CERTAIN CASES IN WHICH THE VANISHING OF THE WRONSKIAN IS A SUFFICIENT CONDITION FOR LINEAR DEPENDENCE*

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PEANO in Mathesis, vol. 9 (1889), p. 75 and p. 110 seems to have been the first to point out that the identical vanishing of the Wronskian of n functions of a single variable is not in all cases a sufficient condition for the linear dependence of these functions.† At the same time he indicated a case in which it *is* a sufficient condition,‡ and suggested the importance of finding other cases of the same sort. Without at first knowing of PEANO's work, I was recently led to this same question, and found a case not included in PEANO's in which the identical vanishing of the Wronskian is a sufficient condition.§ It is my purpose in the present paper to consider these cases and others of a similar nature.

By far the most important case in which the identical vanishing of the Wronskian is a sufficient condition for linear dependence is that in which the functions in question are at every point of a certain region analytic functions, whether of a real or complex variable is, of course, immaterial. This case requires no further treatment here.

We shall therefore be concerned exclusively with the case in which the independent variable x is real. This variable we will suppose to be confined to an interval I which may be finite or infinite, and if limited in one or both directions may or may not contain the end points. In some of the proofs we shall use a subinterval $a \leqq x \leqq b$ of I; ‖ this subinterval we call I'.

Whether the functions are real or complex is immaterial.

We use the symbol $\equiv$ to denote an identity, i. e., an equality which holds at every point of the interval we are considering.

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† It is of course a necessary condition provided the functions have finite derivatives of the first $n-1$ orders at every point of the region in question.

‡ See § 4 of the present paper.

§ See Bulletin of the American Mathematical Society, December, 1900, p. 120, and Annals of Mathematics, second series, vol. 2, p. 93.

‖ We suppose here that a and b are finite quantities.

§ 1. *The Fundamental Theorem.*

We consider first the special case of two functions.

THEOREM I. *Let $u_1(x)$ and $u_2(x)$ be functions of x which at every point of I have finite first derivatives, while u_1 does not vanish in I; then if*

$$u_1u_2' - u_2u_1' \equiv 0, \tag{1}$$

u_1 and u_2 are linearly dependent throughout I, and in particular

$$u_2 \equiv cu_1. \tag{2}$$

For dividing (1) by u_1^2 we have:

$$\frac{d}{dx}\left(\frac{u_2}{u_1}\right) \equiv 0.$$

Therefore:

$$\frac{u_2}{u_1} \equiv c.$$

We pass now to the general case which includes the case just considered.

THEOREM II. *Let $u_1(x), u_2(x), \cdots, u_n(x)$ be functions of x which at every point of I have finite derivatives of the first $n-1$ orders, while the Wronskian of $u_1, u_2, \cdots, u_{n-1}$ does not vanish in I; then if the Wronskian W of $u_1, u_2, \cdots, u_n$ vanishes identically $u_1, u_2, \cdots, u_n$ are linearly dependent throughout I, and in particular:*

$$u_n \equiv c_1u_1 + c_2u_2 + \cdots + c_{n-1}u_{n-1}.$$

In the Wronskian:

$$W = \begin{vmatrix} u_1 & u_2 & \cdots & u_n \\ u_1' & u_2' & \cdots & u_n' \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ u_1^{(n-1)} & u_2^{(n-1)} & \cdots & u_n^{(n-1)} \end{vmatrix},$$

we denote by $W_1, W_2, \cdots, W_n$ the minors corresponding to the elements of the last row. We have then:

$$W_1u_1^{(i)} + W_2u_2^{(i)} + \cdots + W_nu_n^{(i)} \equiv 0 \quad (i=0, 1, \cdots, n-1).$$

Differentiating each of the first $n-1$ of these identities and subtracting from it the one next following we get:

$$W_1'u_1^{(i)} + W_2'u_2^{(i)} + \cdots + W_n'u_n^{(i)} \equiv 0 \quad (i=0, 1, \cdots, n-2).$$

