A SYMBOLIC TREATMENT OF THE THEORY OF INVARIANTS OF QUADRATIC DIFFERENTIAL QUANTICS OF n VARIABLES*

BY

HEINRICH MASCHKE

In the article † *A new method of determining the differential parameters and invariants of quadratic differential quantics* I have shown that the application of a certain symbolic method leads very readily to the formation of expressions remaining invariant with respect to the transformation of quadratic differential quantics. The presentation in that article was only a preliminary one and the work practically confined to the case of two independent variables. In my paper ‡ *Invariants and covariants of quadratic differential quantics of n variables* a more complete treatment was intended and the investigation applied throughout to the case of n variables, leaving aside, however, simultaneous invariant forms of more than one quantic.

The present paper contains in §§ 1–6 and § 8 essentially the content of the last mentioned paper; the greater parts of § 5 and § 8, and all the remaining articles are new, in particular the extensive use of covariantive differentiation.

§ 1. *Definitions. The fundamental theorem.*

To the given quadratic differential quantic

$$A = \sum_{i,k=1}^{n} a_{ik} dx_i dx_k, \tag{1}$$

with $x_1, x_2, \cdots, x_n$ as independent variables, and the a_{ik} $(a_{ki} = a_{ik})$ as functions of these variables, we apply the transformation

$$x_i = x_i(y_1, y_2, \cdots, y_n) \qquad (i = 1, 2, \cdots, n), \tag{2}$$

and obtain

$$A = A' = \sum_{i,k=1}^{n} a'_{ik} dy_i dy_k. \tag{3}$$

* Presented to the Society (Chicago) April 11, 1903, under the title *Invariants and covariants of quadratic differential quantics of n variables.* Received for publication June 20, 1903.

† Transactions of the American Mathematical Society, vol. 1 (1900), pp. 197–204.

‡ The Decennial Publications of the University of Chicago, Chicago, 1903.

Since the differentials dy are connected linearly with the differentials dx by means of the formulas

$$dx_i = \sum_{k=1}^{n} \frac{\partial x_i}{\partial y_k} dy_k \qquad (i=1, 2, \cdots, n), \tag{4}$$

we have at once, denoting the discriminant of A by $|a_{ik}|$,

$$|a'_{ik}| = r^2 |a_{ik}|, \tag{5}$$

where r denotes the determinant of the linear substitution, viz.:

$$r = \left| \frac{\partial x_i}{\partial y_k} \right|. \tag{6}$$

Let now $u, v, \cdots$, be any (arbitrary) functions of $x_1, x_2, \cdots, x_n$; $u', v', \cdots$ the same functions after application of the transformation (2), then we call every function of the coefficients a_{ik} and their derivatives, and of $u, v, \cdots$ and their derivatives, an *invariant expression* of the quadratic differential quantic A, if the expression remains the same, whether formed with the old quantities $a_{ik}, u, v, \cdots$ and their derivatives with respect to x, or with the new quantities $a'_{ik}, u', v', \cdots$ and their derivatives with respect to y.

From $u' = u$ it follows, for instance, that every arbitrary function of x is an invariant expression of A.

If such an invariant expression involves one or more arbitrary functions $u, v, \cdots$ and their derivatives, it is called a *differential parameter;* if it involves no such functions, if it is therefore a function of the a_{ik} and their derivatives alone, it is called an *invariant proper.* *

By the *order* of an invariant expression we shall understand the order of the highest derivative appearing in it.

Suppose now $F^1, F^2, \cdots, F^n$ are any n invariant expressions of A, then we have

$$F^{i'} = F^i \qquad (i=1, 2, \cdots, n),$$

and also

$$\sum_{k=1}^{n} \frac{\partial F^{i'}}{\partial y_k} dy_k = \sum_{k=1}^{n} \frac{\partial F^i}{\partial x_k} dx_k \qquad (i=1, 2, \cdots, n).$$

It follows now at once that

$$\left| \frac{\partial F^{i'}}{\partial y_k} \right| = r \cdot \left| \frac{\partial F^i}{\partial x_k} \right|,$$

and therefore with reference to (5)

$$|a'_{ik}|^{-\frac{1}{2}} \cdot \left| \frac{\partial F^{i'}}{\partial y_k} \right| = |a_{ik}|^{-\frac{1}{2}} \cdot \left| \frac{\partial F^i}{\partial x_k} \right|. \tag{7}$$

* Cf. LUIGI BIANCHI, *Vorlesungen über Differentialgeometrie;* autorisierte deutsche Uebersetzung von MAX LUKAT, Leipzig (1899), p. 39.

This equation defines the last expression as an invariant expression, and so we have the fundamental theorem:

"*If $F^1, F^2, \cdots, F^n$ are any n invariant expressions of A, then*

$$|a_{ik}|^{-\frac{1}{2}} \cdot \left| \frac{\partial F^i}{\partial x_k} \right|$$

is again an invariant expression of A."

§ 2. *Choice of convenient notations.*

Since we shall have in the following to compute continuously with expressions of the type (7), a shorter notation is indispensable. Let us first agree to indicate differentiation by subscripts. F being any quantity whatever we write

$$\frac{\partial F}{\partial x_i} = F_i. \tag{8}$$

We further denote the reciprocal value of the (positive) square root of the discriminant of A — which we always suppose to be different from zero — by the single letter β:*

$$\beta = |a_{ik}|^{-\frac{1}{2}}. \tag{9}$$

The functional determinant of any n quantities F^i $(i = 1, 2, \cdots, n)$ will be denoted by

$$\{F^1, F^2, F^3, \cdots, F^n\},$$

so that we have

$$\{F^1, F^2, F^3, \cdots, F^n\} = \left| \frac{\partial F^i}{\partial x_k} \right|. \tag{10}$$

The product of β into such a functional determinant will be denoted by

$$(F^1, F^2, \cdots, F^n) = \beta \{F^1, F^2, \cdots, F^n\}. \tag{11}$$

The quantities we have almost exclusively to deal with in the sequel are not the functional determinants themselves, but their products into β, and for this reason we use for the latter quantities the simpler symbol ( ) instead of { }.†

Even this notation is in most cases too cumbersome. We write then simply

$$(F^1, F^2, \cdots, F^n) = (F). \tag{12}$$

If it should be necessary to indicate the first, or the first two, three, etc., quantities of such a parenthesis distinctly, we write them in their proper places and let the last letter run out. For instance

* α in my previous paper.

† I therefore withdraw the suggestion made in my previous paper, loc. cit., p. 199, footnote, where the two parentheses ( ) and { } were used in the reversed sense.

$$(a) \text{ means } (a^1, a^2, \cdots, a^n),$$

$$(13) \qquad (b, a) \text{ means } (b, a^2, a^3, \cdots, a^n),$$

$$(b, c, a) \text{ means } (b, c, a^3, a^4, \cdots, a^n).$$

It is understood that the letter a occurring in the coefficients a_{ik} of A has no connection whatever with the letter a occurring, for instance, in (b, c, a).

The quantities in a parenthesis ( ) should be separated by commas. If, however, no misrepresentation can occur, the commas may be omitted:

$$(bca) = (b, c, a), \text{ etc.}$$

In our new notation the last theorem of § 1 is now this: "*If* $F^1, F^2, \cdots, F^n$ *are invariant expressions of* A, *then* (F) *is also an invariant expression of* A;" we shall call it an *invariantive constituent.*

§ 3. *The symbolic method.*

If we define

$$(14) \qquad f_i f_k = a_{ik},$$

we have

$$(15) \qquad A = \sum_{ik} a_{ik} dx_i dx_k = [\sum_i f_i dx_i]^2.$$

The expression $\sum_i f_i dx_i$ appears, if we think of the notation $f_i = \partial f / \partial x_i$ agreed upon in (8) as the complete differential of a (symbolic) function of the n variables x.

If expressions of higher than the first dimensions in the coefficients a_{ik} are to be formed symbolically, we have to use different symbols $f, \phi, \cdots$.

