DENUMERANTS OF DOUBLE DIFFERENTIANTS*

BY

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I. *Introduction.*

1.† In 1856 ‡ CAYLEY gave (without rigorous proof) a formula for the number of linearly independent yx-differentiants (as I call them) of a given type belonging to a single binary quantic,—a formula that he derived from the *assumption* that the necessary and sufficient conditions, as they naturally appear, are linearly independent; this formula was first proved in 1877 § by SYLVESTER, who extended it to a system of any number of binary quantics. The formula as originally given by CAYLEY expresses the number in question as the excess of the number of terms in the general form of the type of the differentiants over the number of terms in the general form of a certain other type, precisely as in formula (37) of this paper (when x and y are the only variables); but the number of terms in a general *binary* form is readily expressible as the sum of the numbers of partitions of a certain kind of certain numbers, and it was the formula in terms of such partition-numbers that SYLVESTER proved and extended. About 1885 it occurred to me, while giving a course of lectures on invariants at the Johns Hopkins University, that CAYLEY's original formula and SYLVESTER's proof of it,—indeed, the latter's extension of it, when expressed in the original terms,—are valid for yx-differentiants (*simple* differentiants, as I call them below), even if x and y are only two out of *any number* of variables. In 1892 || I discovered operators that produce all the differentiants of a given type and certain combinations of characters, namely, all forms that are at once yx-, zx-, sx-, $\cdots$, vx-differentiants and all forms that are at once xy-, xz-, xs-, $\cdots$, xv-differentiants. Suspecting that the method by which these operators were obtained could be utilized for the discovery and proof of a formula for the number of linearly independent forms of any given type that possess any given combination of differentiant characters, I have devoted much time during the

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† I shall refer to these numbered *sections* as § 1, etc. For Table of Contents see p. 70.

‡ *A Second Memoir upon Quantics*, Philosophical Transactions, London, vol. 146, p. 107.

§ Philosophical Magazine, ser. 5, vol. 5, March, 1878.

|| Mathematische Annalen, vol. 41 (1893), pp. 485–489; Proceedings of the London Mathematical Society, vol. 23 (1892), pp. 265-272.

last fourteen years to attempts to obtain such formulæ. I have succeeded in the cases of *double* differentiants of all kinds, as the present paper will show; the extension of the formulæ to triple and higher differentiants is almost self-evident, but the proof of the extended formulæ presents a difficulty that I have not yet overcome.

When I began these attempts I was not aware of what DERUYTS had done in a similar line;* but his work differs from mine in essential respects, both as to methods and results. He has, to be sure, given a formula for the denumerant of the semi-invariants (chain-differentiants of order 0 — that is, not involving the variables explicitly) which is *formally* identical with the formula (51) of this paper for three variables, but my formula is proved for xyz-differentiants of *any* type, even if this type implies any number of variables other than x, y and z, and the explicit presence of all these variables in the differentiant, while his formula supposes that the variables do not enter explicitly and that x, y, and z are the only variables in the system. Moreover, I have obtained formulæ for other combinations of differentiant characters not considered by DERUYTS. But the most important differences in our work will be found in the methods used. DERUYTS has employed the methods of "sources" and of "symbolic representation," while my main object has been to obtain formulæ by direct consideration of the "actual" forms, without recourse to those (from my point of view) indirect methods. At all events, it must be for the advantage of mathematical science that various methods should be exploited.

2. We assume a system of quantics (homogeneous polynomials), which we shall call simply *the quantics*, finite in number and of finite orders m, m', m'', $\cdots$, respectively, in a finite number n of variables $x, y, z, \cdots$. Any such quantic of order m is assumed in the form

$$(1) \qquad \sum \frac{m!}{g!\ h!\ i!\ \ldots} a_{g,h,i\ldots}\, x^g y^h z^i \cdots,$$

where the summation extends to all integral values (positive and 0) of $g, h, i, \cdots$, *whose sum is* m. For convenience, we speak of $a_{g,h,i,\ldots}$ freed from the corresponding polynomial coefficient as a *coefficient* of the quantic. If it were necessary to represent the coefficients of the different quantics, we should denote those of the first quantic by a, those of the second by b, those of the third by c, etc., each with its proper suffixes;† the coefficients of the quantics will be used as so many sets of additional variables (though not so called) having no determined numerical values. To each coefficient is assigned a *weight* in each var-

* *Essai d'une théorie générale des formes algébriques*, Mémoires de la société royale des sciences de Liège, ser. 2, vol. 17 (1891).

† Of course the sum of the suffixes of any coefficient is equal to the order of the quantic to which that coefficient belongs.

iable, namely, the *weight of any coefficient in either variable* is the sum of all its suffixes excepting that which denotes the power of the variable in question in the term of the quantic to which the coefficient belongs; thus, in (1) the weight of $a_{g,h,i,\dots}$ in z is $m-i$. The *weight of any variable* in itself is 1 and its weight in any other variable is 0. The *weight of any product* of variables and coefficients in any variable is the sum of the weights of its factors in that variable; thus the weight of any power of a variable or coefficient is the weight of that variable or coefficient multiplied by the exponent of the power.

The *order* of any product of powers of variables and coefficients is the sum of the exponents of the powers of the variables and its *degree* in the coefficients of either quantic is the sum of the exponents of the powers of such coefficients in it. If the terms of a polynomial in the variables and coefficients of the quantics are all of one order and of one degree in the coefficients of *each* quantic of the system (even if of different degrees in the coefficients of the different quantics), the polynomial is *homogeneous* of that order and those degrees. If the terms of a polynomial are all of one weight in each variable (even if of different weights in the different variables), the polynomial is *isobaric* of those weights in the several variables. *Any polynomial is a sum of homogeneous isobaric polynomials.* The order, degrees in the coefficients of the several quantics, and weights in the several variables, of a homogeneous isobaric polynomial, or *form*, together constitute the *type* of that form. One consequence (and advantage) of the definition of *weight* here given (which has not been adopted by all writers) is that each of the quantics is isobaric in each of the variables of a weight equal to its order. The numbers that characterize the type of any form are connected by a simple relation; namely, if the quantics are of orders $m, m', m'', \cdots$, respectively, in n variables, $x, y, z, \cdots$, and any form of degrees $j, j', j'', \cdots$, in the coefficients of these quantics, respectively, and of order ϑ in the variables, is of weights $w_x, w_y, w_z, \cdots$ in these variables, respectively, then

$$(2) \qquad w_x + w_y + w_z + \cdots = (n-1)\sum mj + \vartheta,$$

where $\sum mj = mj + m'j' + m''j'' + \cdots$; so that the weight in either variable is completely determined when the other characteristics of its type are given.

If a linear transformation is imposed upon the variables, that is, if the n variables $x, y, z, \cdots$ are expressed as homogeneous linear functions of n new variables $x', y', z', \cdots$, and these expressions are substituted for $x, y, z, \cdots$ in any one of the quantics, this becomes a quantic of the same order in $x', y', z', \cdots$, whose coefficients are homogeneous linear functions of the original coefficients of the quantic and homogeneous functions of the parameters of transformation of a degree equal to the order of the quantic: the coefficients of the new quantic in $x', y', z', \cdots$ are the *transformed coefficients* of the quantic in question. It is assumed that the determinant of the parameters of transformation is not 0, so that the new variables x', y', z', $\cdots$ can be expressed as homogeneous linear functions of the old variables x, y, z, $\cdots$. Any polynomial in the variables and the coefficients of the quantics is transformed by replacing the variables by the expressions of the corresponding new variables in terms of the old ones and the coefficients of the quantics by the expressions of the corresponding transformed coefficients in terms of the original coefficients. A homogeneous polynomial is thus transformed into a homogeneous polynomial of the same order and the same degrees; but the weights are not generally preserved by an arbitrary linear transformation,—indeed an isobaric form does not remain isobaric in general.

The linear transformation by which the variable x is increased by a multiple of the variable y, that is, the transformation $x = x' + \lambda y'$, $y = y'$, $z = z'$, $\cdots$, shall be called a *yx-shear;* and similarly for other pairs of variables. A shear involves a single parameter (λ, in the example just given), which we shall generally suppose to have an arbitrary value. A polynomial in the variables and the coefficients of the quantics that is not altered by an *arbitrary* yx-shear (that is, by a yx-shear whose parameter is arbitrary) shall be called a *yx-differentiant*, and similarly for other pairs of variables. A polynomial subject to only one such condition is a *simple* differentiant; if subject to more than one such condition, it is a *multiple* differentiant, namely, double, triple, etc., according to the number of *independent* conditions of this kind to which it is subject (we shall see that certain combinations of simple differentiant conditions imply others). The aggregate of differentiant conditions to which a polynomial is subject constitute its *character*. The object of this paper is the determination of the number of linearly independent *double* differentiants of a given type, more specifically, the number of xy- and xz-differentiants, of xz- and yz-differentiants, of xy- and yz-differentiants (which are, as we shall see, also xz-differentiants), and of xy- and zs-differentiants. There are as many kinds of simple shears and, therefore, of simple differentiants and simple characters as there are pairs of variables, if we have regard to the sequence of variables in the pairs (so that xy and yx are regarded as different pairs), that is $n(n-1)$ kinds.

A very important class of multiple differentiants is that defined by a set of simple characters such that the second letter of each defining pair of variables is the same as the first letter of the next pair, when the pairs are taken in the proper sequence; such differentiants may be called *chain-differentiants* (SYLVESTER called them "seminvariants"). Any chain-differentiant character is completely determined by the letters in it and their sequence in the chain, so that it may be unambiguously designated by this sequence; thus an xy-, yz-, zs-, $\cdots$, tu-differentiant may be called simply an $xyzs \cdots tu$-differentiant. Any simple differentiant character may be regarded as a chain-differentiant character of two letters. Of course, the number of letters in any given differentiant character has no particular relation to the whole number of variables in the quantics and differentiants in question.

The name "differentiant" is derived from the fact that a yx-differentiant not explicitly involving the variables is a function of the *differences* of the roots of the equations in $x:y$ formed by equating the quantics to 0, if x and y are the only variables ($n = 2$).

3. The result of imposing the yx-shear $x = x' + \lambda y'$, $y = y'$, $z = z'$, $\cdots$ on any polynomial ϕ is

$$\phi + \lambda \cdot \widehat{yx} \cdot \phi + \frac{\lambda^2}{2!} \cdot \widehat{yx}^2 \cdot \phi + \frac{\lambda^3}{3!} \cdot \widehat{yx}^3 \cdot \phi + \cdots, \tag{3}$$

the operator $\widehat{yx}$ being defined by *

$$\widehat{yx} \cdot \phi \equiv \sum\sum h \cdot a_{g+1,\, h-1,\, i, \ldots} \cdot \frac{\partial \phi}{\partial a_{g,\, h,\, i, \ldots}} - y \cdot \frac{\partial \phi}{\partial x}, \tag{4}$$

where the double summation extends to the coefficients $a_{g,\,h,\,i,\ldots}$, $b_{g,\,h,\,i,\ldots}$, $c_{g,\,h,\,i,\ldots}$, $\cdots$ of all the quantics and to all values of the suffixes of the coefficients of each quantic whose sum is the order of that quantic. From (3) it is evident that the necessary and sufficient condition that ϕ shall be a yx-differentiant is the identity

$$\widehat{yx} \cdot \phi \equiv 0\,; \tag{5}$$

that is, the condition is that the multipliers of the different products of powers of the variables and of the coefficients of the quantics that occur in $\widehat{yx} \cdot \phi$ (these multipliers are homogeneous linear functions of the multipliers of such products in ϕ) shall all be 0, a condition that gives as many homogeneous linear equations between the multipliers of the terms of ϕ as there are terms of $\widehat{yx} \cdot \phi$, but it does not yet appear that these equations are linearly independent. From (4) it appears that $\widehat{yx}$ leaves unchanged the order and degrees of any *form*, as well as its weights in all the variables excepting x and y, diminishes the weight in x by 1, and increases the weight in y by 1, so that (5) must be satisfied by each of the homogeneous isobaric parts of ϕ of different types if it is satisfied by ϕ as a whole; that is, every yx-differentiant is a sum of homogeneous isobaric yx-differentiants, — and the same is true of differentiants of any simple or multiple character. It appears from (5) that a yx-differentiant is a polynomial that is annihilated by the *shear-operator* $\widehat{yx}$.

Being concerned with linearly independent differentiants alone, we shall henceforth assume that every polynomial with which we have to do is homogeneous and isobaric, that is, is a *form*. Moreover, the forms to be considered at any one time will all be obtained by applying operators analogous to $\widehat{yx}$ to forms of one and the same given type; the orders and degrees of all forms thus obtained

* The sign $\equiv$ is used throughout this paper to denote *identity*.

are the same, as are also the weights in the several variables excepting those variables that occur in the shear-operators used in obtaining the forms, and the sum of the weights in these excepted variables is the same for all the forms. In giving the type of a form it will, then, suffice to give its weights in those variables in which the forms to be considered are not all of the same weight; thus, if the forms considered are all of a given order, of given degrees, and of given weights in all the variables excepting x, y, and z,—so that the weights in these three variables are the only characteristics of the type that vary from form to form,—the type of any particular form may be represented by (w_x, w_y, w_z), where w_x, w_y, w_z are its weights in x, y, z, respectively; and similarly for any number of variables. It is to be observed that the expression of the type of a form by its weights in certain variables does not imply anything with regard to the whole number of variables, but only that the weights in all the variables that do not enter into this expression of the type are the same for all the forms considered. To avoid confusion, we suppose the variables to be arranged in a certain sequence x, y, z, $\cdots$ and we write the weights in any expression of a type in the corresponding sequence.

4. By a *complete* system of forms of given type or types and given differentiant characters, subject to other conditions or not, we mean a system of such *linearly independent* forms satisfying the given conditions that *any* form of the given type or types satisfying the conditions can be expresed as a linear function of the forms of the system. The forms of any complete system of *one* type can, evidently, be replaced by the same number of linearly independent linear functions of them, chosen arbitrarily, without affecting the completeness of the system or the character of its forms (because the shear-operators are distributive over their operands). By the *rank* of any form relatively to any given shear-operator we mean the exponent of the highest power of that operator that does not annihilate the form; thus, ϕ is of rank r quâ* $\widehat{xy}$ if $\widehat{xy}^r \cdot \phi \not\equiv 0$ but $\widehat{xy}^{r+1} \cdot \phi \equiv 0$. A complete system of forms of a given type and given characters shall be called *reduced* quâ $\widehat{xy}$ if its forms are so taken as to include the greatest possible number of forms of rank not greater than r quâ $\widehat{xy}$ for *each* value of r from 0 to the highest rank of any form † of the system, inclusive. A complete system of forms of a given type and given characters reduced quâ $\widehat{xy}$ can be constructed thus: out of the (N) forms ψ of *any* complete system of the given type and characters construct the greatest possible number (N_0) of linearly independent linear functions ϕ_0 of rank 0 quâ $\widehat{xy}$ and select as many $(N'_0 = N - N_0)$ other forms ψ_0 of the system that are linearly independent of

* *Quâ* signifies relatively to, with respect to, according to.

