QUATERNION SPACE*

BY

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INTRODUCTION.

The recent paper by STRINGHAM in these Transactions (vol. 2, p. 183) has suggested the preparation of a consecutive development of my methods of treating quaternion space as they finally took form in the last of the series of my papers to which STRINGHAM refers; particularly since those papers are published in different journals, and for the most part in briefest abstract.

While our results have many points of contact with each other and with the results of previous authors, as is shown by STRINGHAM'S very thorough compilation of literature on the subject, yet our methods and points of view are entirely different. Also some results, particularly as to the quaternion form of the most general rigid displacement, were obtained by STRINGHAM long before I turned my attention to the subject.

STRINGHAM deals analytically with the equations of loci, and develops the geometry by the interpretation of those equations. I use a more synthetic method, which interprets the quaternion symbols themselves instead of the equations between them. This divergence between the analytic and the synthetic methods of employing the quaternion analysis constitutes the general difference between the methods of CAYLEY and of TAIT.

CLIFFORD has stated this synthetic view in his *Further Note on Biquaternions* (*Mathematical Papers*). CLIFFORD makes, however, an artificial limitation in that paper when he says that "we may regard the *last* symbol of a product as either a concrete number or a symbol of operation; but all others must be regarded as symbols of operation." I held this view in my first paper (Bulletin, Nov., 1897) and was forced by it to the development of two systems of quaternions (the direct and the contra systems). Immediately after the publication of that paper, I discovered that both systems became one and the same system, by the abandonment of the restriction that only the *last* symbol could be concrete.

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* Presented to the Society (Chicago) December 27, 1900, under the title: *Quaternions as numbers of fourfold space*. Received for publication May 4, 1901.

STRINGHAM points out that my theory of contra and direct parallel great circles in the hypersphere of fourfold space corresponds exactly to CLIFFORD'S theory of right and left parallels in elliptic space, in which "great circles" correspond to "straight lines," and "great spheres" to "planes." I may say, further, that CLIFFORD'S right and left vector translations correspond to my contra and direct turns; his motor twist, to my contra-direct turn. CLIFFORD'S *right* and *left* refer to right-hand and left-hand screw movements; my *contra* and *direct* refer to the position of the operating symbol on *the right* or on *the left* of the concrete symbol. These positions of the operating factor (of which that on the right is not recognized by CLIFFORD) correspond to operative powers that are equivalent to the screw movements of CLIFFORD. I shall, therefore, hereafter use the terms *right* and *left* instead of *contra* and *direct*, to denote these positions of the multiplier.

COMPLEX NUMBER SPACE.

1. This preliminary discussion of n-fold space is made for the sake of completeness of development and point of view. Most of the results, and probably all, have been obtained by the usual analytical treatment of equations of loci. The method of dealing synthetically with complex numbers as the concrete elements corresponding to geometric elements, without reference to equations of loci, may possibly be new. There is no assumption that the geometric terms *point*, *direction*, *line*, *locus*, *etc.*, refer to anything else than complex numbers. Nevertheless, the results given will be sufficient to show that an exact correspondence may be developed between any threefold space and euclidean space. Proofs are omitted, because they depend only upon the simplest principles of the linear composition of independent complex units.

2. The space considered is the totality of points

$$i_1x_1 + i_2x_2 + \cdots + i_nx_n,$$

where the x's are variable *real* numbers, and the i's are given *independent* complex units, so that two points

$$i_1x_1' + \cdots + i_nx_n', \quad i_1x_1'' + \cdots + i_nx_n''$$

are the same only if

$$x_1' = x_1'', \cdots, x_n' = x_n''.$$

3. A contained m-fold ($m < n$) is a locus

$$a + a_1x_1 + a_2x_2 + \cdots + a_mx_m,$$

where $a_1, a_2, \cdots, a_m$ are independent complex numbers and a is any complex number. The *directions* of this locus are the differences (displacements) between its points, viz.,

$$a_1x_1 + a_2x_2 + \cdots + a_mx_m.$$

In particular the direction from a to b is the difference $b - a$. The context must show whether a given complex number is taken as a point or as a direction (or as a line as in art. 9).

