TWO PLANE MOVEMENTS GENERATING QUARTIC SCROLLS*

BY

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The object of this paper is to give the results of a study of the point and line loci of two plane movements. The former have received some consideration as the references given below will show, but the latter it is believed have not been described before.

I. The Degenerate Case of Three-bar Movement whose Centrodes are Limaçons.

§ 1. *Definition of the movement. Point loci.*

In space regarded as fixed let a system of rectangular coördinates x, y, z be established, whose origin is O. Denote the xy-plane by σ. Upon it moves a plane σ_1 carrying two other planes, the three being mutually perpendicular. By means of them we establish a system of coördinates x_1, y_1, z_1 whose origin will be denoted by O_1. The axes of x_1 and y_1 are taken in σ_1. Denote the point $(a, 0, 0)$ of fixed space by A, and $(a_1, 0, 0)$ of the moving space by A_1, where a and a_1 are positive. The movement to be studied is defined by the conditions that A_1 is to remain at the distance a_1 from O, and O_1 at the distance a from A. This definition implies that the limaçon

$$\left(x_1^2 + y_1^2 - \frac{2a^2a_1}{a^2 - a_1^2}x_1\right)^2 = \left(\frac{2aa_1^2}{a^2 - a_1^2}\right)^2(x_1^2 + y_1^2),$$

in σ_1, rolls upon the related one:

$$\left(x^2 + y^2 - \frac{2aa_1^2}{a_1^2 - a^2}x\right)^2 = \left(\frac{2a^2a_1}{a_1^2 - a^2}\right)^2(x^2 + y^2),$$

in σ.† These centrodes will be denoted by c_1 and c respectively. The real double points of the centrodes are O_1 and O, and A_1 and A are foci.

* Extract from a paper presented to the Society December 28, 1899, under the title: *Upon plane movements whose point loci are of order not greater than four.* Received for publication April 26, 1900.

† S. Roberts, Proceedings London Mathematical Society, vol. 3 (1871), p. 95.
Burmester, *Lehrbuch der Kinematik*, vol. 1, pp. 306–308.

The loci of the points of σ_1 are bicircular unicursal quartics and pedals of conics. The equation of the locus of the point (x_1, y_1) is

$$\{a_1(x^2+y^2)+a(xx_1+yy_1-2a_1x)\}^2+(a^2-a_1^2)(x_1y-xy_1)^2 - \{a(x_1^2+y_1^2)+a_1(xx_1+yy_1-2ax_1)\}^2=0;$$

the point

$$x=-\frac{ax_1(x_1^2+y_1^2-2a_1x_1)}{a_1(x_1^2+y_1^2)};\quad y=-\frac{ay_1(x_1^2+y_1^2-2a_1x_1)}{a_1(x_1^2+y_1^2)},$$

is its real double point.*

The forms of the point loci are given by the following theorem, which is not difficult to demonstrate. When $a < a_1$ (in which case c_1 has a real conjugate point) all points within c_1 describe curves consisting of two loops one within the other and joined at a node. The locus of any point without c_1 is a closed loop with conjugate point within, and that of any point upon c_1 a loop with cusp pointing inward. All nodes fall upon σ between the two loops of c, all cusps upon c, and all conjugate points either within the smaller loop of c or without the curve according as the tracing point is within or without the circle $x_1^2+y_1^2-2a_1x_1=0$. Any point upon this circle describes a curve whose double point falls at O. The circle $x^2+y^2-2ax=0$ is twice completely described by O_1 and the locus of A_1 is $x^2+y^2=a_1^2$ once completely described and $(x-a)^2+y^2=0$. When $a > a_1$ (in which case c_1 has a real node) all points between the two loops of c_1 describe curves having two loops mutually exterior joined at a node. The locus of any point within the smaller loop of c_1 or without the curve consists of a closed loop with a conjugate point without, and that of any point upon c_1 a loop with a cusp pointing outward. All nodes fall upon σ within c, all cusps upon c, and all conjugate points without c and either within or without the circle $x^2+y^2-2ax=0$ according as the tracing point is within the smaller loop of c_1 or without c_1. Any point upon the circle $x_1^2+y_1^2-2a_1x_1=0$ describes a curve whose double point falls at O. An arc (only) of the circle $x^2+y^2-2ax=0$ is twice described by O_1 and the locus of A_1 is $x^2+y^2=a_1^2$ once completely described and $(x-a)^2+y^2=0$.

