial a_{il}}{\partial x_k} - \frac{\partial a_{kl}}{\partial x_i}\right) = \begin{bmatrix} kl \\ i \end{bmatrix},$$

of the triple index symbol.

We see therefore that, the other conditions given above being satisfied, we can also admit the connections $f_i f_{kl}$ as well as the connections $f_i f_k$.

For example, from theorem I we see that the expression $a\big(f, a(f, u)\big)$ is formally invariant; it also satisfies the conditions for actual existence. Indeed we find

$$a\big(f, a(f, u)\big) = a^2[f_1(f_{12}u_2 - f_{22}u_1) - f_2(f_{11}u_2 - f_{12}u_1)] + \cdots.$$

The complete computation leads to the result:

(21) $$a\big(f, a(f_1 u)\big) = \Delta_2 u.$$

I give the values of some other symbolic expressions:

(22) $$a^3(f, u)(\phi, u)\big(f, a(\phi, u)\big) = \tfrac{1}{2}\nabla(u, \Delta_1 u),$$

(23) $$a^4(\phi, u)(\psi, u)\Big(f, a(\phi, u)\Big)\Big(f, a(\psi, u)\Big) = \tfrac{1}{4}\Delta_1(\Delta_1 u),$$

(24) $$a^2(f, \phi)\Big(a(f, u), a(\phi, u)\Big) = 2\Delta_{22}u, *$$
$$a^2\Big(f, a(\phi, u)\Big)\Big(\phi, a(f, u)\Big) = \Delta_2^2 u - 2\Delta_{22}u,$$

(25) $$(f, \phi)\Big(f, a(\phi, u)\Big) = 0.$$

For the computation with symbolic expressions the following identity is useful:

(26) $$(a, b)(c, d) + (a, c)(d, b) + (a, d)(b, c) = 0.$$

Also the relation (25) (which becomes an identity only when actual values are substituted for the symbols) often serves to reduce complicated expressions.

As an example the invariant $\Delta_1 u \cdot \Delta_1 v - \nabla^2(u, v)$ will be transformed. We have

$$\Delta_1 u \cdot \Delta_1 v - \nabla^2(u, v) = a^4[(f, u)^2(\phi, v)^2 - (f, u)(f, v)(\phi, u)(\phi, v)]$$
$$= a^4(f, u)(\phi, v)[(f, u)(\phi, v) - (f, v)(\phi, u)].$$

The quantity in the bracket reduces by means of (26) to $(f, \phi)(u, v)$. Hence

$$\Delta_1 u \cdot \Delta_1 v - \nabla^2(u, v) = a^4(f, \phi)(f, u)(\phi, v)(u, v).$$

If to the right side of this equation we add the expression obtained from it by interchanging the equivalent symbols f, ϕ and divide by 2, we obtain:

$$\tfrac{1}{2}a^4(f, \phi)(u, v)[(f, u)(\phi, v) - (f, v)(\phi, u)].$$

Hence:

$$\Delta_1 u \cdot \Delta_1 v - \nabla^2(u, v) = \tfrac{1}{2}a^4(f, \phi)^2(u, v)^2 = \frac{1}{D}(u, v)^2.$$

As a second example of symbolic computation I give a proof of the formula:

$$2\Delta_2 u \cdot \nabla(u, \Delta_1 u) - \Delta_1(\Delta_1 u) = 4\Delta_1 u \cdot \Delta_{22} u. †$$

We have from (21), (22), (23):

$$2\Delta_2 u \cdot \nabla(u, \Delta_1 u) - \Delta_1(\Delta_1 u) =$$
$$4a^4(\phi, u)\Big(f, a(\phi, u)\Big)\Big[(f, u)\Big(\psi, a(\psi, u)\Big) - (\psi, u)\Big(f, a(\psi, u)\Big)\Big].$$

* For the definition of $\Delta_{22}u$ see BIANCHI, loc. cit., p. 48.

† This formula is given by BIANCHI, loc. cit., p. 48 without proof. Compare also HESSENBERG, loc. cit., p. 143.

The application of (26) reduces the quantity in the brackets to

$$(f, \psi)\Big(u, a(\psi, u)\Big).$$

Again by (26) we obtain:

$$\Big(f, a(\phi, u)\Big)\Big(u, a(\psi, u)\Big) = (u, f)\Big(a(\psi, u), a(\phi, u)\Big) + \Big(f, a(\psi, u)\Big)\Big(u, a(\phi, u)\Big).$$

The second term of this sum vanishes on account of (25) if we multiply by $(\phi, u)(f, \psi)$, and now we have:

$$2\Delta_2 u \cdot \nabla(u, \Delta_1 u) - \Delta_1(\Delta_1 u) = -4a^4(f, u)(\phi, u)(f, \psi)\Big(a(\psi, u), a(\phi, u)\Big).$$

Interchanging here the equivalent symbols ϕ, ψ, adding the resulting expression to the right side, and dividing by 2, we obtain on the right side the factor $(\phi, u)(f, \psi) - (\psi, u)(f, \phi)$ which reduces, according to (26), to the product $(\phi, \psi)(f, u)$. The desired result is now obtained:

$$\begin{aligned} 2\Delta_2 u \cdot \nabla(u, \Delta_1 u) - \Delta_1(\Delta_1 u) &= 2a^4(f, u)^2(\phi, \psi)\Big(a(\phi, u), a(\psi, u)\Big) \\ &= 4\Delta_1 u \cdot \Delta_{22} u. \end{aligned}$$

The symbolic expressions for the second derivatives of a_{ik} appear as aggregates of the form $f_{ik} f_{lm}$ and $f_i f_{klm}$. But it is impossible to express conversely each of these two quantities separately in terms of the a_{ik} and their derivatives. There exist only certain connections of the $f_{ik} f_{lm}$ and $f_i f_{klm}$ which admit of such a representation, e. g.,

$$f_{11} f_{22} - f_{12}^2 = \tfrac{1}{2}\left(\frac{\partial^2 a_{11}}{\partial x_2^2} - 2\frac{\partial^2 a_{12}}{\partial x_1 \partial x_2} + \frac{\partial^2 a_{22}}{\partial x_1^2}\right),$$

and others. A similar remark holds good also for the case of all the higher derivatives of the a_{ik}.

