ON A FUNDAMENTAL PROPERTY OF A MINIMUM IN THE CALCULUS OF VARIATIONS AND THE PROOF OF A THEOREM OF WEIERSTRASS'S*

BY

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The notion of the minimum of an integral in the calculus of variations is analogous to that of the minimum of a function of a real variable. The function, $f(x)$, is thought of as continuous and it is said to have a minimum at the point $x = a$ when, for all points of the neighborhood of a with the one exception of a itself, the value of $f(x)$ is greater than $f(a)$; i. e., when the relation holds:

$$f(x) > f(a), \quad 0 < |x - a| \leqq h.$$

This is, however, only a partial description of the behavior of the function in the neighborhood of the point a, as is shown by a simple example of a discontinuous function for which the above relation is true. Let the function $\phi(x)$ be defined as follows:

$\phi(x) = 1$, when x is irrational;

$\phi(x) = 1/q$, when $x = \pm p/q$, where p/q is a positive fraction in its lowest terms (in particular, an integer).

$\phi(0) = 0$.

Then $\phi(x)$ has a minimum at the point $x = 0$, for

$$\phi(x) > \phi(0), \quad x \neq 0;$$

and yet, in the neighborhood of an arbitrarily chosen value of x, the lower limit of $\phi(x)$ is 0.

A continuous function $f(x)$ has, then, a further property in the neighborhood of a point $x = a$ at which it is a minimum, which may be expressed as follows:

Let δ be any positive quantity less than h. Then there exists a positive quantity ϵ such that the lower limit of $f(x)$ for values of x in the above neighborhood of a for which $|x - a| > \delta$ is at least as great as $f(a) + \epsilon$; or

$$f(x) \geqq f(a) + \epsilon, \quad \delta < |x - a| \leqq h.$$

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In the calculus of variations the integral

$$I=\int_{x_0}^{x_1}F(x,y,y')\,dx$$

or, in the parametric form,

$$I=\int_{t_0}^{t_1}F(x,y,x',y')\,dt$$

is said to be made a minimum by the functions $x=\phi(t)$, $y=\psi(t)$ corresponding to a curve C if its value J for C is less than its value I for any other curve $\bar{C}$ of a specified class (K) of curves which all lie in the neighborhood T of C (Fig. 1):

$$I>J.$$

WEIERSTRASS* has given a sufficient condition that this may be the case. The example of the function $\phi(x)$ above considered raises, however, the question: *Assuming that Weierstrass's sufficient condition is fulfilled, so that no curve of the class* (K) *distinct from* C *will give the integral* I *so small a value as* J, *may we not still have a set of curves belonging to this class,* $\bar{C}_1$, $\bar{C}_2$, $\cdots$, *which do not cluster about* C *as their limit and which have the property that, if* I_1, I_2, $\cdots$ *denote respectively the values of the integral* I *formed for these curves,*

$$\lim_{n=\infty} I_n = J\,?$$

In particular, it is conceivable that these curves might approach a curve $\bar{C}$ of the class (K) as their limit, distinct from C. For this curve, the value of I would, of course, not be $\lim\limits_{n=\infty} I_n = J$.

This question I answer in the negative, showing in § 1 by means of the theorem of § 2 that, if $\mathfrak{T}$ denotes an arbitrarily small neighborhood of C lying wholly within T, the lower limit of I formed for all the curves of the class (K) that do not lie wholly within $\mathfrak{T}$ is greater than J by a positive quantity ϵ, so that

$$I\geqq J+\epsilon,$$

when $\bar{C}$ does not lie wholly in $\mathfrak{T}$.

By means of these results it is possible to deal with a problem of importance concerning the scope of the class of curves (K). What curves shall be admitted to this class? The larger the class (K), the more general the property of the integral which is made a minimum by C; for the integral is thus compared

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* In his lectures. For a statement of the condition in the parametric case, see § 1 below. This is in substance KNESER'S form of the sufficient condition: KNESER, *Lehrbuch der Variationsrechnung*, chap. 3. Other forms of the sufficient condition, in particular, HILBERT'S form, have recently been treated in a paper by the writer: Annals of Mathematics, 2nd series, vol. 2, no. 3 (1901), p. 105.

with the curves of a more extended class. Hence it is natural to seek to enlarge the class (K) until limitations are reached which are dictated by the *nature of the problem*, not by the *exigencies of the proof*. For example, consider the integral

$$I = \int_{t_0}^{t_1} \sqrt{x'^2 + y'^2}\, dt.$$

This integral represents the length of the curve $\bar{C}$ when $\bar{C}$ has a continuous tangent. But the notion of length is not restricted to curves of this class. In fact, a curve may have a length and yet fail to have a tangent in a set of points everywhere dense along the curve. For, the length of a curve may be defined as the upper limit of the lengths of the inscribed polygons, in case a finite upper limit exists, i. e., as

$$\bar{L} \sum \sqrt{\Delta x_i^2 + \Delta y_i^2},$$

where $\Delta x_i = \phi(\tau_{i+1}) - \phi(\tau_i)$, $\Delta y_i = \psi(\tau_{i+1}) - \psi(\tau_i)$, and the vertices $\tau_0 = t_0$, $\tau_1, \cdots \tau_n = t_1$ are chosen in all possible ways.* JORDAN has shown that the necessary and sufficient condition that such an upper limit exist is that the functions ϕ, ψ both belong to the class of functions with limited variation (fonctions à variation bornée).†

The above generalization of the integral that represents the length of a curve with continuous tangent suggests a corresponding generalization of the general integral

$$I = \int_{t_0}^{t_1} F(x, y, x', y')\, dt.$$

Let $\bar{C}$ be a continuous curve lying in T and connecting the points A and B. Since $\bar{C}$ will not in general have a tangent, the proper integral I will not in general have a meaning. Inscribe a polygon in $\bar{C}$, as in the case of the arc integral, and, remembering that the function F is homogeneous of the first degree in its last two arguments, form the sum

* This is in substance SCHEEFFER'S definition. The form, which is a particularly elegant one is due to PEANO : *Applicazioni geometriche del calcolo infinitesimale*, 1887, p. 161 ; *Lezioni di analisi infinitesimale*, vol. 1, 1893, p. 171 ; Annali di matematica, 2nd ser., vol. 23 (1895), p. 153. It is analogous to PEANO'S definition of the upper definite integral ; cf. *Lezioni* (just cited), § 103 ; Annali, loc. cit. See also HILBERT'S definition at the close of this paper.