§ 2. *Line loci.*

The equations and forms of the loci of the straight lines carried by σ_1 and oblique to it, can be readily determined by the method already used by the author.† As the equations can be immediately written down we omit them and

*S. ROBERTS, Proceedings London Mathematical Society, vol. 2 (1869), pp. 133–135 and vol. 3 (1871), pp. 88–99.

CAYLEY, ibid., vol. 3 (1871), pp. 100–106.

†American Journal of Mathematics, vol. 21 (1899), pp. 257–269.

give the following theorems on the forms of the scrolls. The projection of the generatrix upon σ_1 is denoted by l' and the distance of l' from O_1 by p.

When $p \neq 0$ the line-loci are unicursal quartic scrolls having a nodal cubical ellipse. They are of the sixteen types shown in the following table.

Type.	Value of $a:a_1$.	Position of l'.	Holgate's* F_1^4 subform
I	$a<a_1$	l' intersects c_1 in four imaginary points.	17
II	$a<a_1$	l' intersects c_1 in two real distinct and two imaginary points.	11
III	$a<a_1$	l' intersects c_1 in two coincident and two imaginary points.	13
IV	$a<a_1<2a$	l' is tangent to c_1 where it is convex inward.	4
V	$a<a_1<2a$	l' intersects c_1 in four real distinct points.	1
VI	$a<a_1<2a$	l' is the double tangent of c_1.	3
VII	$a<a_1<2a$	l' is tangent to c_1 where it is concave inward and intersects it in two real distinct points.	2
VIII	$a<a_1<2a$	l' is tangent to c_1 at an inflexion.	8
IX	$a_1=2a$	l' has a contact of 3d order with c_1.	10
X	$a>a_1$	l' intersects c_1 in four imaginary points.	17
XI	$a>a_1$	l' intersects c_1 in two real distinct and two imaginary points.	11
XII	$a>a_1$	l' intersects c_1 in two coincident and two imaginary points.	13
XIII	$a>a_1$	l' is tangent to the inner loop of c_1.	4
XIV	$a>a_1$	l' intersects c_1 in four real distinct points.	1
XV	$a>a_1$	l' is the double tangent of c_1.	3
XVI	$a>a_1$	l' is tangent to the outer loop of c_1 and intersects it in two other real distinct points.	2

When $p = 0$ the line-loci are unicursal quartic scrolls having a nodal circle and an intersecting nodal straight line. They are of four types as follows:

Type.	Value of $a:a_1$	Position of l'.	Holgate's F_3^4 subform.
XVII	$a<a_1$		3
XVIII	$a>a_1$	l' intersects both loops of c_1.	1
XIX	$a>a_1$	l' intersects the outer loop of c_1 in two points.	1
XX	$a>a_1$	l' is tangent to c_1 at its node.	4

II. Conchoidal Movement.

§ 3. *Definition of the movement. Point loci.*

To define the movement, we establish as before a system of rectangular coördinates x, y, z in fixed space, another system x_1, y_1, z_1 in the moving space, and then condition σ_1 to move upon σ so that the axis of x_1 always passes through

*This refers to the "species" and "subforms" of the classification of unicursal quartic scrolls as given in the two articles: Holgate, *On certain ruled surfaces of the fourth order*, American Journal of Mathematics, vol. 15 (1893), pp. 344–386; *Note* etc., ibid., vol 22 (1900), pp. 27–30; where a general description of the above surfaces can be found.

