THE QUARTIC CURVE AS RELATED TO CONICS*

BY

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A characteristic feature of the plane curve of even order, $2n$, is the one-to-one correspondence set up by it between curves of class n and curves of order n. Given a fundamental curve of order $2n$, every curve of class n has a definite polar curve of order n, every curve of order n has " associated " with it a definite curve of class n whose polar it is.

In the following this correspondence will be studied with special reference to the quartic. Some simultaneous irrational invariantive forms of the system of two quartics will be considered, the method used being an extension to the ternary domain of that employed by HILBERT † for binary forms. A generalization of the configuration known as the self-polar triangle for $n = 1$ will be treated.

The first section will be devoted to an investigation of an error contained in a statement of CLIFFORD,‡ an error originally pointed out by Professor MORLEY. The facts there obtained suggested the more general inquiry which follows.

Forms of order n will be denoted in general by f^n, g^n, etc., or symbolically by α_x^n, β_x^n, $\cdots$ ($\alpha_x \equiv \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3$), those of class n, by F^n, G^n, $\cdots$ or by a_ξ^n, b_ξ^n, $\cdots$; and $c_{\iota\kappa l}$ will be the general expression for an invariantive form of f^4 of degree ι, order κ and class l.

§1. A Special Class of Quartic Curves.

In the collected works of Clifford, p. 117, the following statement is found: The quartic f^4 has a contravariant of degree 5 and class 4 ($\phi \equiv c_{504}$), the evectant of the invariant B § which is possessed of this property, that every conic, c_ξ^2, has a polar as to f^4 whose polar as to ϕ is again c_ξ^2.

We shall prove first

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§ The invariant B is the determinant of sixth order

$$|a_{\iota\kappa, lm}| \qquad (\iota\kappa, lm = 11, 22, 33, 23, 31, 12)$$

whose vanishing is the condition that a conic apolar to f^4 exists.

(I) *For the general quartic, $f^4 \equiv \alpha_x^4$ there exists no quartic, $F^4 \equiv a_\xi^4$ which has with regard to f^4 the above described property.*

For if F^4 and f^4 are so related and c_ξ^2 is an arbitrary line conic, the equation

$$(1) \qquad c_a^2 \alpha_a^2 a_\xi^2 \equiv c_\xi^2$$

must hold identically for all values of the ξ's and c's. In strictness a factor λ should be introduced before c_ξ^2 in (1), a factor, however, necessarily independent of c_ξ^2 and therefore supposed incorporated with the coefficients of F.

Equating coefficients of $\xi_\iota \xi_\kappa$ in (1) we have six equations of the type

$$(2) \qquad c_a^2 \alpha_a^2 a_{\iota\kappa} = c_{\iota\kappa},$$

which are to be identities in the quantities $c_{\iota\kappa}$. If in each equation of the type (2) we equate coefficients of c_{mn} a system of six non-homogeneous equations is obtained. The determinant of this system in the six unknowns

$$a_{mn11}, \quad a_{mn22}, \quad a_{mn33}, \quad 2a_{mn23}, \quad 2a_{mn31}, \quad 2a_{mm12}$$

is the invariant B which for the general quartic is not zero. If the minor of $a_{\iota\kappa, lm}$ in B be denoted by $A_{\iota\kappa, lm}$ the solution of the system gives

$$(3) \qquad a_{\iota\kappa, mn} = \frac{A_{\iota\kappa, mn}}{\mu_{\iota\kappa, mn} B},$$

where

$$\mu_{\iota\kappa, mn} = 1 \quad \text{if} \quad \iota = \kappa \quad \text{and} \quad m = n,$$

$$\mu_{\iota\kappa, mn} = 2 \quad \text{if} \quad \iota \neq \kappa \quad \text{or} \quad m \neq n,$$

$$\mu_{\iota\kappa, mn} = 4 \quad \text{if} \quad \iota \neq \kappa \quad \text{and} \quad m \neq n.$$

But the symbolic coefficients a are such that $a_{\iota\kappa, mn} = a_{mn, \iota\kappa} = a_{m\iota, n\kappa}$. Hence must

$$(4) \qquad \frac{A_{\iota\kappa, mn}}{\mu_{\iota\kappa, mn}} = \frac{A_{mn, \iota\kappa}}{\mu_{mn, \iota\kappa}} = \frac{A_{m\iota, n\kappa}}{\mu_{m\iota, n\kappa}}.$$

Since B is a symmetrical determinant we have

$$A_{\iota\kappa, lm} = A_{lm, \iota\kappa} \neq A_{l\iota, m\kappa}.$$

Relations (4) then, contrary to hypothesis, do impose conditions on the quartic f, and this proves (I). Conditions (4) in full are

$$(5) \qquad A_{1212} = 4A_{1122}, \qquad A_{1313} = 4A_{1133}, \qquad A_{2323} = 4A_{2233}, \qquad A_{1231} = 2A_{1123}$$

$$A_{1223} = 2A_{2231}, \qquad A_{3123} = 2A_{3312}, \qquad B \neq 0.$$

(II) *If the quartic f satisfies conditions* (5), *a quartic F, namely* ϕ, *the evectant of the invariant B, actually exists, which has with f the property ascribed to it by* CLIFFORD.

Evidently F actually exists, for in (4) its coefficients have been so determined that (1) is an identity for both $\xi_{\iota\kappa}$ and $c_{\iota\kappa}$. The contravariant ϕ is

$$
\begin{vmatrix}
\alpha_{1111} & \alpha_{1122} & \cdots & \alpha_{1112} & \xi_1^2 \\
\alpha_{2211} & \alpha_{2222} & \cdots & \alpha_{2212} & \xi_2^2 \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\alpha_{1211} & \alpha_{1222} & \cdots & \alpha_{1212} & \xi_1\xi_2 \\
\xi_1^2 & \xi_2^2 & \cdots & \xi_1\xi_2 & 0
\end{vmatrix}
\equiv
\begin{aligned}
& A_{1111}\xi_1^4 + 2A_{1112}\xi_1^3\xi_2 + 2A_{1113}\xi_1^3\xi_3 \\
& + (A_{1212} + 2A_{1122})\xi_1^2\xi_2^2 \\
& + 2(A_{1123} + A_{1213})\xi_1^2\xi_2\xi_3 + \cdots,
\end{aligned}
$$

and has been studied by SCHERRER. * Applying conditions (5) to the expanded form of ϕ the coefficient of $\xi_\iota\xi_\kappa\xi_m\xi_n$ becomes the previously determined $a_{\iota\kappa mn}$ of F to within the factor B.

An analogous property which does belong to the *general* quartic is this:

(III) *The cubic polar of any point as to the general quartic f has a linear polar as to ϕ which is the original point.*

For the proof, see § 2, at the end.

A quartic f satisfying conditions (5) will be called a "special" quartic and some of the properties of such quartics will now be derived. Let first

$$
\frac{A_{\iota j,\kappa l}}{\mu_{\iota j,\kappa l}} \equiv A'_{\iota j,\kappa l}.
$$

The quartic envelope ϕ may then be written symbolically A'^4_ξ.

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* Annali, ser. 2, vol. 10. If the polar of c_ξ^2 as to f^4 is γ_x^2, c_ξ^2 will be called the conic "associate" to γ_x^2. If the polar of c_ξ^2 as to f^4 is the line η taken twice, c_ξ^2 will be called the conic "associate" to the line η.

SCHERRER shows that, for a given η, the form

$$
\begin{vmatrix}
a_{11\,11} & \cdots & a_{11\,12} & \xi_1^2 \\
\cdot & \cdot & \cdot & \cdot \\
a_{12\,11} & \cdots & a_{12\,12} & \xi_1\xi_2 \\
\eta_1^2 & \cdots & \eta_1\eta_2 & 0
\end{vmatrix}
\equiv \phi(\xi,\eta) = \phi(\eta,\xi)
$$

gives the conic associate to the line η. Also $\phi(\xi,\gamma)$ gives the conic associate to the conic γ_x^2.

Therefore ϕ, or $\phi(\xi,\xi)$, is the locus of lines which touch their own associate conics (self-associate lines).

It is easily seen that *the locus of lines whose associate conics are apolar to a given point conic is the associate of the point conic.*

Hence $c_a^2 a_x^2$ has a polar as to $\phi(\xi,\eta)$ (η variable) which is Bc_ξ^2. That is, *the form $\phi(\xi,\eta)$ has a property similar to that required of a_ξ^4 by* CLIFFORD, as may be seen also from an equation formed analogously to (3).