† JORDAN, *Cours d'analyse*, vol. 1, 2d ed., 1893, § 67. An example of a continuous curve, $y = \Phi(x)$, which has a length, but which does not have a tangent for any rational value of the abscissa can easily be given. Let $f(x) = x \sin \log x$, $x \neq 0$; $f(0) = 0$, and let $\phi(x) = f(\sin \pi x)$. Then the function $\Phi(x)$, given by the series

$$\Phi(x) = a_1\phi(x) + a_2\phi(2!\,x) + a_3\phi(3!\,x) + \cdots,$$

where the coefficients a_n are suitably chosen ($a_n = 1/n!^2$ will suffice), is a continuous function of limited variation. It is readily seen that this function, however, fails to have a derivative when x is rational. — The graph of WEIERSTRASS'S continuous function that has no derivative is an example of a continuous curve that does not have a length.

$$\sum F\left(x_i, y_i, \frac{\Delta x_i}{\Delta t_i}, \frac{\Delta y_i}{\Delta t_i}\right)\Delta t_i = \sum F(x_i, y_i, \Delta x_i, \Delta y_i).$$

If this sum converges toward one and the same limit when $n = \infty$ no matter how the polygons be chosen, provided merely that their sides all converge toward 0, then this curve shall be included in the class (K) and the value of the limit of this sum shall be taken as the generalized integral I. If $\bar{C}$ has a continuous tangent, this value is the same as that of the proper integral I. Thus we have a natural limitation for the class (K) of curves $\bar{C}$ to be admitted to consideration. WEIERSTRASS, to whom the generalization here considered is due, states in his lectures the theorem that when his sufficient condition for a minimum is fulfilled, the integral will still be a minimum, even when its definition is thus generalized and its value for C is compared with its value for any curve of this class (K). I have never seen a proof of this theorem. In § 3 of the present paper a proof is given.

The theorem of § 1 is proven only for the simplest integral of the calculus of variations; but it is true for other integrals, for example, for the double integral

$$\int\int F(x, y, z, p, q)\, dx\, dy,$$

and it probably holds in general. In the case of a weak minimum, the neighborhood T of C must be replaced by the *engere Nachbarschaft* * of C.

In the proof which HILBERT has recently given of DIRICHLET'S Principle † a continuous function z of the variables x, y is defined as $\lim_{n=\infty} \bar{F}_n(x, y)$ and it has to be shown that z satisfies LAPLACE'S equation at all interior points of the region. The theorem of § 1, stated for double integrals, may be employed to establish this point. A corresponding step has to be taken in the proof of the existence of a shortest line on a surface, and the theorems of this paper are applicable in this case, too.

§ 1. A CHECK FOR THE VALUE OF THE INTEGRAL I EXTENDED ALONG A CURVE $\bar{C}$ NOT LYING WHOLLY WITHIN THE NEIGHBORHOOD $\mathfrak{T}$ OF THE EXTREMAL $\bar{C}$.

1. *The quantity H, expressed in terms of the curvilinear coördinates of the field.* — Consider the integral

$$I = \int_{t_0}^{t_1} F(x, y, x', y')\, dt$$

* ZERMELO, *Untersuchungen zur Variationsrechnung*, Berlin dissertation, 1894.

† Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 8 (1899) p. 184.

extended along the curve

$$\bar{C}: \quad x = \phi(t), \quad y = \psi(t), \quad t_0 \leqq t \leqq t_1,$$

that connects the fixed points A, with the coördinates (x_0, y_0), $(t = t_0)$, and B, with the coördinates (x_1, y_1), $(t = t_1)$. Here, ϕ, ψ shall be continuous functions of t for the values in question, and shall have continuous first derivatives that do not vanish simultaneously for any of the above values of t. The function F shall, together with all its partial derivatives of the first and those of the second order that present themselves, be a single valued continuous function of the four arguments (x, y, x', y') regarded as independent variables, where (x, y) is any point of a certain region of the (x, y)-plane, within which the curve $\bar{C}$ must lie, and x', y' have any values except the one pair $(0, 0)$; furthermore, since the integral is to depend only on the curve $\bar{C}$, not on the particular choice of the parameter t, the function F shall be homogeneous of the first degree in the last two arguments. Finally, we assume that F is positive for all values of the arguments considered.

A necessary condition for a minimum is that the curve $\bar{C}$ satisfy the differential equations

$$F_x - \frac{d}{dt}F_{x'} = 0, \quad F_y - \frac{d}{dt}F_{y'} = 0, \qquad \left(F_x = \frac{\partial F}{\partial x}, \text{ etc.}\right).$$

We assume that this condition is fulfilled by the extremal* C, and that C is a curve which does not cut itself between A and B; furthermore, that a field exists surrounding C. More precisely, we assume that a one-parameter family of curves exists:

$$x = \xi(t, a), \quad y = \eta(t, a),$$

where the functions ξ, η have the following properties:

(a) These functions, together with their first partial derivatives and the cross derivatives ξ_{ta}, η_{ta}, are continuous functions of the independent varibles (t, a) at all points of the region

$$R: \qquad \bar{t}_0 \leqq t \leqq \bar{t}_1, \quad |a - a_0| \leqq \kappa,$$

where $\bar{t}_0 < t_0$, $t_1 < \bar{t}_1$, and $\kappa > 0$; $\bar{t}_0$, $\bar{t}_1$, κ are constants. When $a = a_0$, the equations represent a curve without multiple points, which passes through A and B; and when a has any constant value, the corresponding curve is an extremal. In particular, when $a = a_0$, the curve is the extremal C.

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* Any curve that satisfies these equations is denoted by KNESER as an extremal; *Lehrbuch der Variationsrechnung*, p. 24.

(*b*) The determinant

$$\Delta = \begin{vmatrix} \xi_t & \xi_a \\ \eta_t & \eta_a \end{vmatrix} \neq 0,$$

no matter for what point of R it be formed.

Let κ be chosen small enough so that all the curves will be free from multiple points. Two curves corresponding to distinct values of a not only will not intersect each other, but, if the two values of a differ from each other by an infinitesimal α, the distance $PP' = \beta$ from a point P of the one curve to the point of intersection P' of the normal at P with the other curve will be an infinitesimal that is uniformly of the first order; in other words,

$$m\alpha \leqq \beta \leqq M\alpha,$$

where m, M are positive constants, i. e., independent of a, t, α.

The proof of these statements follows the same lines as the proof in a similar case considered in the writer's paper on *Sufficient Conditions in the Calculus of Variations*.* Kneser † has given a proof by means of power series on the assumption that all the functions considered are analytic. This assumption is unfortunate, inasmuch as it introduces restrictions without being accompanied by any simplification.

By means of the family of extremals $a =$ const., a system of curvilinear coordinates of the field is introduced (Kneser, loc. cit.), $v = a$ being taken as one of the coördinates and

$$u = \int_{t'}^{t} F\big(\xi(t, a), \eta(t, a), \xi_t(t, a), \eta_t(t, a)\big)\, dt$$

being chosen as the other. The lower limit of integration, t', is so taken that the point $x = \xi(t', a)$, $y = \eta(t', a)$ lies on the curve through A which cuts all the extremals $a =$ const. transversely. In terms of the new coördinates let

$$F(x, y, dx, dy) = G(u, v, du, dv),$$

so that

$$I = \int_{t_0}^{t_1} F(x, y, x', y')\, dt = \int_{t_0}^{t_1} G(u, v, u', v')\, dt,$$

where I is taken along the curve $\bar{C}$. The quantity, H, the study of which is the main object of this paragraph, is then given by the formula

$$H = I - J = \int_{t_0}^{t_1} [G(u, v, u', v') - u']\, dt.$$

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*Annals of Mathematics, 2nd series, vol. 2, No. 3 (April, 1901), p. 105.