Let us add these identities together after having multiplied the i-th of them

$(i = 1, 2, \cdots, n-1)$ by the first minor of W_n corresponding to $u_1^{(i-1)}$. This gives:

$$W_1' W_n - W_n' W_1 \equiv 0 .$$

Now since by hypothesis W_n does not vanish in I we have by theorem I:

$$W_1 \equiv - c_1 W_n .$$

In the same way:

$$W_2 \equiv - c_2 W_n ,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot$$

$$W_{n-1} \equiv - c_{n-1} W_n .$$

Therefore the identity

$$W_1 u_1 + W_2 u_2 + \cdots + W_n u_n \equiv 0 ,$$

can be written:

$$W_n(- c_1 u_1 - c_2 u_2 \cdots c_{n-1} u_{n-1} + u_n) \equiv 0 ,$$

and, since W_n does not vanish, our theorem follows at once.*

§ 2. *A Generalization for the Case of Two Functions.*

THEOREM III.† *Let u_1 and u_2 be functions of x which at every point of I have finite derivatives of the first k orders $(k \geqq 1)$, while $u_1, u_1', u_1'', \cdots, u_1^{(k)}$ do not all vanish at any one point of I; then if*

$$u_1 u_2' - u_2 u_1' \equiv 0 ,$$

u_1 and u_2 are linearly dependent, and in particular:

$$u_2 \equiv c u_1 .$$

This theorem will evidently be established if we can prove it for every finite and perfect subinterval I' of I. We will therefore in our proof consider only the interval I'.

There cannot be more than a finite number of points in I' where $u_1 = 0$. For if there were these points would have at least one limiting point x_0 in I', and since u_1 is continuous it would vanish at x_0. By ROLLE's theorem there would also be an infinite number of points where $u_1' = 0$ and these points would have x_0 as limiting point, and owing to the continuity of u_1' we should have $u_1'(x_0) = 0$. Proceeding in the same way we see that $u_1'', u_1''', \cdots, u_1^{(k-1)}$ would all vanish at x_0. That $u_1^{(k)}$ would also vanish at x_0 must be shown in a slightly

* This proof is merely a slight modification of the one given by FROBENIUS, Crelle, vol. 76 (1873), p. 238. Cf. also HEFFTER, *Lineare Differentialgleichungen*, p. 233.

† The special case $k = 1$ of this theorem was given by PEANO, l. c. Cf. also Annals of Mathematics, second series, vol. 2, p. 92.

different manner since we do not know that $u_1^{(k)}$ is continuous. This follows at once, however, from the fact that $u_1^{(k-1)}$ would vanish in every neighborhood of x_0. We thus see that if u_1 vanished at an infinite number of points in I' there would be a point x_0 where $u_1, u_1', \cdots, u_1^{(k)}$ all vanish, and this is contrary to hypothesis.

The points at which $u_1 = 0$ therefore divide the interval I' into a finite number of pieces throughout each of which theorem I tells that u_2 is a constant multiple of u_1, and owing to the continuity of u_1 and u_2 this relation must also hold at the extremities of the piece in question. It remains to show that this constant is the same for all the pieces. It will evidently be sufficient to consider two adjacent pieces separated by the point p. Suppose that in the piece to the left of p we have

$$u_2 = c_1 u_1,$$

and in the piece to the right,

$$u_2 = c_2 u_1.$$

Since the derivatives of u_1 and u_2 at p may be found either by differentiating to the right or to the left we have:

$$\left.\begin{aligned} u_2^{(i)}(p) &= c_1 u_1^{(i)}(p), \\ u_2^{(i)}(p) &= c_2 u_1^{(i)}(p), \end{aligned}\right\} \qquad (i=1, 2, \cdots, k).$$

Therefore

$$(c_1 - c_2) u_1^{(i)}(p) = 0 \qquad (i=1, 2, \cdots, k).$$

Now, since $u_1(p) = 0$, there must be at least one of the derivatives $u_1', u_1'', \cdots, u_1^{(k)}$ which does not vanish at p. Therefore

$$c_1 = c_2,$$

and our theorem is proved.

§ 3. *Two Extensions to the case of n Functions.*

THEOREM IV.* *Let $u_1, u_2, \cdots, u_n$ be functions of x which at every point of I have finite derivatives of the first $n - 2 + k$ orders ($k \geqq 1$), while the Wronskian of $u_1, u_2, \cdots, u_{n-1}$ and its first k derivatives do not all vanish at any one point of I; then if the Wronskian of $u_1, u_2, \cdots, u_n$ is identically zero $u_1, u_2, \cdots, u_n$ are linearly dependent, and in particular:*

$$u_n \equiv c_1 u_1 + c_2 u_2 + \cdots + c_{n-1} u_{n-1}.$$

The proof of this theorem is, in the main, the same as that of theorem II. We will therefore only point out the two points of difference.