† We employ this abbreviated expression instead of the more cumbrous "rank of the forms that are of highest rank."

each other and the ϕ_0's as may be necessary to constitute a complete system with the ϕ_0's; out of these (N_0') other forms ψ_0 construct the greatest possible number (N_1) of linearly independent linear functions ϕ_1 of rank 1 quâ $\widehat{xy}$ and select as many $(N_1' = N_0' - N_1)$ other forms ψ_1 from among the ψ_0's that are linearly independent of each other and the ϕ_1's as may be necessary to constitute a complete system with the ϕ_0's and ϕ_1's; out of the (N_1') forms ψ_1 construct the greatest possible number (N_2) of linearly independent linear functions ϕ_2 of rank 2 quâ $\widehat{xy}$ and select as many $(N_2' = N_1' - N_2)$ other forms ψ_2 from among the ψ_1's that are linearly independent of each other and the ϕ_2's as may be necessary to constitute a complete system with the ϕ_0's, ϕ_1's and ϕ_2's; and so proceed until the forms ψ_{k-1} last selected are all expressible as linear functions of the forms ϕ_k last constructed. The forms $\phi_0, \phi_1, \phi_2, \cdots, \phi_k$ thus constructed constituted a complete system of forms of the given type and given characters reduced quâ $\widehat{xy}$. Of course, the forms ψ_{k-1} are forms ϕ_k.

The peculiarity of a system *reduced* quâ $\widehat{xy}$ is that any linear function of the forms of such a system is of exactly the rank quâ $\widehat{xy}$ of the forms of highest rank in it. For no linear function of forms of a system reduced quâ $\widehat{xy}$ that are all of the rank r can be annihilated by $\widehat{xy}^r$ (otherwise, the system would not contain the greatest possible number of forms of rank less than r, that is, would not be reduced), but every linear function whose terms are all of the rank r is annihilated by $\widehat{xy}^{r+1}$; so that the result of applying $\widehat{xy}^r$ to a linear function of forms of the reduced system of which some are of rank r and others of rank less than r is the same as the result of applying $\widehat{xy}^r$ to the aggregate of those terms of the linear function whose rank is r alone, and this result is not identically 0.

If ϕ is a form of rank r quâ $\widehat{xy}$, then $\widehat{xy}^r \cdot \phi \not\equiv 0$ but $\widehat{xy}^{r+1} \cdot \phi \equiv 0$, that is, $\widehat{xy}^r \cdot \phi$ is an xy-differentiant (of rank 0 quâ $\widehat{xy}$). The form derived from any form ϕ by operating upon it with the power of $\widehat{xy}$ whose exponent is the rank quâ $\widehat{xy}$ of the form ϕ in question shall be called the *derivative* quâ $\widehat{xy}$ of that form ϕ; the derivative quâ $\widehat{xy}$ of any form is, then, an xy-differentiant, and the derivatives quâ $\widehat{xy}$ of the forms of a system of *one* type *reduced* quâ $\widehat{xy}$ are *linearly independent* xy-differentiants (namely, the derivatives of forms of the same type that are of different ranks are themselves of different types, so that a linear relation between the forms of a reduced system implies a linear relation between the derivatives of forms that are of one rank, and this again implies a linear function of these forms that is of lower rank than they, which is impossible, as above shown).

5. In the present investigation we are concerned chiefly with what we may call *denumerants*, that is, expressions for the numbers of linearly independent forms of given types that satisfy certain conditions. A denumerant shall be

represented by N followed by the type with the conditions attached to the N as suffixes and exponents. Conditions shall be expressed thus:

that the forms in question have certain differentiant characters shall be indicated by the pairs and sequences of variables that define these characters, attached to N as suffixes (thus, the suffix xy shall indicate that the forms are xy-differentiants);

that the forms are of given ranks relatively to certain shear-operators in a system *reduced* relatively to each of these operators shall be indicated by the highest powers of the respective operators that do not annihilate the forms, attached to N as exponents (thus, the exponent $\widehat{xy}^r$ indicates that the forms are of rank r quâ $\widehat{xy}$ in a system *reduced* quâ $\widehat{xy}$);

other conditions will generally be attached to N as suffixes;

several suffixes or exponents attached to the same N shall be separated by commas; and

the absence of suffix or exponent shall indicate that no condition such as would generally be written in that place is imposed.

Thus,

$N(w_x, w_y, \cdots)$ is the number of linearly independent forms of the type $(w_x, w_y, \cdots)$, which is also the number of terms in the *general* form of this type;

$N_{xy}(w_x, w_y, \cdots)$ is the number of linearly independent xy-differentiants of the type $(w_x, w_y, \cdots)$;

$N_{xy,\, xz}(w_x, w_y, w_z, \cdots)$ is the number of linearly independent xy- and xz-differentiants of the type $(w_x, w_y, w_z, \cdots)$;

$N^{\widehat{xy}^r}(w_x, w_y, \cdots)$ is the number of forms of a complete system of type $(w_x, w_y, \cdots)$ *reduced* quâ $\widehat{xy}$ that are of rank r quâ $\widehat{xy}$;

$N^{\widehat{yz}^r}_{xy,\, xz}(w_x, w_y, w_z, \cdots)$ is the number of xy- and xz-differentiants of rank r quâ $\widehat{yz}$ in a complete system of type $(w_x, w_y, w_z, \cdots)$ *reduced* quâ $\widehat{yz}$;

$N_{xyz}(w_x, w_y, w_z, \cdots)$ is the number of linearly independent xyz-differentiants of the type $(w_x, w_y, w_z, \cdots)$;

$N_K(w_x, w_y, \cdots)$ is the number of linearly independent forms of the type $(w_x, w_y, \cdots)$ that satisfy the conditions K (whatever they may be);

$N_{K,\, xy}(w_x, w_y, \cdots)$ is the number of linearly independent xy-differentiants of the type $(w_x, w_y, \cdots)$ that satify the further conditions K; etc. Evidently, the exponent $\widehat{xy}^0$ is equivalent to the suffix xy, so that

$N^{\widehat{yz}^0}_{xy,\, xz}(w_x, w_y, w_z) = N_{xyz}(w_x, w_y, w_z)$.

The conditions to which the forms under consideration are subject are generally that the forms shall have certain differentiant characters and that certain numerical characteristics (rank, etc.) of the forms shall have given values or values lying between given limits; it will be convenient to speak of the former as *differentiant* conditions and of the latter as *numerical* conditions; there can even be no objection to speaking of the differentiant characters assumed and the values or limitations of value of the numerical characteristics as themselves *conditions*, because they determine the conditions.

If ϕ is a form of type $(w_x, w_y, \cdots)$, $\widehat{xy}\cdot\phi$ is a form of type $(w_x+1, w_y-1, \cdots)$, — provided it is not identically 0; accordingly, using the symbol $\overline{xy}$ to represent the operator that changes the *type* of any form ϕ into the *type* of $\widehat{xy}\cdot\phi$, we have (as definition of the symbol $\overline{xy}$).

$$\overline{xy}\cdot(w_x, w_y, \cdots) = (w_x+1, w_y-1, \cdots),$$

so that

$$\overline{xy}^k\cdot(w_x, w_y, \cdots) = (w_x+k, w_y-k, \cdots).$$

The types obtained from any given type by applying powers of $\overline{xy}$ whose exponents are positive integers or 0 shall be called types *subordinate* to the given type quâ $\widehat{xy}$; thus, the types $(w_x+k,\ w_y-k, \cdots)$ for all positive integral values of k and 0 are types *subordinate* to $(w_x, w_y, \cdots)$. Of course the same notation and nomenclature applies to the *type-operators* $\overline{xy}$, etc., that correspond to *all* shear-operators $\widehat{xy}$, etc.

Any denumerant may be regarded as a *function of the type*, and we shall apply the type-operators $\overline{xy}$, etc., to any denumerant as to a function of the type, observing that they affect only the *type*, but neither the *suffixes* nor the *exponents* of N; thus,

$$\overline{xy}^k\cdot N_{xz}(w_x, w_y, w_z) = N_{xz}(w_x+k, w_y-k, w_z),$$

$$(1-\overline{xy})(1-\overline{xz})\cdot N(w_x, w_y, w_z) = N(w_x, w_y, w_z) - N(w_x+1, w_y-1, w_z)$$
$$- N(w_x+1, w_y, w_z-1) + N(w_x+2, w_y-1, w_z-1), \text{ etc.}$$

It is to be observed that the operators $\overline{xy}$, etc., like the operators $\widehat{xy}$, etc., are distributive over their operands, but, unlike the latter, the former operators are *commutative*, so that, in their effects on any function of a type,

$$\overline{xy}\cdot\overline{yx} = 1, \qquad \overline{xy}\cdot\overline{yz} = \overline{yz}\cdot\overline{xy} = \overline{xz}, \text{ etc.}$$

As $\widehat{xy}^k\cdot\phi \equiv 0$ if k is greater than the weight of ϕ in y, we must take $\overline{xy}^k\cdot N(w_x, w_y, \cdots) = 0$ if $w_y < k$, whatever suffixes and exponents N may have; that is, the result of applying any product of powers of type-operators to a denumerant must be taken to be 0 if it makes the weight in any variable negative.* But it may be that some less value of k than w_y+1 will make $\overline{xy}^k\cdot N(w_x, w_y, \cdots) = 0$, because, with the notation of § 2, the weight in y of any form can be 0 only if the weights in each of the other variables is as least as great as $\sum mj$ for the form. The result of applying a product of powers of type-operators to a denumerant may be 0 also because there are no forms of the

* Any form of weight 0 in either variable has, evidently, every differentiant character that is defined by a pair in which the variable in question occupies the *second* place.

resultant type that satisfy the conditions imposed by the suffixes and exponents of N in the case in question; thus,

$$\overline{yz}^k \cdot N_{xy,\,xz}(w_x, w_y, w_z) = N_{xy,\,xz}(w_x, w_y + k, w_z - k) = 0$$

if $w_x - w_y < k$, because there are no xy-differentiants whose weight in y is greater than that in x, as we shall see in § 8.

6. We shall make use of the notations

$$(6) \qquad \alpha^{(i)} = \alpha(\alpha-1)(\alpha-2)\cdots(\alpha-i+1), \qquad i^{(i)} = i!$$

for *any* value of α and any *positive integer* i; from this follows

$$(7) \qquad \alpha^{(i)} = \alpha(\alpha-1)^{(i-1)} = \frac{\alpha^{(i+1)}}{\alpha - i},$$

by means of which the symbol $\alpha^{(i)}$ can be extended to negative integral values of i and 0; thus, still for a positive integer i,

$$(8) \quad \alpha^{(-i)} = \frac{1}{(\alpha+1)(\alpha+2)(\alpha+3)\cdots(\alpha+i)} = \frac{1}{(\alpha+i)^{(i)}}, \qquad \alpha^{(0)} = 1.$$

We shall write also, for any *integer* i (positive, negative or 0),

$$(9) \qquad \frac{\alpha^{(i)}}{i!} = \binom{\alpha}{i},$$

which is the coefficient of x^i in the development of $(1+x)^\alpha$ according to positive or negative powers of x, according as i is positive or negative. In particular, if α is a *positive integer* or 0,

$$(10) \qquad \binom{\alpha}{i} = 0 \text{ unless } 0 \leqq i \leqq \alpha;$$

but if α is a *negative integer*,

$$(11) \qquad \binom{\alpha}{i} = 0 \text{ for } \alpha < i \leqq -1;$$

while

$$(12) \qquad i! = \infty \text{ if } i \leqq -1.$$

It is evident from (7) and (8) that

$$(13) \quad (-\alpha)^{(i)} = (-1)^i(\alpha+i-1)^{(i)}, \qquad \alpha^{(-i)} = (-1)^i \frac{1}{(-\alpha-1)^{(i)}}.$$

If α and i are both *integers*,

$$(14) \qquad \alpha^{(i)} = 0 \text{ if } 0 \leqq \alpha < i, \text{ and only then.}$$

We shall have occasion to use the formula of summation

$$(15)\quad S_{p,q}^{m,n}=\sum_{i=0}^{n}(-1)^i\cdot\binom{n}{i}\cdot\frac{(p-i)^{(m+n-1)}}{(q+i)^{(m)}}=\frac{\binom{p-n}{m-1}\cdot(p+q-m+1)^{(n)}}{\binom{q+n}{m+n}\cdot(m+n)},$$

where m, n, p, q are given integers, of which n is not negative; from which follows

$$(16)\qquad S_{p,q}^{m,n}\neq 0 \text{ if } 1\leqq m\leqq q \text{ and } m+n-1\leqq p.$$

7. The results of applying the shear-operators defined by (4) for different pairs of letters successively are connected by these simple relations, where ϕ is any form of type $(w_x, w_y, w_z, w_s, \cdots)$:

$$(17)\quad \begin{cases} \widehat{xy}\cdot\widehat{yx}\cdot\phi=\widehat{yx}\cdot\widehat{xy}\cdot\phi+(w_x-w_y)\cdot\phi, & \widehat{xy}\cdot\widehat{xz}\cdot\phi=\widehat{xz}\cdot\widehat{xy}\cdot\phi,\\ \widehat{xy}\cdot\widehat{zx}\cdot\phi=\widehat{zx}\cdot\widehat{xy}\cdot\phi+\widehat{zy}\cdot\phi, & \widehat{xy}\cdot\widehat{zy}\cdot\phi=\widehat{zy}\cdot\widehat{xy}\cdot\phi,\\ \widehat{xy}\cdot\widehat{yz}\cdot\phi=\widehat{yz}\cdot\widehat{xy}\cdot\phi-\widehat{xz}\cdot\phi, & \widehat{xy}\cdot\widehat{zs}\cdot\phi=\widehat{zs}\cdot\widehat{xy}\cdot\phi, \end{cases}$$

where x, y, z, s are any four different variables. The last three identities of (17) show that any two shear-symbols and their powers are commutative unless they have a common letter that occupies *different* places in them. Repetitions of the first three formulæ of (17) give

$$(18)\quad \begin{aligned} \widehat{xy}^{\alpha}\cdot\widehat{yx}^{\beta}\cdot\phi&\equiv\sum_{i=0}^{\alpha,\beta}\frac{\alpha^{(i)}\beta^{(i)}}{i!}(w_x-w_y+\alpha-\beta)^{(i)}\cdot\widehat{yx}^{\beta-i}\,\widehat{xy}^{\alpha-i}\cdot\phi,\\ \widehat{xy}^{\alpha}\cdot\widehat{zx}^{\beta}\cdot\phi&\equiv\sum_{i=0}^{\alpha,\beta}\frac{\alpha^{(i)}\beta^{(i)}}{i!}\cdot\widehat{zx}^{\beta-i}\,\widehat{zy}^{i}\,\widehat{xy}^{\alpha-i}\cdot\phi,\\ \widehat{xy}^{\alpha}\cdot\widehat{yz}^{\beta}\cdot\phi&\equiv\sum_{i=0}^{\alpha,\beta}(-1)^i\frac{\alpha^{(i)}\beta^{(i)}}{i!}\cdot\widehat{yz}^{\beta-i}\,\widehat{xz}^{i}\,\widehat{xy}^{\alpha-i}\cdot\phi, \end{aligned}$$

where each sum extends to all integral values of i from 0 to the smaller of the two numbers α and β.