For instance

$$a_{12}^2 = f_1 f_2 \phi_1 \phi_2, \qquad a_{11} a_{22} = f_1 f_1 \phi_2 \phi_2, \text{ etc.}$$

The symbolic functions $f, \phi, \cdots$ (we shall simply call them symbols) appear now, as every arbitrary function $u, v, \cdots$ of the variables x does, as invariant expressions of A.

If now we form invariantive constituents containing the symbols $f, \phi, \cdots$ or $f^1, f^2, \cdots$ and any number of arbitrary functions $u, v, \cdots$, then every product of these constituents will represent an expression which, according to the fundamental theorem of § 1, will represent an expression which is *formally* invariant. But from the same reasoning as in algebra it follows that these products will at once represent *actual* invariant expressions, as soon as the symbols $f, \phi, \cdots$ occur in such connections as to permit actual meaning, e. g., in the connection $f_i f_k$, $\phi_i \phi_k$, etc. The connections of this type, however, are not the only ones.

Some of the elements of the different invariantive constituents may be constituents themselves, as for instance in the examples

$$\big((fa), b\big)\cdot(fc) \qquad \text{or} \qquad \big((fa), (fb), c\big),$$

where f denotes a symbol. In such cases also the higher derivatives of the symbol f will occur, and it is then the question, whether or not these combinations have actual meaning (they do in the above examples). Thus we have the following theorem:

Every product of invariantive constituents, the elements of which are symbols or arbitrary functions or both, or again invariantive constituents of the same character, represents an invariant expression of A, provided that every symbol occurs precisely twice and in such a connection as to permit actual interpretation in terms of the a_{ik} and their derivatives.

With regard to these symbolic invariant expressions the following two theorems are evidently true:

The value of an invariant expression in symbolic form is not changed if two equivalent symbols are interchanged, and:

An invariant expression in symbolic form vanishes if by the interchange of two equivalent symbols its sign is changed.

Covariants can now also be formed easily. In the first place, the complete differential of every invariant expression represents immediately a linear covariant of A. Let us denote for simplicity

$$F_1 dx_1 + F_2 dx_2 + \cdots F_n dx_n = F_x. \tag{16}$$

A single symbol and also any invariantive constituent with the subscript x represents then, at least formally, a linear covariant — let us call it a *covariantive constituent.* We therefore obtain covariants of any degree by forming products of these covariantive and invariantive constituents. Thus we have the theorem:

Every product of covariantive and invariantive constituents represents an actual covariant of A, provided that every symbol occurs precisely twice and in such a connection as to permit actual interpretation in terms of the coefficients a_{ik} and their derivatives. The degree of the covariant is determined by the number of the covariantive constituents as factors in the product.

§ 4. *Some important invariant expressions.*

The simplest possible invariant proper is $(f)^2$, with $f^1, f^2, \cdots f^n$ as equivalent symbols. But this invariant reduces to a constant. To show it we compute first the product

$$P = f^1_1 f^2_2 \cdots f^n_n \{f^1, f^2, \cdots, f^n\}.$$

If we multiply the first row of the functional determinant by f^1_1, the second by f^2_2, etc., we find at once

$$P = |a_{ik}| = \beta^{-2}.$$

If, on the other hand, we permute in P the equivalent symbols f^i in all possible ways, and add the products so obtained, which are all equal, together, we have

$$n!\,P = \{f^1, f^2, \cdots, f^n\}^2.$$

Since now

$$(f)^2 = \beta^2 \{f^1, f^2, \cdots, f^n\}^2,$$

we have the important formula

$$(f)^2 = n!. \tag{17}$$

Let us next compute the invariant expression $(uf)^2$, where u denotes an arbitrary function and $f^2, f^3, \cdots, f^n$ as before, equivalent symbols. For this purpose, and also for the following, we need a proper notation for the minors of some of our determinants. We denote

(18) by A_{ik}* the minor of the element a_{ik} in the determinant $|a_{ik}|$,

and further

(19) by $F^{i,k}$ the minor of the element f^i_k, in the determinant $\{f^1, f^2, \cdots, f^n\}$.

In this notation we have

$$(uf)^2 = \beta^2 \sum_{ik} F^{1,i} \cdot F^{1,k} u_i u_k.$$

But

$$F^{1,i}F^{1,k} = (n-1)!\,f_1 f_2 \cdots f_{i-1} f_{i+1} \cdots f_n \cdot \begin{vmatrix} f^2_1, f^2_2, \cdots, f^2_{k-1}, f^2_{k+1}, \cdots, f^2_n \\ \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot \\ f^n_1, f^n_2, \cdots, f^n_{k-1}, f^n_{k+1}, \cdots, f^n_n \end{vmatrix},$$

or

$$F^{1,i}F^{1,k} = (n-1)!\,A_{ik},$$

and also

$$F^{\lambda,i}F^{\lambda,k} = (n-1)!\,A_{ik}. \tag{20}$$

Hence

$$\frac{1}{(n-1)!}(uf)^2 = \beta^2 \sum_{ik} A_{ik} \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_k},\dagger \tag{21}$$

which is the well-known first differential parameter $\Delta_1 u$. Thus we have

$$(uf)^2 = (n-1)!\,\Delta_1 u. \tag{22}$$

---

* BIANCHI's quantities A_{ik} (cf. loc. cit., p. 37) have a slightly different signification. His A_{ik} is in our notation $= \beta^2 A_{ik}$.

† BIANCHI, loc. cit., p. 41.

In the same way we obtain

$$\frac{1}{(n-1)!}(uf)(vf) = \beta^2 \sum_{ik} A_{ik} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_k},^*$$

or

$$(uf)(vf) = (n-1)!\,\Delta(u, v). \tag{23}$$

Numerous other differential parameters can be formed: e. g. $(uvf)^2$, etc.

In all these examples of invariant expressions only the coefficients a_{ik} themselves occur, not their derivatives. In order to derive invariant expressions involving also derivatives of the a_{ik}, we have to express these in terms of our symbols.

We deduce from (14) by differentiation

$$\begin{aligned} \frac{\partial a_{ik}}{\partial x_l} &= f_i f_{kl} + f_k f_{il}, \\ \frac{\partial a_{kl}}{\partial x_i} &= f_k f_{li} + f_l f_{ki}, \\ \frac{\partial a_{li}}{\partial x_k} &= f_l f_{ik} + f_i f_{lk}, \end{aligned} \tag{24}$$

which gives at once

$$f_i f_{kl} = \tfrac{1}{2}\left[\frac{\partial a_{ik}}{\partial x_l} + \frac{\partial a_{il}}{\partial x_k} - \frac{\partial a_{kl}}{\partial x_i}\right]. \tag{25}$$

The expression on the right side is precisely CHRISTOFFEL's † so-called triple index symbol denoted briefly by $\left[\begin{matrix} kl \\ i \end{matrix}\right]$.‡

We have thus the important theorem: *The symbolic product $f_i\, f_{kl}$ has actual meaning for every system of values i, k, l; it is equal to the triple index symbol*

$$f_i f_{kl} = \left[\begin{matrix} kl \\ i \end{matrix}\right]. \tag{26}$$

We see, then, further, that the following combinations permit actual interpretation in the second derivatives of the a_{ik}:

$$f_{im} f_{kl} + f_i f_{klm} = \left[\begin{matrix} kl \\ i \end{matrix}\right]_m,$$

* BIANCHI, loc. cit., p. 41.

† CHRISTOFFEL, *Ueber die Transformation der homogenen Differentialausdrücke des zweiten Grades*, Crelle's Journal, vol. 70, p. 48.