(IV) *If f is a special quartic, the relation of f and ϕ is reciprocal, i. e., the ϕ of ϕ is f.*

For ϕ being $A_\xi'^4$ we may write the ϕ of ϕ in determinant form. The A''s may then be changed into the A's by multiplying the last three rows and columns of the determinant by 2. Since the minor of $A_{ij,\kappa l}$ is $B^4 a_{ij,\kappa l}$, we have on developing,

$$\phi \text{ of } \phi = \frac{B^4}{64} f.$$

The reciprocity between f and ϕ becomes more marked with the study of their allied forms. Let ψ be the envelope of lines whose associate * conics degenerate into a pair of points which lie on a locus S and let D be the envelope of lines into which the polar conics of Hessian points break up. Then from the definition of f as a special quartic and of the various loci and envelopes we have at once:

(V) *If f is a special quartic the Hessian of ϕ is the ψ of f and the ψ of ϕ is the Hessian of f; also the D of ϕ is the S of f and the S of ϕ is the D of f.*

Theorems (IV) and (V), exemplifying the reciprocity between f and ϕ, may be brought under a more general theorem. For this we denote by I any invariantive form associated to a quantic Q given in point (or line) coördinates, line and point coördinates being supposed interchanged in I if Q is given in line (or point) coördinates. We have then

(VI) *If f is a special quartic and if the I_1 of ϕ is the I_2 of f then the I_2 of ϕ is the I_1 of f to within a factor $(B^4/64)^n$ where n is the degree of I_1 in the coefficients of ϕ.*

For, taking ϕ in the form $A_\xi'^4$, the I_1 of ϕ is a certain function of the A''s, x's and ξ's. The I_2 of f is what the I_1 of ϕ becomes when the A''s are written in terms of the a's. Hence the I_2 of ϕ is what the I_1 of ϕ becomes when in each $A'_{mn,pq}$ of I_1, a_{ijkl} is replaced by A'_{ijkl} (line and point coördinates being interchanged). Thus $A'_{mn,pq}$ becomes $B^4 a_{mn,pq}/64$.

We shall now prove

(VII) *The necessary and sufficient condition that f be a special quartic is the identical vanishing of a certain covariant conic C of degree 5.*

Conditions (5) have already been found to be necessary and sufficient. They may be shown to be identical with the condition of (VII). The general quartic f has two independent C_{520}, C_1 and C_2, the first being obtained by operating with σ † (C_{204}) on H † (C_{360}); the second by operating with T † (C_{306}) on f^2. The sought conic C will be a linear combination of C_1 and C_2.

For simplicity let f be referred to a triangle of reference chosen as follows: Take for vertex ξ_3 a point of intersection of the Hessian, H, of f and the

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* See definition in the preceding footnote.

· † $\sigma \equiv (a\beta\xi)^4$; $H =$ Hessian; $T \equiv (a\beta\xi)^2(\beta\gamma\xi)^2(\gamma a\xi)^2$.

Steinerian, Σ, of f. Take for ξ_2 the point of Σ which corresponds to ξ_3 considered as a point on H; and for ξ_1 the point of H which corresponds to ξ_3 considered as a point on Σ. Then

$$\xi_2 \xi_3^2 :: f \qquad \text{and} \qquad \xi_3 \xi_1^2 :: f$$

($::$ is a symbol for the expression " is apolar to ").

The quartic must take the form

$$f \equiv ax_1^4 + bx_2^4 + cx_3^4 + 6hx_1^2x_2^2 + 12mx_1x_2^2x_3 + 4a_2x_1^3x_2$$
$$+ 4b_1x_2^3x_1 + 4b_3x_2^3x_3 + 4c_1x_3^3x_1.$$

We find then for the coefficient of x_1^2 in C_1,

$$-a^2cb_3^2 + 6acma_2b_3 - 9achm^2 - 9abhc_1^2 + 32am^3c_1 + 8ab_1^2c_1^2$$
$$+ 5h^3c_1^2 - 12ha_2b_1c_1^2 + 8ba_2^2c_1^2,$$

and for the coefficient of x_1^2 in C_2,

$$-19a^2cb_3^2 + 2ab_1^2c_1^2 - 21abhc_1^2 + 114acma_2b_3 - 21achm^2 + 8am^3c_1$$
$$+ 72ha_2b_1c_1^2 + 2ba_2^2c_1^2 - 55h^3c_1^2 - 150cm^2a^2.$$

Let now C be defined* as follows:

(6) $$150\,C = 19\,C_1 - C_2.$$

The coefficient of x_1^2 in C is then

$$ba_2^2c_1^2 - achm^2 - abhc_1^2 + 4am^3c_1 + ab_1^2c_1^2 + h^3c_1^2 - 2ha_2b_1c_1^2 + cm^2a_2^2.$$

But this is equal to $A_{2323} - 4A_{2233}$. For the coefficient of x_2x_3 in C a similar calculation gives $2(2A_{1123} - A_{1213})$. We have then

$$C \equiv (A_{2323} - 4A_{2233})x_1^2 + (A_{3131} - 4A_{3311})x_2^2 + (A_{1212} - 4A_{1122})x_3^2$$
$$+ 2(2A_{1123} - A_{1213})x_2x_3 + 2(2A_{2231} - A_{1223})x_3x_1 + 2(2A_{3312} - A_{2331})x_1x_2.$$

This form of C renders (VII) evident.

(VIII) *If f is a special quartic the two covariant conics of the fifth degree are identical.*

This conic C possesses other interesting properties.

(IX) *If the quartic f has an apolar conic, C is the apolar conic expressed in point coördinates.*

For if f has an apolar conic, $B = 0$ and six quantities $p_{\iota\kappa}$ can be determined such that $A_{\iota j,\,\kappa l} = p_{\iota j}p_{\kappa l}$. Moreover (CLEBSCH, Crelle's Journal, vol. 49),

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*The only previous mention of C occurs in SALMON (*Higher Plane Curves*) where it is merely described and used as a particularly simple combination of the covariants C_1 and C_2.

$$\phi = (p_{11}\xi_1^2 + p_{23}\xi_2\xi_3 + \cdots)^2$$

is the square of the apolar conic. The equation of the latter in point coördinates is

$$x_1^2(\tfrac{1}{4}p_{23}p_{23} - p_{22}p_{33}) + \cdots + 2x_2x_3(\tfrac{1}{2}p_{11}p_{23} - \tfrac{1}{4}p_{12}p_{13}) + \cdots,$$

or

$$\tfrac{1}{4}[(A_{2323} - 4A_{2233})x_1^2 + \cdots + 2(2A_{1123} - A_{1213})x_2x_3 + \cdots],$$

i. e., it is the C found above. We have further

The necessary and sufficient conditions that f have a triple point are $B = 0$ and $C \equiv 0$. * ϕ is then the triple point taken four times.

Let a conic whose polar as to f has a polar as to ϕ which is the original conic be called a "fixed" conic (f no longer being taken as a special quartic). Recalling that the conic "associate" to a given point conic γ_x^2 is obtained by replacing the column of ξ's in the determinant expression for ϕ by the coefficients γ, a formula may be derived which will indicate some possible variations in the number of fixed conics.

If the combination: $\tfrac{1}{2}$ (polar of γ as to ϕ) $- 6$ (associate of γ as to f) be formed, we obtain

$$\xi_1^2[(A_{1212} - 4A_{1122})\gamma_{22} + (A_{3131} - 4A_{1133})\gamma_{33} + 2(A_{1231} - 2A_{1123})\gamma_{23}] + \cdots$$
$$+ 2\xi_2\xi_3[(A_{1213} - 2A_{1123})\gamma_{11} - (A_{2323} - 4A_{2233})\gamma_{23} - (A_{3123} - 2A_{3312})\gamma_{31}$$
$$- (A_{2312} - 2A_{2231})] + \cdots.$$

This resulting conic is the "intermediate" of γ and C. The intermediate of two conics α_x^2 and β_x^2 is $(\alpha\beta\xi)^2$, the locus of lines cutting the two conics in harmonic pairs of points. We have then

(XI) *For the general quartic f and any conic γ_x^2 the following relation holds:*

(7) $$(\gamma \text{ on } \phi) = 2I_\gamma + 12A_\gamma\dagger,$$

where I_γ is the intermediate of γ and C, A_γ is the associate of γ.

If $I_\gamma \equiv 0$ or if $I_\gamma \equiv \mu A_\gamma$ then $(\gamma \text{ on } \phi) \equiv \lambda A_\gamma$, i. e., γ is a fixed conic. But in general $I_\gamma \not\equiv \mu A_\gamma$; hence if every conic γ is to be a fixed conic, $I_\gamma \equiv 0$ or $C \equiv 0$, a verification of (VII).

