AN ELEMENTARY PROOF OF A THEOREM OF STURM*

BY

MAXIME BÔCHER

We shall have to deal with two real solutions y_1 and y_2 of the differential equation:

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = 0$$

where p and q are throughout an interval $a \leqq x \leqq b$ real and continuous functions of the real variable x. † One of the most important of STURM'S results (Liouville's Journal, vol. 1 (1836), p. 106) is that, if y_1 and y_2 are linearly independent, between two successive roots of one lies one and only one root of the other.

The following generalization is (implicitly at least) contained in STURM'S paper, and from it what I have called STURM'S theorems of comparison for a single equation ‡ follow at once. It is my object in the present note to prove this theorem by a simple and elementary method which makes use only of a single property of y_1 and y_2, namely that a necessary and sufficient condition for their linear dependence is that $y_1y_2' - y_2y_1'$ should vanish at some point of the interval ab. §

The theorem in question may be stated as follows, and when it is so stated the method of proof is at once suggested:

Suppose that y_1 vanishes neither at a nor at b, and that y_2, if it does not vanish at a, satisfies the relation:

$$\frac{y_2'(a)}{y_2(a)} > \frac{y_1'(a)}{y_1(a)},$$

and, if it does not vanish at b, satisfies the relation:

* Presented to the Society February 23, 1901. Received for publication January 14, 1901.

† We may, if we wish, allow p and q to have a finite number of discontinuities in this interval, provided that in the neighborhood of each discontinuity p and q remain finite, or, if they become infinite, do so in such a way that $|p|$ and $|q|$ can be integrated up to these points. See Bulletin of the American Mathematical Society, March, 1899, p. 276.

‡ Bulletin of the American Mathematical Society, May, 1898, pp. 366–367.

§ In a slightly different form I applied this method four years ago to the simple case stated above. See Bulletin of the American Mathematical Society, March, 1897, p. 210.

$$\frac{y_2'(b)}{y_2(b)} < \frac{y_1'(b)}{y_1(b)},$$

*and finally that y_2 does not vanish in the interval $a < x < b$; then y_1 vanishes once and only once in this interval.**

We first prove that y_1 must vanish at least once. For if it did not the function:

$$f(x) = \frac{y_2'(x)}{y_2(x)} - \frac{y_1'(x)}{y_1(x)}$$

would be continuous throughout the interval $a < x < b$. Moreover if $y_2(a) \neq 0$ we have $f(a) > 0$, if $y_2(a) = 0$ we have $f(a) = +\infty$. In any case therefore $f(x)$ is positive in the neighborhood of a. In the same way we see that $f(x)$ is negative in the neighborhood of b. Accordingly $f(x)$ must vanish at some point between a and b. Therefore at this point:

$$y_1 y_2' - y_2 y_1' = 0,$$

and hence y_1 and y_2 are linearly dependent. This however is clearly not the case.

If now y_1 vanished more than once in the interval $a < x < b$, let $x = \alpha$ and $x = \beta$ be two successive points in this interval where $y_1 = 0$. Then, interchanging y_1 and y_2, and applying the part of the theorem we have proved to the interval $\alpha < x < \beta$, we see that y_2 vanishes at least once between α and β, and this is contrary to hypothesis.

RAPALLO, ITALY, *December* 31, 1900.

* If $y_2(a) = y_2(b) = 0$ this theorem reduces to the simple theorem stated above.