‡ BIANCHI, loc. cit., p. 43.

$$(27)\qquad f_{ir}f_{ks}-f_{kr}f_{is}=\begin{bmatrix} ir \\ k \end{bmatrix}_s-\begin{bmatrix} is \\ k \end{bmatrix}_r,$$

$$f_rf_{iks}-f_sf_{ikr}=\begin{bmatrix} ik \\ r \end{bmatrix}_s-\begin{bmatrix} ik \\ s \end{bmatrix}_r.$$

The simplest symbolic invariant expression involving the second derivatives of the symbols f is $\big((uf),f\big)$. The computation, which will be given at the end of § 6, leads to the result

$$\frac{1}{(n-1)!}\big((uf),f\big)=\beta^2\cdot\sum_{ik}A_{ik}\frac{\partial^2 u}{\partial x_i\partial x_k}-\beta^4\cdot\sum_{iklm}{}'A_{ik}A_{lm}\begin{bmatrix} ik \\ l \end{bmatrix}\frac{\partial u}{\partial x_m}.$$

This expression is the second differential parameter $\Delta_2 u$.* Hence we have

$$(28)\qquad \big((uf),f\big)=(n-1)!\,\Delta_2 u.$$

§ 5. *Relations between symbolic expressions.*

For our further computation with symbolic expressions it is necessary to deduce a number of fundamental relations. As a matter of convenience (not of definition) we shall as a rule denote symbols by the letters $a, b, c, \cdots, f, \phi, \cdots$, and any functions of the x (not necessarily symbols) by the letters $u, v, \cdots$ and $x, y, \cdots$.

Differentiating formula (17) with regard to x_i, we have the fundamental formula

$$(29)\qquad (f)(f)_i=0.$$

Let now

(30) $[f]$ stand for any alternating function of $f^1, f^2, \cdots, f^n$,

e. g., for (f) or for $(f)_i$ or any higher ordinary or covariantive (see § 7) derivative of (f), and let us form the symbolic product

$$f_1^1(uf)[f].$$

We find

$$f_1^1\{uf\}[f]=\begin{vmatrix} f_1^1u_1, & u_2, & \cdots, & u_n \\ f_1^1f_1^2, & f_2^2, & \cdots, & f_n^2 \\ \cdot & \cdot & \cdot & \cdot \\ f_1^1f_1^n, & f_2^n, & \cdots, & f_n^n \end{vmatrix}\cdot[f]=\begin{vmatrix} f_1^1u_1, & u_2, & \cdots, & u_n \\ 0, & f_2^2, & \cdots, & f_n^2 \\ \cdot & \cdot & \cdot & \cdot \\ 0, & f_2^n, & \cdots, & f_n^n \end{vmatrix}\cdot[f]$$

because the product $f_1^1f_1^i[f]$ changes its sign if the equivalent symbols f^1 and f^i are interchanged, and must therefore vanish; i. e.,

$$f_1^1f_1^i[f]=0.$$

* BIANCHI, loc. cit., p. 47.

It now follows further that

$$f_1^1\{uf\}[f] = (n-1)!\,u_1 f_1^1 f_2^2 \cdots f_n^n [f] = \frac{(n-1)!}{n!} u_1 \{f\}[f],$$

and

$$f_1^1(uf)[f] = \frac{1}{n} u_1(f)[f].$$

The same method can be applied when we operate with f_i^1 instead of f_1^1, so that we also have

$$f_i^1(uf)[f] = \frac{1}{n} u_i(f)[f],$$

or, changing the notation

$$f_i(ua)[fa] = \frac{1}{n} u_i(fa)\,[fa]. \tag{31}$$

If now we specify the symbol $[f]$ according to (30) we obtain the two equations

$$f_k(fa)(ua) = (n-1)!\,u_k, \tag{32}$$

$$f_k(fa)_i(ua) = 0. \tag{33}$$

With $v^2, v^3, \cdots, v^n$ as arbitrary functions these formulas can at once be extended to

$$(fa)(fv)(ua) = (n-1)!(uv), \tag{34}$$

$$(fa)_i(fv)(ua) = 0. \tag{35}$$

A similar method serves to reduce the expression

$$f_1^1 f_2^2 (uvf)[f].$$

We find

$$f_1^1 f_2^2\{uvf\}[f] = \begin{vmatrix} f_1^1 u_1, & f_2^2 u_2, & u_3, & \cdots, & u_n \\ f_1^1 v_1, & f_2^2 v_2, & v_3, & \cdots, & v_n \\ f_1^1 f_1^3, & f_2^2 f_2^3, & f_3^3, & \cdots, & f_n^3 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ f_1^1 f_1^n, & f_2^2 f_2^n, & f_3^n, & \cdots, & f_n^n \end{vmatrix} \cdot [f]$$

$$= \begin{vmatrix} f_1^1 u_1, & f_2^2 u_2, & u_3, & \cdots, & u_n \\ f_1^1 v_1, & f_2^2 v_2, & v_3, & \cdots, & v_n \\ 0, & 0, & f_3^3, & \cdots, & f_n^3 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0, & 0, & f_3^n, & \cdots, & f_n^n \end{vmatrix} \cdot [f].$$

Further

$$f_1^1 f_2^2 \{uvf\}[f] = (n-2)!(u_1v_2 - u_2v_1) f_1^1 f_2^2 f_3^3 \cdots f_n^n [f]$$
$$= \frac{(n-2)!}{n!}(u_1v_2 - u_2v_1)\{f\}[f],$$

and changing notation

$$f_i\phi_k(uva)[f\phi a] = \frac{1}{n(n-1)}(u_iv_k - u_kv_i)(f\phi a)[f\phi a]. \tag{36}$$

Specifying again the symbol $[f\phi a]$ we deduce

$$f_i\phi_k(uva)(f\phi a) = (n-2)!(u_iv_k - u_kv_i), \tag{37}$$

$$f_i\phi_k(uva)(f\phi a)_l = 0. \tag{38}$$

From the formulas thus obtained a great number of others can be deduced. A number partly to be applied in the following articles are here listed. They involve ϵ, η as abbreviations:

$$\epsilon = \frac{1}{(n-1)!}, \qquad \eta = \frac{1}{(n-2)!}. \tag{39}$$

$$(fu)(xa)[fa] = \frac{1}{n}(xu)(fa)[fa]. \tag{40}$$ From (31).

$$[f_{ik}(xa) + f_i(xa)_k][fa] = \frac{1}{n}[x_{ik}(fa) + x_i(fa)_k][fa]. \tag{41}$$

From (31) by differentiation, and the observation that also

$$f_i(xa)[fa]_k = \frac{1}{n}x_i(fa)[fa]_k.$$

$$[f_i(xa)_k - f_k(xa)_i][fa] = \frac{1}{n}[x_i(fa)_k - x_k(fa)_i][fa]. \tag{42}$$ From (41).

$$\big((xa), f, u\big)[fa] = \frac{1}{n}\big((fa), x, u\big)[fa]. \tag{43}$$ From (42).

$$[(xa)_i(fu) + (xa)(fu)_i][fa]$$
$$= \frac{1}{n}[(xu)_i(fa) + (xa)(fu)_i][fa]. \tag{44}$$ From (40).

$$[f_i(xa)_k - f_k(xa)_i](fa) = 0. \tag{45}$$ From (42).

$$\epsilon[(xa)_i(fu) + (xa)(fu)_i](fa) = (xu)_i. \tag{46}$$ From (44).

$$[(xa)_i(fu) + (xa)(fu)_i](fa)_k = \frac{1}{n}(xu)(fa)_i(fa)_k. \tag{47}$$ From (44).

(48) $f_i\big((fa), u\big)(xa) = 0.$ From (33).

(49) $(fu)\big((fa), v\big)(xa) = 0.$ From (48).

(50) $\big((fa), f, u\big)(xa) = 0.$ From (33).

(51) $\big((fa), u\big)(fa) = 0.$ From (29).

(52) $\big((xa), f, u\big)(fa) = 0.$ From (43) and (51).

(53) $x_{ik} = \epsilon f_i\,(fa)(xa)_k + \epsilon f_{ik}(fa)(xa).$ From (41).

(54) $f_i(xa)_k(fa) = f_k(xa)_i(fa).$ From (53).

(55) $f_i(fa)_l(xa)_k = \dfrac{1}{n}\,x_i(fa)_k(fa)_l - f_{ik}(fa)_l(xa).$ From (41).

(56) $\big((fa), (xa), f, u\big) = 0.$ From (55).