The *general* quartic has, as we shall see later, six fixed conics. Since $C \not\equiv 0$, for these conics $I_\gamma \equiv \mu A_\gamma$. We have then as a first consequence of formula (7),

1° *The six fixed conics of the general quartic f with regard to ϕ are the intermediates of their polars as to f with C.*

From (7) follows also a proposition whose dual is stated by SCHERRER (loc. cit.):

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* Actually only 4 conditions, since the covariant conic furnishes only 3 independent relations.
† "γ on ϕ" is the polar of γ with regard to ϕ.

2° *The third polar of a line L as to ϕ is the polar point of L to its associate conic.*

The following facts also may be derived from (7):

3° *If f has a cusp the cuspidal tangent is a double line of ϕ* (possibly an inflexional tangent).

4° *If ϕ has a double line L, L touches also T, ψ and C_{12}.* *

The condition that C degenerate is the vanishing of SALMON's invariant E_1, by definition.

5° *If the invariant E_1 of the quartic f vanishes there is a pencil of fixed conics.*

For if C is the two lines $\alpha\beta$ meeting at a point P, any two lines through P and harmonic with α and β form a conic whose intermediate as to $\alpha\beta$ vanishes. These conics form a pencil which includes α^2 and β^2 and may be written $\alpha^2 + \lambda\beta^2$. The associate pencil is $A_{\alpha^2} + \lambda A_{\beta^2}$.

6°. *If C is the square of a line L, a web of fixed conics exists.*

For then the I_γ for any conic γ is the two points in which γ meets L. If in particular $\gamma \equiv LM$, where M is any line of the plane, $I_\gamma \equiv 0$, whence it follows that LM is a fixed conic. The web A_{LM} (projective to the net of associate conics) contains a single infinity of pairs of points ab such that the polar of ab as to f has L for a part. These points ab are then "conjugate points on G_L" a covariant curve of third order and degree defined originally by CAPORALI † as the Jacobian of the net of polar conics of couples of points of L.

Besides the conics of the web a finite number of other fixed conics exists, viz: the pairs of "conjugate points as to f" which lie on L. For by definition † a pair of conjugate points as to f is a pair which lie on their polar conic γ. If the pair also lie on L, γ meets L in this pair, i. e., I_γ and A_γ are each this pair of points and $(\gamma$ on $\phi) = \lambda A_\gamma$.

§ 2. *Some Simultaneous Irrational Invariantive Forms of the System: f^n, F^n*

The discussion in the preceding section of the number of fixed conics associated to f^4 and its contravariant ϕ suggests this more general problem:

Given a quartic $f \equiv \alpha_x^4$ and a quartic $F = a_\xi^4$, do there exist and if so, in what number, conics whose polars as to f have each a polar as to F identical with the original conic?

Such conics will also be called "fixed" conics. If c_ξ^2 is a fixed conic it must satisfy the identical equation

(1) $c_a^2 \alpha_a^2 a_\xi^2 \equiv \lambda c_\xi^2$.

Equating coefficients of ξ we have six homogeneous equations in the six unknowns $c_{\iota\kappa}$ whose determinant must then vanish, i. e.,

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* Defined by SCHERRER, loc. cit.

† CAPORALI, *Memorie di Geometria*, pp. 344–5.

$$\Delta_2^4(\lambda) \equiv \begin{vmatrix} a_{11}\,\alpha_{11}\,a_a^2 - \lambda & a_{11}\,\alpha_{22}\,a_a^2 & \cdots & 2a_{11}\,\alpha_{31}\,a_a^2 & 2a_{11}\,\alpha_{12}\,a_a^2 \\ a_{22}\,\alpha_{11}\,a_a^2 & a_{22}\,\alpha_{22}\,a_a^2 - \lambda & \cdots & 2a_{22}\,\alpha_{31}\,a_a^2 & 2a_{22}\,\alpha_{12}\,a_a^2 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{31}\,\alpha_{11}\,a_a^2 & a_{31}\,\alpha_{22}\,a_a^2 & \cdots & 2a_{31}\,\alpha_{31}\,a_a^2 - \lambda & 2a_{31}\,\alpha_{12}\,a_a^2 \\ a_{12}\,\alpha_{11}\,a_a^2 & a_{12}\,\alpha_{22}\,a_a^2 & \cdots & 2a_{12}\,\alpha_{31}\,a_a^2 & 2a_{12}\,\alpha_{12}\,a_a^2 - \lambda \end{vmatrix} = 0.$$

The notation $\Delta_2^4(\lambda)$ indicates that the λ-equation is formed with regard to the fixed conics of quartics. Since f and F are given as general, $\Delta_2^4(\lambda) = 0$ must be viewed as a sextic in λ with roots $\lambda_1, \lambda_2, \cdots, \lambda_6$ such that any root put for λ in (1) makes the resulting system consistent for the determination of the ratios of the coefficients $c_{\iota\kappa}$ which will satisfy (1).

Allowing f and F to interchange their roles, $\Delta_2^4(\lambda)$ is unaltered; therefore to a root λ_ι corresponds also a point conic which is a fixed conic of F and f. Call the six fixed line conics, $c_{\xi:1}^2, \cdots, c_{\xi:6}^2$, the fixed point conics, $\gamma_{x:1}^2, \cdots, \gamma_{x:6}^2$ such that $c_{\xi:\iota}^2$ and $\gamma_{x:\iota}^2$ each correspond to the root λ_ι. If the polar of $c_{\xi:\iota}^2$ as to f be called $\bar{\gamma}_{x:\iota}^2$, then $\bar{\gamma}_{x:\iota}^2$ has for polar as to F, $\lambda_\iota\,c_{\xi:\iota}^2$ which again has for polar as to f, $\lambda_\iota\,\bar{\gamma}_{x:\iota}^2$. Hence $\bar{\gamma}_{x:\iota}^2 \equiv \gamma_{x:\iota}^2$ or $\gamma_{x:\iota}^2$ is the polar of $c_{\xi:\iota}^2$ as to f. Similarly $c_{\xi:\iota}^2$ is the polar of $\gamma_{x:\iota}^2$ as to F.

(I) *Corresponding to each root of the equation* $\Delta_2^4(\lambda) = 0$, *there exist two fixed conics* $c_{\xi:\iota}^2$ *and* $\gamma_{x:\iota}^2$ *which are such that they stand in the relation of pole and polar to both* f *and* F.

The forms $c_{\xi:\iota}^2$ and $\gamma_{x:\kappa}^2$ are defined by the identities

$$c_{a:\iota}^2\,a_a^2\,a_\xi^2 \equiv \lambda_\iota\,c_{\xi:\iota}^2, \qquad \gamma_{a:\kappa}^2\,a_a^2\,\alpha_x^2 \equiv \lambda_\kappa\,\gamma_{x:\kappa}^2.$$

Operating on the first identity with $\gamma_{x:\kappa}^2$, on the second with $c_{\xi:\iota}^2$ we have the identities

$$c_{a:\iota}^2\,a_a^2\,\gamma_{a:\kappa}^2 \equiv \lambda_\iota\,c_{\gamma:\iota\kappa}^2, \qquad \gamma_{a:\kappa}^2\,a_a^2\,c_{a:\iota}^2 \equiv \lambda_\kappa\,c_{\gamma:\iota\kappa}^2.$$

Subtracting,

$$0 = (\lambda_\iota - \lambda_\kappa)\,c_{\gamma:\iota\kappa}^2.$$

In general $\lambda_\iota \neq \lambda_\kappa$ for $\iota \neq \kappa$, therefore $c_{\gamma:\iota\kappa}^2 = 0$.

(II) *Among the fixed conics of the general quartics* f *and* F *exist the following apolarity relations:* $c_{\xi:\iota}^2 :: \gamma_{x:\kappa}^2$ $(\iota \neq \kappa)$.

Suppose now that a linear identity exists:

$$(2) \qquad\qquad \rho_1\,c_{\xi:1}^2 + \cdots + \rho_6\,c_{\xi:6}^2 \equiv 0.$$

If we multiply the identity (1) for $\lambda = \lambda_\iota$ by ρ_ι and sum for $\iota = 1, 2, \cdots, 6$ we have by virtue of (2) the identity

$$\lambda_1\rho_1\,c_{\xi:1}^2 + \cdots + \lambda_6\rho_6\,c_{\xi:6}^2 \equiv 0.$$

In the same way we obtain four more identities. These can subsist only if the determinant

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_6 \\ \cdot & \cdot & \cdot & \cdot \\ \lambda_1^5 & \lambda_2^5 & \cdots & \lambda_6^5 \end{vmatrix} = \prod_{\iota \neq \kappa} (\lambda_\iota - \lambda_\kappa) = 0 \,,$$

i. e., only if two roots of $\Delta_2^4(\lambda)$ are equal.