1. We must use theorem III instead of theorem I to establish the relation:

* The special case $k = 1$ of this theorem was given by the writer, l. c.

$$W_1 \equiv -c_1 W_n.$$

2. From the identity:

$$W_n(-c_1u_1 - c_2u_2 \cdots c_{n-1}u_{n-1} + u_n) \equiv 0,$$

we can now infer only that *at the points where* $W_n \neq 0$,

$$u_n = c_1u_1 + \cdots + c_{n-1}u_{n-1}.$$

In order to prove that this equation also holds at the points where $W_n = 0$, we notice first that these points in any finite and perfect subinterval I' of I are finite in number as otherwise there would be (cf. the proof of theorem III) a point of I' where W_n, W_n', $\cdots$, $W_n^{(k)}$ all vanish. All the points where W_n vanishes are therefore isolated, and since the equation

$$u_n = c_1u_1 + \cdots + c_{n-1}u_{n-1}$$

holds everywhere except at these points it must on account of the continuity of the u's hold at these points also. Thus our theorem is proved.

A little reflection on the results so far obtained will suggest the question whether the theorem of the last section might not be extended to the case of n functions by requiring, not as we have just done, that W_n, W_n', $\cdots$, $W_n^{(k)}$, do not all vanish at any point of I, but that u_1 and a certain number of its derivatives shall not all vanish at any point of I. The following example shows, however, not only that the theorem thus suggested is not true, but that even when *no one of the u's vanishes at any point of I* the identical vanishing of the Wronskian is not necessarily a sufficient condition for linear dependence when we have more than two functions.

Example. Consider the three functions:

$$u_1 = \begin{cases} 1 + e^{-\frac{1}{x^2}} & (x \neq 0), \\ 1 & (x = 0), \end{cases} \qquad u_2 = \begin{cases} 1 + e^{-\frac{1}{x^2}} & (x > 0), \\ 1 & (x = 0), \\ 1 - e^{-\frac{1}{x^2}} & (x < 0), \end{cases} \qquad u_3 = 1.$$

These three functions are obviously linearly independent in any interval including both positive and negative values of x. Moreover no one of them vanishes for any real value of x. Yet the Wronskian of u_1, u_2, u_3 is identically zero.

The following theorems V and VI, which run somewhat along the lines just indicated, are, however, true:

Theorem V. *Let $u_1, u_2, \cdots, u_n$ be functions of x which at every point of I have finite derivatives of the first $n-1$ orders, while no function (other than zero) of the form:*

$$g_1u_1 + g_2u_2 + \cdots + g_nu_n$$

(*the g's being constants*) *vanishes together with its first* $n-1$ *derivatives at any point of* I; *then if the Wronskian of* $u_1, u_2, \cdots, u_n$ *vanishes at any point* p *of* I *these functions are linearly dependent.*

From the fact that the Wronskian vanishes at p follows the existence of n constants $c_1, c_2 \cdots, c_n$ not all zero and such that

$$c_1 u_1^{(i)}(p) + c_2 u_2^{(i)}(p) + \cdots + c_n u_n^{(i)}(p) = 0 \quad (i = 0, 1, \cdots, n-1),$$

i. e., the function $c_1 u_1 + c_2 u_2 + \cdots + c_n u_n$ vanishes together with its first $n-1$ derivatives at the point p, and must therefore be identically zero. Thus our theorem is proved.

THEOREM VI. *Let* $u_1, u_2, \cdots, u_n$ *be functions of* x *which at every point of* I *have finite derivatives of the first* k *orders* ($k > n-1$), *while no function* (*other than zero*) *of the form*

$$g_1 u_1 + g_2 u_2 + \cdots + g_n u_n,$$

(*the g's being constants*) *vanishes together with its first* k *derivatives at any point of* I; *then if the Wronskian of* $u_1, u_2, \cdots, u_n$ *vanishes identically these functions are linearly dependent.*

We prove this theorem first on the supposition that the Wronskian of $u_1, u_2, \cdots, u_{n-1}$ does not vanish identically.* In this case there exists a point p of I where the Wronskian of $u_1, u_2, \cdots, u_{n-1}$ does not vanish. Since this last named Wronskian is continuous it is different from zero throughout the neighborhood of p. We see then by applying II that there exist n constants $c_1, c_2, \cdots, c_n$ not all zero and such that the function

$$c_1 u_1 + c_2 u_2 + \cdots + c_n u_n$$

is zero throughout the neighborhood of p. Accordingly this fuction vanishes together with its first k derivatives at p, and therefore vanishes identically. Thus our theorem is proved in this special case.