(57) $f_i(\phi u)(xya)[f\phi a] = \dfrac{1}{n(n-1)}[x_i(yu) - y_i(xu)](f\phi a)[f\phi a].$ From (36).

(58) $(f\phi u)(xya)[f\phi a] = \dfrac{2}{n(n-1)}(xyu)(f\phi a)[f\phi a].$ From (36).

(59) $(fu)(\phi v)(xya)[f\phi a]$

$$= \frac{1}{n(n-1)}[(xu)(yv) - (xv)(yu)](f\phi a)[f\phi a].$$ From (36).

(60) $\eta f_i(f\phi a)(\phi u)(xya) = x_i(yu) - y_i(xu).$ From (57).

(61) $\eta(f\phi u)(f\phi a)(xya) = 2(xyu).$ From (58).

(62) $\eta(fu)(\phi v)(f\phi a)(xya) = (xu)(yv) - (xv)(yu).$ From (59).

(63) $f_i(\phi u)(f\phi a)_k(xya) = 0.$ From (57).

(64) $(f\phi u)(f\phi a)_i(xya) = 0.$ From (58).

(65) $(fu)(\phi v)(f\phi a)_i(xya) = 0.$ From (59).

We notice that some of these equations contain in every term connections of the form $f_{ik}f_{rs}$, e. g., equation (55). Indeed the different terms of this and similar equations have no actual meaning. Nevertheless the formulas are formally correct and can be used with safety for reduction work.

§ 6. *The quadratic covariant expression and the triple index symbols of the second kind.*

The quantity $f_i(ua)_k(fa)$ which according to (54) remains unchanged when the two indices i and k are interchanged gives rise to the quadratic covariant expression

$$(66)\qquad (fa)f_x(ua)_x = \sum_{ik}(fa)f_i(ua)_k dx_i dx_k,$$

whose coefficients can be shown to be equal to the "*covariant second derivatives of the function u*" in BIANCHI's terminology.*

For this purpose we have first to compute the triple index symbol of the second kind, †

$$(67)\qquad \left\{ \begin{matrix} ik \\ l \end{matrix} \right\} = \beta^2 \sum_m A_{lm} \left[\begin{matrix} ik \\ m \end{matrix} \right].$$

From (20) and (26) we have

$$(n-1)! \left\{ \begin{matrix} ik \\ l \end{matrix} \right\} = \beta^2 f^1_{ik} F^{1,l} \cdot \sum_m f^1_m F^{1,m} = \beta f^1_{ik} F^{1,l} \cdot (f),$$

$$(68)\qquad \left\{ \begin{matrix} ik \\ l \end{matrix} \right\} = \epsilon\beta f^1_{ik} F^{1,l} \cdot (f),$$

which is the required symbolic representation.

We derive from this equation

$$\sum_m u_m \left\{ \begin{matrix} ik \\ m \end{matrix} \right\} = \epsilon\beta f^1_{ik}(f) \sum_m u_m F^{1,m},$$

which gives, with a slight change of notation,

$$(69)\qquad \sum_m u_m \left\{ \begin{matrix} ik \\ m \end{matrix} \right\} = \epsilon f_{ik}(ua)(fa).$$

Now BIANCHI's covariant second derivatives of u are defined as follows: ‡

$$(70)\qquad u^{(ik)} = \frac{\partial^2 u}{\partial x_i \partial x_k} - \sum_m \frac{\partial u}{\partial x_m} \left\{ \begin{matrix} ik \\ m \end{matrix} \right\}.$$

Hence

$$(71)\qquad u^{(ik)} = u_{ik} - \epsilon f_{ik}(ua)(fa),$$

and by means of (53)

$$(72)\qquad u^{(ik)} = \epsilon f_i(ua)_k(fa),$$

which formula verifies the above statement concerning the coefficients of the quadratic covariant expression (66).

This formula leads now also to the proof of (28).

BIANCHI defines §

* Loc. cit., p. 46.

† CHRISTOFFEL, loc. cit., p. 49, and BIANCHI, loc. cit., p. 43.

‡ Loc. cit., p. 46.

§ Loc. cit., p. 47, formula (24).

(73) $$\Delta_2 u = \beta^2 \sum_{ik} A_{ik} u^{(ik)},$$

which gives

$$(n-1)!\Delta_2 u = \beta^2 (fa) \sum_{ik} A_{ik} f_i (ua)_k.$$

But from (21)

$$\beta^2 \sum_{ik} A_{ik} f_i (ua)_k = \epsilon (f\phi)\big((ua), \phi\big),$$

and from (34)

$$(fa)(f\phi)\big((ua), \phi\big) = (n-1)!\big((ua), a\big),$$

therefore

$$(n-1)!\Delta_2 u = \big((ua), a\big).$$

§ 7. *Covariantive differentiation.*

For the formation of invariants and covariants, which involve explicitly derivatives higher than the first, it turns out to be of the greatest advantage to use instead of ordinary differentiation another process which might be called *covariantive differentiation.** We shall use for its notation upper indices in parentheses.

Let x stand for any quantity not involving derivatives, then the first covariantive derivative is the same as the first ordinary derivative:

(74) $$x^{(\lambda)} = x_\lambda.$$

The second covariantive derivative, suggested by formula (71) is defined as follows

(75) $$x_i^{(k)} = x^{(ik)} = x_{ik} - \epsilon f_{ik} (fa)(xa).$$

On account of (53) the following definition might also be used

(76) $$x^{(ik)} = \epsilon f_i (xa)_k (fa).$$

We have then from (70)

(76*a*) $$\sum_\lambda x_\lambda \begin{Bmatrix} ik \\ \lambda \end{Bmatrix} = x_{ik} - x^{(ik)}.$$

It further follows from (54) that

(77) $$x^{(ik)} = x^{(ki)}.$$

If a product is to be differentiated covariantively we apply the same rule as in ordinary differentiation, i. e.,

(78) $$[x_i y_k]^{(\lambda)} = x_i y^{(k\lambda)} + y_k x^{(i\lambda)}.$$

---

*See the remark on CHRISTOFFEL's process at the introduction to § 9. The same process has been called by RICCI "dérivation covariante;" Bulletin des Sciences Mathématiques, ser. 2, vol. 16 (1892), p. 175.

We now have to determine the covariantive derivative of an invariant constituent (f). In order to avoid however too lengthy formulas we carry the work through at first for the case $n = 3$.

To the terms of the equation

$$\{f\phi\psi\}^{(\lambda)} = \{f^{(\lambda)}\phi\psi\} + \{f\phi^{(\lambda)}\psi\} + \{f\phi\psi^{(\lambda)}\}$$

we apply (75) so that

$$\{f^{(\lambda)}\phi\psi\} = \{f_\lambda\phi\psi\} - \epsilon(\chi a)(fa)\{\chi_\lambda\phi\psi\}.$$

Hence

$$\{f\phi\psi\}^{(\lambda)} = \{f\phi\psi\}_\lambda - \epsilon\,[(fa)\{\chi_\lambda\phi\psi\} + (\phi a)\{f\chi_\lambda\psi\} + (\psi a)\{f\phi\chi_\lambda\}](\chi a).$$

Using now the identity

$$79)\qquad \begin{vmatrix} a_1, & b_1, & c_1, & d_1 \\ a_2, & b_2, & c_2, & d_2 \\ a_3, & b_3, & c_3, & d_3 \\ (au), & (bu), & (cu), & (du) \end{vmatrix} \equiv 0,$$

and setting $a = f$, $b = \phi$, $c = \psi$, $d = \chi_\lambda$, $u = a$ we reduce the quantity in the bracket to

$$\{f\phi\psi\}(\chi_\lambda a).$$

But

$$\{\chi a\}\{\chi_\lambda a\} = \frac{1}{n}\{\chi a\}\{\chi a\}_\lambda,$$

because χ and every one of the a's are equivalent symbols. With reference to (29) and (17) we find now

$$(\chi_\lambda a)(\chi a) = -(n-1)!\,\frac{\beta_\lambda}{\beta},$$

and therefore

$$\beta\{f\phi\psi\}^{(\lambda)} = \beta\{f\phi\psi\}_\lambda + \beta_\lambda\{f\phi\psi\},$$

i. e.,

$$\beta\{f\phi\psi\}^{(\lambda)} = (f\phi\psi)_\lambda.$$

For n variables the proof is quite analogous by using the identity

$$(80)\qquad \begin{vmatrix} a_1^1 & a_1^2 & a_1^3 & \cdots & a_1^n \\ a_2^1 & a_2^2 & a_2^3 & \cdots & a_2^n \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ (a^1u) & (a^2u) & (a^3u) & \cdots & (a^nu) \end{vmatrix} \equiv 0.$$

Thus we have also for the general case of n variables the result

$$\beta\{f\}^{(\lambda)} = (f)_\lambda. \tag{81}$$

In order to determine

$$(f)^{(\lambda)} = \beta^{(\lambda)}\{f\} + \beta\{f\}^{(\lambda)}$$

we have to find the value of $\beta^{(\lambda)}$.