the fixed point $(0, a, 0)$ and the point $(0, a_1, 0)$ remains upon the axis of x. The loci of the points of σ_1 are unicursal quartics having one double point at infinity on the axis of x. The equation of the locus of (x_1, y_1) is

$$y^2[x^2 + (y - a)^2] - [x(y_1 - a_1) + x_1(y - a)]^2 + 2x_1y_1xy - 2y_1y(y_1 - a_1)(y - a) + y_1^2[x_1^2 + (y_1 - a_1)^2] = 0.*$$

Besides the one mentioned the curve has the two double points:

$$x = \frac{x_1}{2(y_1 - a_1)}\left[a \mp \sqrt{a^2 + 4y_1(y_1 - a_1)}\right],$$
$$y = \tfrac{1}{2}\left[a \pm \sqrt{a^2 + 4y_1(y_1 - a_1)}\right].$$

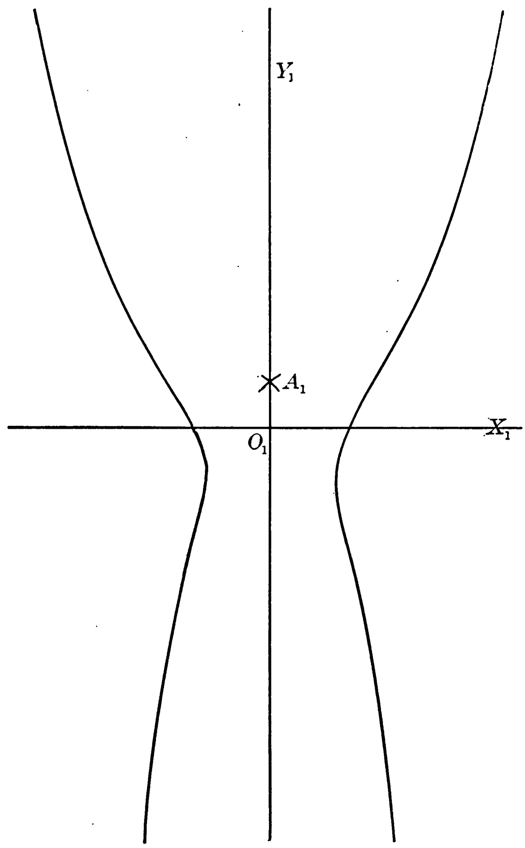

FIG. 1. c_1 for $a > a_1$.

* This equation in another form is due to S. ROBERTS (Proceedings of the London Mathematical Society, vol. 2 (1869), pp. 127–133), who appears to have been the first to generalize the movement of the conchoid mechanism of Nicomedes to the one we are studying. The movement of the mechanism of Nicomedes is obtained by making $a_1 = 0$. Any point upon O_1y_1 then describes a conchoid.

For $a = a_1$ the curves degenerate to cubics. As I have already treated that case (American Journal of Mathematics, vol. 21 (1899), pp. 267–269), it is omitted here.

The fixed centrode c and the moving one c_1 are respectively,

$$a_1^2[x^2 + (y - a)^2] - [x^2 - a(y - a)]^2 = 0,$$
$$a^2[x_1^2 + (y_1 - a_1)^2] - [x_1^2 - a_1(y_1 - a_1)]^2 = 0.$$

The centrode c_1 (Figs. 1, 2) is of the fourth order, symmetrical with respect to the axis of y_1, having a real double point at A_1 and a real tac-node at infinity on