In order to prove the theorem in general we first notice that if $u_1 \equiv 0$ the u's are surely linearly dependent. If u_1 is not identically zero, consider in succession the Wronskians of u_1, u_2, of u_1, u_2, u_3, of u_1, u_2, u_3, u_4, etc. Suppose the first of these which vanishes identically is the Wronskian of $u_1, u_2, \cdots, u_m$ ($m \leqq n-1$). Then since the Wronskian of $u_1, u_2, \cdots, u_{m-1}$ does not vanish identically, the special case of our theorem which we have already proved shows that $u_1, u_2, \cdots, u_m$ are linearly dependent. Accordingly $u_1, u_2, \cdots, u_n$ are linearly dependent, and our theorem is proved.

Theorems V and VI admit of immediate application to the theory of linear differential equations, as the following theorem shows.

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* The proof of this part of the theorem has been modified since the paper was presented to the Society by making it depend on II instead of on the lemmas of § 5.

THEOREM VII. *Let* $p_1, p_2, \cdots, p_n$ *be functions of* x *which at every point of* I *are continuous, and let* $y_1, y_2, \cdots, y_k (k \leqq n)$ *be functions of* x *which at every point of* I *satisfy the differential equation:*

$$y^{(n)} + p_1 y^{(n-1)} + \cdots + p_n y = 0 ;$$

then the identical vanishing of the Wronskian of $y_1, y_2, \cdots, y_k$ *(or in the case* $k = n$ *the vanishing of this Wronskian at a single point of* I*) is a sufficient condition for the linear dependence of* $y_1, y_2, \cdots, y_k$.

This theorem follows at once from theorems V and VI when we recall the fact that a solution of the above written differential equation which vanishes together with its first $n - 1$ derivatives at a point of I is necessarily identically zero.

§ 4. *Discussion of Peano's Theorems.* *

One of PEANO'S results, as has already been stated, is the special case $k = 1$ of theorem III. Apart from this PEANO'S results cover no case which is not also covered by the fundamental theorem of § 1. I propose to show this in the present section.

For this purpose we first establish the following:

LEMMA. *Let* u_1 *and* u_2 *be functions of* x *which at every point of* I *have finite first derivatives, while*

$$u_1 u_2' - u_2 u_1' \equiv 0 ;$$

if a point p *exists in* I *at which* $u_2 = 0$, *while in every neighborhood of* p *lie points where* $u_2 \neq 0$, *then* $u_1(p) = 0$.

For if $u_1(p) \neq 0$ we could, on account of the continuity of u_1, mark off a neighborhood of p throughout which u_1 does not vanish, and throughout which therefore by theorem I

$$u_2 = c u_1 .$$

Since at p $u_1 \neq 0$ and $u_2 = 0$ we must have $c = 0$, but this would make u_2 vanish throughout the neighborhood of p, and this is contrary to hypothesis.

PEANO deduces the following theorem in the case of two functions. This theorem includes as a special case the theorem to which theorem III reduces when $k = 1$, and appears at first sight to go beyond it.

PEANO'S FIRST THEOREM. *Let* u_1 *and* u_2 *be functions of* x *which at every point of* I *have finite first derivatives, while* u_1, u_2, u_1', u_2' *do not all vanish at any point of* I; *then if*

$$u_1 u_2' - u_2 u_1' \equiv 0 ,$$

u_1 *and* u_2 *are linearly dependent.*

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* See, besides the notes in Mathesis referred to at the beginning of this article, a paper by PEANO: Rendiconti della Accademia dei Lincei, ser. 5, vol. 6, 1° sem. (1897), p. 413.