† loc. cit., chap. 3.

This quantity will be positive, and hence the integral I will be made a minimum by C, when the WEIERSTRASSIAN invariant F_1 (KNESER, p. 52) is always positive in R, and this we assume to be the case. This completes the sufficient condition for a minimum.

2. *The determination of a lower limit (Abschätzung) for the value of H for the simplest class of curves $\bar{C}$.* — Consider first the case that $u' > 0$ throughout the whole interval $t_0 \leqq t \leqq t_1$. Then u may be chosen as the parameter t and, setting

$$G(u, v, 1, s) = g(u, v, s),$$

we have

$$[G(u, v, u', v') - u'] \, dt = [g(u, v, s) - 1] \, du.$$

KNESER shows that

$$g(u, v, 0) = 1, \quad g_s(u, v, 0) = 0,$$

so that

$$H = \tfrac{1}{2} \int_{u_0}^{u_1} g_{ss}(u, v, \vartheta s) s^2 \, du,$$

the coördinates of A and B being respectively (u_0, v_0) and (u_1, v_0). The function $g_{ss}(u, v, s)$ is a continuous function of its three arguments, when s has any value whatever and (u, v) is any point of the region

$$T: \qquad \bar{u}_0 \leqq u \leqq \bar{u}_1, \qquad |v - v_0| \leqq \bar{\kappa},$$

where the constants $\bar{u}_0$, $\bar{u}_1$, $\bar{\kappa}$ are so chosen that to each point of T corresponds a point of R, while the arc AB of C lies wholly within T. This function is positive for all such values of the arguments, since the invariant F_1 is always positive. It may, however, have 0 for its lower limit when $|s| = \infty$. Let the minimum value of this function, when (u, v) is an arbitrary point of T and $|s| \leqq \mu$ (μ being a constant to be determined later) be denoted by M.* If we impose on the curve $\bar{C}$ the further restriction that s shall, at no one of its points, exceed μ in absolute value, then we have

$$H \geqq M \int_{u_0}^{u_1} s^2 \, du.$$

Now mark off an arbitrary neighborhood of the arc AB of C lying within the field T and consisting of the region

$$\mathfrak{T}: \quad u_0 - L \leqq u \leqq u_1 + L, \quad |v - v_0| \leqq L; \quad L < 3\sqrt{2}(u_1 - u_0).$$

* It will be necessary to consider later the minimum value of the function $\frac{1}{2}\bar{g}_{ss}(u, v, s)$ where $G(u, v, -1, s) = \bar{g}(u, v, s)$, for the same values of the arguments, and so we will choose M once for all as the smaller of these two minimum values.

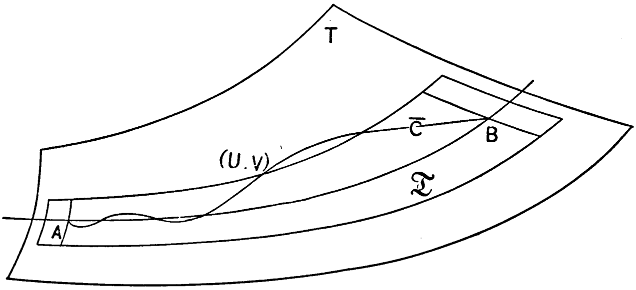

FIG. 1.

If the curve $\overline{C}$, restricted as above, does not lie wholly within $\mathfrak{T}$ and (U, V) is a point of $\overline{C}$ lying on the boundary of $\mathfrak{T}$, so that $|V| = L$, then, by the theorem of § 2

$$\int_{u_0}^{U} s^2\,du \geqq \frac{L^3}{54(u_1-u_0)^2},$$

and since

$$\int_{u_0}^{u_1} s^2\,du > \int_{u_0}^{U} s^2\,du,$$

it follows that

$$(A_1) \qquad H > \frac{ML^3}{54(u_1-u_0)^2}.$$

3. *Continuation; the general case.* — We have thus obtained the desired inequality, or approximate evaluation, (Abschätzung) for H for the simplest class of curves $\overline{C}$. The most general curve

$$\overline{C}: \qquad x = \phi(t), \quad y = \psi(t); \qquad t_0 \leqq t \leqq t_1,$$

connecting A and B, having at each point (extremities included) a continuous tangent, and lying in T, but not wholly in $\mathfrak{T}$, can be broken up into a finite number of pieces of the following three classes:

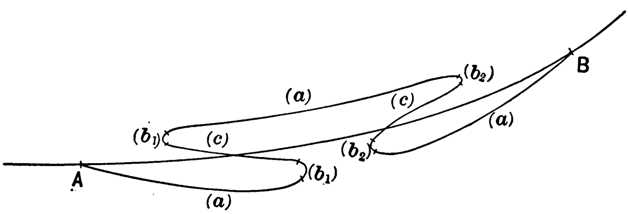

FIG. 2.

$$(a) \qquad u' > 0, \quad |s| \leqq \mu;$$

$$(b) \quad \begin{cases} (b_1) & v' > 0 \text{ and, when } u' \neq 0, \quad |s| > \tfrac{1}{2}\mu; \\ (b_2) & v' < 0 \text{ and, when } u' \neq 0, \quad |s| > \tfrac{1}{2}\mu; \end{cases}$$

$$(c) \qquad u' < 0, \quad |s| \leqq \mu.$$

Instead of $\frac{1}{2}\mu$, any other positive quantity less than μ would answer. — The integral H can then be decomposed in a corresponding manner into the three sums:

$$(1) \qquad H = \Big(\sum_{(a)} + \sum_{(b)} + \sum_{(c)}\Big)\int [G(u, v, u', v') - u'] dt.$$

We proceed to obtain approximate evaluations for these sums, and hence for H.