Two directions, a_1, $a_1 x_1$ are *like* directions in the same or in opposite *senses* according as x_1 is positive or negative.

4. A set of points and directions is an *independent* set when the directions from one point to each of the others form with the given directions an independent set of complex numbers. This definition applies to a set of points alone and to a set of directions alone.

5. The *elements* of an m-fold are its points and directions. An m-fold is determined by any set of $m + 1$ independent elements, of which one at least is a point. Directions alone cannot determine a space.

6. The *minimum space* of a given k-fold and l-fold is the m-fold of smallest number m which contains the given spaces. The *convergence* of the given spaces is the number c of their independent *common* elements and is given by the formula

$$c = k + l + 1 - m.$$

The *divergence* of the given spaces is the number d of independent elements besides the common elements which are necessary to determine the lesser space. If $k \lesseqgtr l$, we have

$$d = k + 1 - c = m - l.$$

7. Two spaces of convergence c intersect, if at all, in the $(c - 1)$-fold determined by the c independent common elements. When the common elements are all directions, the two spaces do not intersect. For example, two planes (two-folds) of minimum four-fold space have $c = 1$; i. e., they intersect in one point with no common direction, or they have one direction in common and do not intersect.

8. *Parallels.* Two spaces are *parallel* when one space contains all the directions of the other. Hence

If two parallel spaces intersect, then one space contains the other.

Two distinct parallel spaces are spaces of unit divergence, which do not intersect.

For example, parallel straight lines (one-folds) are straight lines which are coplanar and non-intersecting.

Through a given point one and only one k-fold can be drawn parallel to a given k-fold; and two k-folds parallel to the same k-fold are parallel to each other.

9. *Lines.* The locus

$$a + xp \qquad (0 \leqq x \leqq 1)$$

is a segment of the straight line $a + xp$ between a and $a + p$. Considered as generated from a to $a + p$ by increasing x, its direction is p; its *length* is

$$Tp = \sqrt{(x_1^2 + x_2^2 + \cdots + x_n^2)},$$

where

$$p = i_1 x_1 + i_2 x_2 + \cdots + i_n x_n.$$

This segment considered as to length and direction will be called *the line p;* considered also as to initial point, it is *the line p drawn from a.*

10. Lines are added according to the displacement law $AB + BC = AC$, viz.,

$$(b - a) + (c - b) = (c - a).$$

Also, since $T(2p) = 2Tp$, etc., we have ordinary principles of measurement upon a straight line; while the inequality $T(p + q) \leqq Tp + Tq$ determines principles of shortest geometric distance.

11. *Angles.* The angle between the directions (or lines) p and q will be, concretely, the locus of directions

$$p \cos \phi + q \sin \phi \qquad (0 \leqq \phi \leqq \pi/2).$$

The magnitude of this angle, in notation $\angle(p, q)$, will be defined so that we have euclidean relations in the parallelogram whose sides are p, q and diagonal $p + q$, viz.,

$$T(p + q)^2 = Tp^2 + Tq^2 + 2\,Tp\,Tq \cos \angle(p, q).$$

If $p = i_1 x_1 + i_2 x_2 + \cdots$, $q = i_1 y_1 + i_2 y_2 + \cdots$, this formula gives

$$\cos \angle(p, q) = \frac{(x_1 y_1 + x_2 y_2 + \cdots)}{Tp \cdot Tq}, \quad = \frac{S \cdot pKq}{Tp \cdot Tq},$$

by an extension of quaternion notation.

12. HAMILTON calls the angle θ (between 0 and π) whose cosine is Sq / Tq, *the angle of q;* and calls the unit vector $\alpha = UVq$, *the axis of q.* In notation

$$Uq = \cos \theta + \alpha \sin \theta = e^{\alpha\theta}.$$

The above definition of angle becomes then in quaternions

$$\angle(p, q) = \angle pKq = \angle pq^{-1} = \cdots;$$

so that in particular, *two lines are perpendicular when their quotient is a vector.*

13. The function

$$S \cdot pKq = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n,$$

whose vanishing is the condition of perpendicularity of the lines or directions p, q, is linear and symmetric in these complex variables, and in consequence we easily find:

A direction perpendicular to each of several directions is perpendicular to all of their linear compositions.