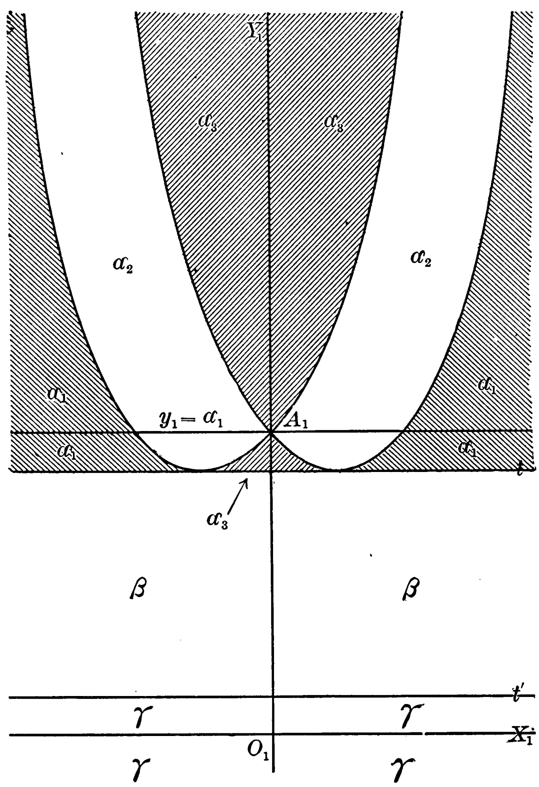

FIG. 2. c_1 for $0 < a < a_1$.

the axis of y_1. The double point at A_1 is a node or a conjugate point according as a_1 is greater or less than a. The line at infinity is tangent at the tac-node and it counts as two double tangents. The other two double tangents are:

$$y_1 = \tfrac{1}{2}\left(a_1 \pm \sqrt{a_1^2 - a^2}\right).$$

They are real or imaginary according as $a_1 > a$ or $a_1 < a$. When the double tangents are real the points of contact of that having the positive value of the radical (which we will denote by t) are real, and the points of contact of the other t' are imaginary. When $a > a_1$ the curve has four real inflexions.

When a is zero c_1 degenerates to $x_1^2 = a_1(y_1 - a_1)$ taken twice (Fig. 3). For $a = a_1$ it consists of $x_1 = 0$ taken twice and $x_1^2 = 2a_1y_1 - a_1^2$.

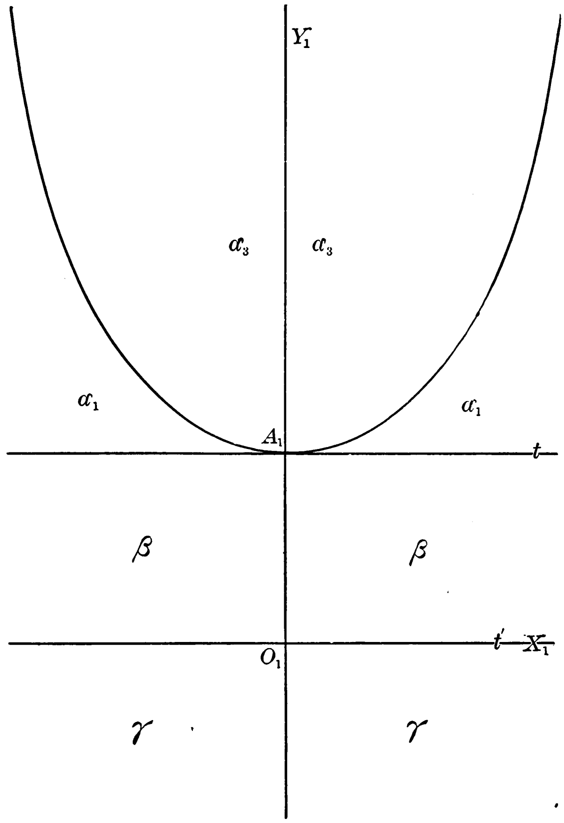

FIG. 3. c_1 for $a = 0$.

The forms of the point loci are shown by the following theorems, the demonstrations of which are omitted.

The locus of any point (x_1, y_1) of σ_1 lies between its tangents:

$$y = \pm \sqrt{x_1^2 + (y_1 - a_1)^2},$$

and has the asymptotes $y = \pm (y_1 - a_1)$ which are tangent to it at its infinitely distant node. The asymptotes are coincident for $y_1 = a_1$ when the curve has a tac-node at infinity. In particular the locus of A_1 consists of $x^2 + (y - a)^2 = 0$ and $y = 0$ each taken twice.