The most important example of this kind is

$$a^2(\phi, \psi)\Big(a(f, \phi), a(f, \psi)\Big).$$

Derivatives of f of order k higher than the second do not occur, but the terms which are quadratic in the f_{ik} are all contained in the expression

$$\Big((f, \phi), (f, \psi)\Big),$$

an expression which can also be written in this form:

$$\frac{\partial (f, \phi)(f, \psi)_2}{\partial x_1} - \frac{\partial (f, \phi)(f, \psi)_1}{\partial x_2},$$

and here in the expressions which are to be differentiated, i. e., in $(f, \phi)(f, \psi)$, the coefficients f occur only in the connection $f_i f_{kl}$. Therefore the above symbolic invariant represents an actual invariant of A, and since it does not contain arbitrary functions, a differential invariant. The result is indeed:

$$a^2(\phi, \psi)\Big(a(f, \phi), a(f, \psi)\Big) = 2K,$$

where K is the expression* for *the curvature* of A. In passing we derive by comparison with (24) the formula:

$$K = \Delta_{22}(f).$$

§3. *Covariants.*

The complete differential of every invariant represents immediately a linear covariant of A. Covariants of higher degrees are therefore obtained at once by forming products of these symbolic linear covariants and (if necessary) also invariants, in such a way that every symbol $f, \phi, \cdots$ occurs precisely twice and so that the quadratic terms of the derivatives of the symbols admit of an actual interpretation in terms of the a_{ik} and their derivatives.

The simplest quadratic covariant is:

$$\left(\frac{\partial a(f, u)}{\partial x_1}\, dx_1 + \frac{\partial a(f, u)}{\partial x_2}\, dx_2\right)(f_1 dx_1 + f_2 dx_2).$$

This covariant is however irrational because the quantity a occurs only once as a factor. The simplest rational quadratic covariant is the following:

$$a(f, \phi)\left(\frac{\partial a(f, u)}{\partial x_1}\, dx_1 + \frac{\partial a(f, u)}{\partial x_2}\, dx_2\right)(\phi_1 dx_1 + \phi_2 dx_2);$$

it is identical with a covariant which BIANCHI† calls the second covariant differential of u.

As a further example I mention:

$$2a(f, \phi)\left(\frac{\partial a(f, F)}{\partial x_1}\, dx_1 + \frac{\partial a(f, F)}{\partial x_2}\, dx_2\right)(\phi_1 dx_1 + \phi_2 dx_2)(F_1 dx_1 + F_2 dx_2),$$

a cubic simultaneous covariant of A and $B = \sum b_{ik} dx_i dx_k$, where f, ϕ are symbols of A, and F is a symbol of B. This is one of the cubic covariants mentioned by BIANCHI, loc. cit., p. 55, and by KNOBLAUCH, loc. cit., p. 197.

* See BIANCHI, loc. cit., pp. 50–53, and KNOBLAUCH, *Einleitung in die Theorie der krummen Flächen*, Leipzig, 1888, pp. 174–179.

† loc. cit., p. 46.

§ 4. *Differential quantics in general.*

The preceding deductions can be extended without much difficulty from the case of binary differential quantics to those of n variables (and also from quadratic quantics to those of higher orders). So we have at once for ternary quantics:

$$\Delta_1 u = \tfrac{1}{2}\, a^2(f,\, \phi\,,\, u)^2,$$
$$\nabla(u\,, v) = \tfrac{1}{2}\, a^2(f,\, \phi\,,\, u)\,(f,\, \phi\,,\, v),$$
$$\Delta_2 u = \tfrac{1}{2}\, a\Big(f,\, \phi\,,\, a(f,\, \phi\,,\, u)\Big),$$

and here the generalization for n variables is evident.

The invariant forms which are analogous to the curvature are in the ternary case:

$$a^2(\phi\,,\, \psi\,,\, \chi)\,\Big(a(f, \phi\,, \psi),\, a(f,\, \chi\,,\, \omega),\, \omega\Big),$$

with $f,\, \phi\,,\, \psi\,,\, \chi\,,\, \omega$ as equivalent symbols, and in the general case:

$$a^2(\phi^1, \phi^2, \cdots \phi^n)\Big(a(f, \phi^1, \phi^2, \cdots \phi^{n-1}),\, a(f, \phi^n, \psi^1, \psi^2, \cdots \psi^{n-2}),\, \psi^1, \psi^2, \cdots \psi^{n-2}\Big),$$

with $f,\, \phi^1,\, \phi^2,\, \cdots\, \phi^n,\, \psi^1,\, \psi^2,\, \cdots\, \psi^{n-2}$ as equivalent symbols. It is easily seen that here the terms quadratic in f_{ik} occur only in the connection:

$$f_{il} f_{km} - f_{im} f_{kl} = \tfrac{1}{2}\left(\frac{\partial^2 a_{il}}{\partial x_k \partial x_m} + \frac{\partial^2 a_{km}}{\partial x_i \partial x_l} - \frac{\partial^2 a_{im}}{\partial x_i \partial x_l} - \frac{\partial^2 a_{kl}}{\partial x_i \partial x_m} \right).$$

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