A set of mutually perpendicular directions is an independent set; and any m-fold contains any number of sets of m determining directions which are thus mutually perpendicular. The locus of directions perpendicular to all the directions of a given r-fold will be the directions of a certain $(n-r)$-fold.

14. *Orthogonals.* An r-fold and an $(n-r)$-fold such that every direction of one is perpendicular to every direction of the other will be called *orthogonal spaces.* The convergence of such orthogonal spaces is

$$c = r + (n - r) + 1 - n = 1.$$

Hence,

Orthogonal spaces intersect in one and only one point; and through a given point one and only one space can be drawn orthogonal to a given space.

Two spaces orthogonal to the same space are parallel to each other.

15. The orthogonal projection of a point upon a given space is therefore a definite point; and the line joining the point with its projection is the *unique* perpendicular from the point to the given space, and the *shortest line* from the given point to the given space.

16. *The orthogonal projection of a line upon a given space is another line generated by the orthogonal projection of the generating point of the given line; and such projection is one of the two components of the line in the given and an orthogonal space.*

17. *Transformations.* The transposition of the points of one figure into the points of another is a *transformation.* The transformation is *continuous* when the transposition takes place by a continuous and simultaneous change of value of all numbers of the figure from their initial to their final values. A continuous transformation is *rigid*, when it leaves invariant the distances between all pairs of points. In consequence, straight lines remain straight lines, planes remain planes, etc., in every rigid transformation, because shortest distances remain such; also the magnitudes of angles are unaltered, and in particular all perpendicular and orthogonal relations, and relative senses* of arrangement of points, lines, etc., are unchanged, because triangles, tetrahedrons, etc., remain rigid.

18. *Translations.* The addition of a given number to all points of a figure does not change distances or directions between points; and this will be called a rigid *translation* of the *magnitude*, *direction*, and *sense* of the given number.

19. *Multiplications.* The multiplication of all points of a given figure on the left or right by a given number p is a transformation which depends upon the multiplication table. The point 0 is an invariant point or *center.* The left multi-

* This part is rigorously an extension of a threefold conception rather than a theorem.

plication p, followed by the translation $a - pa$ which together leave the point a invariant, will be called *the left multiplication p about the center a;* in this transformation the point r becomes the point $a + p(r - a)$.

20. *Dilations.* A right or left multiplication p may be separated into its tensor and versor parts Tp, Up, which are interchangeable. The tensor multiplication, Tp, increases all lengths in the ratio $Tp : 1$, and leaves directions and their senses unaltered. It is *a dilation about the center, of magnitude Tp.*

Quaternion Space.

21. *Right and Left Turns.* A right or left versor multiplication may be called a *right or left turn about the center.* It is in the quaternion analysis, to which we now confine ourselves, *a rigid displacement,* on account of the tensor property $T \cdot pq = Tp \cdot Tq$; for $T(pr - ps) = T \cdot p(r - s) = T(r - s)$, since $Tp = 1$. Also, the transformation is made continuous by continuous variation of p from 1 to its final value.

In what follows, results are *dual* with respect to the terms *right* and *left*; i. e., one should repeat all statements and proofs which are given with the interchange of *right* and *left* terms and significations.

22. *A left turn of given axis a and variable angle θ has for its invariant linear spaces a certain system of planes through the center such that one and only one invariant plane contains a given direction. The angular displacement in every invariant plane is of magnitude θ in a fixed sense determined by the axis; and the path curves of points are circles in these planes about the center.* (See art. 12 in connection with this article.)

For since $\angle(r, qr) = \angle qrr^{-1} = \angle q$, therefore by a left turn q every direction r receives the angular displacement $\angle q = \theta$. There can therefore be no invariant lines (except for the special values $\theta = 0$ or π). Hence there can be no invariant spaces not through the center; for if there were such a space then the unique perpendicular upon it from the center would be an invariant line. Similarly, there can be no invariant threefold through the center, since a threefold has an unique orthogonal line at the center. It remains to consider planes through the center.