8. Any form ϕ of type (w_x, w_y) is annihilated by a certain lowest power of $\widehat{xy}$ and by a certain lowest power of $\widehat{yx}$; let α and β be the exponents of the highest powers of $\widehat{xy}$ and $\widehat{yx}$, respectively, that do not annihilate ϕ (the ranks of ϕ quâ $\widehat{xy}$ and $\widehat{yx}$, respectively), so that

$$\widehat{xy}^{\alpha}\cdot\phi\not\equiv 0,\qquad \widehat{xy}^{\alpha+1}\cdot\phi\equiv 0,\qquad \widehat{yx}^{\beta}\cdot\phi\not\equiv 0,\qquad \widehat{yx}^{\beta+1}\cdot\phi\equiv 0;$$

then, by the first formula of (18) as applied to $\widehat{xy}^{\alpha+\beta+1}\cdot\widehat{yx}^{\beta+1}\cdot\phi$ and $\widehat{yx}^{\alpha+\beta+1}\cdot\widehat{xy}^{\alpha+1}\cdot\phi$,

$$0 \equiv (\alpha+\beta+1)^{(\beta+1)} \cdot (w_x - w_y + \alpha)^{(\beta+1)} \cdot \widehat{xy}^{\alpha} \cdot \phi$$

and

$$0 \equiv (\alpha+\beta+1)^{(\alpha+1)} \cdot (w_y - w_x + \beta)^{(\alpha+1)} \cdot \widehat{yx}^{\beta} \cdot \phi\,;$$

therefore,

$$(w_x - w_y + \alpha)^{(\beta+1)} = 0 \qquad \text{and} \qquad (w_y - w_x + \beta)^{(\alpha+1)} = 0,$$

that is, by (14),

$$w_x - w_y + \alpha \leqq \beta \qquad \text{and} \qquad w_y - w_x + \beta \leqq \alpha,$$

so that

(19) $$\beta = w_x - w_y + \alpha \qquad \text{or} \qquad \alpha = w_y - w_x + \beta.$$

If ϕ is an xy-differentiant, $\alpha = 0$ and, therefore, $\beta = w_x - w_y$, so that

(20) $$w_y \leqq w_x$$

and

A. Every xy-differentiant is of at least as great a weight in x as in y.

If ϕ is an xy-differentiant and $w_x = w_y$, then $\beta = 0$, so that

B. Every xy-differentiant whose weights in x and y are equal is also a yx-differentiant.

From the third identity of (17) follows that $\widehat{xz} \cdot \phi \equiv 0$ if both $\widehat{xy} \cdot \phi \equiv 0$ and $\widehat{yz} \cdot \phi \equiv 0$; that is,

C. Every xy- and yz-differentiant is also an xz-differentiant (xyz-differentiant). From this theorem it appears that every chain-differentiant has every simple differentiant character that is defined by two letters of the corresponding chain, taken in the sequence in which they occur.

It is evident that

D. If two given shear-operators are commutative, the result of applying any power of the first to a form that is annihilated by the second is also a form annihilated by that second operator;

that is, the repeated application of one of two *commutative* shear-operators does not affect the simple differentiant character defined by the other. Similarly, from the second and third formulae of (18), together with theorem D, it appears that,

E. If ϕ is an xy- and xz-differentiant, so is also $\widehat{yz}^k \cdot \phi$: and, *if ϕ is an xz- and yz-differentiant, so is also $\widehat{xy}^k \cdot \phi$;*

provided, in each case, the result is not identically 0.

A polynomial that has *all* the simple differentiant characters that are defined by pairs of variables taken from a certain set (including both orders of each pair) may be called a *complete* differentiant relatively to that set of variables. In the paper in vol. 41 of the Annalen to which I have referred, I have proved that every *isobaric* complete differentiant in *all* the variables is a *covariant* (or invariant); but the proof shows also that an isobaric complete differentiant in *any set* of variables is simply multiplied by a power of the determinant of transformation when the variables of *that set alone* are subjected to any linear transformation.

From (19) follows also that

F. If ϕ is a form of weights w_x and w_y in x and y, respectively, and of rank r quâ $\widehat{xy}$, then $w_y - w_x \leqq r$ (of consequence only if $w_x < w_y$).

Theorems B and C show that an isobaric chain-differentiant is a complete differentiant in the variables defining the chain provided it is of one and the same weight in those variables (which, by theorem A, requires only that it shall be of the same weight in the first and last variables of the chain).

9. In the papers cited in § 1* I have shown that the operators

$$(21)\quad \{v;\ u, \cdots, z, y, x\} \equiv \sum(-1)^{\sigma}\cdot\frac{(w_x - w_v + \nu)!\ \widehat{vx}^{\alpha}\cdot\widehat{vy}^{\beta}\cdot\widehat{vz}^{\gamma}\cdots\widehat{vu}^{\epsilon}\cdot\widehat{uv}^{\epsilon}\cdots\widehat{zv}^{\gamma}\cdot\widehat{yv}^{\beta}\cdot\widehat{xv}^{\alpha}}{\alpha!\ \beta!\ \gamma!\ \cdots\epsilon!\ (\sigma + w_x - w_v + \nu)!}$$

and

$$(22)\quad \{x, y, z, \cdots, u;\ v\} \equiv \sum(-1)^{\sigma}\cdot\frac{(w_v - w_x + \nu)!\ \widehat{xv}^{\alpha}\cdot\widehat{yv}^{\beta}\cdot\widehat{zv}^{\gamma}\cdots\widehat{uv}^{\epsilon}\cdot\widehat{vu}^{\epsilon}\cdots\widehat{vz}^{\gamma}\cdot\widehat{vy}^{\beta}\cdot\widehat{vx}^{\alpha}}{\alpha!\ \beta!\ \gamma!\ \cdots\epsilon!\ (\sigma + w_v - w_x + \nu)!}$$

where ν is the number of variables $x, y, z, \cdots, u$ (without v), the summations extend to all integral but not negative values of $\alpha, \beta, \gamma, \cdots, \epsilon$, and $\sigma = \alpha + \beta + \gamma + \cdots + \epsilon$ for *each* term, have, when applied to a form ϕ of weights $w_x, w_y, w_z, \cdots, w_u, w_v$ in $x, y, z, \cdots, u, v$, respectively (whatever the number of other variables and the weights of ϕ in them may be), these properties:

If ϕ is an xy-, xz-, $\cdots$, xu-differentiant, $\{v;\ u, \cdots, z, y, x\}\cdot\phi$ *is an xy-, xz-, $\cdots$, xu-, xv-differentiant of the same type as ϕ* (if it is not identically 0), *is the most general differentiant of that type and these characters if ϕ is the most general differentiant of its type and the characters stated, and is ϕ itself if that form is an xy-, xz-, $\cdots$, xu-, xv-differentiant.*

If ϕ is a yx-, zx-, $\cdots$, ux-differentiant, $\{x, y, z, \cdots, u;\ v\}\cdot\phi$ *is a yx-, zx-, $\cdots$, ux-, vx-differentiant of the same type as ϕ* (if it is not identically 0), *is the most general differentiant of that type and these characters if ϕ is the most general differentiant of its type and the characters stated, and is ϕ itself if that is a yx-, zx-, $\cdots$, ux-, vx-differentiant.*

The summation in (21) is naturally limited to values of $\alpha, \beta, \gamma, \cdots, \epsilon$ such that $\sigma \leqq w_v$, and that in (22) to values of $\alpha \leqq w_x$, $\beta \leqq w_y$, $\gamma \leqq w_z$, $\cdots$, $\epsilon \leqq w_u$.

In particular, if ϕ is any form of type (w_x, w_y),†

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*E. g., Mathematische Annalen, vol. 41 (1893), pp. 485–489.

†If ϕ is a form involving only the *coefficients* of the quantics (that is, a form of order 0) for which $w_x = w_y$, this formula is equivalent to that for $[\phi]$ given by HILBERT, Mathematische Annalen, vol. 36 (1890), p. 523.

$$(23)\qquad \{y;x\}\cdot\phi \equiv \sum_{a=0}^{w_y}(-1)^a\cdot\frac{(w_x-w_y+1)!}{a!\,(w_x-w_y+a+1)!}\cdot\widehat{yx}^a\cdot\widehat{xy}^a\cdot\phi$$

is an xy-differentiant of the same type as ϕ (if it is not identically 0), *is the most general xy-differentiant of that type if* ϕ *is the most general form of the type, and is* ϕ *itself if that is an xy-differentiant.*

In applying (21), (22) and (23) to any particular form ϕ, it is to be observed that the values of $w_x, w_y, w_z, \cdots, w_u, w_v$ to be used are the weights of *that form.*

If ϕ is any form of type $(w_x, w_y, w_z, \cdots, w_u, w_v)$,

$$(24)\quad [\phi]' \equiv \{v;\, u, \cdots, z, y, x\}\cdot\{u;\, \cdots, z, y, x\}\cdots\{z;\, y, x\}\cdot\{y;\, x\}\cdot\phi$$

is an xy-, xz-, $\cdots$, xu, xv-differentiant of that type, and

$$(25)\quad [\phi]_1 \equiv \{x, y, z, \cdots, u;\, v\}\cdot\{x, y, z, \cdots;\, u\}\cdots\{x, y;\, z\}\cdot\{x;\, y\}\cdot\phi$$

is a yx-, zx-, $\cdots$, ux, vx-differentiant of that type.

II. *Determination of denumerants in general.*

Let xy be any one of the differentiant characters included in the conditions for the denumerant, let the aggregate of all the other conditions be represented by K, and let (a, b) be any given type (to specify the weights in x and y alone). If the shear-operator $\widehat{xy}$, or differentiant character xy, is so related (or unrelated) to the conditions K that the form $\widehat{xy}^k\cdot\phi$ satisfies the conditions K for every form ϕ that satisfies them and every value of k for which $\widehat{xy}^k\cdot\phi$ is not identically 0, we shall say that the operator $\widehat{xy}$, or the differentiant character xy, *does not interfere* with the conditions K.

10. If the differentiant character xy does not interfere with the conditions K, the forms $\widehat{xy}^r\cdot\phi$ derived from the forms ϕ of a complete system of type (a, b) satisfying the conditions K and *reduced* quâ $\widehat{xy}$ by operating on each such form ϕ with the power of $\widehat{xy}$ whose exponent is the rank of that form ϕ quâ $\widehat{xy}$ are *linearly independent* xy-differentiants of the types $(a+r, b-r)$ subordinate to the type (a, b) quâ $\widehat{xy}$, by § 4. Therefore, the number of linearly independent xy-differentiants of any type $(a+r, b-r)$ subordinate to (a, b) quâ $\widehat{xy}$ is at least as great as the number of forms of rank r in a complete system of type (a, b) satisfying the conditions K *reduced* quâ $\widehat{xy}$; that is, for $0 \leqq r \leqq r'$,

$$(26)\qquad N_K^{\widehat{xy}r}(a, b) \leqq N_{K,\,xy}(a+r, b-r),$$

from which follows

$$(27)\qquad N_K(a, b) = \sum_{r=0}^{r'} N_K^{\widehat{xy}r}(a, b) \leqq \sum_{r=0}^{r'} N_{K,\,xy}(a+r, b-r),$$

where r' is the greatest value of r for any possible type $(a+r,\, b-r)$ subordinate to $(a,\, b)$ quâ $\widehat{xy}$.

11. If, for *each* value of r from 0 to r', the greatest value of r for any possible type subordinate to $(a,\, b)$ quâ $\widehat{xy}$, there exists a *determinate* * distributive operator ω_r such that the forms $\omega_r\psi_r$ obtained from a complete system of xy-differentiants ψ_r of *all* types $(a+r,\, b-r)$ subordinate to $(a,\, b)$ quâ $\widehat{xy}$ that satisfy the conditions K are *linearly independent* † forms of type $(a,\, b)$ that satisfy the conditions K, then the number of linearly independent forms of type $(a,\, b)$ that satisfy the conditions K is at least as great as the number of linearly independent xy-differentiants of all types $(a+r,\, b-r)$ subordinate to $(a,\, b)$ quâ $\widehat{xy}$ that satisfy the conditions K: that is,

$$(28)\qquad \sum_{r=0}^{r'} N_{K,\,xy}(a+r,\, b-r) \leqq \sum_{r=0}^{r'} N_K^{\widehat{xy}^r}(a,\, b) = N_K(a,\, b).$$

12. If the conditions of *both* § 10 and § 11 are satisfied, we have, by (27) and (28),

$$(29)\qquad \sum_{r=0}^{r'} N_{K,\,xy}(a+r,\, b-r) = \sum_{r=0}^{r'} N_K^{\widehat{xy}^r}(a,\, b) = N_K(a,\, b),$$

and, because (29) cannot be satisfied unless the sign of equality is to be taken in (26) for *every* value of r from 0 to r', inclusive,

$$(30)\qquad N_{K,\,xy}(a+r,\, b-r) = N_K^{\widehat{xy}^r}(a,\, b) \quad \text{for } 0 \leqq r \leqq r'.$$

In this case, for any *particular* value of r, the forms $\widehat{xy}^r\cdot\phi$ of § 10, being, as we have there seen, linearly independent xy-differentiants of type $(a+r,\, b-r)$ that satisfy the conditions K, and being, by (30), just equal in number to the number of such linearly independent xy-differentiants, in themselves constitute a complete system of linearly independent xy-differentiants of type $(a+r,\, b-r)$ that satisfy the conditions K: for any *particular* value of r, the forms ψ_r of § 11 are, then, linearly independent linear functions of the forms $\widehat{xy}^r\cdot\phi$ of § 10, and vice versâ, and the former forms may be replaced by the latter, whenever it may be convenient; that is, when we are considering simply a complete system of linearly independent xy-differentiants of a type $(a+r,\, b-r)$ subordinate to $(a,\, b)$ quâ $\widehat{xy}$ satisfying the conditions K, we may take these xy-differentiants to be the forms $\widehat{xy}^r\cdot\phi$ derived from the forms ϕ of rank r in a complete system of type $(a,\, b)$, to which $(a+r,\, b-r)$ is subordinate quâ $\widehat{xy}$,

* By a *determinate* operator ω_r I mean one that depends on the type $(a,\, b)$, the conditions K, the differentiant character xy, and the value of r, alone, and not at all on the particular form ψ_r to which it is to be applied.