When $a > a_1$ and $y_1 \neq a_1$ the locus has two branches which come from the direction of positive x from tangency with different asymptotes, pass through the finite part of the plane (where they cross and form a real node N of Fig. 4), and go to infinity in the direction of the negative x-axis each approaching the asymptote from which the other started. The third double point of the locus is in the finite part of the plane, and is a node (M of Fig. 4), cusp, or conjugate point according as (x_1, y_1) is without, on, or between the branches of c_1.

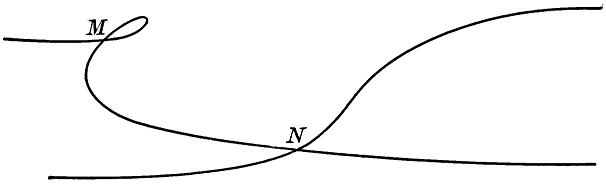

FIG. 4. $a > a_1 ; y_1 \neq a_1$.

When $0 < a < a_1$ and $y_1 \neq a_1$ the locus of (x_1, y_1) consists of two branches. One comes from the direction of positive x from contact with an asymptote into the finite part of the plane and returns to infinity in the direction of positive x and tangency with the other asymptote. The other branch comports itself similarly but comes from the direction of negative x_1 (Fig. 5). The plane σ_1 is

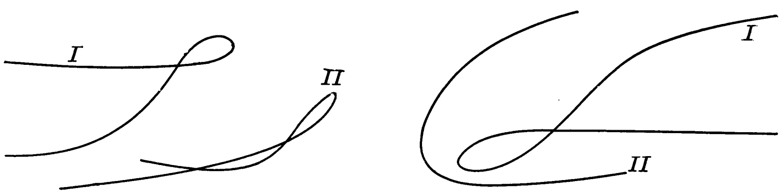

FIG. 5. $a < a_1 ; y_1 \neq a_1$.

divided into three regions α, β, γ by the two double tangents t, t' of c_1 and the line at infinity (Fig. 2). The region α is between t and the line at infinity and contains the real branches of c_1 and the point A_1. The region γ is between t' and the line at infinity, and contains the axis of x_1. The region α is divided into three regions α_1, α_2, α_3 as shown in Fig. 2. The points of α_1 generate curves with two real nodes in addition to that at infinity. They are of two kinds (Fig. 5, I and II) having a node on each branch or two on one according as the trac-

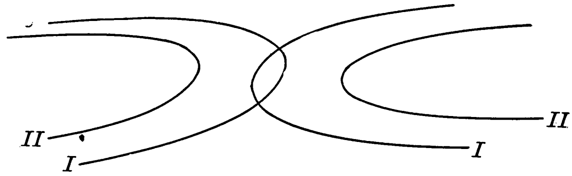

FIG. 6. $a < a_1$.

ing point is not or is between t and $y_1 = a_1$. The loci of points in α_2 have a real node and a real conjugate point, those in α_3 two real conjugate points. On the boundary between α_1 and α_2 points trace curves with one real cusp and one real node, and on the boundary between α_2 and α_3 the loci have one real cusp and one real conjugate point. The points of t between a_1 and β trace curves which have a tac-node on one branch, which becomes a ramphoid cusp when a point of contact of c_1 and t is taken. Lastly, the portion of t between a_3 and β traces curves with double conjugate points.

The curves of the region β have two non-intersecting branches (II, Fig. 6) and those of γ have two intersecting branches forming two real nodes (I, Fig. 6). On the boundary t' the two branches touch.