If an invariant plane through the center contain a direction r then it must contain the direction $e^{a\theta}r$, and such a plane will therefore contain the direction ar as one of the linear compositions of $e^{a\theta}r = r \cos\theta + ar \sin\theta$, with r; so that such a plane can only be the plane (r, ar) through the center. Further, any direction of this plane is $e^{a\phi}r$, which transforms into the direction $e^{a\theta}e^{a\phi}r = e^{a(\phi+\theta)}r$ of the same plane by an angular displacement θ in the sense of the plane (from r to ar). Finally, the path of a point in an invariant plane (from the invariance of distances) is at a constant distance from the center, i. e., it is a circle about the center.

23. *A series of parallel planes of given angular sense are directionally invariant planes of a left turn of one and only one axis* α, *whose sense of turn is that of the planes. Such axis* α *will be called the left axis of each plane. Reversal of angular sense reverses the corresponding left axis.*

For, since $qp^{-1} \cdot p = q$,

$$\alpha = UV \cdot qp^{-1}$$

is the axis of a left turn which leaves the plane of directions (p, q) invariant and displaces its directions in its angular sense (from p to q). If β be the axis of *any* such left turn, then we see, on remembering that the axis turns through a right angle in the sense of the turn, that

$$\beta p = \alpha p \quad \text{or} \quad \beta = \alpha .$$

For the reversed plane of directions (q, p), the corresponding left axis is

$$UV \cdot pq^{-1} = - UV \cdot qp^{-1} = - \alpha .$$

(The unit vectors of reciprocal numbers are opposites.)

24. Two planes with the same or opposite left axes will be called *left parallel planes in the same or opposite senses.** From article 22 we have, *one and only one plane can be drawn through a given straight line left parallel to a given plane.*

Transferred to the hypersphere, which intersects lines and planes through its center in points and great circles, this result is: *One and only one great circle may be drawn through a given point left parallel to a given great circle.*

25. *There is one and only one plane of directions and only one angular sense, corresponding to given right and left axes, viz., such a plane must contain the vector direction that bisects the angle between the given axes.*

Let α, β be the given left and right axes, and r any direction of any plane corresponding to these axes; then we must have

$$\alpha r = r\beta . \dagger$$

Since $\alpha^2 = \beta^2 = -1$, an obvious solution is $r = \alpha + \beta$, the vector bisector of the angle (α, β). (This is a vector δ perpendicular to α when $\beta = -\alpha$.) Thus a required plane and sense is $(\alpha + \beta, \alpha \overline{\alpha + \beta})$. There cannot be more than two independent solutions for r; for if there were three, r_1, r_2, r_3, then their linear composition would determine a threefold solution (r_1, r_2, r_3), and $\alpha(r_1, r_2, r_3)$ would determine a fourth independent solution, since no threefold is invariant for left turn α. Thus all values of r would be solutions (in particular $r = \pm 1$), which is impossible.

* STRINGHAM calls these left (and also the right) parallel planes *isoclines*.

† This is STRINGHAM'S equation of the plane in question, which passes through the origin; but without reference to angular sense.

26. Let $\{\alpha, \beta\}$ signify a plane whose left axis is α, and right axis, β, viz., a plane $(\alpha + \beta, \alpha\alpha + \beta)$. The opposite plane is then $\{-\alpha, -\beta\}$; the *orthogonal plane* is $\{\alpha, -\beta\}$ or $\{-\alpha, \beta\}$ according to angular sense (since $\alpha + \beta, \alpha(\alpha + \beta), (\alpha - \beta), \alpha(\alpha - \beta)$ are mutually perpendicular). Also $\{\alpha, \alpha\} = (1, \alpha)$ and $\{\alpha, -\alpha\} = (\delta, \alpha\delta)$, a plane of vector directions perpendicular to α. To correspond with conventions of TAIT, the *left* turn α must be a *right handed* turn about the axis α in the plane of vector space which is perpendicular to α.

27. Two orthogonal planes are seen to be both right and left parallel planes; if right parallel in the same sense then they are left parallel in opposite senses.