Consider a term of the first sum

$$H_{a_i} = \int_{t_i}^{t_{i+1}} [G(u, v, u', v') - u'] \, dt.$$

Here, $u = \phi(t)$, $u' > 0$ and hence the variable of integration, t, can be replaced by u. We will introduce, however, not u but w, where

$$w = u + a_i$$

and a_i is a constant to be determined later. We have then

$$(2) \qquad H_{a_i} = \int_{w_i}^{w_{i+1}} \tfrac{1}{2} g_{ss}(u, v, s) s^2 dw \; \geqq \; M \int_{w_i}^{w_{i+1}} s^2 dw.$$

Next consider a term of the third sum

$$H_{c_k} = \int_{t_k}^{t_{k+1}} [G(u, v, u', v') - u'] dt$$

and replace t by w, where

$$w = -u + a_k$$

and a_k is a constant to be determined later. Let

$$G(u, v, -1, s) = \bar{g}(u, v, s).$$

Then, as before,

$$\bar{g}(u, v, 0) = 1, \quad \bar{g}_s(u, v, 0) = 0.$$

Hence

$$(3) \quad H_{c_k} = \int_{w_k}^{w_{k+1}} [\tfrac{1}{2}\bar{g}_{ss}(u, v, \vartheta s) s^2 + 2] dw \; \geqq \; M \int_{w_k}^{w_{k+1}} s^2 dw + 2(w_{k+1} - w_k).$$

Finally, consider a term of the second sum for which $v' > 0$. Replace t by v, and denote the smaller of the minimum values of the two functions

$$G(u, v, r, 1), \quad G(u, v, r, -1),$$

when (u, v) is any point of T and $r = du/dv (= s^{-1}$ when $u' \neq 0$ and $v' \neq 0)$, $|r| \leqq 1$, by N. Then, since the maximum value of $|r|$ is $2/\mu$, if μ is so chosen (1) that $2/\mu \leqq 1$ and (2) that $2/\mu < N$, we obtain for H_{b_j} the following relation.

$$
\begin{aligned}
H_{b_j} &= \int_{t_j}^{t_{j+1}} [G(u, v, u', v') - u'] dt = \int_{v_j}^{v_{j+1}} [G(u, v, r, 1) - r] dv \\
&> \left(N - \frac{2}{\mu}\right)(v_{j+1} - v_j).
\end{aligned}
\tag{4}
$$

The same formula also holds for a term for which $v' < 0$, since in that case the left hand member is positive, the right hand member negative. Furthermore, the same reasoning shows that, when $v' < 0$, the quantity $v_{j+1} - v_j$ then becoming negative,

$$
H_{b_j} > -\left(N - \frac{2}{\mu}\right)(v_{j+1} - v_j). \tag{4_1}
$$

We are now ready to obtain the final approximate evaluation of the quantity H. Assume that the curve $\bar{C}$ leaves the strip $\mathfrak{T}$, and let (U, V) be a point of $\bar{C}$ on the boundary of $\mathfrak{T}$. Two cases arise:

$$
u_0 \leqq U \leqq u_1, \quad |v| = L; \tag{A}
$$

$$
U > u_1 \quad \text{or} \quad U < u_0. \tag{B}
$$

Case (A). We begin by disposing of a simple special case, namely, that in which

$$
\sum_k 2(w_{k+1} - w_k),
$$

the summation applying to those arcs of $\bar{C}$ which belong to class (c), exceeds an arbitrarily preassigned quantity, which we take for simplicity as L. Observe that the equation of the boundary of $\mathfrak{T}$ for which u is a maximum is $u = u_1 + L$. Here, we have at once that

$$
H > L. \tag{A_2}
$$

This case being disposed of, we assume that

$$
\sum_k 2(w_{k+1} - w_k) \leqq L
$$

and that the point (U, V) is the boundary of an interval of some one of the three classes. This condition can be fulfilled by cutting an interval in two, if necessary; both pieces belonging, then, to the same class. We now determine the constants a_i, a_k in such a manner that those intervals

$$
w \leqq w \leqq w_{i+1}, \quad w_k \leqq w \leqq w_{k+1}
$$

which correspond to the arcs of $\bar{C}$ lying between A and $(U,\ V)$ will follow each other without overlapping, their ends just touching; the initial value of w in the first interval shall be chosen as u_0. Denote the final value of w in the last interval by $\mathfrak{w}$. We have, then, from (2) and (3), suppressing whatever terms of the form $2(w_{k+1}-w_k)$ may arise on the right hand side of (3), — this only strengthens the inequality: —

$$\text{(5)}\qquad \Big(\sum_{(a)}+\sum_{(c)}\Big)\int[G(u,\,v,\,u',\,v')-u']\,dt > M\int_{w_0}^{\mathfrak{w}} s^2\,dw.$$

The function s that here enters is a single valued function of w continuous at all points of the interval $w_0 \leqq w \leqq \mathfrak{w}$ with the exception of a finite number of points; and s approaches a limit when w approaches any one of these exceptional points from one side only. Let

$$f(w)=\int_{u_0}^{w} s\,dw.$$

Then $v=f(w)$ is a continuous curve made up (1) of the pieces of class (a) carried in the field of the curvilinear coördinates $(u,\,v)$ to a new part of the field;* (2) of the pieces of class (c), first reflected in a line $u=$ constant (or $v=$ constant) and then carried like the former pieces to a new part of the field. Let $f(\mathfrak{w})$ be denoted by W; $f(u_0)=0$. We have the relation:

$$\text{(6)}\qquad W+\sum(v_{j+1}-v_j)=V.$$

Collecting and combining results we infer from (4), (4_1), (5) that†

$$\text{(7)}\qquad H > M\int_{u_0}^{\mathfrak{w}} s^2\,dw+\Big(N-\frac{2}{\mu}\Big)|x|,$$

where

$$x=\sum(v_{j+1}-v_j).$$

Relation (7) we now proceed to transform by means of the theorem of § 2, which enables us to write the inequality

$$\int_{u_0}^{\mathfrak{w}} s^2\,dw \geqq \frac{|W|^3}{54(u_1+L-u_0)^2},$$

provided that $|W|<3\sqrt{2}(u_1+L-u_0)$; a condition that will surely be satisfied when $|W|\leqq L$. We will assume this to be the case, since otherwise we could choose $0<\mathfrak{w}_1<\mathfrak{w}$ so that $|f(\mathfrak{w}_1)|=L$, and then we should have

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* We may think of the field T as mapped on a rectangle by means of the equations $u=x$, $v=y$. The curves in question thus go over into curves lying in this rectangle and the transformation is then an ordinary euclidean rigid motion. The corresponding actual transformations in the field T now become apparent to the intuition. They are given by the formulas:

$$\text{(1)}\quad u=w-a_i,\quad v=v;\qquad\qquad \text{(2)}\quad u=-w-a_k,\quad v=v.$$

† In obtaining this relation *either* (4) *or* (4_1) is to be used for all values of j.

$$\int^{w} s^2\,dw > \int_{u_0}^{w_1} s^2\,dw \ \geqq\ \frac{L^3}{54\,(u_1+L-u_0)^2},$$

whence relation (A_3) would follow at once.

We have then, since $W = V - x$,

$$H > \frac{M\,|V-x|^3}{54\,(u_1+L-u_0)^2} + \left(N - \frac{2}{\mu}\right)|x|, \tag{8}$$

or, writing

$$\frac{M}{54\,(u_1+L-u_0)^2} = A, \quad \left(N-\frac{2}{\mu}\right) = B,$$

$$H > A\,|V-x|^3 + B\,|x|. \tag{8'}$$

A and B are both positive quantities.