(III) *Among the fixed conics of the general quartics f and F can exist no linear identities.*

It follows then that $c_{\xi:\iota}^2$ is not $:: \gamma_{x:\iota}^2 (\iota = 1, 2, \cdots, 6)$.

(IV) *The coefficients of the different powers of λ in the equation $\Delta_2^4(\lambda) = 0$ are simultaneous invariants of f and F,* i. e., *the roots λ_ι are simultaneous irrational invariants of f and F.*

Proof of this theorem is obtained from the equation defining λ, and will be supplied so readily that we may omit it here; so, too, the proof of the following theorem.

(V) *The fixed conics $c_{\xi:\iota}^2$ and $\gamma_{x:\iota}^2$ belonging to f and F are simultaneous irrational covariants and contravariants of f and F.*

The equation $\Delta_2^4(\lambda) = 0$ arranged in powers of λ may be written

$$\Delta_2^4(\lambda) \equiv \lambda^6 - I_1 \lambda^5 + I_2 \lambda^4 - I_3 \lambda^3 + I_4 \lambda^2 - I_5 \lambda + I_6 = 0 \,,$$

where I_κ is a simultaneous invariant of f and F of degree κ in the coefficients of each. A general expression for I_κ may readily be obtained. For this we use temporarily a new notation. Let $g \equiv a_\xi^2 a_x^2 = 0$ be a *general* connex * $(2, 2)$ where $a_{\iota\kappa} a_{lm}$ represents the arbitrary coefficient of $x_\iota x_\kappa \xi_l \xi_m$. The polar of c_ξ^2 as to g is $c_a^2 a_\xi^2$. If $c_a^2 a_\xi^2 \equiv \lambda c_\xi^2$ we have the analogue of the fixed conic previously treated, and λ will then satisfy an equation $\Delta(\lambda) = 0$ which may be obtained from $\Delta_2^4(\lambda)$ by suppression of the symbolic factor a_a^2.

Writing $g \equiv a_\xi^2 a_x^2 = b_\xi^2 \beta_x^2 = c_\xi^2 \gamma_x^2 = \cdots$ we have first that a seven-rowed determinant vanishes identically:

$$D \equiv \begin{vmatrix} a_a^2 & a_\beta^2 & a_\gamma^2 & \cdots & a_\zeta^2 & a_\xi^2 \\ b_a^2 & b_\beta^2 & b_\gamma^2 & \cdots & b_\zeta^b & b_\xi^2 \\ c_a^2 & c_\beta^2 & c_\gamma^2 & \cdots & c_\zeta^2 & c_\xi^2 \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ f_a^2 & f_\beta^2 & f_\gamma^2 & \cdots & f_\zeta^2 & f_\xi^2 \\ \alpha_x^2 & \beta_x^2 & \gamma_x^2 & \cdots & \zeta_x^2 & x_\xi^2 \end{vmatrix} \equiv 0 \text{ (identity in } x \text{ and } \xi),$$

---

*The author hopes in future to treat some analogous problems for the general connex by the method of section 4.

since there must be a linear identity among any seven ternary quadrics.   If now
we write for the κ-rowed determinant

$$J_\kappa \equiv \begin{vmatrix} a_\alpha^2 & a_\beta^2 & a_\gamma^2 & \cdots \\ b_\alpha^2 & b_\beta^2 & b_\gamma^2 & \cdots \\ c_\alpha^2 & c_\beta^2 & c_\gamma^2 & \cdots \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix}$$

we have, taking into account the permutability of symbols,

(α)

$$D \equiv x_\xi^2 J_6 - \frac{6!}{5!} a_\xi^2 \alpha_x^2 J_5 + \frac{6!}{4!} a_\xi^2 \alpha_b^2 \beta_x^2 J_4 - \frac{6!}{3!} a_\xi^2 \cdots \gamma_x^2 J_3 + \frac{6!}{2!} a_\xi^2 \cdots \delta_x^2 J_2$$
$$- \frac{6!}{1} a_\xi^2 \cdots \epsilon_x^2 J_1 + 6! \, a_\xi^2 \cdots \zeta_x^2 \equiv 0.$$

Since $c_\alpha^2 a_\xi^2 \equiv \lambda c_\xi^2$,

(β) $$c_\alpha^2 a_\beta^2 b_\xi^2 \equiv \lambda^2 c_\xi^2, \qquad c_\alpha^2 a_\beta^2 b_\gamma^2 c_\xi^2 \equiv \lambda^3 c_\xi^2, \qquad \cdots$$

Replacing, then, in the identity (α) the symbols x by c and making use of (β),

$$c_\xi^2 \left[J_6 - \frac{6!}{5!} J_5 \lambda + \frac{6!}{4!} J_4 \lambda^2 - \frac{6!}{3!} J_3 \lambda^3 + \frac{6!}{2!} J_2 \lambda^4 - \frac{6!}{1} J_1 \lambda^5 + 6! \lambda^6 \right] \equiv 0.$$

Hence λ satisfies the equation

$$\lambda^6 - J_1 \lambda^5 + \frac{1}{2!} J_2 \lambda^4 - \frac{1}{3!} J_3 \lambda^3 + \frac{1}{4!} J_4 \lambda^2 - \frac{1}{5!} J_5 \lambda + \frac{1}{6!} J_6 = 0.$$

Returning to the original notation by introducing the factor a_α^2 we have I_κ expressed by means of the κ-rowed determinant

$$I_\kappa = \frac{1}{\kappa!} a_\alpha^2 b_\beta^2 c_\gamma^2 \cdots \begin{vmatrix} a_\alpha^2 & a_\beta^2 & a_\gamma^2 & \cdots \\ b_\alpha^2 & b_\beta^2 & b_\gamma^2 & \cdots \\ c_\alpha^2 & c_\beta^2 & c_\gamma^2 & \cdots \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix}.$$

These properties of the invariants admit of immediate generalization and we
have

THEOREM (A).   *Given two q-ary n-ics in complementary variables which,
symbolically, are $f^n \equiv \alpha_{x:1}^n = \alpha_{x:2}^n = \cdots$ and $F^n \equiv a_{\xi:1}^n = a_{\xi:2}^n = \cdots$, there
exist $N(\rho)$ q-ary ρ-ics ($\rho \geqq n/2$), F^ρ, such that the polar of F^ρ as to f^n has
a polar as to F^n which is identically $\lambda_\iota F^\rho$. These $N(\rho)$ quantities λ_ι satisfy*

an equation in λ of degree $N(\rho)$ whose coefficients are simultaneous rational invariants of the same degree in the coefficients of f^n and F^n.

If the coefficient of λ to the highest power, be 1, of the next power $-I_1$, etc., the general form of I_κ is

$$I_\kappa = \frac{1}{\kappa!}\, a_{a:1,1}^{n-\rho} \cdots a_{a:\kappa,\kappa}^{n-\rho}\,\begin{vmatrix} a_{a:1,1}^{\rho} & a_{a:1,2}^{\rho} & \cdots & a_{a:1,\kappa}^{\rho} \\ a_{a:2,1}^{\rho} & a_{a:2,2}^{\rho} & \cdots & a_{a:2,\kappa}^{\rho} \\ \cdot & \cdot & \cdots & \cdot \\ a_{a:\kappa,1}^{\rho} & a_{a:\kappa,2}^{\rho} & \cdots & a_{a:\kappa,\kappa}^{\rho} \end{vmatrix}.$$

The series of invariants so formed closes with $I_{N(\rho)}$, $N(\rho)$ being the number of terms in the general q-ary ρ-ic.

Returning to the equation $\Delta_2^4(\lambda) = 0$ for the case of two quartics, the vanishing of the constant term, i. e., the invariant $I_6 = 0$ is the condition that a polar conic (in this case, any conic) of the one shall be an apolar conic of the other and it breaks up into the product of the invariants B formed for each quartic.

(VI) *The various invariants in the equation $\Delta_2^4(\lambda) = 0$ are the results of the successive performance of the operation*

$$O \equiv \tfrac{2}{6}\left[\frac{\partial^2}{\partial a_{1111}\,\partial \alpha_{1111}} + \tfrac{1}{4}\frac{\partial^2}{\partial a_{1112}\,\partial \alpha_{1112}} + \tfrac{1}{4}\frac{\partial^2}{\partial a_{1113}\,\partial \alpha_{1113}} + \tfrac{1}{6}\frac{\partial^2}{\partial a_{1122}\,\partial \alpha_{1122}} \right.$$
$$\left. + \tfrac{1}{12}\frac{\partial^2}{\partial a_{1123}\,\partial \alpha_{1123}} + \cdots \right]$$

upon the final invariant I_6.