All planes of left axis α are included in the form

$$(a) \qquad \{\alpha, \pm\lambda\}$$

where λ is an arbitrary unit vector; from which it is seen that such planes occur in orthogonal pairs.

28. *A left turn of axis α and a right turn of axis β have one orthogonal pair of invariant planes in common*, $\{\alpha, \beta\}$ *and* $\{\alpha, -\beta\}$ *and only one* (art. 27); *in the plane* $\{\alpha, \beta\}$ *the two turns are in the same sense (the sense of the plane); in the plane* $\{\alpha, -\beta\}$, *the two turns are opposed in sense (the left turn with the sense of the plane).*

The Double Turn $e^{\alpha\theta}(\ )e^{\beta\phi}$.*

29. The invariant planes of the double turn of axes α, β are evidently the common orthogonal pair of invariant planes of the component turns, viz., $\{\alpha, \beta\}$, $\{\alpha, -\beta\}$. The path curves in these planes are circles about the center and the angular displacement in the first plane is $\theta + \phi$, and in the second it is $\theta - \phi$, in the senses of those planes (art. 28).

30. For a given point of initial position r outside an invariant plane a locus or path curve under the double turn $e^{\alpha\theta}(\ )e^{\beta\phi}$ is

$$e^{\alpha\theta}\, r\, e^{\beta\phi}$$

for a continuous simultaneous variation of θ, ϕ from $0, 0$ to their final values. For independent variations of θ, ϕ, the above locus is an invariant surface, generated by all possible path curves of the point r, which we call *the hypercircular ring through the point r, of cyclic planes* $\{\alpha, \beta\}$, $\{\alpha, -\beta\}$.

31. The above ring is a surface in the threefold boundary of the hypersphere of radius Tr, with its center at the center of turning. The equation $\phi = \phi_1$,

* The *right* and *left turn*, in the hypersphere CLIFFORD'S *motor twist*. STRINGHAM calls it a *rotation*, because path curves in the invariant planes are circles. In the present development, which proceeds much farther than STRINGHAM'S, it appears that path curves are generally spirals, as in twists; whereas the path curves of rotations are concentric circles in each of a system of parallel planes.

where ϕ_1 is a constant, determines a great circle section of this ring with a left axis α, so that the family of curves $\phi = const.$ is a generating system of left parallel great circles. Similarly $\theta = \theta_1$ determines a great circle section of the ring with a right axis β; and the family $\theta = const.$ is a generating system of right parallel great circles. Two generating circles of the same system cannot intersect because they are right or left parallel; but two generating circles of opposite systems always intersect in the point $e^{\alpha\theta} r e^{\beta\phi}$ corresponding to the given values of θ, ϕ on each circle. The ring is ruled by its two systems of generating circles in "great arc parallelograms." (See art. 41.)

32. It will be proved in art. 35 that $e^{\alpha\psi}(\ )e^{\beta\psi}$ is a rotation about the invariant plane $\{\alpha, -\beta\}$ as a fixed axis. Thus we may interpret the identity:

$$e^{\alpha\psi}\left(e^{\alpha\theta} r e^{\beta\phi}\right)e^{\beta\psi} = e^{\alpha(\theta+\psi)} r e^{\beta(\phi+\psi)},$$

to mean that the ϕ generating circle of the left parallel system rotates about the axis of revolution into the $\phi + \psi$ generating circle of the same system. By changing $e^{\beta\psi}$ into $e^{-\beta\psi}$ we make $\{\alpha, \beta\}$ the axis of revolution. Hence

Either generating system of circles of a hypercircular ring is a system of revolution about either cyclic plane of the ring as axis of revolution.