† And, therefore, *non-vanishing* forms: for a vanishing form may be regarded as a linear function of any forms, having all its coefficients 0.

satisfying the conditions K and *reduced* quâ $\widehat{xy}$. Again, in this same case, the forms $\omega_r \psi_r$ of § 11 for *all* values of r, being, as we have there seen, linearly independent forms of type (a, b) that satisfy the conditions K, and being, by (29), just equal in number to the number of such linearly independent forms, in themselves constitute a complete system of linearly independent forms of type (a, b) that satisfy the conditions K; the forms ϕ of § 10 are, then, linearly independent linear functions of the forms $\omega_r \psi_r$ of § 11 for *all* values of r, and vice versâ: whenever we are considering simply a complete system of linearly independent forms of type (a, b) that satisfy the conditions K, we may, then, take them to be the forms $\omega_r \psi_r$ obtained from complete systems of linearly independent xy-differentiants ψ_r of *all* types $(a + r, b - r)$ subordinate to (a, b) quâ $\widehat{xy}$ that satisfy the conditions K: but it is to be observed that we have *not* shown that the forms ψ_r, although they may be replaced by the same number of *arbitrary* linearly independent linear functions of them for each value of r, can be so taken that the forms $\omega_r \psi_r$ for *all* values of r shall constitute a system *reduced* quâ $\widehat{xy}$,—however, it would be so if the forms $\omega_r \psi_r$ were of rank r for *each* value of r, as a little consideration will show.

13. If the differentiant character xy does not interfere with the conditions K, so that (26) and (27) hold, and if, for each value of r from 0 to r' [the greatest value of r for which $(a + r, b - r)$ is a possible type subordinate to the *given* type (a, b) quâ $\widehat{xy}$], there exists a determinate distributive operator ω_r such that the form $\omega_r \psi_r$ is a form of type (a, b) that satisfies the conditions K for every xy-differentiant ψ_r of type $(a + r, b - r)$ that satisfies the same conditions, and also such that

$$\widehat{xy}^r \cdot \omega_r \psi_r \equiv S_r \cdot \psi_r, \tag{31}$$

where S_r is a *non-vanishing* constant, then the forms $\omega_r \psi_r$ obtained from complete systems of xy-differentiants of *all* types $(a + r, b - r)$ subordinate to (a, b) quâ $\widehat{xy}$ that satisfy the conditions K are *linearly independent*. For, by (31),

$$\widehat{xy}^k \cdot \omega_r \psi_r \equiv S_r \cdot \widehat{xy}^{k-r} \cdot \psi_r \equiv 0 \qquad \text{if} \qquad r < k,$$

so that any linear relation between forms $\omega_r \psi_r$ for which the greatest value of r is k would be turned by the operator $\widehat{xy}^k$ into a linear relation between the forms ψ_k with non-vanishing coefficients, whereas no such relation exists; therefore, such operators ω_r satisfy the conditions of § 11, so that their existence proves (28) and, because (26) and (27) hold, also (29) and (30); then, by § 12, the linearly independent xy-differentiants of any type $(a + r, b - r)$ subordinate to the given type (a, b) quâ $\widehat{xy}$ that satisfy the conditions K may be taken to be the forms $\widehat{xy}^r \cdot \phi$ derived from the forms ϕ of rank r in a complete system of type (a, b) satisfying the conditions K and *reduced* quâ $\widehat{xy}$.

In this case, for any *given* value of k from 0 to r', inclusive, the forms $\widehat{xy}^k\cdot\phi$ of type $(a+k,\ b-k)$ obtained from the forms ϕ *of rank as great as* k are *linearly independent*, for any linear relation between such of them as are obtained from forms ϕ of rank h and less quâ $\widehat{xy}$ $(k\leqq h\leqq r')$ would be turned by the operator $\widehat{xy}^{h-k}$ into a linear relation between the linearly independent forms $\widehat{xy}^h\cdot\phi$ derived from forms ϕ of rank h quâ $\widehat{xy}$. Again, the types $(a+r,\ b-r)$ subordinate to $(a,\ b)$ quâ $\widehat{xy}$ for *all* values of r *as great as* k are also the types subordinate to $(a+k,\ b-k)$ quâ $\widehat{xy}$, and the xy-differentiants $\widehat{xy}^r\cdot\phi\equiv\widehat{xy}^{r-k}\cdot\widehat{xy}^k\cdot\phi$ derived from the forms ϕ that are of rank r quâ $\widehat{xy}$ are the xy-differentiants derived from the forms $\widehat{xy}^k\cdot\phi$ that are of rank $r-k$ quâ $\widehat{xy}$, so that the latter forms constitute a system *reduced* quâ $\widehat{xy}$. Finally, the operators $\widehat{xy}^k\cdot\omega_r$ for *all* values of r *as great as* k turn the linearly independent xy-differentiants $\widehat{xy}^r\cdot\phi$ into *linearly independent* forms of type $(a+k,\ b-k)$ and rank $r-k$ quâ $\widehat{xy}$ that satisfy the conditions K and for which, by (31),

$$\widehat{xy}^{r-k}\cdot\widehat{xy}^k\cdot\omega_r\cdot\widehat{xy}^r\cdot\phi\equiv\widehat{xy}^r\cdot\omega_r\cdot\widehat{xy}^r\cdot\phi\equiv S_r\cdot\widehat{xy}^r\cdot\phi,$$

which is what (31) becomes when the system of forms ϕ is replaced by the system of forms $\widehat{xy}^k\cdot\phi$, so that formulæ (29) and (30) hold also if the type $(a,\ b)$ is replaced by the type $(a+k,\ b-k)$ and, of course, the limit r' of r by the limit $r'-k$ of $r-k$. The resultant formulæ are, evidently, equivalent to

$$(32)\qquad \sum_{r=k}^{r'}N_{K,\,xy}(a+r,\ b-r)=N_K(a+k,\ b-k)$$

and

$$(33)\qquad N_{K,\,xy}(a+r,\ b-r)=N_K^{\widehat{xy}^{r-k}}(a+k,\ b-k)\quad\text{for}\quad k\leqq r\leqq r'.$$

On subtracting (32) from (29), member from member, we get, for $1\leqq k\leqq r'$,

$$(34)\quad \sum_{r=0}^{k-1}N_{K,\,xy}(a+r,\ b-r)=N_K(a,\ b)-N_K(a+k,\ b-k)=(1-\overline{xy}^k)\cdot N_K(a,\ b),$$

and, in particular, for $k=1$,

$$(35)\quad N_{K,\,xy}(a,\ b)=N_K(a,\ b)-N_K(a+1,\ b-1)=(1-\overline{xy})\cdot N_K(a,\ b),$$

by which (if the conditions for its validity are satisfied) the number of linearly independent forms of *given* type that satisfy the conditions K *and have the additional differentiant character xy* is determined when the numbers of linearly independent forms of *certain* types that satisfy the conditions K are known. In other words (35) is a formula for *the addition*, in a denumerant, *of a differentiant character* to other conditions with which it does not interfere.

14. Whatever differentiant and numerical conditions may be given, it seems easy to select one of the differentiant characters that shall not interfere with the

aggregate of the other conditions and, for each of the possible values of r, to find an operator ω_r that shall have all the properties stated in § 13, *excepting the last*, that is, that the constant S_r shall be *different from* 0 for each value of r. At least, that is the only difficulty that has presented itself to me in my attempts to determine the denumerants for triple and higher differentiants.

Evidently, the reasoning of §§ 11–13 requires that there shall be forms of type (a, b) that satisfy the conditions K *if there are forms of any type subordinate to* (a, b) *quâ* $\widehat{xy}$ *that satify them* and we shall find that, at least in the cases we consider, the operators ω_r that naturally suggest themselves do not satisfy (31) with a *non-vanishing* constant S_r for *all* the values of r in question unless $b \leqq a$, which, by theorem A of § 8, is a necessary, though not sufficient, condition that there shall be xy-differentiants of the type (a, b). In other words, we shall find it impossible to use (35) unless the type (a, b) is such that the condition of theorem A of § 8 is satisfied by the weights in *all* the pairs of variables that define the differentiant characters for the denumerant to be determined. But this is really no limitation to the use of (35), because the left member of this formula implies these relations between the weights, or else its value is 0.

15. The numerical conditions included in K may be of very different kinds, of which it would be impossible to state all that it might be useful to take into account, but we ought to mention one class that we shall have occasion to consider. Numerical conditions satisfied by the *derivatives* of a system of forms relatively to a given shear-operator may be regarded as conditions satisfied by the forms of the system. In particular, that the *rank* quâ $\widehat{yz}$ of the *derivative* quâ $\widehat{xy}$ of any form ϕ of a complete system of type (a, b) satisfying certain other conditions shall *lie within given limits* is admissible as a numerical condition to restrict the system further, and will be satisfied by the *derivatives* of the restricted system; the xy-differentiants ψ_r of types $(a + r, b - r)$ must then be taken to satisfy the same condition, namely, that their *ranks* quâ $\widehat{yz}$ *shall lie within the given limits*. It may be that the conditions K can be separated into two sets, of which the second set contains only conditions that are satisfied by *all* forms ϕ of type (a, b) that satisfy the conditions of the first set *and by their derivatives* quâ $\widehat{xy}$*:* the conditions of the second set need be *specified* only for the xy-differentiants ψ_r of types $(a + r, b - r)$.

For example, if the differentiant character xy does not interfere with the conditions K (representing only the conditions of the *first* set); if the rank quâ $\widehat{yz}$ of the form $\widehat{xy}^k \cdot \phi$ lies within limits l_k to l'_k dependent only on the given type (a, b) and the value of k, inclusive of both limits, for *every* form of type (a, b) that satisfies the conditions K and for *every* value of k for which $\widehat{xy}^k \cdot \phi$ is not identically 0 (the one condition of the *second* set); and if, for *each* value of r

from 0 to r' [the greatest value of r for which $(a+r, b-r)$ is a type subordinate to (a, b) quâ $\widehat{xy}$], inclusive, there exists a determinate distributive operator ω_r such that the forms $\omega_r \psi_r$ obtained from the xy-differentiants ψ_r *whose ranks* quâ $\widehat{yz}$ *lie within the limits* l_r *to* l'_r, inclusive, in complete systems of *all* types $(a+r, b-r)$ subordinate to (a, b) quâ $\widehat{xy}$ satisfying the conditions K and *reduced* quâ $\widehat{yz}$ are forms of type (a, b) that satisfy the conditions K and the identity (31), where S_r is a non-vanishing constant; then, by (29),

$$\sum_{r=0}^{r'} \sum_{\rho=l_k}^{l'_k} N_{K, xy}^{\widehat{yz}\rho}(a+r, b-r) = N_K(a, b). \tag{36}$$

III. *Simple differentiants.*

16. In this section we consider only forms of a given order in the variables, of given degrees in the coefficients of the several quantics, and of given weights in the several variables *excepting two*, say x and y. For any such form of type (w_x, w_y) we have, by (2), $w_x + w_y = \sigma$, where σ is a constant, that is, is the same number for *all* the forms considered. It will be convenient to represent w_x temporarily by a simpler symbol a (or a'): then, $w_y = \sigma - a$ (or $\sigma - a'$).

For xy-differentiants of a *given* type $(a, \sigma - a)$ there are *no* conditions excepting the differentiant character xy and the types subordinate to $(a, \sigma - a)$ quâ $\widehat{xy}$ are the types $(a + r, \sigma - a - r)$ for which $0 \leqq r \leqq \sigma - a$ (it may even be that r has a smaller upper limit than $\sigma - a$). If $\psi_{a'-a}$ is any xy-differentiant of one of these subordinate types, say $(a', \sigma - a')$ for $r = a' - a$, where $a \leqq a' \leqq \sigma$ and $\frac{1}{2}\sigma \leqq a'$, by (20), then $\widehat{yx}^{a'-a} \cdot \psi_{a'-a}$ is a form of type $(a, \sigma - a)$ for which, by (18), because $\widehat{xy} \cdot \psi_{a'-a} \equiv 0$,

$$\widehat{xy}^{a'-a} \cdot \widehat{yx}^{a'-a} \cdot \psi_{a'-a} \equiv (a'-a)!\,(2a' - \sigma)^{(a'-a)} \cdot \psi_{a'-a},$$

which is not identically 0 if $\sigma - a' \leqq a \leqq a' \leqq \sigma$: therefore, by (35), if $\frac{1}{2}\sigma \leqq a \leqq \sigma$,

$$N_{xy}(a, \sigma - a) = (1 - \overline{xy}) \cdot N(a, \sigma - a)$$

or, if we replace a by w_x and $\sigma - a$ by w_y,

$$N_{xy}(w_x, w_y) = (1 - \overline{xy}) \cdot N(w_x, w_y) \qquad \text{for} \qquad w_y \leqq w_x. \tag{37}$$

This is CAYLEY's formula as extended to a system of any number of quantics in any number of variables.

Formula (37) shows that the number of linearly independent linear equations between the coefficients of the general form ϕ of type (w_x, w_y) implied by the identity $\widehat{xy} \cdot \phi \equiv 0$ is just the number of terms in the general form of type $\overline{xy} \cdot (w_x, w_y)$, that is, that the linear equations implied by this identity are *linearly independent*, as Cayley *assumed*. This proves also that, if ϕ is the *general* form of type (w_x, w_y), then $\widehat{xy}\cdot\phi$ is the *general* form of type (w_x+1, w_y-1) and, therefore, $\widehat{xy}^k\cdot\phi$ is the most *general* form of type (w_x+k, w_y-k) for *every* value of k for which this type exists.