For O affects only the diagonal terms of $\Delta_2^4(0)$ and acting on each gives a result unity.

The only objection to the immediate extension of (VI) to the $\Delta_\rho^n(\lambda)$ equation is that the effect of the corresponding operator O upon the various elements of the diagonal of $\Delta_\rho^n(0)$ is not obviously the same numerical constant for each element. This objection does not hold, for the general diagonal term is for ternary forms

$$\frac{\rho!}{\kappa!\,\lambda!\,\mu!}\, a_1^\kappa a_2^\lambda a_3^\mu\, \alpha_1^\kappa \alpha_2^\lambda \alpha_3^\mu\, a_a^{n-\rho} \qquad (\kappa+\lambda+\mu=\rho).$$

Expanded, this becomes

(a) $$\sum \frac{\rho!}{\kappa!\,\lambda!\,\mu!}\,\frac{(n-\rho)!}{k!\,l!\,m!}\, a_1^{\kappa+k} a_2^{\lambda+l} a_3^{\mu+m}\, \alpha_1^{\kappa+k} \alpha_2^{\lambda+l} \alpha_3^{\mu+m},$$

the summation extending to all values of k, l, m such that $k + l + m = n - \rho$. The operator O has its various terms affected by the reciprocals of the polynomial coefficients of the form of order n. The effect of operating upon the general diagonal term (a) with O is then

(b)
$$\sum_{k+l+m=n-\rho} \frac{\rho!}{\kappa!\,\lambda!\,\mu!} \frac{(n-\rho)!}{k!\,l!\,m!} \frac{(\kappa+k)!\,(\lambda+l)!\,(\mu+m)!}{n!}.$$

The sum (b) is independent of the choice of κ, λ, μ if $\kappa+\lambda+\mu=\rho$, i. e., (b) is independent of the particular diagonal term chosen. For the coefficient of x^k in $(1-x)^{-\kappa-1}$ is

$$\frac{(\kappa+k)!}{\kappa!\,k!}.$$

Hence the coefficient of $x^k y^l z^m$ in the expansion of

$$(1-x)^{-\kappa-1}(1-y)^{-\lambda-1}(1-z)^{-\mu-1} \quad \text{is} \quad \frac{(\kappa+k)!\,(\lambda+l)!\,(\mu+m)!}{k!\,\kappa!\,\lambda!\,l!\,\mu!\,m!},$$

then (b) is to a factor $\rho!\,(n-\rho)!/n!$ the sum of the coefficients of terms homogeneous and of degree $k+l+m=n-\rho$ in that expansion. For $z=y=x$, to the same factor, (b) is the coefficient of $x^{n-\rho}$ in the expansion of $(1-x)^{-(\kappa+\lambda+\mu)-3}$, a coefficient independent of the choice of κ, λ, μ, if always $\kappa+\lambda+\mu=\rho$. This proof is independent of the number of letters k, l, m so that (VI) admits of extension also to forms with more variables. We have then

THEOREM (B). *The invariant $I_{N(\rho)}$ of theorem (A) when equated to zero is the condition that a polar $(n-\rho)$-ic of f^n exist which $::$ F^n and vice versa. If $\rho=n/2$, $I_{N(\rho)}$ breaks up into two factors, each of which equated to zero gives the condition that one of the forms f^n and F^n have an apolar ρ-ic. The other invariants I_κ of the series are the results of the successive application of the operator*

$$O \equiv c \sum_{k+l+m+\ldots=n} \frac{k!\,l!\,m!\cdots}{(k+l+m+\cdots)!} \frac{\partial^2}{\partial a_1^k a_2^l a_3^m \cdots \partial a_1^k a_2^l a_3^m \cdots},$$

where c is a constant so adjusted as to make the result of O upon a diagonal term of $\Delta_\rho^n(\lambda)$ unity.

Returning to the fixed conics of f^4 and F^4, their coefficients may be determined by putting $\lambda=\lambda_\iota$ and solving any five of the resulting equations. But for symmetry we may, following Hilbert, consider the modified system *

(3)
$$
\begin{aligned}
&c_{11}^\iota(a_{11}\alpha_{11}a_a^2-\lambda_\iota)+c_{22}^\iota a_{11}\alpha_{22}a_a^2 &&+\cdots+c_{12}^\iota\cdot 2a_{11}\alpha_{12}a_a^2 &&+x_1^2 u &&=0,\\
&c_{11}^\iota a_{22}\alpha_{11}a_a^2 &&+c_{22}^\iota(a_{22}\alpha_{22}a_a^2-\lambda_\iota)+\cdots+c_{12}^\iota\cdot 2a_{22}\alpha_{12}a_a^2 &&+x_2^2 u &&=0,\\
&\quad\cdot\quad\cdot\quad\cdot\quad\cdot &&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot &&\cdot\quad\cdot\quad\cdot\quad\cdot &&\cdot\quad\cdot\quad\cdot\\
&c_{11}^\iota a_{12}\alpha_{11}a_a^2 &&+c_{22}^\iota a_{12}\alpha_{22}a_a^2 &&+\cdots+c_{12}^\iota(2a_{12}\alpha_{12}a_a^2-\lambda_\iota)+x_1 x_2 u &&=0,\\
&c_{11}^\iota \xi_1^2 &&+c_{22}^\iota \xi_2^2 &&+\cdots+c_{12}^\iota\cdot 2\xi_1\xi_2 &&-c_{\xi:\iota}^2 &&=0.
\end{aligned}
$$

_____

* Here and below it seems desirable to use two notations interchangeably, writing the particular index of the c_{ik} sometimes above, as in c_{12}^ι, sometimes below as in $c_{\xi:\iota}^2$.

Solving for the unknowns c_{ik}^ι, u and $c_{\xi:\iota}^2$ we have

$$c_{11}^\iota : c_{22}^\iota : \cdots : c_{12}^\iota : u : c_{\xi:\iota}^2 = \Delta_{c_{11}^\iota} : \Delta_{c_{22}^\iota} : \cdots : \Delta_{c_{12}^\iota} : \Delta_u : \Delta_{c_{\xi:\iota}^2},$$

Δ_κ denoting the minors of the matrix of the system. Then $u \backsim \Delta_u \backsim \Delta_2^4(\lambda_\iota) = 0$, i. e., the system (3) is equivalent to the original system. Putting

$$\Delta_{c_{\xi:\iota}^2} \equiv \Delta_2^4(x, \xi, \lambda_\iota)$$

we have

$$c_{\xi:\iota}^2 \backsim \Delta_2^4(x, \xi, \lambda_\iota) \equiv \begin{vmatrix} a_{11}\alpha_{11}a_\alpha^2 - \lambda_\iota & a_{11}\alpha_{22}a_\alpha^2 & \cdots & 2\,a_{11}\alpha_{12}a_\alpha^2 & x_1^2 \\ a_{22}\alpha_{11}a_\alpha^2 & a_{22}\alpha_{22}a_\alpha^2 - \lambda_\iota & \cdots & 2\,a_{22}\alpha_{12}a_\alpha^2 & x_2^2 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{12}\alpha_{11}a_\alpha^2 & a_{12}\alpha_{22}a_\alpha^2 & \cdots & 2\,a_{12}\alpha_{12}a_\alpha^2 - \lambda_\iota & x_1 x_2 \\ \xi_1^2 & \xi_2^2 & \cdots & 2\,\xi_1\xi_2 & 0 \end{vmatrix}.$$

But dually

$$\gamma_{x:\iota}^2 \backsim \Delta_2^4(x, \xi, \lambda_\iota),$$

therefore

$$\Delta_2^4(x, \xi, \lambda_\iota) \backsim \gamma_\iota c_\iota.$$

Since c_ι on f is γ_ι and γ_ι on F is c_ι we have

$$(\gamma_\iota)^2 \backsim \Delta_2^4(x, \alpha_{\iota\kappa}a_x^2, \lambda_\iota) \text{ and } (c_\iota)^2 \backsim \Delta_2^4(a_{\iota\kappa}a_\xi^2, \xi, \lambda_\iota).$$

These squares will appear in another connection. The bordered matrix $\Delta_2^4(x, \xi, \lambda)$ is a connex $(2, 2)$ which for $\lambda = \lambda_\iota$ degenerates, i. e., breaks up into the product $c_{\xi:\iota}^2 \cdot \gamma_{x:\iota}^2$. Arranged in powers of λ,

$$\Delta_2^4(x, \xi, \lambda) \equiv K_0\lambda^5 + K_1\lambda^4 + \cdots + K_5,$$

where K_ι is an invariantive connex $(2, 2)$ of degree ι in the coefficients of f^4 and of F^4. K_0 is x_ξ^2. K_5 is $\Delta_2^4(x, \xi, 0)$ and for a given x is the line conic whose polar as to f has for polar as to F the square of the point x, with a dual meaning for a given ξ.