The covariantive differentiation of the equation

$$n!\,\beta^{-2} = \{a\}^2$$

leads to

$$-2n!\,\beta^{-3}\beta^{(\lambda)} = 2\{a\}\{a\}^{(\lambda)}.$$

This reduces by means of (79) to

$$\frac{2}{\beta^2}(a)(a)_{(\lambda)} = 0, \text{ i. e.,}$$

$$\beta^{(\lambda)} = 0. \tag{82}$$

With respect to covariantive differentiation the quantity β represents a constant.

Now we have from (81) and (82) the important result

$$(f)^{(\lambda)} = (f)_\lambda. \tag{83}$$

By means of this and the preceding theorems and definitions the third and higher covariantive derivatives can now be formed without any further difficulty.

One of the advantages of computing with covariantive derivatives lies in the fact that the product of a first and a second covariantive derivative of a symbol vanishes.

Indeed from

$$f_k f^{(il)} = \epsilon f_k \phi_i (fa)_l (\phi a)$$

it follows by means of (33) that

$$f_k f^{(il)} = 0, \tag{84}$$

and also

$$(fu) f^{(ik)} = 0. \tag{85}$$

The following formulas can now easily be established:

$$(f)(f)^{(ik)} = (f)(f)_{ik} = -(f)_i (f)_k. \tag{86}$$

$$[f^{(ik)}(xa) + f_i(xa)_k][fa] = \frac{1}{n}[x^{(ik)}(fa) + x_i(fa)_k][fa]. \quad \text{From (31).} \tag{87}$$

$$\big((xa), f, u\big)_{(d)}[fa] = \frac{1}{n}\big((fa), x, u\big)_{(d)}[fa]. \quad \text{From (43),} \tag{88}$$

where the suffix (d) indicates ordinary or covariantive differentiation with respect to any number of variables.

(89) $[f_i(xa)_k - f_k(xa)_i]_{(d)}[fa] = \frac{1}{n}[x_i(fa)_k - x_k(fa)_i]_{(d)}[fa]$. From (42).

(90) $f^{(ik)}(fa)_\lambda(xa) = f_{ik}(fa)_\lambda(xa)$.

(91) $(fa)^{(ik)}f_l(xa) = (fa)_{ik}f_l(xa)$.

(92) $f_i(fa)_l(fa)_k = \frac{1}{n}x_i(fa)_k(fa)_l - f^{(ik)}(fa)_l(xa)$. From (55).

(93) $f_i(fa)_l(xa)_k = -[f^{(ik)}(fa)_l + f_i(fa)^{(kl)}](xa)$. From (33).

(94) $f_l(fa)^{(ik)}(xa) = -\frac{1}{n}x_l(fa)_i(fa)_k$. From (92) and (93).

(95) $f_i[(fa)_l(xa)_k - (fa)_k(xa)_l] = [f^{(il)}(fa)_k - f^{(ik)}(fa)_l](xa)$. From (92).

(96) $f_k((fa),(xa),u) = (xa)_k((fa),f,u) - \frac{1}{n}(fa)_k((fa),x,u)$. From (92).

(97) $f_k((fa),(xa),u) = (xa)_k((fa),f,u) - (fa)_k((xa),f,u)$. From (96).

(98) $f_k((fa),(xa),u) = (xa)(f(fa),u)_k + (fa)_k(f,(xa),u)$.

(99) $(f\phi u)((fa),(\phi a),v) = (f\phi v)((fa),(\phi a),u)$. From (97).

(100) $n(n-1)[(f_i\phi_k - f_k\phi_i)^{(l)}(xya) + (f_i\phi_k - f_k\phi_i)(xya)_l][f\phi a]$
$= 2[(x_iy_k - x_ky_i)^{(l)}(f\phi a) + (x_iy_k - x_ky_i)(f\phi a)_l][f\phi a]$. From (36).

(101) $\eta(f_i\phi_k - f_k\phi_i)(f\phi a)(xya)_\lambda = 2(x_iy_k - x_ky_i)^{(\lambda)}$. From (100).

(102) $n(n-1)[(f_i\phi_k - f_k\phi_i)^{(\lambda)}(xya) + (f_i\phi_k - f_k\phi_i)(xya)_\lambda](f\phi a)_\mu$
$= 2(x_iy_k - x_ky_i)(f\phi a)_\lambda(f\phi a)_\mu$. From (100).

(103) $[(f_i\phi_k - f_k\phi_i)^{(\lambda)}(xya) + (f_i\phi_k - f_k\phi_i)(xya)_\lambda]$
$= -(xya)(f_i\phi_k - f_k\phi_i)^{(\lambda)}(f\phi a)_\mu$. From (38).

(104) $n(n-1)(f_i\phi_k - f_k\phi_i)^{(\lambda)}(f\phi a)_\mu(xya)$
$= -2(x_iy_k - x_ky_i)(f\phi a)_\lambda(f\phi a)_\mu$. From (102) and (103).

The following formulas involving third covariantive derivatives are of special importance for the next article.

(105) $x^{(irs)} = \epsilon[\phi_i(\phi a)_s(xa)_r + \phi_i(\phi a)(xa)^{(rs)}]$. From (76).

(106) $x^{(irs)} - x^{(isr)} = \epsilon\phi_i[(\phi a)_s(xa)_r - (\phi a)_r(xa)_s]$.

We notice that in third covariantive derivatives the order of differentiation is not arbitrary.

(107) $x^{(irs)} - x^{(isr)} = \epsilon[\phi^{(is)}(\phi a)_r - \phi^{(ir)}(\phi a)_s](xa)$. From (106) and (92).

(108) $x^{(irs)} - x^{(isr)} = \epsilon(\phi^{(irs)} - \phi^{(isr)})(\phi a)(xa)$. From (107) and (84).

(109) $x^{(irs)} - x^{(isr)} = \epsilon^2[(\phi a)_r(\phi b)_s - (\phi a)_s(\phi b)_r]\psi_i(\psi b)(xa)$. From (107).

(110) $x^{(irs)} - x^{(isr)} = \epsilon^2[\psi_s(\phi a)_r - \psi_r(\phi a)_s](\psi b)(\phi b)_i(xa)$. From (107).

(111) $[x^{(irs)} - x^{(isr)}](fa)[fa] = n(f^{(irs)} - f^{(isr)})(xa)[fa]$.

(112) $(f^{(irs)} - f^{(isr)})(xa)(fa)_\lambda = 0$. From (111).

(113) $f_k(f^{(isr)} - f^{(irs)}) = f^{(ir)}f^{(sk)} - f^{(is)}f^{(rk)}$. From (84).

(114) $f^{(k\lambda)}(f^{(irs)} - f^{(isr)}) = 0$. From (76) and (112).