The truth of this theorem will be established, and at the same time it will be proved that it covers no case which is not also covered by the special case $k=1$ of theorem III, if we can show that either there is no point of I where u_1 and u_1' both vanish, or there is no point of I where u_2 and u_2' both vanish. Assume then that there is a point where u_2 and u_2' both vanish. Here we distinguish between two cases:

(a) $u_2 \equiv 0$. Here $u_2 = u_2' = 0$ at every point of I, and therefore there can be no point in I where $u_1 = u_1' = 0$.

(b) u_2 is not identically zero. Then there exists a point p in I at which $u_2 = u_2' = 0$, but in whose every neighborhood lie points where $u_2 \neq 0$. Therefore by the above lemma $u_1(p) = 0$. We must therefore have $u_1'(p) \neq 0$. Accordingly there exists an ϵ such that throughout the interval $p < x < p + \epsilon$, and also throughout the interval $p > x > p - \epsilon$, u_1 does not vanish. Let us choose that one of these intervals in which lie points where $u_2 \neq 0$. By theorem I we have at every point of this interval, and therefore on account of the continuity of u_1 and u_2 also at p,

$$u_2 = cu_1,$$

where $c \neq 0$ as otherwise u_2 would vanish at every point of this interval. From this last equation we infer that

$$u_2'(p) = cu_1'(p).$$

Therefore since $u_2'(p) = 0$ and $c \neq 0$ we get $u_1'(p) = 0$. We are thus led to a contradiction, and therefore the case (b) cannot occur.

PEANO'S SECOND THEOREM. *Let $u_1, u_2, \cdots, u_n$ be functions of x which at every point of I have finite derivatives of the first $n-1$ orders, while the Wronskians of these functions taken $n-1$ at a time do not all vanish at any point of I; then if the Wronskian of $u_1, u_2, \cdots, u_n$ vanishes identically $u_1, u_2, \cdots, u_n$ are linearly dependent.*

We will establish this theorem, and at the same time show that it covers no case which is not also covered by the fundamental theorem II, by proving that there must be one of the Wronskians $W_1, W_2, \cdots, W_n$ (to use the notation employed in the proof of theorem II) which does not vanish at any point of the interval I. Suppose each of these W's vanished in I. They cannot all vanish identically. Suppose that W_n is one of those which does not vanish identically. Then there exists a point p at which $W_n = 0$ but in whose every neighborhood lie points where $W_n \neq 0$.

Now by the reasoning used in the proof of theorem II we see that:

$$W_i' W_n - W_i W_n' \equiv 0 \qquad (i=1, 2, \cdots, n-1).$$

Therefore, by our lemma, W_i vanishes at p $(i = 1, 2, \cdots, n-1)$ and this is contrary to hypothesis since W_n also vanishes at p.

§ 5. *A Theorem concerning Wronskians.*

I have now completed what I have to say on the subject of linear dependence. There remains however a theorem concerning Wronskians which I have found useful in the course of my work, although in the form which I have finally given to this paper no use has been made of it.

Before stating this theorem we will first establish two lemmas which we shall use in its proof.

Consider a matrix M of $n + m$ rows and n columns. Denote by D_i the n-rowed determinant obtained from M by striking out all of its $m + 1$ last rows except the $(n - 1 + i)$-th row. Denote by M' the matrix obtained from M by striking out its last $m + 1$ rows. Denote by Δ_i the $(n - 1)$-rowed determinant obtained from M' by striking out its i-th column.

LEMMA I. *If $D_1 = D_2 = \cdots = D_{m+1} = 0$, and if $\Delta_1, \Delta_2, \cdots, \Delta_n$ are not all zero, then all the n-rowed determinants of M are zero.*

For denoting the element of M which stands in the i-th row and j-th column by a_{ij}, we have:

$$a_{i1}\Delta_1 - a_{i2}\Delta_2 + \cdots + (-1)^{n-1}a_{in}\Delta_n = 0 \qquad (i = 1, 2, \cdots, n+m),$$

and these form a set of $n + m$ homogeneous linear equations satisfied by the n Δ's which by hypothesis are not all zero.

LEMMA II. *Let $u_1, u_2, \cdots, u_n$ be functions of x which at every point of I have finite derivatives of the first k orders $(k \geqq n)$, while their Wronskian vanishes identically; then, except at points where the Wronskian of $u_1, u_2, \cdots, u_{n-1}$ is zero, all the n-rowed determinants of the matrix:*

$$\begin{vmatrix} u_1 & u_2 & \cdots & u_n \\ u_1' & u_2' & \cdots & u_n' \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ u_1^{(k)} & u_2^{(k)} & \cdots & u_n^{(k)} \end{vmatrix}$$

are zero.