And in general if the $\Delta_\rho^n(\lambda)$ of theorem (A) be bordered with variables we obtain the connex (ρ, ρ), $\Delta_\rho^n(x, \xi, \lambda)$ which has properties entirely analogous to those of $\Delta_2^4(x, \xi, \lambda)$.

Neither is the generality of theorem (A) diminished by the assumption $\rho \gtreqless n/2$, for the fixed $(n - \rho)$-ics are the polars of the fixed ρ-ics as to f^n and F^n.

We have found in the preceding section that under certain conditions f^4 and its contravariant ϕ possessed an infinite number, a pencil or even a web of fixed conics. For every fixed conic of f^4 and F^4 the identity (1) holds in which λ is a root of $\Delta_2^4(\lambda) = 0$. But $\Delta_2^4(\lambda) = 0$ has always six roots, some of which may coincide. Having substituted a root in (1) the fixed conic is in gen-

eral uniquely determined by the resulting equations. If, however, the chosen root λ_ι is such as to make every five-rowed minor of $\Delta_2^4(\lambda)$ vanish, the fixed conic may be determined from any four equations but no longer uniquely. Instead of using four equations consider the system (3) further modified by the addition to each of the first six equations of a term $y_\iota y_\kappa v$ and the adjunction of the equation $c_{\eta\,:\,\iota}^2 = 0$.

Then $u \backsim \Delta_2^4(y, \eta, \lambda_\iota)$ and $v \backsim \Delta_2^4(x, \eta, \lambda_\iota)$, both of which are identically zero, i. e., the modified system is equivalent to the original system. Further, $c_{\xi\,:\,\iota}^2 \backsim \Delta_2^4(x, y, \xi, \eta, \lambda_\iota)$ where

$$\Delta_2^4(x, y, \xi, \eta, \lambda_\iota) \equiv \begin{vmatrix} a_{11} a_{11} a_a^2 - \lambda_\iota & \cdots & 2a_{11} a_{12} a_a^2 & y_1^2 & x_1^2 \\ \cdot & \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot & \cdot & \cdot & \cdot \\ a_{12} a_{11} a_a^2 & \cdots & 2a_{12} a_{12} a_a^2 - \lambda_\iota & y_1 y_2 & x_1 x_2 \\ \eta_1^2 & \cdots & 2\eta_1 \eta_2 & 0 & 0 \\ \xi_1^2 & \cdots & 2\xi_1 \xi_2 & 0 & 0 \end{vmatrix}.$$

Moreover $\Delta_2^4(x, y, \xi, \eta, \lambda_\iota) \backsim c_{\xi\,:\,\iota}^2$ entirely irrespective of what x and y may be and therefore $\backsim c_{\xi\,:\,\iota}^2 \cdot \phi(y, x)$. From the dual nature of the problem must then

$$\Delta(x, y, \xi, \eta, \lambda_\iota) \backsim c_{\xi\,:\,\iota}^2 \cdot \gamma_{x\,:\,\iota}^2.$$

The adjunction of the equation $c_{\eta\,:\,\iota}^2 = 0$ requires that $c_{\xi\,:\,\iota}^2$ touch the line η. Hence corresponding to a root λ_ι of $\Delta_2^4(\lambda) = 0$ which makes all the five-rowed minors of Δ_2^4 vanish, there exists a whole pencil of conics c_ι; for to *any* given line η corresponds *one* conic $c_{\xi\,:\,\iota}^2$ which will touch it. Also to any given point y corresponds one conic γ_ι which passes through it. Such a root λ_ι is necessarily a double root of $\Delta_2^4(\lambda) = 0$, though to a double root only one fixed conic may correspond.

In the same way if a root of $\Delta_2^4(\lambda)$, necessarily a triple root, makes all four-rowed minors vanish there will correspond to this root a web of line conics and a net of point conics, the product of the web and net being given by the thrice bordered determinant, $\Delta_2^4(x, y, z, \xi, \eta, \zeta, \lambda_\iota)$.

If finally a sextuple root of $\Delta_2^4(\lambda)$ makes all the elements of $\Delta_2^4(\lambda)$ vanish, the root is necessarily the bilinear invariant of f and F. Every conic will be a fixed conic and f and F will stand in the reciprocal relation of f and ϕ discussed in § 1. All these bordered determinants possess the same invariantive character as $\Delta_2^4(x, \xi, \lambda)$.

The degenerations of $\Delta_1^4(\lambda)$, (and in general of $\Delta_p^n(\lambda)$) are entirely similar to those of $\Delta_2^4(\lambda)$. However for the *general* quartic f^4 infinitely many F^4 exist for which *every* point and line are "fixed." Among these are every C_{504} of f. For let

$$f = \alpha_x^4 = \beta_x^4; \quad \sigma \equiv (\alpha\beta\xi)^4 \equiv a_\xi^4 = b_\xi^4; \quad \bar{S} \equiv (ab\xi)^4; \quad \bar{\phi} \equiv (\alpha\bar{S}\xi)^4, \quad A \equiv a_\alpha^4,$$

then according to MAISANO (Giornale di Mat., vol. 19) every C_{504} belongs to the pencil $A\sigma - \lambda\bar{\phi}$. The final polar of a point p_ξ as to f and the pencil is

$$A p_\alpha a_\alpha^3 a_\xi + \lambda p_\alpha (\alpha\beta\bar{S})^3 (\beta\bar{S}\xi) \equiv A p_\alpha (\alpha\beta\gamma)^3 (\beta\gamma\xi) + \lambda p_\alpha (abc)^3 (\beta\bar{S}\xi).$$

MAISANO shows that every symbolic expression which contains the apparent factors $(\alpha\beta\gamma)^3$ or $(abc)^3$ contains the actual factors $(\alpha\beta\gamma)^4$ or $(abc)^4$ respectively. Hence the final polar is of the form

$$p_\xi \left[\mu A (\alpha\beta\gamma)^4 + \nu (abc)^4 \right].$$

We have proved then (III) of § 1.   Also

(VII) $B = 0$ *is the condition that every polar cubic of $f :: \phi$ and vice versa.* The corresponding fact for f, σ and $A = 0$ is stated by MAISANO (loc. cit.).

§ 3. *Systems of Conics Connected with the Quartic.*

To a certain extent point and line conics are related to a given quartic f as are points and lines to a given conic.   Thus

(I) *If* $c_{\xi:1}^2 :: $ *polar conic of* $c_{\xi:2}^2$, $c_{\xi:2}^2 :: $ *polar conic of* $c_{\xi:1}^2$.   For, by the hypothesis, $c_{\xi:1}^2 \cdot c_{\xi:2}^2 :: f$, a perfectly mutual relation between c^1 and c^2.

The system of conics

$$\left. \begin{array}{c} \gamma_{x:1}^2, \ \gamma_{x:2}^2, \ \cdots, \ \gamma_{x:6}^2 \\ c_{\xi:1}^2, \ c_{\xi:2}^2, \ \cdots, \ c_{\xi:6}^2 \end{array} \right\}$$

shall be called an apolar system if $\gamma_{x:\iota}^2 :: c_{\xi:\kappa}^2$ for $\iota \neq \kappa$ and $\gamma_{x:\iota}^2$ not $:: c_{\xi:\iota}^2$. Such a system is an immediate extension of the three points and three lines which form a triangle and may be obtained by arbitrarily choosing six linearly independent point conics.   The line conics will then be determined by the apolarity conditions.   Thus taking point conics with, line conics without, polynomial coefficients we have

$$c_{\xi:\kappa}^2 = \Gamma_{11}^\kappa \xi_1^2 + \Gamma_{23}^\kappa \xi_2 \xi_3 + \cdots,$$

where Γ_{mn}^κ is the minor of γ_{mn}^κ in

$$\begin{vmatrix} \gamma_{11}^1 & \cdots & \gamma_{12}^1 \\ \cdot & \cdot & \cdot \\ \gamma_{11}^6 & \cdots & \gamma_{12}^6 \end{vmatrix} \equiv \Delta.$$

II. *If in two systems each consisting of six point conics and six line conics the point conics of each are the polars, as to a quartic, of the line conics in the other system, then if either one is an apolar system, so also is the other.*

This follows immediately from (I).