33. If r_1, r_2 be the orthogonal projections of a line (or point) r upon the cyclic planes of the above ring, then $r = r_1 + r_2$, and hence

$$e^{\alpha\theta} r e^{\beta\phi} = e^{\alpha\theta} r_1 e^{\beta\phi} + e^{\alpha\theta} r_2 e^{\beta\phi}.$$

This resolution of any point of the ring (or line from the center to that point) into components in the cyclic planes must therefore be its orthogonal projections on those planes; and, since the loci of these projections are circles about the center of radii Tr_1, Tr_2, *the orthogonal projection of a hypercircular ring on either cyclic plane is a circle about the center.*

34. The hypercircular ring is CLIFFORD's *surface of zero curvature generated by a series of parallel lines meeting a fixed line.* The planes of the circles of either generating system of a hypercircular ring form what STRINGHAM calls either an *ordinal* or a *cardinal* system of planes.* There is only *one type* of linear one dimensional systems of right or left parallel planes (isoclines); and STRINGHAM's division into two types, *ordinal* and *cardinal*, is a distinction founded on common properties of the one type. The ordinal property is that of article 31, the cardinal property is that of article 32. The properties of the hypercircular ring were presented by me in 1898 before the Chicago Section of the American Mathematical Society, and before the American Association as part of a complete development of the types of the invariant figures, path curves and surfaces, of all conformal transformations. †

*Transactions of the American Mathematical Society, vol. 2 (1901), p. 211.

†Bulletin of the American Mathematical Society, November, 1898, p. 93.

35. *Rotations.* The double turn $e^{\alpha\theta}(\ )e^{\beta\theta}$ is characterized by a *fixed* plane $\{\alpha, -\beta\}$ (displacement $\theta - \theta = 0$); while *any* orthogonal plane $\{\alpha, \beta\}$ is invariant, since it intersects the fixed plane in a fixed point. The path curves in each plane $\{\alpha, \beta\}$ are therefore circles about its fixed point as center; and the angular displacement is 2θ. This double turn is therefore *a rotation;* the fixed plane is the *plane axis;* the orthogonals to the fixed plane are *equators.*

36. A *left* rotation is one in which the angular sense of its plane axis is determined by its component left turn. The left rotation about the axis $(p, q) = \{\alpha, -\beta\}$ through twice the angle (p, q) is given by

$$qp^{-1}(\ )q^{-1}p = e^{\alpha\theta}(\ )e^{\beta\theta},$$

where

$$\alpha = UV\cdot qp^{-1}, \quad \beta = UV\cdot q^{-1}p, \quad \theta = \angle(p, q).$$

With the same axis and angle, the corresponding right rotation is

$$pq^{-1}(\ )p^{-1}q = e^{-\alpha\theta}(\ )e^{-\beta\theta}.$$

37. Taking $p = 1$ in the preceding rotations we obtain the left and right rotations $q(\ )q^{-1}$, $q^{-1}(\ )q$ about the axis $(1, q) = (1, Vq)$ through twice the angle $(1, q)$, which is HAMILTON's *twice the angle of* q. The equator lies in vector space, perpendicular to Vq; and vector space rotates within itself about Vq as axis. The left rotation is *right-handed* about the axis.

38. The resultant of two left rotations of vector space $q(\ )q^{-1}$ and $r(\ )r^{-1}$ is $rq(\ )q^{-1}r^{-1} = \overline{rq}(\ )\overline{rq}^{-1}$ a left rotation. It thus follows that any two equal angles in vector space, with a common vertex, may be brought into coincidence by a single left rotation.

Turning motion of planes.

39. In left turning a plane about a center, *the left axis of the plane is left-rotated in vector space about the axis of the turn through twice its angle, and the right axis of the plane remains fixed.*

For let (q, r) be any plane, which becomes the plane (pq, pr) by a left turn p; then the axes of the given plane are (art. 23)

$$\alpha = UV\cdot rq^{-1}, \quad \beta = UV\ q^{-1}r,$$

and the axes of the transformed plane are

$$\alpha' = UV\cdot prq^{-1}p^{-1} = p\alpha p^{-1}, \quad \beta' = UV\ q^{-1}r = \beta.$$

40. We see now in applying a left turn to any figure of lines and planes through the center that *all lines through the center describe equal left parallel plane angles in the same sense, while the successive positions of any plane are right parallel in the same sense.*