Incidentally, formula (37) shows that, of the general forms of two types that differ only in the weights in two variables, that has the more terms in which the weights in the two variables in question are the more nearly equal (unless they have the same number of terms, in which case the general forms of all intermediate types,—types subordinate to one and to which the other is subordinate,—have that same number of terms).

Again, formula (37) gives the number of linearly independent *covariants* or *invariants* of any system of *binary* quantics if x and y are the only variables and $w_x = w_y$, by theorems A and B of § 8, because every xy-differentiant for which these conditions are satisfied is a *complete* differentiant in x and y and, therefore, a *covariant* or *invariant.*

IV. *Double differentiants.*

In §§ 17–22 we shall consider only forms of a *given* order in the variables, of *given* degrees in the coefficients of the several quantics, and of *given* weights in the several variables *excepting three*, say x, y, and z. The type of such a form shall be regularly denoted by (w_x, w_y, w_z), representing only the weights in x, y, and z, in this sequence, so that, by (2), $w_x + w_y + w_z = \sigma$, where σ has the same value for *all* forms considered; but, in the deduction of formulae, it will be more convenient to represent two of these three weights by simple letters, and for this purpose we shall put either $w_x = a$, $w_y = b$ and $w_z = \sigma - a - b$, —or $w_x = \sigma - b - c$, $w_y = b$, and $w_z = c$, —accenting a, b, or c when we have occasion to represent more than one value of either.

17. If χ is an xz- and yz-differentiant of type $(\sigma - b' - c, b', c)$,— where, by theorem A of § 8, $c \leqq b' \leqq \sigma - 2c$,— so that $\widehat{xz}\cdot\chi \equiv 0$ and $\widehat{yz}\cdot\chi \equiv 0$, then, by (17) and (18),

$$\widehat{xz}\cdot\widehat{xy}^k\cdot\chi \equiv \widehat{xy}^k\cdot\widehat{xz}\cdot\chi \equiv 0 \quad\text{and}\quad \widehat{yz}\cdot\widehat{xy}^k\cdot\chi \equiv \widehat{xy}^k\cdot\widehat{yz}\cdot\chi + k\cdot\widehat{xy}^{k-1}\cdot\widehat{xz}\cdot\chi \equiv 0,$$

so that the differentiant character xy *does not interfere* with the differentiant characters xz and yz. The *derivatives* of xz- and yz-differentiants of the *given* type $(\sigma - b' - c, b', c)$ quâ $\widehat{xy}$ are, then, xyz-differentiants of types $(\sigma - b - c, b, c)$ subordinate to $(\sigma - b' - c, b', c)$ quâ $\widehat{xy}$ for which $0 \leqq c \leqq b \leqq b' \leqq \sigma - b - c$ and $b \leqq \frac{1}{2}(\sigma - c)$, by theorems A and F of § 8. If $\psi_{b'-b}$ is any xyz-differentiant of type $(\sigma - b - c, b, c)$,— for which, in the notation of §§ 10–13, $r = b' - b$ and $0 \leqq r \leqq b' - c$,—then, by (17) and (18),

$$\widehat{xz}\cdot\widehat{yx}^{b'-b}\cdot\psi_{b'-b} \equiv \widehat{yx}^{b'-b}\cdot\widehat{xz}\cdot\psi_{b'-b} + (b'-b)\cdot\widehat{yx}^{b'-b-1}\cdot\widehat{yz}\cdot\psi_{b'-b} \equiv 0$$

and

$$\widehat{yz}\cdot\widehat{yx}^{b'-b}\cdot\psi_{b'-b}\equiv\widehat{yx}^{b'-b}\cdot\widehat{yz}\cdot\psi_{b'-b}\equiv 0,$$

so that $\widehat{yx}^{b'-b}\cdot\psi_{b'-b}$ is an xz- and yz-differentiant of type $(\sigma-b'-c, b', c)$, if not identically 0, for which, by (18),

$$\widehat{xy}^{b'-b}\cdot\widehat{yx}^{b'-b}\cdot\psi_{b'-b}\equiv(b'-b)!\ (\sigma-2b-c)^{(b'-b)}\cdot\psi_{b'-b},$$

where the multiplier of $\psi_{b'-b}$ is a *non-vanishing* constant if $b\leqq b'\leqq\sigma-b-c$. Therefore, if $0\leqq c\leqq b'\leqq\frac{1}{2}(\sigma-c)$,

$$N_{xz,\,yz}(\sigma-b'-c, b', c)=\sum_{b=c}^{b'}N_{xyz}(\sigma-b-c, b, c),$$

by (28), and

$$N_{xyz}(\sigma-b'-c, b', c)=(1-\overline{xy})\cdot N_{xz,\,yz}(\sigma-b'-c, b', c),$$

by (35), while, if $0\leqq c\leqq b\leqq b'\leqq\frac{1}{2}(\sigma-c)$,

$$N_{xz,\,yz}^{\widehat{xy}^{b'-b}}(\sigma-b'-c, b', c)=N_{xyz}(\sigma-b-c, b, c),$$

by (30); or, when $\sigma-b'-c$ is replaced by w_x, b' by w_y, c by w_z, and b by $b'-r$, if $w_z\leqq w_y\leqq w_x$,

$$N_{xz,\,yz}(w_x, w_y, w_z)=\sum_{r=0}^{w_y-w_z}N_{xyz}(w_x+r, w_y-r, w_z)\tag{38}$$

and

$$N_{xyz}(w_x, w_y, w_z)=(1-\overline{xy})\cdot N_{xz,\,yz}(w_x, w_y, w_z),\tag{39}$$

while, if $w_z\leqq w_y\leqq w_x$ and $0\leqq r\leqq w_y-w_z$,

$$N_{xz,\,yz}^{\widehat{xy}^r}(w_x, w_y, w_z)=N_{xyz}(w_x+r, w_y-r, w_z).\tag{40}$$

Also, by § 12, the forms $\widehat{yx}^{b'-b}\cdot\psi_{b'-b}$ obtained from *complete* systems of xyz-differentiants of *all* types $(\sigma-b-c,\ b,\ c)$ subordinate to the *given* type $(\sigma-b'-c, b', c)$ quâ xy constitute a *complete* system of xz- and yz-differentiants of the latter type if $0\leqq c\leqq b'\leqq\frac{1}{2}(\sigma-c)$.

18. If χ is an xy- and xz-differentiant of type $(a, b', \sigma-a\dot{-}b')$,—where, by theorem A of § 8, $\sigma-2a\leqq b'\leqq a$,—so that $\widehat{xy}\cdot\chi\equiv 0$ and $\widehat{xz}\cdot\chi\equiv 0$, then, by (17) and (18),

$$\widehat{xy}\cdot\widehat{yz}^k\cdot\chi\equiv\widehat{yz}^k\cdot\widehat{xy}\cdot\chi-k\cdot\widehat{yz}^{k-1}\cdot\widehat{xz}\cdot\chi\equiv 0\qquad\text{and}\qquad\widehat{xz}\cdot\widehat{yz}^k\cdot\chi\equiv\widehat{yz}^k\cdot\widehat{xz}\cdot\chi\equiv 0,$$

so that the differentiant character yz *does not interfere* with the differentiant characters xy and xz. The *derivatives* of xy- and xz-differentiants of the *given* type $(a,\ b',\ \sigma-a-b')$ quâ $\widehat{yz}$ are, then, xyz-differentiants of types $(a,\ b,\ \sigma-a-b)$ subordinate to $(a,\ b',\ \sigma-a-b')$ quâ $\widehat{yz}$ for which

$\sigma - a - b \leqq b' \leqq b \leqq a \leqq \sigma - b$ and $\frac{1}{2}(\sigma - a) \leqq b$, by theorems A and F of § 8. If $\psi_{b-b'}$ is any xyz-differentiant of type $(a, b, \sigma - a - b)$,—for which, in the notation of §§ 10–13, $r = b - b'$ and $0 \leqq r \leqq a - b'$,—then, by (17) and (18),

$$\widehat{xy} \cdot \widehat{zy}^{b-b'} \cdot \psi_{b-b'} \equiv \widehat{zy}^{b-b'} \cdot \widehat{xy} \cdot \psi_{b-b'} \equiv 0$$

and

$$\widehat{xz} \cdot \widehat{zy}^{b-b'} \cdot \psi_{b-b'} \equiv \widehat{zy}^{b-b'} \cdot \widehat{xz} \cdot \psi_{b-b'} - (b - b') \cdot \widehat{zy}^{b-b'-1} \cdot \widehat{xy} \cdot \psi_{b-b'} \equiv 0,$$

so that $\widehat{zy}^{b-b'} \cdot \psi_{b-b'}$ is an xy- and xz-differentiant of type $(a, b', \sigma - a - b')$, if not identically 0, for which, by (18),

$$\widehat{yz}^{b-b'} \cdot \widehat{zy}^{b-b'} \cdot \psi_{b-b'} \equiv (b - b')!\ (a + 2b - \sigma)^{(b-b')} \cdot \psi_{b-b'},$$

where the multiplier of $\psi_{b-b'}$ is a *non-vanishing* constant if $\sigma - a - b \leqq b' \leqq b$. Therefore, if $\frac{1}{2}(\sigma - a) \leqq b' \leqq a \leqq \sigma - b'$,

$$N_{xy,\,xz}(a, b', \sigma - a - b') = \sum_{b=b'}^{a} N_{xyz}(a, b, \sigma - a - b),$$

by (28), and

$$N_{xyz}(a, b', \sigma - a - b') = (1 - \overline{yz}) \cdot N_{xy,\,xz}(a, b', \sigma - a - b'),$$

by (35), while, if $\frac{1}{2}(\sigma - a) \leqq b' \leqq b \leqq a \leqq \sigma - b$,

$$N_{xy,\,xz}^{\widehat{yz}^{b-b'}}(a, b', \sigma - a - b') = N_{xyz}(a, b, \sigma - a - b),$$

by (30), or when a is replaced by w_x, b' by w_y, $\sigma - a - b'$ by w_z, and b by $b' + r$, if $w_z \leqq w_y \leqq w_x$,

$$(41) \qquad N_{xy,\,xz}(w_x, w_y, w_z) = \sum_{r=0}^{w_x - w_y} N_{xyz}(w_x, w_y + r, w_z - r)$$

and

$$(42) \qquad N_{xyz}(w_x, w_y, w_z) = (1 - \overline{yz}) \cdot N_{xy,\,xz}(w_x, w_y, w_z),$$

while, if $w_z \leqq w_y \leqq w_x$ and $0 \leqq r \leqq w_x - w_y$,

$$(43) \qquad N_{xy,\,xz}^{\widehat{yz}^r}(w_x, w_y, w_z) = N_{xyz}(w_x, w_y + r, w_z - r).$$

Also, by § 12, the forms $\widehat{zy}^{b-b'} \cdot \psi_{b-b'}$ obtained from *complete* systems of xyz-differentiants of *all* types $(a, b, \sigma - a - b)$ subordinate to the given type $(a, b', \sigma - a - b')$ quâ $\widehat{yz}$ constitute a *complete* system of xy- and xz-differentiants of the latter type if $\frac{1}{2}(\sigma - a) \leqq b' \leqq a \leqq \sigma - b'$.

19. If ϕ is an xy-differentiant of type $(a', b', \sigma - a' - b')$, where, by theorem A of § 8, $b' \leqq a' \leqq \sigma - b'$, so that $\widehat{xy} \cdot \phi \equiv 0$, then, by (17),

$$\widehat{xy} \cdot \widehat{xz}^k \cdot \phi \equiv \widehat{xz}^k \cdot \widehat{xy} \cdot \phi \equiv 0,$$

that is, the differentiant character xz *does not interfere* with the differentiant character xy. The derivatives of xy-differentiants of *given type* $(a', b', \sigma - a' - b')$ quâ $\widehat{xz}$ are, then, xy- and xz-differentiants of types $(a, b', \sigma - a - b')$ for which $\sigma - a - a' \leqq b' \leqq a' \leqq a \leqq \sigma - b'$, by theorems A and F of § 8. If $\chi_{a-a'}$ is any xy- and xz-differentiant of type $(a, b', \sigma - a - b')$, —for which, in the notation of §§ 10 — 13, $r = a - a'$ and $0 \leqq r \leqq \sigma - a' - b'$, —then, by § 9, $\{y : x\} \cdot \widehat{zx}^{a-a'} \cdot \chi_{a-a'}$ is an xy-differentiant of type $(a', b', \sigma - a' - b')$, if not identically 0.