When V and x have opposite signs, the inequality (8′) will only be strengthened by replacing x by 0. Since $|V| = L$,

$$H > AL^3. \tag{A_3}$$

When V and x have the same sign, we can replace (8′) by the inequality

$$H > A\,(L-\xi)^3 + B\xi, \tag{9_1}$$

when $|x| = \xi < L$; and by the inequality

$$H > A\,(\xi - L)^3 + B\xi, \tag{9_2}$$

when $\xi \geqq L$. To get the desired approximate evaluation, replace ξ in each of these formulas by a value that will make the right hand side of the inequality a minimum. The curve

$$y = A\,(L-\xi)^3 + B\xi$$

has its minimum value in the interval $0 \leqq \xi < L$ at the point

$$\xi = L - \sqrt{B/3A},$$

provided that this is a point of the interval, i. e., provided $L \geqq \sqrt{B/3A}$; otherwise, at the point $\xi = 0$. In the first case the minimum value is

$$B[L - \tfrac{2}{3}\sqrt{B/3A}];$$

in the second case it is AL^3.

When $\xi \geqq L$, the minimum is given by putting $\xi = L$, and is BL.

Collecting results, we have

$$(A_4)\qquad \begin{cases} H > B(L - \tfrac{2}{3}\sqrt{B/3A}), & \text{when } 0 \leqq \xi < L, \quad \sqrt{B/3A} \leqq L; \\ H > AL^3, & \text{when } 0 \leqq \xi < L, \quad \sqrt{B/3A} > L; \\ H > BL, & \text{when} \quad \xi \geqq L. \end{cases}$$

It remains to consider *Case* (B). It is sufficient to assume that $U > u_1$ for the case that $U < u_0$ can be dealt with in a similar manner, the points A and B merely interchanging their rôles. The curve $\bar{C}$ in this case crosses the curve $u = u_1$ (let $t = \tau_1$, be the value of t corresponding to such a point) and meets one of the three lines

$$v = L, \qquad u_1 < u \leqq u_1 + L;$$

$$v = -L, \quad u_1 < u \leqq u_1 + L;$$

$$v_0 - L \leqq v \leqq v_0 + L, \quad u = u_1 + L,$$

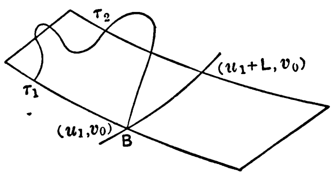

FIG. 3.

in the point (U, V) (let $t = \tau_2$ be the value of t corresponding to such a point). Let the minimum distance from the point B to the above three lines be denoted by D. Then there exists a positive constant, k, such that

$$D > kL,$$

no matter what value L may have subject to the conditions of the problem we are considering.*

We may write H in the form:

$$H = \int_{t_0}^{\tau_1} [G(u, v, u', v') - u']\, dt + \int_{\tau_1}^{\tau_2} G(u, v, u', v')\, dt$$
$$+ \int_{\tau_2}^{t_1} G(u, v, u', v')\, dt.$$

The first of these integrals is positive or zero, since it is the difference between the integral I taken along a curve $\bar{C}$ drawn in the field from A to the curve $u = u_1$, which cuts the extremals of the field transversely, and the integral I taken from A to B along C. The second integral is also positive since G is positive. Discard these first two integrals and then introduce in the third as parameter the arc s of $\bar{C}$. We have then

$$(A_5) \qquad H > \int_{\sigma_2}^{s_1} G(u, v, \cos\phi, \sin\phi)\, ds > k\Gamma L,$$

* For the region about A a corresponding pair of quantities D', k' must be considered, and so we will choose D, k at once each as the smaller one of the two quantities to be considered.

where $\Gamma > 0$ denotes the minimum value of $G(u, v, \cos\phi, \sin\phi)$ when (u, v) is any point of T and ϕ has any value whatever.

This completes the approximate evaluation. The final result may be stated as follows:

If L is any positive quantity so chosen that no point of the region $\mathfrak{T}$ will be exterior to the region T and that $L < 3\sqrt{2}(u_1 - u_0)$, and if E denotes the smallest of the quantities entering on the right hand sides of the inequalities for H: (A_1), (A_2), (A_3), (A_4), (A_5); then the value of the integral I extended along any curve $\bar{C}$ whatsoever that does not lie wholly within $\mathfrak{T}$ exceeds the value of this integral extended along C by a quantity greater than E; or

$$H > E.$$

for such a curve $\bar{C}$.

The curves $\bar{C}$ considered in the foregoing investigation were assumed to have a continuous tangent throughout their whole course inclusive of their extremities. It is easy to see that the above results still hold for curves $\bar{C}$ made up of a finite number of pieces, each of which enjoys the above properties. In particular, $\bar{C}$ may be made up of a finite number of pieces of straight lines.

§ 2. The Fundamental Theorem.

4. Fundamental Theorem. *Let $f(x)$ be a single valued continuous function of x in the interval $a \leqq x \leqq b$, and let $f(x)$ have a continuous derivative $f'(x)$ at all points of this interval. Let*

$$|f(l) - f(a)| = L > 0, \qquad a < l \leqq b,$$

Then

$$\int_a^l f'(x)^2\, dx \geqq \frac{L^3}{54(b-a)^2},$$

provided that

$$L < 3\sqrt{2}(b - a).$$

*The theorem still holds under the more general hypothesis that $f(x)$ is continuous throughout the interval $a \leqq x \leqq b$ and that this interval can be broken up into a finite number of subintervals throughout each of which, inclusive of its extremities, $f(x)$ has a continuous derivative.**

We may restrict ourselves to the case that $f(l) - f(a) > 0$, since the case that $f(l) - f(a) < 0$ can be referred to the former case by setting $f(x) = -\phi(x)$. We may furthermore assume that $f'(x) \geqq 0$, since, if $f'(x) < 0$ in parts of the interval, we may define a function $F(x)$ by the conditions:

* Since this theorem merely serves the purpose of a lemma for the proof of the main theorems of this paper, no attempt is made to state the theorem in more general form than is needed for present purposes.

$$F(x) = f(x) \text{ for values of } x \text{ for which } f'(x) \geqq 0;$$
$$F(x) = 0 \qquad \text{for values of } x \text{ for which } f'(x) < 0,$$

and this function will have the property that

$$F(l) - F(a) = \bar{L} > L.$$

If l' be so chosen that $F(l') = L$ and $0 < l' < l$, then

$$\int_a^l f'(x)^2\,dx > \int_a^l F'(x)^2\,dx > \int_a^{l'} F'(x)^2\,dx \geqq \frac{L^3}{54(b-a)^2}.$$

Thus the theorem will hold in all cases if it holds when $f(l) > f(a)$ and $f'(x) \geqq 0$, $a \leqq x \leqq b$.