We first prove this lemma in the case $k = n$. Here the determinant obtained from the above matrix by striking out the next to the last row is simply the derivative of the Wronskian of $u_1, u_2, \cdots, u_n$, and therefore also vanishes identically. The truth of our lemma thus follows at once from lemma I.

In order to prove the lemma in the general case we use the method of mathematical induction, and assume that the lemma has been proved when $k = k_1 - 1$. We wish to prove that the lemma also holds when $k = k_1$. Let us denote by M the above matrix when k has the value k_1, and by N the matrix obtained from M by striking out its last row; and let p be any point of I where the

Wronskian of $u_1, u_2, \cdots, u_{n-1}$ does not vanish. If then we can prove that the determinant:

$$D = \begin{vmatrix} u_1 & u_2 & \cdots & u_n \\ u_1' & u_2' & \cdots & u_n' \\ \cdot & \cdot & \cdot & \cdot \\ u_1^{(n-2)} & u_2^{(n-2)} & \cdots & u_n^{(n-2)} \\ u_1^{(k_1)} & u_2^{(k_1)} & \cdots & u_n^{(k_1)} \end{vmatrix},$$

vanishes at p, it will follow at once from lemma I that all the n-rowed determinants of M vanish at p, since this is true of all the n-rowed determinants of N. In order to prove that D vanishes at p let us consider the $(k_1 - n + 1)$-th derivative of the Wronskian of $u_1, u_2, \cdots, u_n$. This derivative will of course vanish identically. If we compute its value we find that it consists of the sum of a number of n-rowed determinants of which D is one while the others are all determinants of the matrix N, and therefore vanish at p. Thus we see that D vanishes at p, and our lemma is proved.

THEOREM VIII. *Let $u_1, u_2, \cdots, u_{n+1}$ be functions of x which at every point of I have continuous derivatives of the first n orders; then if the Wronskian of $u_1, u_2, \cdots, u_n$ vanishes identically the Wronskian of $u_1, u_2, \cdots, u_{n+1}$ will vanish identically.*

Denote by M the matrix obtained from the Wronskian

$$W = \begin{vmatrix} u_1 & u_2 & \cdots & u_{n+1} \\ u_1' & u_2' & \cdots & u_{n+1}' \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ u_1^{(n)} & u_2^{(n)} & \cdots & u_{n+1}^{(n)} \end{vmatrix}$$

by striking out the last column. Then lemma II tells us that all the n-rowed determinants of M vanish except at the points where the Wronskians $\Delta_1, \Delta_2, \cdots, \Delta_n$ of the functions $u_1, u_2, \cdots, u_n$ taken $n-1$ at a time all vanish. Accordingly $W = 0$ except at these points. Let p be any such point of I. Our theorem will be proved if we can show that W vanishes at p.

We must distinguish two cases:

(*a*) $\Delta_1, \Delta_2, \cdots, \Delta_n$ do not all vanish identically throughout the neighborhood of p. There are therefore points in every neighborhood of p where the Δ's are not all zero, and where therefore $W = 0$; accordingly W must also vanish at p since it is a continuous function of x.*

(*b*) The Δ's all vanish identically throughout the neighborhood of p. Before

* This is the only point in the proof where use is made of the assumption that the nth derivatives of the u's are continuous. Would not the theorem still be true without this assumption?

proving in general that $W = 0$ for points of class (b) we will prove it in the simple case $n = 2$. Here we have two Δ's: $\Delta_1 = u_2$, $\Delta_2 = u_1$. Since these vanish identically in the neighborhood of p all the elements of the first two columns of W vanish at p, and therefore W vanishes at p.

We will now complete our proof by the method of mathematical induction by assuming that the theorem has been proved when we have less than $n + 1$ functions. Since each of the Δ's is the Wronskian of $n - 1$ of the functions $u_1, u_2, \cdots, u_n$ it follows that throughout the neighborhood of p the Wronskian of any n of the $n + 1$ functions $u_1, u_2, \cdots, u_{n+1}$ must vanish. Accordingly W also vanishes at p, as we see by expanding it according to the elements of its last row.

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