41. In the hypersphere, about the given center, the *left turn* becomes a *left parallel translation* of all points through equal arc distances in the same sense, while successive positions of any great circle are right parallel in the same sense. If we join by a great arc any point of a given great circle to any point of another right parallel great circle, the transversing arc may be taken as the path curve of a left turn which displaces one great circle into the other; and in this turn, all points of the turning circle describe great arcs that are equal and left parallel to the given path curve. In other words, two right parallel great circles are everywhere equally distant by left parallel arc measurement. Also, in this motion the translated great circle generates the right parallel system of a hypercircular ring, while its points describe the left parallel generating system. Hence if the entire ring be subjected to the left turn of its left generating system then its surface advances with the left parallel generating circles as path curves, and the right parallel generating circles as successive positions of any one.* This shows in particular, that opposite sides and angles of great arc parallelograms are equal, and that adjacent angles are supplementary. A ring is determined by any parallelogram; two adjacent sides determine two parallelograms and two rings, viz., if a be a vertex, and b, c adjacent vertices, the opposite vertex is $ba^{-1}c$ or $ca^{-1}b$.

42. We need not repeat the proof in the Bulletin for November, 1897 (p. 57fg.), that equidistant translation of all points of a rigid arc is either right or left parallel translation. It is not necessary for that proof that *all* points be given as equidistantly translated; it is sufficient if only *three points* of the arc be so translated, no two of them being diametrically opposite.

Angles between planes.

43. The angle between the left axes of two planes will be called their *left axial angle.* We have

The axial angles of two planes are unaltered in magnitude by the rigid displacements of single or double turns. (Art. 39.)

One pair of planes is rigidly reconcilable with another pair of planes, in repect to their directions and angular senses, when and only when the magnitudes of corresponding axial angles are equal. (Arts. 38, 39.)

The most general rigid displacement is the resultant of a double turn and a translation.

44. The *plane angle* between two planes which have a common direction is the angle whose sides are formed by turning such a common direction through a right angle in the sense of turn of each plane. Such planes through the center meet in a line, and this is the usual plane angle of their diedral angle.

* The left turn by which right parallel arcs are reconcilable is a particular right-handed twist; this displacement property is the foundation for CLIFFORD'S term *right-parallel lines*.

Planes may be separated by parallel translation of either so as to have only a common direction and no point of intersection, without altering their plane angle. In the hypersphere about the center, the great circles of diametric planes that intersect in a line will intersect in a point, and the sides of each plane angle will be tangential directions of the corresponding circles at their point of intersection, in the senses of the respective circles; and such plane angle will be *the angle between the great circles.*

45. *If two planes transverse (contain a common direction), then their axial angles are equal to each other and to their plane angle. Conversely, if the axial angles of two planes are equal to each other, then the planes transverse.*

For, let $\{\alpha, \beta\}$ $\{\alpha', \beta'\}$ be given planes which contain a common direction r; then their plane angle is

$$\angle(\alpha r, \alpha' r) = \angle \alpha' r(\alpha r)^{-1} = \angle \alpha'\alpha^{-1} = \angle(\alpha, \alpha') = \angle(r\beta, r\beta') = \angle(\beta, \beta').$$

Conversely, let $\angle(\alpha, \alpha') = \angle(\beta, \beta') = \theta$; then construct transversing planes with plane angle θ; their axial angles will each be θ, as just shown, and they will therefore be rigidly reconcilable in directions and senses with the given planes, by art. 43.

46. Two transversing planes are *perpendicular* when their plane angle is a right angle. *A common perpendicular to two planes* is a plane that transverses each perpendicularly.

47. *An orthogonal pair of common perpendiculars to two given planes can be drawn through a given point; and there is only one such pair if the given planes are neither right nor left parallel. If the given planes are left parallel there is a continuous system of right parallel orthogonal pairs of common perpendiculars.*

Let $\{\alpha, \beta\}$, $\{\alpha', \beta'\}$ be the given planes and let $\{\alpha'', \beta''\}$ be any common perpendicular to them. Then

$$\angle(\alpha, \alpha'') = \angle(\beta, \beta'') = \pi/2, \quad \angle(\alpha', \alpha'') = \angle(\beta', \beta'') = \pi/2,$$

and therefore

$$\alpha'' = \pm UV\alpha\alpha', \quad \beta'' = \pm UV\cdot\beta\beta',$$

which determines an unique orthogonal pair of common perpendiculars if α, α' are not parallel and β, β' are not parallel. If α, α' are parallel, then α'' may be any unit vector perpendicular to α, and to each of these values of α'' corresponds $\beta'' = \pm UV\cdot\beta\beta'$. If the given planes are orthogonal it will be seen that any common transversal is a common perpendicular to each.