§ 8. *The quadrilinear covariant.*

In § 6 we found a quadratic covariant expression. It can be shown that the lowest covariant proper, i. e. one which does not contain any arbitrary function u, is quadrilinear and of the second order. This covariant occurs in RIEMANN's paper: *Commentatio mathematica*, etc., whose second part is devoted practically to the analytic deduction of the propositions established in the famous paper: *Ueber die Hypothesen welche der Geometrie zu Grunde liegen.* The quadrilinear covariant constitutes the numerator of a fraction which RIEMANN defines as the general measure of curvature.*

CHRISTOFFEL arrived quite independently † at the same covariant; it forms the basis of his deductions of the conditions for the equivalence of two quadratic differential quantics.

The covariant is defined in the four sets of differentials, $d^{(1)}x$, $d^{(2)}x$, $d^{(3)}x$, $d^{(4)}x$ as

(115) $$G_4 = \sum_{ikrs} (ikrs)\, d^{(1)}x_i\, d^{(2)}x_k\, d^{(3)}x_r\, d^{(4)}x_s, \ddagger$$

where $(ikrs)$ is the *quadruple index symbol:*

(116) $$(ikrs) = \frac{\partial \begin{bmatrix} ir \\ k \end{bmatrix}}{\partial x_s} - \frac{\partial \begin{bmatrix} is \\ k \end{bmatrix}}{\partial x_r} + \beta^2 \sum_{lm} A_{lm} \left\{ \begin{bmatrix} is \\ m \end{bmatrix} \begin{bmatrix} rk \\ l \end{bmatrix} - \begin{bmatrix} ir \\ m \end{bmatrix} \begin{bmatrix} sk \\ l \end{bmatrix} \right\}. \S$$

* *Riemann's gesammelte Werke*, herausgegeben von H. WEBER, 2d ed. (Leipzig, 1892), pp. 403, 412.

† In 1869. RIEMANN's paper *Commentatio mathematica* was written in 1861, but not published until 1876, ten years after his death, by WEBER-DEDEKIND.

‡ CHRISTOFFEL, loc. cit., p. 58; BIANCHI, loc. cit., p. 50.

§ RIEMANN, loc. cit., pp. 402, 411; CHRISTOFFEL, loc. cit., p. 54; BIANCHI, loc. cit., p. 51.

If now we introduce symbols, we have

$$(ikrs) = f_{ir}f_{ks} - f_{is}f_{kr} + \frac{\beta^2}{(n-1)!}\sum_{lm} F^{1,l}\cdot F^{1,m}\cdot[f_m f_{is}\phi_l\phi_{rk} - f_m f_{ir}\phi_l\phi_{sk}],$$

which expression, considering that

$$\beta\sum\phi_l F^{1,l} = (\phi f) \text{ and } \beta\sum_m f_m F^{1,m} = (f),$$

is transformed into

$$\begin{aligned}(ikrs) &= f_{ir}f_{ks} - f_{is}f_{kr} + \epsilon(f_{ik}\phi_{kr} - f_{ir}\phi_{ks})(\phi a)(fa)\\ &= f_{ir}[f_{ks} - \epsilon\phi_{ks}(\phi a)(fa)] - f_{is}[f_{kr} - \epsilon\phi_{kr}(\phi a)(fa)]\\ &= f_{ir}f^{(ks)} - f_{is}f^{(kr)}.\end{aligned}$$

This on account of (76) and (85) finally leads to

$$(ikrs) = f^{(ir)}f^{(ks)} - f^{(is)}f^{(kr)}, \tag{117}$$

or also from (113),

$$(ikrs) = f_k(f^{(isr)} - f^{(irs)}). \tag{118}$$

This is the simple symbolic representation of the quadruple index symbol $(ikrs)$. By means of the formulas (105)–(110) we deduce from it the following expressions:

$$(ikrs) = \epsilon f_i\phi_k[(fa)_r(\phi a)_s - (fa)_s(\phi a)_r]. \tag{119}$$

$$(ikrs) = \epsilon(fa)_r(\phi a)_s(f_i\phi_k - f_k\phi_i). \tag{120}$$

$$2(ikrs) = \epsilon(f_i\phi_k - f_k\phi_i)[(fa)_r(\phi a)_s - (fa)_s(\phi a)_r]. \tag{121}$$

$$(ikrs) = \epsilon^2[(\phi b)_r(\phi a)_s - (\phi a)_r(\phi b)_s](\psi b)(fa)\psi_i f_k. \tag{122}$$

$$2(ikrs) = \epsilon^2(fa)(\psi b)(\psi_i f_k - \psi_k f_i)[(\phi b)_r(\phi a)_s - (\phi a)_r(\phi b)_s]. \tag{123}$$

$$(ikrs) = \epsilon^2(\psi b)(fa)[\psi_r(\phi a)_s - \psi_s(\phi a)_r]f_k(\phi b)_i. \tag{124}$$

$$2(ikrs) = \epsilon^2(\psi b)(fa)[\psi_r(\phi a)_s - \psi_s(\phi a)_r][f_k(\phi b)_i - f_i(\phi b)_k]. \tag{125}$$

To each of the parentheses appearing on the right sides of these expressions (119)–(125) we now apply formula (37), and thus we find seven expressions for the coefficients $(ikrs)$ of G_4, which lead at once to as many symbolic representations of G_4, furnishing at the same time the proof that G_4 is a covariant of A.

It will be sufficient to write one of these seven formulas down. From (119) we deduce

$$(ikrs) = \epsilon\eta\big((fa)(\phi a)b\big)(f'\phi' b)f_i\phi_k f'_r\phi'_s, \tag{126}$$

and

$$(n-1)!(n-2)!\,G_4 = \big((fa)(\phi a)b\big)(f'\phi' b)f_{x^1}\phi_{x^2}f'_{x^3}\phi'_{x^4}. \tag{127}$$

§ 9. *Higher covariants.*

For the general theory and for CHRISTOFFEL's algebraic theory in particular there are of paramount importance besides G_4 certain covariants $G_5, G_6, \cdots$ which are linear in $5, 6, \cdots$ sets of differentials. The process by which CHRISTOFFEL deduces the coefficients of $G_{\mu+1}$ from those of G_μ appears rather complicated. We shall see, however, that *this process consists simply in one single covariantive differentiation.*

CHRISTOFFEL denotes the coefficients of G_μ by

$$(i_1 i_2 \cdots i_\mu)$$

and defines the coefficient of $G_{\mu+1}$ by

$$(ii_1 i_2 \cdots i_\mu) = \frac{\partial (i_1 i_2 \cdots i_\mu)}{\partial x_i} - \sum_\lambda \left[\begin{Bmatrix} i_1 i \\ \lambda \end{Bmatrix} (\lambda i_2 \cdots i_\mu) + \begin{Bmatrix} i_2 i \\ \lambda \end{Bmatrix} (i_1 \lambda \cdots i_\mu) + \cdots \right]^* . \tag{128}$$

Let us suppose now that $(i_1 i_2 \cdots i_\mu)$ is representable as a sum of terms of the form

$$T = M u^1_{i_1} u^2_{i_2} \cdots u^\mu_{i_\mu}, \tag{129}$$

where M denotes any quantity for which the law

$$M^{(\lambda)} = M_\lambda$$

holds and where

$$u^1_{i_1} = \frac{\partial u^1}{\partial x_{i_1}} \text{ etc.}$$

Applying now to every T formula (128) we obtain

$$T_i - M \left[u^1_{i_2} \cdots u^\mu_{i_\mu} \sum_\lambda u^1_\lambda \begin{Bmatrix} i_1 i \\ \lambda \end{Bmatrix} + u^1_{i_1} u^3_{i_3} \cdots u^\mu_{i_\mu} \sum u^2_\lambda \begin{Bmatrix} i i_2 \\ \lambda \end{Bmatrix} + \cdots \right],$$

which reduces by means of (76^a) to

$$T_i - M \left[u^1_{i_1 i} u^2_{i_2} \cdots u^\mu_{i_\mu} + u^1_{i_1} u^2_{i_2 i} \cdots u^\mu_{i_\mu} + \cdots - (u^{1(i_1 i)} u^2_{i_2} \cdots u^\mu_{i_\mu} + \cdots) \right],$$

i. e., to $T^{(i)}$ since $M^{(i)} = M_i$.