Now, by § 18, if $\frac{1}{2}(\sigma - a) \leqq b' \leqq a \leqq \sigma - b'$, we may take a complete system of xy- and xz-differentiants $\chi_{a-a'}$ of type $(a, b', \sigma - a - b')$ in such manner that any one of them of rank $b - b'$ quâ $\widehat{yz}$ can be represented as $\chi_{a-a'} \equiv \widehat{zy}^{b-b'} \cdot \psi_{b-b'}$, where $\psi_{b-b'}$ is an xyz-differentiant of some type $(a, b, \sigma - a - b)$ subordinate to $(a, b', \sigma - a - b')$ quâ $\widehat{yz}$ for which $b' \leqq b \leqq a$ and $b \leqq \sigma - a$. Then,

$$\widehat{xy} \cdot \chi_{a-a'} \equiv 0, \qquad \widehat{xz} \cdot \chi_{a-a'} \equiv 0,$$

$$\widehat{xy} \cdot \psi_{b-b'} \equiv 0, \qquad \widehat{xz} \cdot \psi_{b-b'} \equiv 0, \qquad \widehat{yz} \cdot \psi_{b-b'} \equiv 0,$$

and, therefore,

$$\widehat{xz} \cdot \widehat{zy}^{b-b'+\alpha} \cdot \psi_{b-b'} \equiv 0 \quad \text{and} \quad \widehat{xz}^{\beta} \cdot \widehat{zx}^{a-a'-\alpha} \cdot \widehat{zy}^{b-b'+\alpha} \cdot \psi_{b-b'} \equiv 0 \text{ if } a - a' - \alpha < \beta,$$

so that, by (17) and (18),

$$\begin{aligned} \widehat{xy}^{\alpha} \cdot \widehat{zx}^{a-a'} \cdot \chi_{a-a'} &\equiv (a - a')^{(\alpha)} \cdot \widehat{zx}^{a-a'-\alpha} \cdot \widehat{zy}^{\alpha} \cdot \chi_{a-a'} \\ &\equiv (a - a')^{(\alpha)} \cdot \widehat{zx}^{a-a'-\alpha} \cdot \widehat{zy}^{b-b'+\alpha} \cdot \psi_{b-b'}, \end{aligned}$$

$$\begin{aligned} \widehat{xz}^{a-a'} \cdot \widehat{yx}^{\alpha} \cdot \widehat{xy}^{\alpha} \cdot \widehat{zx}^{a-a'} \cdot \chi_{a-a'} &\equiv (a - a')^{(\alpha)} \cdot \widehat{yz}^{\alpha} \cdot \widehat{xz}^{a-a'-\alpha} \cdot \widehat{xy}^{\alpha} \cdot \widehat{zx}^{a-a'} \cdot \chi_{a-a'} \\ \equiv (a - a')!\,(a - a')^{(\alpha)} \cdot (2a + b' - \sigma - \alpha)^{(a-a'-\alpha)} \cdot \widehat{yz}^{\alpha} \cdot \widehat{zy}^{b-b'+\alpha} \cdot \psi_{b-b'} \\ \equiv (a - a')!\,(a - a')^{(\alpha)} \cdot (2a + b' - \sigma - \alpha)^{(a-a'-\alpha)} \cdot (b - b' + \alpha)^{(\alpha)} \\ \times (a + b + b' - \sigma)^{(\alpha)} \cdot \widehat{zy}^{b-b'} \cdot \psi_{b-b'} \\ \equiv \frac{(a - a')!\,(a + b + b' - \sigma)!}{(a + a' + b' - \sigma)!\,(b - b')!} \cdot (a - a')^{(\alpha)} \cdot \frac{(2a + b' - \sigma - \alpha)!\,(b - b' + \alpha)!}{(a + b + b' - \sigma - \alpha)!} \cdot \chi_{a-a'}, \end{aligned}$$

and, by (23),

$$\widehat{xz}^{a-a'} \cdot \{y ; x\} \cdot \widehat{zx}^{a-a'} \cdot \chi_{a-a'} \equiv S_{a-a'} \cdot \chi_{a-a'}, \tag{44}$$

where, by (15)

$$\begin{aligned} S_{a-a'} = \frac{(a - a')!\,(a' - b' + 1)!\,(a + b + b' - \sigma)!}{(b - b')!\,(a + a' + b' - \sigma)!} \cdot \sum_{\alpha=0}^{a-a'} (-1)^{\alpha} \\ \times \binom{a - a'}{\alpha} \cdot \frac{(2a + b' - \sigma - \alpha)^{(a-b)}}{(a' - b' + \alpha + 1)^{(a'-b+1)}} \\ = \frac{(a - a')!\,(a' - b' + 1)!\,(a + b + b' - \sigma)!}{(b - b')!\,(a + a' + b' - \sigma)!}\, S^{a'-b+1,\, a-a'}_{2a+b'-\sigma,\, a'-b'+1}, \end{aligned}$$

which, by (14) and (16), is a *non-vanishing* constant if $\sigma-a-b\leqq b'\leqq b\leqq a'\leqq a$; so that, at least when these conditions are satisfied, the *derivative* of the form $\{y;\, x\}\cdot\overset{\frown}{zx}{}^{a-a'}\cdot\chi_{a-a'}$ quâ $\overset{\frown}{xz}$ is a non-vanishing constant multiple of $\chi_{a-a'}$. Observe that the condition $\sigma-a-b\leqq b'$ is satisfied if $\frac{1}{2}(\sigma-a')\leqq b'\leqq b$.

But, if ϕ is an xy-differentiant of type $(a',\, b',\, \sigma-a'-b')$ and of rank $a-a'$ quâ $\overset{\frown}{xz}$, its derivative quâ $\overset{\frown}{xz}$, namely $\overset{\frown}{xz}{}^{a-a'}\cdot\phi$, is an xy- and xz-differentiant of type $(a,\, b',\, \sigma-a-b')$ and, say, of rank $b-b'$ quâ $\overset{\frown}{yz}$, so that $\overset{\frown}{yz}{}^{b-b'}\cdot\overset{\frown}{xz}{}^{a-a'}\cdot\phi$ is a non-vanishing xyz-differentiant of type $(a,\, b,\, \sigma-a-b)$, where, by theorems A and F of §8, $\sigma-a-a'\leqq b'\leqq a'\leqq a\leqq\sigma-b,\, b'\leqq b\leqq a$, and $\sigma-a-b'\leqq b$; then, by (18), because $\overset{\frown}{xy}\cdot\phi\equiv 0$,

$$\overset{\frown}{xy}{}^{a-a'}\cdot\overset{\frown}{yz}{}^{a+b-a'-b'}\cdot\phi\equiv(-1)^{a-a'}\cdot(a+b-a'-b')^{(a-a')}\cdot\overset{\frown}{yz}{}^{b-b'}\cdot\overset{\frown}{xz}{}^{a-a'}\cdot\phi\not\equiv 0,$$

by (14), while

$$\overset{\frown}{xy}{}^{a-a'+1}\cdot\overset{\frown}{yz}{}^{a+b-a'-b'}\cdot\phi\equiv 0;$$

therefore, $\overset{\frown}{yz}{}^{a+b-a'-b'}\cdot\phi$ is a non-vanishing form of type $(a',\, a+b-a',\, \sigma-a-b)$ and rank $a-a'$ quâ $\overset{\frown}{xy}$, so that, by theorem F of §8, $a+b-2a'\leqq a-a'$, that is, $b\leqq a'$; the types of ϕ and its *successive* derivatives quâ $\overset{\frown}{xz}$ and quâ $\overset{\frown}{yz}$ are, then, so related that $\sigma-a-b\leqq b'\leqq b\leqq a'\leqq a\leqq\sigma-b$.

If, then, $\frac{1}{2}(\sigma-a)\leqq b'\leqq a'\leqq a\leqq\sigma-b'$, the xy- and xz-differentiant $\chi_{a-a'}$ of type $(a,\, b',\, \sigma-a-b')$ subordinate to $(a',\, b',\, \sigma-a'-b')$ quâ $\overset{\frown}{xz}$ and of rank $b-b'$ quâ $\overset{\frown}{yz}$ is turned by the operator $\{y;\, x\}\cdot\overset{\frown}{zx}{}^{a-a'}$ into an xy-differentiant of type $(a',\, b',\, \sigma-a'-b')$ satisfying the condition (44) with a non-vanishing value of the constant $S_{a-a'}$ if $b'\leqq b\leqq a'$; the condition (44) is equivalent to (31) if, in the latter, r is replaced by $a-a'$, $\overset{\frown}{xy}$ by $\overset{\frown}{xz}$, ψ_r by $\chi_{a-a'}$, and ω_r by $\{y;\, x\}\cdot\overset{\frown}{zx}{}^{a-a'}$; and the conditions $b'\leqq b\leqq a'$ are satisfied by *every* xy-differentiant of type $(a',\, b',\, \sigma-a'-b')$ whose derivative quâ $\overset{\frown}{xz}$ of type $(a,\, b',\, \sigma-a-b')$ is of rank $b-b'$ quâ $\overset{\frown}{yz}$. Therefore, by (36), if $\frac{1}{2}(\sigma-a')\leqq b'\leqq a'\leqq\sigma-b'$,

$$\sum_{b=b'}^{a'}\sum_{a=a'}^{\sigma-b} N_{xy,\,xz}^{\overset{\frown}{yz}{}^{b-b'}}(a,\, b',\, \sigma-a-b')=N_{xy}(a',\, b',\, \sigma-a'-b')$$

or, by (43),

$$\sum_{b=b'}^{a'}\sum_{a=a'}^{\sigma-b} N_{xyz}(a,\, b,\, \sigma-a-b)=N_{xy}(a',\, b',\, \sigma-a'-b'), \tag{45}$$

where, however, the upper limit of b is $\sigma-a'$ if $\frac{1}{2}\sigma\leqq a'$, because otherwise there could be no summation quâ a. Evidently, the summations quâ r and ρ in (36) may be performed in either order if the limits are properly taken, and here it is more convenient to sum first with respect to r $(a-a')$ and then with respect to ρ $(b-b')$.

If $\frac{1}{2}(\sigma - a') \leqq b' \leqq a' \leqq \sigma - b' - 1$, (45) holds also if a' is replaced by $a' + 1$, and when we subtract the result of the replacement from (45), member from member, we find

$$(46)\quad \begin{cases} \sum_{b=b'}^{a'} N_{xyz}(a', b, \sigma - a' - b) - \sum_{a=a'+1}^{\sigma-a'-1} N_{xyz}(a, a'+1, \sigma - a - a' - 1) \\ \qquad\qquad = (1 - \overline{xz}) \cdot N_{xy}(a', b', \sigma - a' - b') \end{cases}$$

if $a' \leqq \frac{1}{2}\sigma$, but

$$(47)\quad \sum_{b=b'}^{\sigma-a'} N_{xyz}(a', b, \sigma - a' - b) = (1 - \overline{xz}) \cdot N_{xy}(a', b', \sigma - a' - b')$$

if $\frac{1}{2}\sigma \leqq a'$; the special case $a' = \frac{1}{2}(\sigma - 1)$ requires a' to be taken as the upper limit of b before the replacement of a' by $a' + 1$ and $\sigma - a' - 1 = a'$ afterward, and the result of the subtraction is (46); in the special case $a' = \frac{1}{2}\sigma$, (46) and (47) are identical; the second sum of (46) vanishes if $a' = \frac{1}{2}\sigma$ or $\frac{1}{2}(\sigma - 1)$.

If $\frac{1}{2}(\sigma - a') \leqq b' \leqq a' = \sigma - b'$, that is, if $\frac{1}{2}\sigma \leqq a'$ and $b' = \sigma - a'$, (45) gives directly

$$(48)\qquad N_{xyz}(a', \sigma - a', 0) = N_{xy}(a', \sigma - a', 0),$$

which is consistent with (47) [and with (46) if $b' = a' = \frac{1}{2}\sigma$], —because $\overline{xz} \cdot N_{xy}(a', \sigma - a', 0) = 0$, by § 5, — and true, because every xy-differentiant of weight 0 in z is an xyz-differentiant, by the foot-note to § 5. So that (46) holds if $\frac{1}{2}(\sigma - a') \leqq b' \leqq a' \leqq \sigma - b'$ and $a' \leqq \frac{1}{2}\sigma$, while (47) holds if $\frac{1}{2}(\sigma - a') \leqq b' \leqq a' \leqq \sigma - b'$ and $\frac{1}{2}\sigma \leqq a'$.

If $\frac{1}{2}(\sigma - a') \leqq b' < a' \leqq \sigma - b' - 1$, (46) holds for $a' \leqq \frac{1}{2}\sigma$ and (47) for $\frac{1}{2}\sigma \leqq a'$ when b' is replaced by $b' + 1$, the difference of the formulæ before and after replacement, in either case, being

$$(49)\quad N_{xyz}(a', b', \sigma - a' - b') = (1 - \overline{yz})(1 - \overline{xz}) \cdot N_{xy}(a', b', \sigma - a' - b'),$$

which, because $\overline{yz}$, $\overline{xz}$ and $\overline{yz} \cdot \overline{xz}$ annihilate $N_{xy}(a', \sigma - a', 0)$, includes also (48), — that being the case of (49) for which $a' = \sigma - b'$.

If $\frac{1}{2}(\sigma - a') \leqq b' = a' \leqq \sigma - b'$, that is, if $\frac{1}{3}\sigma \leqq a' \leqq \frac{1}{2}\sigma$ and $b' = a'$, (45) gives

$$\sum_{a=a'}^{\sigma-a'} N_{xyz}(a, a', \sigma - a - a') = N_{xy}(a', a', \sigma - 2a')$$

or, with the replacement of $N_{xyz}(a, a', \sigma - a - a')$ by its expression from (49) for all values of a *greater than* a' [for which, evidently, (49) holds],

$$N_{xyz}(a', a', \sigma - 2a') = N_{xy}(a', a', \sigma - 2a') - (1 - \overline{yz}) \cdot \sum_{a=a'+1}^{\sigma-a'} (1 - \overline{xz}) \\ \times N_{xy}(a, a', \sigma - a - a')$$

$$=N_{xy}(a', a', \sigma-2a')-N_{xy}(a'+1, a', \sigma-2a'-1)+\overline{yz}\cdot N_{xy}(a'+1, a', \sigma-2a'-1)$$
$$=(1-\overline{xz}+\overline{xz}\cdot\overline{yz})\cdot N_{xy}(a', a', \sigma-2a')$$
$$=(1-\overline{yz})(1-\overline{xz})\cdot N_{xy}(a', a', \sigma-2a'),\text{—}$$

because $\overline{yz}\cdot N_{xy}(a', a', \sigma-2a')=N_{xy}(a', a'+1, \sigma-2a'-1)=0$, by theorem A of § 8, — and this result is what (49) becomes for $b'=a'$.* Therefore, (49) holds for $\frac{1}{2}(\sigma-a')\leqq b'\leqq a'\leqq\sigma-b'$, that is, for *all* types of which there can be xyz-differentiants in accordance with theorem A of § 8.

On replacing $N_{xy}(a', b', \sigma-a'-b')$ in (49) by its expression from (37), we have, if $\frac{1}{2}(\sigma-a')\leqq b'\leqq a'\leqq\sigma-b'$,

$$N_{xyz}(a', b', \sigma-a'-b')=(1-\overline{yz})(1-\overline{xz})(1-\overline{xy})\cdot N(a', b', \sigma-a'-b');$$

after the restoration of w_x, w_y, w_z for $a', b', \sigma-a'-b'$, respectively, we have, if $w_z\leqq w_y\leqq w_x$,

$$(50)\qquad N_{xyz}(w_x, w_y, w_z)=(1-\overline{yz})(1-\overline{xz})(1-\overline{xy})\cdot N(w_x, w_y, w_z).$$

By § 5, $\overline{xy}$, $\overline{xz}$, and $\overline{yz}$ are commutative and $\overline{xy}\cdot\overline{yz}=\overline{xz}$; therefore, (50) may be written

$$(51)\qquad \begin{cases} N_{xyz}\cdot(w_x, w_y, w_z)=(1-\overline{xy}-\overline{yz}+\overline{xz})(1-\overline{xz})\cdot N(w_x, w_y, w_z) \\ \qquad =[1-\overline{xy}-\overline{yz}+(\overline{xy}+\overline{yz}-\overline{xz})\cdot\overline{xz}]\cdot N(w_x, w_y, w_z).\end{cases}$$

In particular, if $w_y=w_x$,

$$\overline{yz}\cdot N(w_x, w_x, w_z)=N(w_x, w_x+1, w_z-1)=N(w_x+1, w_x, w_z-1)$$
$$=\overline{xz}\cdot N(w_x, w_x, w_z),$$

so that

$$N_{xyz}(w_x, w_x, w_z)=(1-\overline{xz}+\overline{xz}\cdot\overline{yz})(1-\overline{xy})\cdot N(w_x, w_x, w_z),$$

as we have already seen in the special case of (49) for which $b'=a'$.