Finally, we shall for simplicity confine ourselves to the proof of the theorem under the hypothesis that $f'(x)$ is continuous throughout the whole interval, for the more general case presents no principal difficulty.

5. LEMMA. *If $f'(x)$ is a single-valued, continuous function of x throughout the interval $a \leqq x \leqq \beta$, and if*

$$0 \leqq f'(x) \leqq 1$$

at all points of the interval; if, furthermore,

$$f(\beta) - f(a) = \lambda > 0 \qquad \textit{and if} \qquad \lambda < \beta - a,$$

then

$$\int_a^\beta f'(x)^2\,dx > \tfrac{1}{8}\frac{\lambda^3}{(\beta - a)^2}.$$

Let γ, τ be defined by the equations:

$$\beta - a = \tau,$$
$$\int_a^\beta f'(x)\,dx = f(\beta) - f(a) = \gamma(\beta - a),$$

so that $\gamma\tau = \lambda$, $0 < \gamma < 1$.

Let the points of the interval (a, β) be divided into two sets:

(s_1), the points in which $f'(x) \geqq \frac{1}{2}\gamma$, $\qquad \kappa'$

(s_2), the points in which $f'(x) < \frac{1}{2}\gamma$; $\qquad \tau - \kappa'$

and let κ', $\tau - \kappa'$ denote respectively the internal content of (s_1) and the external content of (s_2).* The quantity κ' varies with different functions $f'(x)$,

* Cf. JORDAN, *Cours d'analyse*, vol. 1, 2d ed., 1893, p. 28, § 36.

and we wish to determine its lower limit. Consider the function $\phi(x)$ defined as follows:

$$\phi(x) = 1, \quad a \leqq x \leqq a + \kappa; \qquad \phi(x) = \tfrac{1}{2}\gamma, \quad a + \kappa < x \leqq \beta,$$

where κ is so chosen that

$$(1) \quad \begin{cases} \displaystyle\int_a^\beta \phi(x)dx = \lambda, \quad \text{i. e.,} \quad \kappa + \tfrac{1}{2}\gamma(\tau - \kappa) = \lambda, \\ \displaystyle\kappa = \frac{\lambda}{2 - \gamma} > \tfrac{1}{2}\lambda. \end{cases}$$

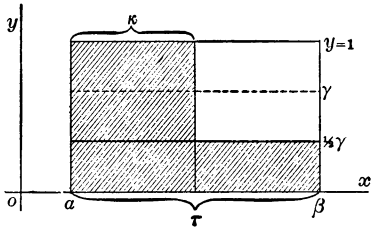

FIG. 4.

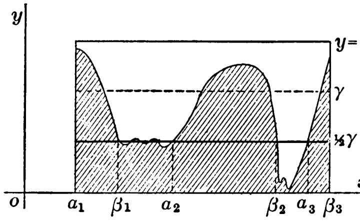

FIG. 5.

This function ϕ is, because of its discontinuity at $x = a + \kappa$, an impossible function $f'(x)$; but there exist functions $f'(x)$ that differ from $\phi(x)$ but slightly, and it seems likely that the number κ thus defined is the lower limit of all the κ''s. This we will now show to be the case.

A positive number ϵ having been chosen at pleasure, it is possible to divide the interval (a, β) into a finite number of subintervals (a_1, β_1), (a_2, β_2), $\cdots$, (a_n, β_n), consisting exclusively of points of (s_1) and such that

$$\kappa' - \epsilon < \sum_{i=1}^{n} (\beta_i - a_i) < \kappa'.$$

The sum of the lengths of the remaining intervals then satisfies the relation:

$$\tau - \kappa' < \sum_{i=0}^{n} (a_{i+1} - \beta_i) < \tau - \kappa' + \epsilon,$$

where $\beta_0 = a$ and $a_{n+1} = \beta$. In particular, it may happen, as in the case indicated in the figure, that $a_1 = a$, so that the first interval, $(a_1 - \beta_0)$, may be suppressed. A similar remark applies to the last interval when $\beta_n = \beta$. Furthermore, the above intervals may be so chosen that, δ being a second arbitrary positive number,

$$f'(x) < \tfrac{1}{2}\gamma + \delta,$$

at all points of each of the intervals (β_i, a_{i+1}). This follows from the uniform continuity of the function $f'(x)$.

From the formula

$$\int_\alpha^\beta f'(x)\,dx = \sum_{i=1}^{n}\int_{\alpha_i}^{\beta_i} f'(x)\,dx + \sum_{i=0}^{n}\int_{\beta_i}^{\alpha_{i+1}} f'(x)\,dx,$$

we now infer, since $f'(x) \leqq 1$ in the interval (x_i, β_i) and $f'(x) < \frac{1}{2}\gamma + \delta$ in the interval (α_{i+1}, β_i), that

$$\lambda < \kappa' + (\tfrac{1}{2}\gamma + \delta)(\tau - \kappa' + \epsilon). \tag{2}$$

From (1) and (2) it follows that

$$\kappa < \kappa' + (\epsilon, \delta),$$

where (ϵ, δ) is a quantity that can be made arbitrarily small by a suitable choice of ϵ and δ. Hence

$$\kappa' \geqq \kappa,$$

and this is the relation that we set out to establish. It is unnecessary to consider whether the lower sign can ever hold.

The proof of the lemma is now given by the relation:

$$\int_\alpha^\beta f'(x)^2\,dx > \sum_{i=1}^{n}\int_{\alpha_i}^{\beta_i} f'(x)^2\,dx > \left(\frac{\gamma}{2}\right)^2(\kappa' - \epsilon),$$

from which it follows that

$$\int_\alpha^\beta f'(x)^2\,dx \geqq \tfrac{1}{4}\gamma^2\kappa > \tfrac{1}{8}\frac{\lambda^3}{\tau^2}, \qquad \text{q. e. d.}$$

6. *Proof of the theorem.* — To prove the theorem let us first assume that the curve $y = f'(x)$ cuts the line $y = 1$ in a finite number of points, and that, in the intervals $(a_1, b_1), (a_2, b_2), \cdots, (a_n, b_n)$,

$$f'(x) \leqq 1, \quad a_i \leqq x \leqq b_i,$$

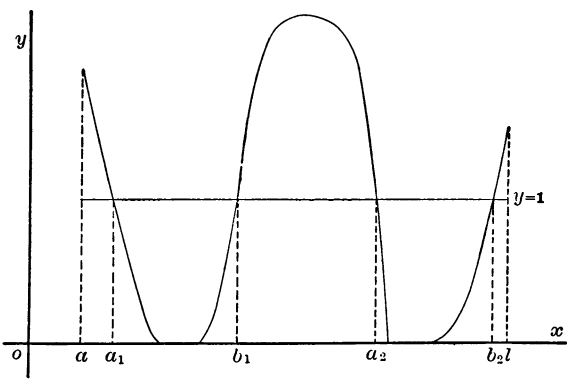

FIG. 6.