In a certain sense the one apolar system is the polar of the other. If it coincides with its polar the system will be called a " self-polar system " of f. In a self polar system certain relations exist among the conics, determined as follows:

If the polar of $C^2_{\xi:\kappa}$, i. e., $\Gamma^2_{\xi:\kappa}$ as to f is $\lambda_\kappa \gamma^2_{x:\kappa}$ then

$$(1) \qquad\qquad \Gamma^2_{\alpha:\kappa} \alpha^2_x \equiv \lambda_\kappa \gamma^2_{x:\kappa} \qquad\qquad (\kappa = 1, 2, \cdots, 6).$$

Equating coefficients of $x_\iota x_j$ for $\kappa = 1, \cdots, 6$ we have a system of six non-homogeneous equations whose determinant is the adjoint of Δ and whose solution gives, to a factor $1/\Delta$,

$$(2) \qquad\qquad \alpha_{\iota j,\, mn} = \sum_{1=\rho}^{6} \lambda_\rho \gamma^\rho_{\iota j} \gamma^\rho_{mn}.$$

Hence already $\alpha_{\iota\kappa,\, lm} = \alpha_{lm,\, \iota\kappa}$. But necessarily

$$\alpha_{1122} = \alpha_{1212} \quad \text{and} \quad \alpha_{1123} = \alpha_{1213}, \quad \cdots.$$

Therefore

$$(3) \qquad \lambda_1 (\gamma^1_{11} \gamma^1_{22} - \gamma^2_{12} \gamma^2_{12}) + \cdots + \lambda_6 (\gamma^6_{11} \gamma^6_{22} - \gamma^6_{12} \gamma^6_{12}) = 0,$$

$$\lambda_1 (\gamma^1_{11} \gamma^1_{23} - \gamma^1_{12} \gamma^1_{13}) + \cdots + \lambda_6 (\gamma^6_{11} \gamma^6_{23} - \gamma^6_{12} \gamma^6_{13}) = 0,$$

etc.

If the point conics have been chosen as repeated lines, equations (3) are identically satisfied. If not, they mean that a linear identity

$$\sum_{1=\rho}^{6} \lambda_\rho (\gamma_\rho \gamma'_\rho \xi)^2 \equiv 0$$

exists among the line equations of the point conics.

(III) *If an apolar system is a self-polar system of a quartic f, a linear identity must necessarily exist among the line equations of the point conics.*

These self-polar systems may be constructed for the given quartic f as follows: Take any conic $c^2_{\xi:1}$ and its polar, $\gamma^2_{x:1}$; take $c^2_{\xi:2} :: \gamma^2_{x:1}$, otherwise arbitrary, and its polar $\gamma^2_{x:2}$; take $c^2_{\xi:3} :: \gamma^2_{x:1}$ and $\gamma^2_{x:2}$ and linearly independent of c_1 and c_2, and its polar $\gamma^2_{x:3}$. Continuing in this way up to 5 we have finally $\dot{c}^6$ completely determined by the five preceding γ's. Its polar γ^6 is also apolar to the five preceding c's by (I). In the choice of the conics c such must be avoided as are apolar to their own polar, else linear identities will exist among the conics. The choice of the system depends then on 15 constants. By (III), a linear identity will exist among the line equations of the point conics. Relations will also exist among the coefficients of the line conics, but these are of higher order.

(IV) *The quartic f is linearly expressible as a sum of the squares of the point conics of any self-polar system.*

For a self-polar system of f, the necessary relations $\big(\S\,3,\,(3)\big)$ hold. From (2) then, theorem (IV) follows at once.

This includes the expression of the quartic as a sum of the fourth powers of linear forms. For it is only necessary to chose the point conics as the squares of lines. The manner of choice is also indicated by the theorem. The first line is chosen at random. Since the square of the second line is to be apolar to the associate conic of the first it must touch it and for the same reason the four other lines must be the four common tangents of the first two associate conics. Also the first two lines must not be self-associate, as was previously noted for the general self-polar system (see SCHERRER, loc. cit.).

(V) *A point quartic f is expressible as a sum of squares of six point conics, and simultaneously a line quartic F as a sum of squares of six line conics, the conics forming a self-polar system of both f and F. This reduction is unique.*

For, from § 2, the two quartics possess one and only one common self-polar system.

The following facts may now be noted:

(*a*) Having fixed the homogeneous coefficients in a self-polar system of f, in the expression

$$f = \sum_{1}^{6} \rho_\kappa \left(\gamma^2_{x\,:\,\kappa}\right)^2$$

the coefficients ρ_κ are such that, to a factor independent of κ, $\left(c^2_{\xi\,:\,\kappa}\,\text{on}\,f\right) \equiv \rho_\kappa \gamma^2_{x\,:\,\kappa}$.

(*b*) The coefficients in the linear identity of § 3, (III) are these same ρ's.

(*c*) The polar reciprocal of c^2_ξ as to $\gamma^2_x (\equiv \gamma'^2_x)$ is

$$\gamma_c\,\gamma'_c\,\gamma_x\,\gamma'_x \equiv P_{c,\,\gamma}.$$

(*d*) The intermediate of c^2_ξ and $(\gamma\gamma'\xi)^2$ is

$$c^2_\gamma\gamma^2_x - P_{c,\,\gamma} \equiv I_{c,\,\gamma}.$$

(*e*) The polar of c^2_ξ as to $(\gamma^2_x)^2$ is

$$\gamma^2_c\,\gamma^2_x + 2P_{c,\,\gamma} = 3\gamma^2_c\,\gamma^2_x - 2I_{c,\,\gamma}.$$

(*f*) $$\rho_1 I_{c,\,\gamma_1} + \rho_2 I_{c,\,\gamma_2} + \cdots = I_{c,\,\rho_1\gamma_1 + \rho_2\gamma_2 + \ldots}.$$

(*g*) $\gamma^2_{c\,:\,\kappa,\,\kappa}$ is the Δ of page 79 and is independent of κ.

Fixing the homogeneous coefficients of the conics of the common self-polar system of f and F we have

$$f \equiv \pi_1(\gamma^2_{x:1})^2 + \cdots + \pi_6(\gamma^2_{x:6})^2,$$

(5)

$$F \equiv \rho_1(c^2_{\xi:1})^2 + \cdots + \rho_6(c^2_{\xi:6})^2,$$

where π_κ and ρ_κ are defined by (a) and (b).

By (d), (e) and (f) the polar of $c^2_{\xi:1}$ as to f is

$$3\pi_1\gamma^2_{c:1,1}\gamma^2_{x:1} - 2I_{c \,:\, \pi_1\gamma_1 + \ldots + \pi_6\gamma_6,\, 1}.$$

But by (d), $I_{c,\gamma}$ is formed with regard to the line equation of γ, and by (b), $\sum \pi_\kappa (\gamma\gamma'\xi)^2 \equiv 0$; also by (g), $3\gamma^2_{c1:1}$ may be neglected and we have that the polar of $c^2_{\xi:1}$ as to f is $\pi_1\gamma^2_{x:1}$. Similarly the polar of $\pi_1\gamma^2_{x:1}$ as to F is $\pi_1\rho_1 c^2_{\xi:1}$. Therefore

(VI) *If according to* (V) § 3, *the quartics f and F are put in the form* (5), *the roots of the $\Delta^4_2(\lambda)$ equation of f and F are such that, to a factor independent of κ,*

$$\lambda_\kappa = \pi_\kappa\rho_\kappa.$$

We may also state

(VII) *Any two point quartics f and f' are simultaneously expressible in but one way as a sum of squares of the same six point conics, the associate conics of these point conics being the same for both f and f', and the point and associate conics together forming a self-polar system of both f and f'. Two distinct linear identities exist among the line equations of the point conics.*

Theorem (III) gives a necessary condition that an apolar system be a self-polar system of a quartic. No other conditions subsist. For a general apolar system depends on 30 constants. If it be a self-polar system of some (undetermined) quartic, the line equation of one point conic is determined linearly in terms of the other five; this cuts out five constants. However, the ratios of five coefficients in the identity which gives the one in terms of the other introduce four more constants. A self-polar system of an undetermined quartic involves therefore 29 constants. But for a given quartic a self-polar system depends on 15 constants; these added to the 14 arbitrary constants involved in the given quartic make up the requisite 29.

Included in the aggregate of self-polar systems of f (15 times infinite) is an aggregate (14 times infinite) of systems which f has in common with line quartics F and an aggregate (also 14 times infinite) which f has in common with other point quartics f^1.

§ 4. *Conics in a Five-dimensional Space.*

It is of interest to derive some of the foregoing results from the representation of conics in the plane S_2 by means of a five-dimensional space S_5. This repre-

sentation has been developed by STUDY and others.* Use will be made chiefly of STUDY's work.