20. In combination with (43), (41), and (37), formula (49) gives

$$(52)\qquad \begin{cases} N_{xy,\,xz}^{\widehat{yz^r}}(w_x, w_y, w_z)=(1-\overline{yz})(1-\overline{xz})\cdot\overline{yz}^r\cdot N_{xy}(w_x, w_y, w_z) \\ \qquad =(1-\overline{yz})(1-\overline{xz})(1-\overline{xy})\cdot\widehat{yz^r}\cdot N(w_x, w_y, w_z)\end{cases}$$

if $w_z\leqq w_y\leqq w_x$ and $0\leqq r\leqq w_x-w_y$, and

$$N_{xy,\,xz}(w_x, w_y, w_z)=(1-\overline{xz})\cdot\sum_{r=0}^{w_x-w_y}(1-\overline{yz})\cdot\overline{yz}^r\cdot N_{xy}(w_x, w_y, w_z)$$
$$=(1-\overline{xz})\cdot[N_{xy}(w_x, w_y, w_z)-\overline{yz}\cdot N_{xy}(w_x, w_x, w_y+w_z-w_x)]$$

*If $a'=\frac{1}{2}\sigma$, the sum in the first expression for $N_{xyz}(a', a', \sigma-2a')$ vanishes and we have $N_{xyz}(\frac{1}{2}\sigma, \frac{1}{2}\sigma, 0)=N_{xy}(\frac{1}{2}\sigma, \frac{1}{2}\sigma, 0)$, which is a particular case of (49), — because $\overline{xz}$ and $\overline{yz}$ annihilate $N_{xy}(\frac{1}{2}\sigma, \frac{1}{2}\sigma, 0)$, by § 5, — and this formula is true, by (48).

$$= (1 - \overline{xz}) \cdot N_{xy}(w_x, w_y, w_z) + \overline{xz} \cdot \overline{yz} \cdot N_{xy}(w_x, w_x, w_y + w_z - w_x),$$

that is, by (37),

$$(53)\begin{cases} N_{xy,\,xz}(w_x, w_y, w_z) \\ \quad = (1 - \overline{xz}) \cdot N_{xy}(w_x, w_y, w_z) + N_{xy}(w_x + 1, w_x + 1, w_y + w_z - w_x - 2) \\ \quad = (1 - \overline{xy})(1 - \overline{xz}) \cdot N(w_x, w_y, w_z) \\ \qquad\qquad + (1 - \overline{xy}) \cdot N(w_x + 1, w_x + 1, w_y + w_z - w_x - 2) \end{cases}$$

if $w_z \leqq w_y \leqq w_x$. If $w_y + w_z < w_x + 2$, the last term of each of the expressions for $N_{xy,\,xz}(w_x, w_y, w_z)$ given by (53) vanishes.

Because the sequence of variables is unessential, (52) gives, if $w_y \leqq w_z \leqq w_x$ and $0 \leqq r \leqq w_x - w_z$,

$$N^{\widehat{yz}^r}_{xy,\,xz}(w_x, w_z, w_y) = (1 - \overline{yz})(1 - \overline{xz})(1 - \overline{xy}) \cdot N(w_x, w_z + r, w_y - r)$$

$$= N^{\widehat{zy}^r}_{xz,\,xy}(w_x, w_y, w_z) = N^{\widehat{yz}^{w_z - w_y + r}}_{xy,\,xz}(w_x, w_y, w_z),$$

by (19), from which follows, when r is replaced by $r + w_y - w_z$,

$$N^{\widehat{yz}^r}_{xy,\,xz}(w_x, w_y, w_z) = (1 - \overline{yz})(1 - \overline{xz})(1 - \overline{xy}) \cdot N(w_x, w_y + r, w_z - r)$$

if $w_y \leqq w_z \leqq w_x$ and $w_z - w_y \leqq r \leqq w_x - w_y$, but this formula is identical with (52); therefore, (52) holds if $w_y \leqq w_x$, $w_z \leqq w_x$, and $w_z - w_y \leqq r \leqq w_x - w_y$, without regard to the relative values of w_y and w_z (of course, r is not negative); that is, (52) holds for *all* types of which there can be xy- and xz-differentiants in accordance with theorem A of § 8 and for *all* possible ranks of such xy- and xz-differentiants quâ $\widehat{yz}$, by theorem F of § 8 and by § 18.

Similarly, (53) gives, if $w_y \leqq w_z \leqq w_x$,

$$\begin{aligned} &N_{xy,\,xz}(w_x, w_z, w_y) \\ &\quad = (1-\overline{xy})(1-\overline{xz}) \cdot N(w_x, w_y, w_z) + (1-\overline{xy}) \cdot N(w_x+1, w_x+1, w_y+w_z-w_x-2) \\ &\quad = N_{xz,\,xy}(w_x, w_y, w_z) \\ &\quad = (1-\overline{xz})(1-\overline{xy}) \cdot N(w_x, w_y, w_z) + (1-\overline{xy}) \cdot N(w_x+1, w_x+1, w_y+w_z-w_x-2), \end{aligned}$$

which is identical with (53), so that (53) holds if $w_y \leqq w_x$ and $w_z \leqq w_x$, without regard to the relative values of w_y and w_z; that is, (53) holds for *all* types of which there can be xy- and xz-differentiants in accordance with theorem A of § 8.

21. If ϕ is a yz-differentiant of type $(\sigma - b' - c', b', c')$, — where, by theorem A of § 8, $c' \leqq b' \leqq \sigma - c'$, — so that $\widehat{yz} \cdot \phi \equiv 0$; then, by (17),

$$\widehat{yz} \cdot \widehat{xz}^k \cdot \phi \equiv \widehat{xz}^k \cdot \widehat{yz} \cdot \phi \equiv 0,$$

that is, the differentiant character xz *does not interfere* with the differentiant character yz; the derivatives of yz-differentiants of *given* type $(\sigma - b' - c', b', c')$ quâ $\overset{\frown}{xz}$ are, then, xz- and yz-differentiants of types $(\sigma - b' - c, b', c)$ for which $0 \leqq c \leqq c' \leqq b' \leqq \sigma - c - c'$, by theorems A and F of § 8; if $\chi_{c'-c}$ is any xz- and yz-differentiant of type $(\sigma - b' - c, b', c)$,—for which, in the notation of §§ 10–13, $r = c' - c$ and $0 \leqq r \leqq c'$,—then, by § 9, $\{z; y\} \cdot \overset{\frown}{zx}^{c'-c} \cdot \chi_{c'-c}$ is a yz-differentiant of type $(\sigma - b' - c', b', c')$, if not identically 0.

Now, by § 17, if $0 \leqq c \leqq b' \leqq \frac{1}{2}(\sigma - c)$, we may take a complete system of xz- and yz-differentiants $\chi_{c'-c}$ of type $(\sigma - b' - c, b', c)$ in such manner that any one of them of rank $b' - b$ quâ $\overset{\frown}{xy}$ can be represented as $\chi_{c'-c} \equiv \overset{\frown}{yx}^{b'-b} \cdot \psi_{b'-b}$, where $\psi_{b'-b}$ is an xyz-differentiant of some type $(\sigma - b - c, b, c)$ subordinate to $(\sigma - b' - c, b', c)$ quâ $\overset{\frown}{xy}$ for which $c \leqq b \leqq b'$; then,

$$\overset{\frown}{xz} \cdot \chi_{c'-c} \equiv 0, \qquad \overset{\frown}{yz} \cdot \chi_{c'-c} \equiv 0,$$

$$\overset{\frown}{xy} \cdot \psi_{b'-b} \equiv 0, \qquad \overset{\frown}{xz} \cdot \psi_{b'-b} \equiv 0, \qquad \overset{\frown}{yz} \cdot \psi_{b'-b} \equiv 0,$$

and, therefore,

$$\overset{\frown}{xz} \cdot \overset{\frown}{yx}^{b'-b+\alpha} \cdot \psi_{b'-b} \equiv 0 \quad \text{and} \quad \overset{\frown}{xz}^{\beta} \cdot \overset{\frown}{zx}^{c'-c-\alpha} \cdot \overset{\frown}{yx}^{b'-b+\alpha} \cdot \psi_{b'-b} \equiv 0 \text{ if } c'-c-\alpha < \beta,$$

so that, by (17) and (18),

$$\begin{aligned} \overset{\frown}{yz}^{\alpha} \cdot \overset{\frown}{zx}^{c'-c} \cdot \chi_{c'-c} &\equiv (-1)^{\alpha} \cdot (c' - c)^{(\alpha)} \cdot \overset{\frown}{zx}^{c'-c-\alpha} \cdot \overset{\frown}{yx}^{\alpha} \cdot \chi_{c'-c} \\ &\equiv (-1)^{\alpha} \cdot (c' - c)^{(\alpha)} \cdot \overset{\frown}{zx}^{c'-c-\alpha} \cdot \overset{\frown}{yx}^{b'-b+\alpha} \cdot \psi_{b'-b}, \end{aligned}$$

$$\overset{\frown}{xz}^{c'-c} \cdot \overset{\frown}{zy}^{\alpha} \cdot \overset{\frown}{yz}^{\alpha} \cdot \overset{\frown}{zx}^{c'-c} \cdot \chi_{c'-c} \equiv (-1)^{\alpha} \cdot (c' - c)^{(\alpha)} \cdot \overset{\frown}{xy}^{\alpha} \cdot \overset{\frown}{xz}^{c'-c-\alpha} \cdot \overset{\frown}{yz}^{\alpha} \cdot \overset{\frown}{zx}^{c'-c} \cdot \chi_{c'-c}$$

$$\begin{aligned} &\equiv (c' - c)!\,(c' - c)^{(\alpha)} \cdot (\sigma - b' - 2c - \alpha)^{(c'-c-\alpha)} \cdot \overset{\frown}{xy}^{\alpha} \cdot \overset{\frown}{yx}^{b'-b+\alpha} \cdot \psi_{b'-b} \\ &\equiv (c' - c)!\,(c' - c)^{(\alpha)} \cdot (\sigma - b' - 2c - \alpha)^{(c'-c-\alpha)} \cdot (b' - b + \alpha)^{(\alpha)} \\ &\qquad\qquad \times (\sigma - b - b' - c)^{(\alpha)} \cdot yx^{b'-b} \cdot \psi_{b'-b} \\ &\equiv \frac{(c' - c)!\,(\sigma - b - b' - c)!}{(\sigma - b' - c - c')!\,(b' - b)!} \cdot (c' - \ \ )^{(\alpha)} \cdot \frac{(\sigma - b' - 2c - \alpha)!\,(b' - b + \alpha)!}{(\sigma - b - b' - c - \alpha)!} \cdot \chi_{c'-c} \end{aligned}$$

and, by (23),

$$\overset{\frown}{xz}^{c'-c} \cdot \{z, y\} \cdot \overset{\frown}{zx}^{c'-c} \cdot \chi_{c'-c} \equiv S_{c'-c} \cdot \chi_{c'-c}, \tag{54}$$

where, by (15),

$$\begin{aligned} S_{c'-c} &= \frac{(c' - c)!\,(b' - c' + 1)!\,(\sigma - b - b' - c)!}{(b' - b)!\,(\sigma - b' - c - c')!} \cdot \sum_{\alpha=0}^{c'-c} (-1)^{\alpha} \cdot \binom{c' - c}{\alpha} \\ &\qquad \times \frac{(\sigma - b' - 2c - \alpha)^{(b-c)}}{(b' - c' + \alpha + 1)^{(b-c'+1)}} \\ &= \frac{(c' - c)!\,(b' - c' + 1)!\,(\sigma - b - b' - c)!}{(b' - b)!\,(\sigma - b' - c - c')!} \cdot S^{b-c'+1,\ c'-c}_{\sigma-b'-2c,\ b'-c'+1}, \end{aligned}$$

which, by (14) and (16), is a *non-vanishing* constant if $c \leqq c' \leqq b \leqq b' \leqq \sigma - b - c$; so that, at least when these conditions are satisfied, the derivative of the form $\{z;\, y\} \cdot zx^{c'-c} \cdot \chi_{c'-c}$ quâ $\widehat{xz}$ is a non-vanishing constant multiple of $\chi_{c'-c}$. Observe that the condition $b' \leqq \sigma - b - c$ is satisfied if $b \leqq b' \leqq \frac{1}{2}(\sigma - c)$.

But if ϕ is a yz-differentiant of type $(\sigma - b' - c',\, b',\, c')$ and of rank $c' - c$ quâ $\widehat{xz}$, its derivative quâ $\widehat{xz}$, namely $\widehat{xz}^{c'-c} \cdot \phi$, is an xz- and yz-differentiant of type $(\sigma - b' - c,\, b',\, c)$ and, say, of rank $b' - b$ quâ $\widehat{xy}$, so that $\widehat{xy}^{b'-b} \cdot \widehat{xz}^{c'-c} \cdot \phi$ is a *non-vanishing* xyz-differentiant of type $(\sigma - b - c,\, b,\, c)$ for which, by theorems A and F of § 8, $0 \leqq c \leqq c' \leqq b' \leqq \sigma - c - c'$, $c \leqq b \leqq b'$, and $b \leqq \sigma - b' - c$; then, by (18), because $\widehat{yz} \cdot \phi \equiv 0$,

$$\widehat{yz}^{c'-c} \cdot \widehat{xy}^{b'+c'-b-c} \cdot \phi \equiv (b' + c' - b - c)^{(c'-c)} \cdot \widehat{xy}^{b'-b} \cdot \widehat{xz}^{c'-c} \cdot \phi \not\equiv 0,$$

by (14), while

$$\widehat{yz}^{c'-c+1} \cdot \widehat{xy}^{b'+c'-b-c} \cdot \phi \equiv 0;$$

therefore, $\widehat{xy}^{b'+c'-b-c} \cdot \phi$ is a non-vanishing form of type $(\sigma - b - c,\, b + c - c',\, c')$ and rank $c' - c$ quâ $\widehat{yz}$, so that, by theorem F of § 8, $2c' - b - c \leqq c' - c$, that is, $c' \leqq b$: the types of ϕ and its *successive* derivatives quâ $\widehat{xz}$ and quâ $\widehat{xy}$ are, then, so related that $0 \leqq c \leqq c' \leqq b \leqq b' \leqq \sigma - b - c$.