while in the remaining intervals, (b_0, a_1), (b_1, a_2), $\cdots$, (b_n, a_{n+1}), where $b_0 = a$, $a_{n+1} = l$,

$$f'(x) > 1, \quad b_i < x < a_{i+1}.$$

(In particular, the curve need not cut this line at all. The result obtained below is still true, and the modification in the proof is but formal.) Observe that we may write down the relation:

$$\sum_{i=0}^{n} [f(a_{i+1}) - f(b_i)] + \sum_{i=1}^{n} [f(b_i) - f(a_i)] = L. \tag{3}$$

The relation on which the proof of the theorem turns is the following:

$$\left\{ \begin{aligned} \int_a^l f'(x)^2\,dx &= \sum_{i=0}^{n} \int_{b_i}^{a_{i+1}} f'(x)^2\,dx + \sum_{i=1}^{n} \int_{a_i}^{b_i} f'(x)^2\,dx \\ &\geqq \sum_{i=0}^{n} [f(a_{i+1}) - f(b_i)] + \tfrac{1}{8} \sum_{i=1}^{n} \frac{[f(b_i) - f(a_i)]^3}{(b_i - a_i)^2}, \end{aligned} \right. \tag{4}$$

the approximate evaluation of the first sum being given by the relation $f'(x)^2 > f'(x)$, and that of the second sum by the lemma of paragraph 5. The terms that compose the last sum we divide into two classes, namely:

(A) The terms for which

$$\frac{f(b_i) - f(a_i)}{b_i - a_i} \geqq \frac{1}{\rho} L,$$

where ρ is a positive quantity to be defined later. Let the values of the index for these terms be denoted by $i_1, i_2, \cdots, i_p$;

(B) The remaining terms. For these

$$f(b_j) - f(a_j) < \frac{1}{\rho} L (b_j - a_j);$$

let the values of the index j for these terms be denoted by $j_1, j_2, \cdots, j_m$. Notice that

$$\sum_{k=1}^{m} [f(b_{j_k}) - f(a_{j_k})] < \frac{1}{\rho} L \sum_{k=1}^{m} (b_{j_k} - a_{j_k}) \leqq \frac{1}{\rho} L (b - a). \tag{5}$$

We now write down relation (4), suppressing all terms of class (B) and replacing each summand in class (A) by a smaller quantity, as follows:

$$\frac{[f(b_{i_k}) - f(a_{i_k})]^3}{(b_{i_k} - a_{i_k})^2} = \left\{ \frac{f(b_{i_k}) - f(a_{i_k})}{b_{i_k} - a_{i_k}} \right\}^2 [f(b_{i_k}) - f(a_{i_k})] \geqq \frac{L^2}{\rho^2} [f(b_{i_k}) - f(a_{i_k})].$$

The result is the relation

$$\int_a^l f'(x)^2\,dx \geqq \sum_{i=0}^{n} [f(a_{i+1}) - f(b_i)] + \frac{L^2}{8\rho^2} \sum_{k=1}^{p} [f(b_{i_k}) - f(a_{i_k})].$$

Next, let ρ be taken large enough so that $L^2/8\rho^2 < 1$. Then

$$\int_a^l f'(x)^2 dx > \frac{L^2}{8\rho^2}\left\{ \sum_{i=0}^{n} [f(a_{i+1}) - f(b_i)] + \sum_{k=1}^{p} [f(b_{i_k}) - f(a_{i_k})] \right\}.$$

The quantity in the brace we now transform by adding and subtracting the terms of class (B). The addition of these terms gives three sums whose total value is precisely L. On the other hand, the inequality will only be strengthened if, instead of

$$\sum_{k=1}^{m} [f(b_{j_k}) - f(a_{j_k})]$$

we subtract the larger quantity given by (5), namely, $\rho^{-1}L(b-a)$. We thus obtain the relation:

$$\int_a^l f'(x)^2 dx > \frac{L^2}{8\rho^2}\left\{ L - \frac{b-a}{\rho} L \right\}.$$

The expression on the right hand side is seen to have its maximum value when $\rho = \frac{3}{2}(b-a)$, and hence it follows that

$$\int_a^l f'(x)^2 dx > \frac{L^3}{54(b-a)^2},$$

provided that the above is an admissible value for ρ; i. e., if it will make $L^2/8\rho^2 < 1$. This will be the case if

$$L < 3\sqrt{2}(b-a),$$

and this relation was made part of the hypothesis of the theorem for this very purpose. Since we are concerned only with small values of L, this restriction is not embarrassing.

We may note in passing that we could obtain a corresponding formula for large values of L by choosing ρ so that $1 - L^2/8\rho^2$ will be an arbitrarily small positive quantity, and then replacing the factor L/ρ in the brace by the larger value $2\sqrt{2}$. The resulting formula (which, indeed, is true for all values of L) is:

$$\int_a^l f'(x)^2 dx > \{L - 2\sqrt{2}(b-a)\}.$$

We have hitherto assumed that the curve $y = f(x)$ meets the line $y = 1$ only in a finite number of points. If this is not the case, then it is possible to divide the interval (a, l) into a finite number of subintervals and to separate these into two classes:

(*a*) Intervals within which the curve and the line do not meet. They shall meet at each extremity of such an interval, the points $x = a$ and $x = l$ excepted.

(*b*) Intervals within which the maximum value of $f'(x)$ is less than $1 + \epsilon$, where ϵ is an arbitrary positive quantity.

Let a function $f_1(x)$ be defined as follows:

$f'_1(x) = f'(x)$ in the intervals (*a*);

$f'_1(x) = f'(x)$ at those points of the intervals (*b*) at which $f'(x) \leqq 1$;

$f'_1(x) = 1$ at the remaining points.

Then, if

$$f_1(l) - f_1(a) = L_1, \quad L - \epsilon(l - a) < L_1 < L.$$

To the function $f'_1(x)$ all the reasoning is applicable that has been used above in the case of the function $f'(x)$, if we introduce the following modification: The curve $y = f_1(x)$ may be cut by the line $y = 1$ in an infinite number of points. The points of this set in parts of whose neighborhoods $f_1(x)$ is positive are, however, finite in number. These points play the rôle of the points of intersection in the former case; and if we denote them by $a_1, b_1, \cdots a_n, b_n$, the former proof will apply. Thus

$$\int_a^l f'_1(x)^2\,dx \geqq \frac{L_1^3}{54(b-a)^2}.$$

But

$$\int_a^l f'(x)^2\,dx > \int_a^l f'_1(x)^2\,dx,$$

and we now readily infer that

$$\int_a^l f'(x)^2\,dx \geqq \frac{L^3}{54(b-a)^2}.$$

This completes the proof of the theorem.

§ 3. The Proof of Weierstrass's Theorem.