To line conics c, c_1, c_2, $\cdots$ in S_2 correspond in S_5 respectively points p, p_1, p_2, $\cdots$; to point conics γ, γ_1, γ_2, $\cdots$ in S_2 correspond in S_5 respectively flat spaces σ, σ_1, σ_2, $\cdots$; to *apolar* γ and c in S_2 correspond in S_5 *incident* σ and p. Denote the general p-manifold in S_5 of dimensions k and order l is M_κ^l; the σ-envelope M_κ^l. To point pairs in S_2 correspond in S_5 the p's of an $M_4^3 \equiv L^3$; on this lies an $M_2^4 \equiv F_2^4$, the p-locus which corresponds to repeated points in S_2. Dually, to line pairs in S_2 correspond in S_5 the σ's of an $\mathrm{M}_4^3 \equiv \Lambda^3$ which is inscribed to an $\mathrm{M}_2^4 \equiv \phi_2^4$, the σ-envelope which corresponds to repeated lines in S_2. The surface F_2^4 occurs as double points in L^3, and Λ^3 contains the envelope ϕ_2^4 as double σ's. Hence the polar quadric of any p as to L^3 contains F_2^4; the polar quadric of any σ as to Λ^3 is inscribed to ϕ_2^4.

(I) *Every quadric M_4^2 containing F_2^4 is a polar quadric of L^3 and may be put in the form* $\lambda_1 F_1 + \cdots + \lambda_6 F_6$ *where* $F_1, \cdots, F_6$ *are linearly independent polar quadrics. Every quadric M_2^4 inscribed to ϕ_2^4 is a polar quadric of Λ^3 and may be put in the form* $\lambda_1 \phi_1 + \cdots + \lambda_6 \phi_6$ *where* $\phi_1, \cdots, \phi_6$ *are linearly independent polar quadrics.*

If the second polar of p_1 as to L^3 is σ_1, then the γ_1 of p_1 is the reciprocal of the c_1 of σ_1. If the second polar of σ_1 as to Λ^3 is p_1, then c_1 is the reciprocal of γ_1. This coördination of p and σ in S_5 which corresponds in S_2 to the *same* conic in complementary variables is equivalent to a *dual quadratic Cremona transformation*, T, with singular surfaces F_2^4 and ϕ_2^4.

Also if the mixed polar of p_1 and p_2 as to L^3 is σ_1, then γ_1 is the intermediate of c_1 and c_2. If the polar of σ_1 and σ_2 as to Λ^3 be p_1 then c_1 is the intermediate of γ_1 and γ_2.

STUDY (Mathematische Annalen, vol. 27, p. 87) proves that

(II) *If an M_4^l contains F_2^4 λ times and an $\mathrm{M}_4^{l'}$ contains ϕ_2^4 λ' times and further M_4^l and $\mathrm{M}_4^{l'}$ correspond to each other by the transformation T, then between the numbers l, l', λ, λ' these equations exist:*

$$l' = 2l - 3\lambda, \qquad l = 2l' - 3\lambda',$$

$$\lambda' = l - 2\lambda, \qquad \lambda = l' - 2\lambda'.$$

If then $l' = 2$ and $\lambda' = 1$, $l = 1$ or we have the

COROLLARY: *An M_4^2 inscribed to ϕ_2^4 is transformed by T into a flat space σ, i. e., every σ_κ of the quadric M_4^2 goes by the dual transformation T into a point p_κ which lies in σ.*

STUDY (Mathematische Annalen, vol. 40, p. 574) proves further

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*STUDY, Mathematische Annalen, vols. 27 (1886) and 40 (1892). SEGRE, Atti Acc. Torino, vol. 20 (1885).

(III) *Through a curve of order $4l$ lying on $F_2'^4$ may be passed a* SINGLE M_4^1 *such that every quadric* M_4^2 *inscribed to* ϕ_2^4 *is apolar to* M_4^1.*

To curves of order l in S_2 (considered as made up of repeated points) correspond on $F_2'^4$ curves of order $2l$. Given then a point quartic f in S_2. To it corresponds on $F_2'^4$ a curve of order 8 which is cut out by a *single* quadric Q of the type described in (III). This same curve however is cut out of $F_2'^4$ by ∞^6 quadrics Q_1', Q_2', $\cdots$. But of all these, Q is the only one whose polar system represents the polar system of f in S_2. Then to the self-polar "bases" or "tetrahedroids" of Q correspond the self-polar systems of f treated in § 3.

(IV) *Given an* M_4^2 *and* M_4^2 *in* S_5, *the necessary and sufficient condition that* $M_4^2 :: M_4^2$ *is that when five σ's of a basis of* M_4^2 *touch* M_4^2, *the sixth also touches* M_4^2.

The proof is entirely analogous to the proof of the corresponding theorem for two conics in S_2.

Having chosen a self-polar basis of the particular quadric Q which cuts out of $F_2'^4$ the quartic f, by (I) a polar quadric of Λ^3 may be found which touches five of the σ's of this basis. From (III) $Q ::$ this polar quadric which by (IV) will then touch all six σ's of the basis. Since this polar quadric is also inscribed to ϕ_2^4, according to (II) Corollary, these six σ's of the basis are transformed by T into points of a certain σ, i. e., among the six points a linear identity exists. Hence

(V) *Among the line equations of the point conics of a self-polar system of f a linear identity exists.* [See § 3 (III).]

As has been said, ∞^6 quadrics Q_1', Q_2', $\cdots$ cut $F_2'^4$ in the same curve as Q. Among the polar systems of these quadrics certain relations exist. Denoting any two by Q_1' and Q_2', every member of the pencil $Q_1' + \lambda Q_2'$ also cuts $F_2'^4$ in that curve and in general only in that curve. But by a proper choice of λ, say λ_0, one of the pencil may be made to go through an additional point of $F_2'^4$, i. e., to contain $F_2'^4$; therefore

$$Q_1' + \lambda_0 Q_2' \equiv \overline{Q}_1,$$

denoting by $\overline{Q}_1$ a quadric which contains $F_2'^4$. By virtue of this identity we have an identity of polar systems or

$$(p \text{ on } Q_1') + \lambda_0 (p \text{ on } Q_2') \equiv (p \text{ on } \overline{Q}_1).$$

But $\overline{Q}_1$ is the polar quadric of some point say p_1 as to L_4^3. Hence

$$(p \text{ on } Q_1') = -\lambda_0 (p \text{ on } Q_2') + (pp_1 \text{ on } L_4^3).$$

Since pp_1 on L_4^3 gives the intermediate of p and p_1 we have

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* The representation of conics just described together with Theorems I, II and III are taken directly from the cited works of STUDY. What follows is an application of STUDY's methods to the proof of some of the results obtained in §§ 1 and 2.

(VI) *The polar systems of Q_1' and Q_2', any two quadrics of the system Q' (which includes the special quadric Q) coincide to within an additive term, viz., the σ which is the intermediate of a fixed point p_1 (fixed except as to variation with Q_1' and Q_2') and the variable point p whose polar is taken.*

The reciprocal of Q is an M_4^2, q, intersecting ϕ_2^4 in σ's which correspond to the lines of the contravariant of f denoted in § 1 by ϕ.

(VII) *In general q is not :: every point quadric containing $F_2'^4$. If it is, f must be the " special " quadric of § 1.*

But by (III) there is a quadric $\bar{q}$ which cuts σ_2^4 in the same σ's as q and whose polar system represents that of the contravariant ϕ. Applying to q and $\bar{q}$ the dual of proposition (VI) we have the formula:

(VIII) $$(\gamma \text{ on } \phi) = I_\gamma + \lambda A_\gamma \quad (\text{See § 1 (XI)}).$$

The covariant conic C is then the dual of the fixed point p_1 in (VI).

The polar system of the quadric q is represented in S_2 by the quadri-quadric form $\phi(\xi, \eta)$ (see footnote, § 1).

This treatment of the quartic as a quadric in S_5 has the further advantage of indicating clearly all *possible* expressions of the quartic as a sum of squares of six conics. The expression by means of " self polar systems " is obtained *only* by means of the bases of the *particular* quadric Q. But every basis of any of the ∞^6 quadrics Q_1', Q_2', $\cdots$ also gives rise to an expression of the quartic as a sum of squares. An enumeration of the constants will verify this. As to the quartic and its contravariant ϕ we have then

(IX) *The contravariant ϕ is expressible as a sum of squares of the line conics of* EVERY *self polar system of the quartic f.*

This system of quadrics in S_5 may also be serviceable in the solution of further problems with regard to the quartic. For example, the quartic f has its contravariant ϕ, ϕ, its covariant f', etc. It would be an advance in the theory of the quartic to determine whether this series of quartics f, f', etc., ever closes.