We have then

$$(ii_1 i_2 \cdots i_\mu) = (i_1 i_2 \cdots i_\mu)^{(i)}. \tag{130}$$

When we apply to the second covariantive derivatives of the u's formula (76) it becomes evident that $T^{(i)}$ consists also of terms of the type (129). The same

* Loc. cit., p. 57.

is therefore true for $(ii_1 i_2 \cdots i_\mu)$. Finally, as (126) shows, $(i_1 i_2 i_3 i_4)$, i. e., our $(ikrs)$ is directly given considering equation (83) in the form (129). Consequently equation (130) is proved for every $\mu > 3$.

For our purpose it is necessary to represent G_μ as a single product of invariantive and covariantive constituents—like G_4 in (127)—i. e., we have to represent the coefficients of G_μ as single products of the form (129) where the factor M consists of a product of invariantive constituents.*

For $\mu = 5$ this representation can be effected as follows: From (130), (118) and (114) we deduce at once for coefficients of G_5 the simple symbolic form

$$(\lambda ikrs) = f_k (f^{(isr\lambda)} - f^{(irs\lambda)}).$$

We obtain by differentiating (119) covariantively

$$(\lambda ikrs) = \epsilon f_i \phi_k [(fa)_r (\phi a)_s - (fa)_s (\phi a)_r]^{(\lambda)} + \epsilon (f_i \phi_k)^{(\lambda)} [(fa)_r (\phi a)_s + (fa)_s (\phi a)_r].$$

The second term on the right side vanishes because, for instance,

$$f^{(i\lambda)} (fa)_r = -f^{(i\lambda r)} (fa) \text{ from (84)},$$

and

$$\phi_k (\phi a)_s (fa) = 0 \text{ from (33)}.$$

From (101) we find

$$[(fa)_r (\phi a)_s - (fa)_s (\phi a)_r]^{(\lambda)} = \eta f'_r \phi'_s (f' \phi' b) \big((fa)(\phi a) b\big)_\lambda,$$

and from (32)

$$\big((fa)(\phi a) b\big)_\lambda = 2 f''_\lambda (f'' c) \Big(\big((fa)(\phi a) b\big) c\Big),$$

hence

$$(131) \qquad (\lambda ikrs) = \epsilon^2 \eta \Big(\big((fa)(\phi a) b\big) c\Big) (f' \phi' b)(f'' c) f_i \phi_k f'_r \phi'_s f''_\lambda,$$

and

$$(132) \qquad (n-1)!\,(n-1)!\,(n-2)!\; G_5 = \Big(\big((fa)(\phi a) b\big) c\Big) (f' \phi' b)(f'' c) f_{x^1} \phi_{x^2} f'_{x^3} \phi'_{x^4} f''_{x^5}.$$

§ 10. *The three invariants of the second order for $n = 3$.*

In the case $n = 2$ there exists only one invariant (proper) of the second order, viz., the Gaussian curvature; for $n = 3$ there are three invariants,† viz.,

---

* I have not yet been able to overcome this difficulty in a satisfactory manner for $\mu > 5$.

† Compare HASKINS, *On the invariants of quadratic differential forms*, Transactions of the American Mathematical Society, vol. 3 (1902), p. 86, footnote.

$$H_1 = \frac{|B_{11}A_{22}A_{33}| + |A_{11}B_{22}A_{33}| + |A_{11}A_{22}B_{33}|}{|A_{11}A_{22}A_{33}|},$$

$$H_2 = \frac{|A_{11}B_{22}B_{33}| + |B_{11}A_{22}B_{33}| + |B_{11}B_{22}A_{33}|}{|A_{11}A_{22}A_{33}|},$$

$$H_3 = \frac{|B_{11}B_{22}B_{33}|}{|A_{11}A_{22}A_{33}|},$$

where the A_{ik} are the quantities (18) and the B_{ik} the quadruple index symbols, viz.:

$$B_{11} = (2323), \qquad B_{12} = (2331), \qquad B_{13} = (2312),$$
$$B_{22} = (3131), \qquad B_{23} = (3112), \qquad B_{33} = (1212).$$

We want to find the symbolic representation of H_1, H_2, H_3.

From

$$|A_{11}A_{22}A_{33}| = \beta^{-4} \qquad \text{and} \qquad \begin{vmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{vmatrix} = \beta^{-2}a_{11}$$

we obtain for the first term of H_1

$$\beta^2 \sum_k a_{1k}B_{1k},$$

which by means of (120) becomes equal to

$$\tfrac{1}{2}\beta^2(fa)_2(\phi a)_3 \begin{vmatrix} f_1 & f_2 & f_3 \\ \phi_1 & \phi_2 & \phi_3 \\ a_{11} & a_{12} & a_{13} \end{vmatrix};$$

and by replacing the a_{ik} by the symbolic products $b_i b_k$, equal to

$$\tfrac{1}{2}\beta b_1(fa)_2(\phi a)_3(f\phi b).$$

Thus we have

$$2H_1 = \beta[b_1(fa)_2(\phi a)_3 + b_2(fa)_3(\phi a)_1 + b_3(fa)_1(\phi a)_2](f\phi\psi),$$

which leads at once to the required representation

$$4H_1 = \big((fa)(\phi a)b\big)(f\phi b). \tag{133}$$

For the computation of H_2 and H_3 we need the following theorem on determinants. The notations P_{ik}, etc., denoting the minors of the elements p_{ik}, etc., in the determinants

$$|p_{ik}|, \qquad |q_{ik}|, \qquad |r_{ik}|$$

the relation holds

$$\begin{vmatrix} P_{11} & P_{12} & P_{13} \\ Q_{11} & Q_{12} & Q_{13} \\ R_{11} & R_{12} & R_{13} \end{vmatrix} = \begin{vmatrix} p_{21} & p_{22} & p_{23} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{vmatrix} \cdot \begin{vmatrix} p_{31} & p_{32} & p_{33} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{vmatrix} - \begin{vmatrix} p_{21} & p_{22} & p_{23} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{vmatrix} \cdot \begin{vmatrix} p_{31} & p_{32} & p_{33} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{vmatrix}. \tag{134}$$

In $$H_2 = \beta^4 \{ |A_{11} B_{22} B_{33}| + |B_{11} A_{22} B_{33}| + |B_{11} B_{22} A_{33}| \}$$

we now replace A_{11} by $f_2\phi_3(f_2\phi_3 - f_3\phi_2)$, etc.,

and the B_{ik} by the quantities given above where, however, we have here to use two different sets of symbols: f', ϕ', a' and f'', ϕ'', a''.

Using the abbreviation

$$(fa)' = (f'a'), \text{ etc.},$$

we find

$$4|A_{11} B_{22} B_{33}| = \beta^4 (f_2\phi_3 - f_3\phi_2)[(fa)'_3(\phi a)'_1 - (fa)'_1(\phi a)'_3]$$
$$\times [(fa)''_1(\phi a)''_2 - (fa)''_2(\phi a)''_1] \cdot \begin{vmatrix} f_2\,\phi_3, & f_3\,\phi_1, & f_1\,\phi_2 \\ f'_2\,\phi'_3, & f'_3\,\phi'_1, & f'_1\,\phi'_2 \\ f''_2\,\phi''_3, & f''_3\,\phi''_1, & f''_1\,\phi''_2 \end{vmatrix},$$

and from this

$$8H_2 = \beta^4 \begin{vmatrix} f_2\phi_3 - f_3\phi_2, & f_3\phi_1 - f_1\phi_3, & f_1\phi_2 - f_2\phi_1 \\ (fa)'_2(\phi a)'_3 - (fa)'_3(\phi a)'_2, & \cdots, & \cdots \\ (fa)''_2(\phi a)''_2 - (fa)''_3(\phi a)''_2, & \cdots, & \cdots \end{vmatrix}$$
$$\times \begin{vmatrix} f_2\,\phi_3, & f_3\,\phi_1, & f_1\,\phi_2 \\ f'_2\,\phi'_3, & f'_3\,\phi'_1, & f'_1\,\phi'_2 \\ f''_2\,\phi''_3, & f''_3\,\phi''_1, & f''_1\,\phi''_2 \end{vmatrix}$$

which can be transformed by proper permutation of equivalent symbols into another expression where the elements $f_2\phi_3$, etc., of the second determinant are replaced by $f_2\phi_3 - f_3\phi_2$, etc. Applying now (134) we have

$$64H_2 = \big[\big((fa)'(\phi a)'f\big)\big((fa)''(\phi a)''\phi\big)$$
$$- \big((fa)'(\phi a)'\phi\big)\big((fa)''(\phi a)''f\big)\big][(f'\phi'f)(f''\phi''\phi)$$
$$- (f'\phi'\phi)(f''\phi''f)],$$

and permuting f and ϕ in the last term

$$32H_2 = \big[\big((fa)'(\phi a)'f\big)\big((fa)''(\phi a)''\phi\big)$$
$$- \big((fa)'(\phi a)'\phi\big)\big((fa)''(\phi a)''f\big)\big](f'\phi'f)(f''\phi''\phi). \tag{135}$$