7. Let the points A, B be joined by an arbitrary curve lying in the field T, but not lying wholly within $\mathfrak{T}$:

$$\bar{C}: \qquad x = \phi(t), \quad y = \psi(t); \qquad t_0 \leqq t \leqq t_1,$$

where ϕ, ψ denote, to begin with, merely single-valued, continuous functions of

t, and let a polygon P be inscribed in $\bar{C}$ with its vertices at the points $\tau_0 = t_0$, $\tau_1, \cdots, \tau_n = t_1$, where $\tau_i < \tau_{i+1}$. Its sides may intersect one another. Form the sum

$$S = \sum_{i=0}^{n-1} F(x_i, y_i, \Delta x_i, \Delta y_i).$$

Let the curve $\bar{C}$ be now restricted to belonging to the class (K) of curves for which S approaches one and the same limit when $n = \infty$, no matter how the vertices of the polygon are chosen, provided merely that its longest side converges towards 0. Denote this limit by $\mathfrak{J}$. Then WEIERSTRASS's theorem is established if we show that

$$\mathfrak{J} \geqq J + \frac{L^3}{54(u_1 + L - u_0)^2}, \tag{1}$$

and this we will now do.

Let ϵ and δ be two arbitrarily chosen positive quantities, and let the polygon P be so chosen that

$$|\mathfrak{J} - S| < \epsilon \tag{2}$$

and that the length of the longest side of P is less than δ. Form the (improper) integral I for the curve $\bar{C} = P$. This will be a proper integral provided we assign to the arguments x', y' definite (but arbitrary) values at the vertices of P. We have then:

$$\begin{aligned} I &= \int_{t_0}^{t_1} F(x, y, x', y')\,dt = \sum_{i=0}^{n-1} \int_{\tau_i}^{\tau_{i+1}} F(x, y, x', y')\,dt \\ &= \sum_{i=0}^{n-1} F\left(\bar{x}_i, \bar{y}_i, \frac{\Delta x_i}{\Delta t_i}, \frac{\Delta y_i}{\Delta t_i}\right)\Delta t_i = \sum_{i=0}^{n-1} F(\bar{x}_i, \bar{y}_i, \Delta x_i, \Delta y_i), \end{aligned} \tag{3}$$

where

$$\bar{x}_i = \phi(t_i + \vartheta_i \Delta t_i) = \phi(t_i) + \zeta_i, \qquad 0 < \vartheta_i < 1,$$

and

$$\bar{y}_i = \psi(t_i + \vartheta'_i \Delta t_i) = \psi(t_i) + \zeta'_i, \qquad 0 < \vartheta'_i < 1.$$

By proper choice of δ, it is possible to make

$$|\zeta_i| < h, \quad |\zeta'_i| < h \qquad (i = 0, 1, \cdots, n-1), \tag{4}$$

where h denotes an arbitrary positive constant.

The function $F(x, y, \cos\phi, \sin\phi)$ is uniformly continuous and hence, by proper choice of h, we can insure that

$$|F(x_i + \zeta_i, y_i + \zeta'_i, \cos a_i, \sin a_i) - F(x_i, y_i, \cos a_i, \sin a_i)| < \epsilon, \tag{5}$$

where $\Delta x_i / p_i = \cos a_i$, $\Delta y_i / p_i = \sin a_i$ and $p_i = \sqrt{\Delta x_i^2 + \Delta y_i^2}$. Multiplying (5) through by p_i and adding from $i=0$ to $i=n-1$, we get, remembering that $F(x_i, y_i, \cos a_i, \sin a_i)p_i = F(x_i, y_i, \Delta x_i, \Delta y_i)$ etc.,

$$|I - S| < \epsilon l_n. \tag{6}$$

Since $F(x, y, \sin\phi, \cos\phi)$ is always positive and is continuous, it has a positive lower limit when x, y, ϕ vary arbitrarily, and hence it is only for curves that have a length that S can converge toward a limit, as is at once evident if the arc S of P is used as the parameter t in (3). Denoting the length of $\bar{C}$ by l, we replace (6) by the relation:

$$|I - S| < \epsilon l. \tag{7}$$

From (2) and (7) it follows that

$$|\mathfrak{J} - I| < \epsilon(l+1).$$

But, from the theorem of § 1,

$$I \geqq J + \frac{L^3}{54(u_1 + L - u_0)}$$

and hence

$$\mathfrak{J} > J + \frac{L^3}{54(u_1 + L - u_0)^2} - \epsilon(l+1).$$

Since ϵ is arbitrary, the truth of the relation (1) is thus established.*

---

* Note of July 6. HILBERT has given a definition of the integrals

$$\int F(x, y, y')\,dx \qquad \int\!\!\int F\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right)dx\,dy,$$

which, like the Weierstrassian definition of the former of these (written in homogeneous form), is a generalization of the ordinary definition for the case that the function y or z does not have a continuous first derivative or continuous first partial derivatives; cf. E. R. HEDRICK, *Ueber den analytischen Character der Lösungen von Differentialgleichungen*, Dissertation, Göttingen, 1901, p. 69. For the integral considered in the present section, HILBERT'S definition is in substance the following. Let the continuous curve $\bar{C}$ be divided as before into n segments by the points $\tau_0 = t_0, \tau_1, \cdots, \tau_n = t_1$, where $\tau_i < \tau_{i+1}$. Through these n points pass an arbitrary curve, which HILBERT assumes to be analytic; it is enough to assume that it is made up of a finite number of pieces, each of which is a continuous curve having a continuous tangent throughout its whole extent, its extremities included. From the integral $I = \int_{t_0}^{t_1} F(x, y, x', y')\,dt$ for this curve. Then when all such curves through the $n+1$ points are considered, the various values of the integral will have a *lower limit*. In the case of the geodesic integral this lower limit is obviously the length of the inscribed polygon above considered. Next, consider the totality of values consisting of these lower limits; to each one of the infinity of modes of division of $\bar{C}$ into n parts corresponds one such value, and $n = 1, 2, \cdots$. The *upper limit* of these values HILBERT defines as the value of the integral in question.

Thus we have, in form at least, a new generalization of the integral I for curves $\bar{C}$ for which this upper limit is finite; and the totality of such curves form a class (K'). *This class coincides, however, with the class (K) and the value of the generalized integral for a curve of the* *new class coincides with the value of the integral generalized according to Weierstrass's definition for the same curve.* The proof of this theorem depends on the fact that the integral I can be made a minimum by a curve c connecting any two points of T, provided that the distance between these points does not exceed a suitably chosen positive constant h, and that these curves have continuous curvature which lies (numerically) uniformly below a suitably chosen positive constant G. Thus, when n is large and the maximum distance between two successive points τ_i, τ_{i+1} is small, the lower limit of the integral will be obtained by employing the curve made up of the n curves connecting the $n+1$ points, each curve being so chosen as to make I a minimum over against all other curves having the same extremities. These curves differ but slightly from right lines in all of their properties that are essential for the above proof.

HARVARD UNIVERSITY,
CAMBRIDGE, MASS., *April*, 1901.