48. *Plane angles.* For convenience consider planes meeting in a center so that transversing planes meet in a line through that point. All planes can be transferred to the center by parallel translation without altering direc-

tional relations. If we draw through the center an orthogonal pair of common perpendiculars to two given planes through the center we obtain a pair of intersections in each common perpendicular, whose included angle is a *plane angle* of the given planes. To be definite, let one plane be double turned about the orthogonal pair of common perpendicular planes as invariant planes into positional and sense coincidence with the other plane; then the intersections of the transferred plane with the invariant planes describe plane angles in the invariant planes; and it is these angles, so described, which are *the plane angles* of the two planes.

49. *The plane angles of two planes are the half sum and half difference of their axial angles.*

If $\{\alpha, \beta\}$, $\{\alpha', \beta'\}$ be the planes, so that $\theta = \angle(\alpha, \alpha')$, $\phi = \angle(\beta, \beta')$ are the axial angles, we take α'' perpendicular the plane of (α, α') on that side which left turns α into α', and take β'' perpendicular to the plane (β, β') on that side which right turns β into β'. The required double turn with the orthogonal pair of common perpendiculars $\{\alpha'', \pm\beta''\}$ as invariant planes, which brings the two planes into coincidence, is then $e^{\frac{1}{2}\alpha''\theta}\ (\ )\ e^{\frac{1}{2}\beta''\phi}$ (art. 39). Hence by art. 29, the required plane angles are $\frac{1}{2}(\theta + \phi)$, $\frac{1}{2}(\theta - \phi)$.

Orthogonal projection of areas.

50. The orthogonal projection of a line q on a plane $\{\alpha, \beta\}$ is a line $\frac{1}{2}(q - \alpha q\beta)$. For let $q = q_1 + q_2$, where $\alpha q_1 = q_1\beta$, $\alpha q_2 = -q_2\beta$, and we find $q_1 = \frac{1}{2}(q - \alpha q\beta)$.

51. The area of a parallelogram (p, q) will be

$$Tp \cdot Tq \cdot \sin \angle(p, q) = TV \cdot qKp.$$

In sign this parallelogram will be considered positive in the plane (p, q) and negative in the plane (q, p), so that it is the ratio of $V \cdot qKp$ to the left axis of the plane, $\pm UV \cdot qKp$.

52. *The area projecting factor between two planes is the product of the cosines of their plane angles.**

For if the area (p, q) be orthogonally projected from the plane

$$(p, q) = \{UV \cdot qKp,\ UV \cdot Kp \cdot q\} = \{\alpha', \beta'\}$$

upon the plane $\{\alpha, \beta\}$, the resulting area is

$$\tfrac{1}{4}\, V \cdot (q - \alpha q\beta) K(p - \alpha p\beta) \cdot \alpha^{-1} = -\tfrac{1}{2} S(\alpha\alpha' + \beta\beta') \cdot TV \cdot qKp.$$

Hence the projecting factor is

$$-\tfrac{1}{2} S(\alpha\alpha' + \beta\beta') = \tfrac{1}{2}(\cos\theta + \cos\phi) = \cos\tfrac{1}{2}(\theta + \phi)\cos\tfrac{1}{2}(\theta - \phi).$$

* STRINGHAM'S *measure of the divergence of the two planes.*

We have arrived at a convenient closing point so far as fundamental ideas are concerned. The treatment of conformal transformations would make a paper by itself and may be prepared later. I am not aware how much has been done in this direction in the ordinary theory of transformations of four real variables, but the quaternion analysis has certainly proven, as I view it, a powerful instrument for investigation in that direction. Also by its use, in connection with space conceptions, it may be said almost to take the place of fourfold eyes.

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