By using (62) for the transformation of the bracket we obtain the final form:

$$32H_2 = \big((fa)'(\phi a)'b'\big)\big((fa)''(\phi a)''b''\big)(bb'b'')(bf\phi)(ff'\phi')(\phi f''\phi''). \tag{136}$$

To compute the third invariant

$$H_3 = \beta^4 |B_{11} B_{22} B_{33}|$$

we take formula (126) for the B_{ik}. On setting, temporarily

$$(\psi\chi b)(\psi'\chi' b')(\psi''\chi'' b'') = (\psi\chi b)_{012}$$

and using the notation

$$\big((fa)(\phi a)b\big)_{012}$$

in a similar sense, we find

$$8H_3 = \beta^4\big((fa)(\phi a)b\big)_{012}\cdot(\psi\chi b)_{012}$$

$$(137)\qquad \cdot\psi_2\chi_3\psi_3'\chi_1'\psi_1''\chi_2'' \begin{vmatrix} f_2\phi_3 - f_3\phi_2, & f_3\phi_1 - f_1\phi_3, & f_1\phi_2 - f_2\phi_1 \\ f_2'\phi_3' - f_3'\phi_2', & \cdots, & \cdots, \\ f_2''\phi_3'' - f_3''\phi_2'', & \cdots, & \cdots, \end{vmatrix}.$$

In this expression we can replace

$$(\psi\chi b)_{012}\cdot\psi_2\chi_3\psi_3'\chi_1'\psi_1''\chi_2''$$

by

$$(138)\qquad \tfrac{1}{6}(\psi\chi b)_{012}\begin{vmatrix} \psi_2\chi_3, & \psi_3\chi_1, & \psi_1\chi_2 \\ \psi_2'\chi_3', & \psi_3'\chi_1', & \psi_1'\chi_2' \\ \psi_2''\chi_3'', & \psi_3''\chi_1'', & \psi_1''\chi_2'' \end{vmatrix}.$$

But

$$\psi_2\chi_3(\psi\chi b) = \beta(b_1A_{11} + b_2A_{12} + b_3A_{13}),$$

hence (138) becomes equal to

$$\tfrac{1}{6}\beta^3|A_{ik}|\cdot\frac{1}{\beta}(bb'b'') = \frac{1}{6\beta^2}(bb'b'').$$

We find now by applying theorem (134) to the determinant in (137)

$$48H_3 = \big((fa)(\phi a)b\big)_{012}(bb'b'')\,[(ff'\phi')(\phi f''\phi'') - (ff''\phi'')(\phi f'\phi')],$$

and finally by permuting f and ϕ in the last term

$$(139)\quad 24H_3 = \big((fa)(\phi a)b\big)\big((fa)'(\phi a)'b'\big)\big((fa)''(\phi a)''b''\big)(ff'\phi')(\phi f''\phi'')(bb'b'').$$

In (133), (136) and (139) we have now the symbolic representation of the three ternary invariants H_1, H_2, H_3.

§ 11. *The three simplest general invariants of the second order.*

The formula for the ternary invariant H_1 can at once be generalized for n variables by letting the letter b stand for the $n-2$ symbols $b^3, b^4, \cdots b^n$. I denote this invariant by R_1, for a reason which will appear presently.

$$(140)\qquad R_1 = \big((fa)(\phi a)b\big)(f\phi b).$$

We have then

$$H_1 = 4R_1 \qquad \text{for} \qquad n = 3,$$

and $2K = R_1$ for $n = 2\,(b = 0)$, where K denotes the Gaussian curvature. To transform H_2 first, we start from (135) and multiply through by $(f'\phi'f)(f''\phi''\phi)$. The first term is then simply equal to R_1^2 and with a slight change of notation in the second term we have

$$32H_2 = R_1^2 - \big((fa)(\phi a)b\big)\big((fa)'(\phi a)'b'\big)(f\phi b')(f'\phi'b).$$

The invariant representing the second term can now be generalized at once by letting the letters b' and b'' run out. We denote it by R_2.

$$R_2 = \big((fa)(\phi a)b\big)\big((fa)'(\phi a)'b'\big)(f\phi b')(f'\phi'b). \tag{141}$$

We have then

$$32H_2 = R_1^2 - R_2 \qquad \text{for} \qquad n = 3.$$

For the transformation of H_3 we deduce from the identity (79) the equation

$$(bb'b'')(ff'\phi') = (bb'f)(b''f'\phi') + (b'b''f)(bf'\phi') + (b''bf)(b'f'\phi'), \tag{142}$$

and also

$$(bb'f)(\phi f''\phi'') = (b'f\phi)(bf''\phi'') - (bf\phi)(b'f''\phi'') + (bb'\phi)(ff''\phi''). \tag{143}$$

Multiplying now (142) by $(\phi f''\phi'')$ and applying formula (143) to the first term of the product, we can, after substitution into (139), permute f and ϕ in the term arising from the last term of (143). By this permutation the last term of (143) becomes equal to the negative term on the left side, so that in the complete expression of H_3 we can use instead of (143) the equation

$$2(bb'f)(\phi f''\phi'') = (b'f\phi)(bf''\phi'') - (bf\phi)(b'f''\phi''),$$

and two others obtained from it by cyclical permutations of b, b', b''. Combining, we can now in H_3 use the equation

$$\begin{aligned}2(bb'b'')(ff'\phi')(\phi f''\phi'') &= (bf''\phi'')(b'f\phi)(b''f'\phi') \\ &\quad + (bf'\phi')(b'f''\phi'')(b''f\phi) + (bf\phi)(b'f'\phi')(b''f''\phi'') \\ &\quad - [(bf\phi)(b'f''\phi'')(b''f'\phi') + (bf'\phi')(b'f\phi)(b''f''\phi'') \\ &\qquad + (bf''\phi'')(b'f'\phi')(b''f\phi)].\end{aligned}$$

This leads to

$$\begin{aligned}48H_3 = \big((fa)(\phi a)b\big)_{012}[(f\phi b')(f'\phi'b'')(f''\phi''b) \\ + (f\phi b'')(f'\phi'b)(f''\phi''b')] + R_1^3 - 3R_1R_2.\end{aligned}$$

The second term becomes equal to the first by the interchange of the quantities with one and two accents. This gives rise to the introduction of the following invariant R_3 which now also holds for every n if we again let the letters b, b', b'', run out.

$$(144)\quad R_3 = \big((fa)(\phi a)b\big)\big((fa)'(\phi a)'b'\big)\big((fa)''(\phi a)''b''\big)(f\phi b')(f'\phi' b'')(f''\phi'' b)\,.$$

We have now

$$48H_3 = 2R_3 + R_1^3 - 3R_1R_2 \qquad \text{for} \qquad n = 3\,.$$

In R_1, R_2, R_3, *given by* (140), (141) *and* (144) *we have then the three simplest invariants of* n *variables.*

THE UNIVERSITY OF CHICAGO,
June, 1903.