When $a = 0$ the two branches of c_1 fall together making the double parabola $x_1^2 = a_1(y_1 - a_1)$ passing through A_1 (Fig. 3). The double tangents t and t' are $y_1 = a_1$ and $y_1 = 0$ respectively. The loci of points of the regions β and γ and the boundary t' have the forms just noted for the case $a \neq 0$, as have also those of a_1 and a_2. The loci of the points of a_1 are of the kind having a node on each branch (Fig. 5, I). The points of c_1 trace curves with two cusps one on each branch, with the exception that the locus of A is degenerate consisting of $x^2 + y^2 = 0$, and $y = 0$ taken twice. The loci of the other points of t have an oscnode at infinity.

§ 4. *Line loci.*

The forms of the unicursal quartic scrolls which are generated by any straight line carried by σ_1 and oblique to it are given by the following theorems.

When l' is parallel to the axis of x_1 the scrolls have three nodal right lines, one of which is a generator at infinity and lies entirely upon the surface. They are of six types as follows:

Type.	Value of $a : a_1$.	Position of l'.	Holgate's Species.	Holgate's Subform.
I	$a > a_1$		F_5^4	4
II	$a < a_1$	l' lies in the region γ.	F_5^4	5
III	$a < a_1$	l' is t'.	F_6^4	
IV	$a < a_1$	l' is in the region of β.	F_7^4	
V	$a < a_1$	l' is t.	F_6^4	
VI	$a < a_1$	l' is in the region of a.	F_5^4	1

When l' passes through A_1 and is not parallel to the axis of x_1 the surfaces have a nodal hyperbola and intersecting nodal straight line. They are of four types.

Type.	Value of $a : a_1$.	Position of l'.	Holgate's F_3^4 subform.
VII	$a > a_1$		2
VIII	$a < a_1$	l' is in the angle made by the tangents to c_1 at A_1 which contains the axis of y_1.	1
IX	$a < a_1$	l' is tangent to c_1.	4
X	$a < a_1$	l' is in the angle made by the tangents to c_1 at A_1 which does not contain the axis of y_1.	1

When l' neither passes through A_1 nor is parallel to the axis of x_1, the scrolls have a nodal cubical hyperbola and are of the following eleven types:

Type.	Value of $a : a_1$..	Position of l'.	Holgate's F_1^4 subform.
XI	$a > a_1$	l' intersects c_1 in two real distinct and two imaginary points.	11
XII	$a > a_1$	l' intersects in four real distinct points.	1
XIII	$a > a_1$	l' is tangent at a point of c_1, between which and infinity there is (in one direction) no point of inflexion upon c_1.	4
XIV	$a > a_1$	l' is tangent to c_1 between two inflexions.	2
XV	$a > a_1$	l' is tangent to c_1 at an inflexion.	8
XVI	$a < a_1$	l' intersects c_1 in four imaginary points.	15 or 16*
XVII	$0 < a < a_1$	l' is tangent to c_1 and intersects in two imaginary points.	12 or 14*
XVIII	$0 < a < a_1$	l' intersects c_1 in two real distinct and two imaginary points.	11
XIX	$0 < a < a_1$	l' is tangent to c_1 and intersects it in two real distinct points.	4 or 6*
XX	$a < a_1$	l' intersects c_1 in four real distinct points.	1 or 5*
XXI	$a = 0$	l' is tangent to c_1.	7

In conclusion it is interesting to note that the only plane movements (excluding those trivial ones by which the direction of no carried straight line is changed) by which any carried straight line generates a ruled surface of order not greater than four, are (as far as an examination of the literature of the subject shows) the two of the present paper, those whose centrodes are equal conics,† that of the ellipsograph,‡ and that of rotation about a fixed axis. This suggests the as yet unsolved problem: to make a complete enumeration of such movements.

ITHACA, N. Y., *April* 14, 1900.

* The method followed does not enable one to determine to which of the two subforms the scroll obtained belongs.

† BLAKE, American Journal of Mathematics, vol. 21 (1899), pp. 257–269.

‡ BLAKE, ibid., vol. 22 (1900), pp. 146–153.