If, then, $0 \leqq c \leqq c' \leqq b' \leqq \frac{1}{2}(\sigma - c')$, the xz- and yz- differentiant $\chi_{c'-c}$ of type $(\sigma - b' - c,\, b',\, c)$ subordinate to $(\sigma - b' - c',\, b',\, c')$ quâ $\widehat{xz}$ and of rank $b' - b$ quâ $\widehat{xy}$ is turned by the operator $\{z;\, y\} \cdot zx^{c'-c}$ into a yz-differentiant of type $(\sigma - b' - c',\, b',\, c')$ satisfying the condition (54) with a non-vanishing value of the constant $S_{c'-c}$ if $c' \leqq b \leqq b'$: the condition (54) is equivalent to (31) if, in the latter, r is replaced by $c' - c$, $\widehat{xy}$ by $\widehat{xz}$, ψ_r by $\chi_{c'-c}$, and ω_r by $\{z;\, y\} \cdot \widehat{zx}^{c'-c}$; and the conditions $c' \leqq b \leqq b'$ are satisfied by *every* yz-differentiant of type $(\sigma - b' - c',\, b',\, c')$ whose derivative quâ $\widehat{xz}$ of type $(\sigma - b' - c,\, b',\, c)$ is of rank $b' - b$ quâ $\widehat{xy}$. Therefore, by (36), if $0 \leqq c' \leqq b' \leqq \frac{1}{2}(\sigma - c')$,

$$\sum_{c=0}^{c'} \sum_{b=c'}^{b'} N_{xz,\, yz}^{\widehat{xy}^{b'-b}}(\sigma - b' - c,\, b',\, c) = N_{yz}(\sigma - b' - c',\, b',\, c')$$

or, by (40),

$$\sum_{c=0}^{c'} \sum_{b=c'}^{b'} N_{xyz}(\sigma - b - c,\, b,\, c) = N_{yz}(\sigma - b' - c',\, b',\, c'). \tag{55}$$

If $1 \leqq c' \leqq b' \leqq \frac{1}{2}(\sigma - c')$, (55) holds also when c' is replaced by $c' - 1$, and when we substract the result of the replacement from (55), member from member, we find

$$\left\{\begin{aligned} &\sum_{b=c'}^{b'} N_{xyz}(\sigma - b - c',\, b,\, c') - \sum_{c=0}^{c'-1} N_{xyz}(\sigma - c - c' + 1,\, c' - 1,\, c) \\ &\qquad = (1 - \overline{xz}) \cdot N_{yz}(\sigma - b' - c',\, b',\, c'). \end{aligned}\right. \tag{56}$$

If $c'=0$, (56) is identical with (55), because $\overline{xz}\cdot N_{yz}(\sigma-b', b', 0)=0$ and the second sum in (56) vanishes by the general principles of summation; therefore, (56) holds for $0\leqq c'\leqq b'\leqq\frac{1}{2}(\sigma-c')$.

If $0\leqq c'<b'\leqq\frac{1}{2}(\sigma-c')$, (56) holds when b' is replaced by $b'-1$, the difference of the formulæ before and after replacement being

$$(57)\quad N_{xyz}(\sigma-b'-c', b', c')=(1-\overline{xy})(1-\overline{xz})\cdot N_{yz}(\sigma-b'-c', b', c').$$

If $0\leqq c'=b'\leqq\frac{1}{2}(\sigma-c')$, that is, if $0\leqq c'\leqq\frac{1}{3}\sigma$ and $b'=c'$, (55) gives

$$\sum_{c=0}^{c'}N_{xyz}(\sigma-c-c', c', c)=N_{yz}(\sigma-2c', c', c');$$

or, with the replacement of $N_{xyz}(\sigma-c-c', c', c)$ by its expression from (57) for all values of c *less than* c' [for which, evidently, (57) holds],

$$\begin{aligned}N_{xyz}(\sigma-2c', c', c')&=N_{yz}(\sigma-2c', c', c')-(1-\overline{xy})\cdot\sum_{c=0}^{c'-1}(1-\overline{xz})\\&\quad\times N_{yz}(\sigma-c-c', c', c)=N_{yz}(\sigma-2c', c', c')-N_{yz}(\sigma-2c'+1, c', c'-1)\\&\quad+\overline{xy}\cdot N_{yz}(\sigma-2c'+1, c', c'-1)=(1-\overline{xz}+\overline{xy}\cdot\overline{xz})\cdot N_{yz}(\sigma-2c', c', c')\\&=(1-\overline{xy})(1-\overline{xz})\cdot N_{yz}(\sigma-2c', c', c'),\text{—}\end{aligned}$$

because $\overline{xy}\cdot N_{yz}(\sigma-2c', c', c')=N_{yz}(\sigma-2c'+1, c'-1, c')=0$, by theorem A of § 8, — and this result is what (57) becomes for $b'=c'$.* Therefore, (57) holds for $0\leqq c'\leqq b'\leqq\frac{1}{2}(\sigma-c')$, that is, for *all* types of which there can be xyz-differentiants in accordance with theorem A of § 8.

On replacing $N_{yz}(\sigma-b'-c', b', c')$ in (57) by its expression from (37), we have, if $0\leqq c'\leqq b'\leqq\frac{1}{2}(\sigma-c')$,

$$N_{xyz}(\sigma-b'-c', b', c')=(1-\overline{xy})(1-\overline{xz})(1-\overline{yz})\cdot N(\sigma-b'-c', b', c');$$

after the restoration of w_x, w_y, w_z for $\sigma-b'-c'$, b', c', respectively, we have, if $w_z\leqq w_y\leqq w_x$,

$$(58)\quad N_{xyz}(w_x, w_y, w_z)=(1-\overline{xy})(1-\overline{xz})(1-\overline{yz})\cdot N(w_x, w_y, w_z),$$

which is identical with (50) and (51). In particular, if $w_y=w_z$,

$$\begin{aligned}\overline{xy}\cdot N(w_x, w_z, w_z)&=N(w_x+1, w_z-1, w_z)\\&=N(w_x+1, w_z, w_z-1)=\overline{xz}\cdot N(w_x, w_z, w_z),\end{aligned}$$

so that

$$N_{xyz}(w_x, w_z, w_z)=(1-\overline{xz}+\overline{xy}\cdot\overline{xz})(1-\overline{yz})\cdot N(w_x, w_z, w_z),$$

as we have already seen in the special case of (57) for which $b'=c'$.

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* If $c'=0$, the sum in the first expression for $N_{xyz}(\sigma-2c', c', c')$ vanishes, but the result is valid, by (55) for $b'=c'=0$.

22. In combination with (40), (38), and (37), formula (57) gives

$$(59)\quad \begin{cases} N^{\widehat{xy}^r}_{xz,\,yz}(w_x,\,w_y,\,w_z) = (1-\overline{xy})(1-\overline{xz})\cdot\overline{xy}^r\cdot N_{yz}(w_x,\,w_y,\,w_z) \\ \qquad\qquad = (1-\overline{xy})(1-\overline{xz})(1-\overline{yz})\cdot\overline{xy}^r\cdot N(w_x,\,w_y,\,w_z) \end{cases}$$

if $w_z \leqq w_y \leqq w_x$ and $0 \leqq r \leqq w_y - w_z$, and

$$N_{xz,\,yz}(w_x,\,w_y,\,w_z) = (1-\overline{xz})\cdot\sum_{r=0}^{w_y-w_z}(1-\overline{xy})\cdot\overline{xy}^r\cdot N_{yz}(w_x,\,w_y,\,w_z)$$
$$= (1-\overline{xz})\cdot[N_{yz}(w_x,\,w_y,\,w_z) - \overline{xy}\cdot N_{yz}(w_x+w_y-w_z,\,w_z,\,w_z)]$$
$$= (1-\overline{xz})\cdot N_{yz}(w_x,\,w_y,\,w_z) + \overline{xy}\cdot\overline{xz}\cdot N_{yz}(w_x+w_y-w_z,\,w_z,\,w_z),$$

that is, by (37),

$$(60)\quad \begin{cases} N_{xz,\,yz}(w_x,\,w_y,\,w_z) \\ = (1-\overline{xz})\cdot N_{yz}(w_x,\,w_y,\,w_z) + N_{yz}(w_x+w_y-w_z+2,\,w_z-1,\,w_z-1) \\ =(1-\overline{xz})(1-\overline{yz})\cdot N(w_x,w_y,w_z)+(1-\overline{yz})\cdot N(w_x+w_y-w_z+2,w_z-1,w_z-1) \end{cases}$$

if $w_z \leqq w_y \leqq w_x$. If $w_z = 0$, the last term in each of the expressions for $N_{xz,\,yz}(w_x,\,w_y,\,w_z)$ vanishes and we have

$$N_{xz,\,yz}(w_x,\,w_y,\,0) = N_{yz}(w_x,\,w_y,\,0) = N(w_x,\,w_y,\,0),$$

as it should be, because every form of weight 0 in z is an xz- and yz-differentiant, by the footnote to § 5.

Because the sequence of variables is unessential, (59) gives, if $w_z \leqq w_x \leqq w_y$ and $0 \leqq r \leqq w_x - w_z$,

$$N^{\widehat{xy}r}_{xz,\,yz}(w_y,\,w_x,\,w_z) = (1-\overline{xy})(1-\overline{xz})(1-yz)\cdot N(w_y+r,\,w_x-r,\,w_z)$$
$$= N^{\widehat{yx}r}_{yz,\,xz}(w_x,\,w_y,\,w_z) = N^{\widehat{xy}^{w_y-w_x+r}}_{xz,\,yz}(w_x,\,w_y,\,w_z),$$

by (19), from which follows, when r is replaced by $r + w_x - w_y$,

$$N^{\widehat{xy}r}_{xz,\,yz}(w_x,\,w_y,\,w_z) = (1-\overline{xy})(1-\overline{xz})(1-\overline{yz})\cdot N(w_x+r,\,w_y-r,\,w_z)$$

if $w_z \leqq w_x \leqq w_y$ and $w_y - w_x \leqq r \leqq w_y - w_z$; but this formula is identical with (59); therefore, (59) holds if $w_z \leqq w_x$, $w_z \leqq w_y$ and $w_y - w_x \leqq r \leqq w_y - w_z$, without regard to the relative values of w_x and w_y (of course r is not negative); that is, (59) holds for *all* types of which there can be xz- and yz-differentiants in accordance with theorem A of § 8 and for *all* possible ranks of such xz- and yz-differentiants quâ $\widehat{xy}$, by theorem F of § 8 and by § 17.

Similarly, (60) gives, if $w_z \leqq w_x \leqq w_y$,

$$N_{xz,\,yz}(w_y,\,w_x,\,w_z) = (1-\overline{xz})(1-\overline{yz})\cdot N(w_y,\,w_x,\,w_z)$$
$$+ (1-\overline{yz})\times N(w_y+w_x-w_z+2,\,w_z-1,\,w_z-1) = N_{yz,\,xz}(w_x,\,w_y,\,w_z)$$
$$= (1-\overline{yz})(1-\overline{xz})\times N(w_x,\,w_y,\,w_z)$$
$$+ (1-\overline{yz})\cdot N(w_y+w_x-w_z+2,\,w_z-1,\,w_z-1),$$

which is identical with (60); therefore (60) holds if $w_z \leqq w_x$ and $w_s \leqq w_y$, without regard to the relative values of w_x and w_y, that is, for *all* types of which there can be *xz*- and *yz*-differentiants in accordance with theorem A of § 8. Formulae (50) or (58), (53) and (60) express the denumerants of *all* kinds of double differentiants whose defining pairs involve only *three* of the variables.

23. The expression for the denumerant of double differentiants of a given type whose two defining pairs involve *four* different variables is found in the same manner in which that for simple differentiants was found in § 16. We consider only forms of a *given* order in the variables, of *given* degrees in the coefficients of the several quantics, and of *given* weights in each of the variables *excepting four*, say x, y, z, and s; in representing the type of such a form, we shall write the weights in these four variables alone, following the sequence of the variables as just given, thus: (w_x, w_y, w_z, w_s), so that, by (2), $w_x+w_y+w_z+w_s$ has the same value for *all* forms of our system. In determining the expression for the denumerant $N_{xy,\,zs}(w_x, w_y, w_z, w_s)$, we shall confine our attention to the *zs*-differentiants that are of *given* weights w_z and w_s in z and s, respectively, — where, by theorem A of § 8, $w_s \leqq w_z$, — and, in writing the type of such a *zs*-differentiant, shall express the weights in x and y alone, in this sequence, as (w_x, w_y). If four weights are written in the type, it is to be understood that they are the weights in x, y, z, and s, respectively: but if only two weights are written, they are the weights in x and y, respectively, and the weights in z and s are constant for all forms whose types are so written. In the latter representation of the type, we shall replace w_x by a (or a') and w_y by $\sigma - a$ (or $\sigma - a'$), where σ has the same value for *all* the forms considered.

If ϕ is a *zs*-differentiant of given type $(a, \sigma - a)$, $\widehat{zs} \cdot \phi \equiv 0$ and, by (17), $\widehat{zs} \cdot \widehat{xy}^k \cdot \phi \equiv \widehat{xy}^k \cdot \widehat{zs} \cdot \phi \equiv 0$, so that the differentiant character *xy does not interfere* with the differentiant character *zs*. The types subordinate to $(a, \sigma - a)$ quâ $\widehat{xy}$ are types $(a', \sigma - a')$ for which $a \leqq a' \leqq \sigma$ (in the notation of § 10, the *zs* character is the only condition K and $r = a' - a$). If $\psi_{a'-a}$ is an *xy*- and *zs*-differentiant of type $(a', \sigma - a')$ subordinate to $(a, \sigma - a)$ quâ $\widehat{xy}$, where $a \leqq a' \leqq \sigma$ and $\frac{1}{2}\sigma \leqq a'$, by theorem A of § 8, then $yx^{a'-a} \cdot \psi_{a'-a}$ is a form of type $(a, \sigma - a)$ for which $\widehat{zs} \cdot \widehat{yx}^{a'-a} \cdot \psi_{a'-a} \equiv \widehat{yx}^{a'-a} \cdot \widehat{zs} \cdot \psi_{a'-a} \equiv 0$, by (17), and

$$\widehat{xy}^{a'-a} \cdot \widehat{yx}^{a'-a} \cdot \psi_{a'-a} \equiv (a' - a)!\,(2a' - \sigma)^{(a'-a)} \cdot \psi_{a'-a},$$

by (18), that is, $\widehat{yx}^{a'-a} \cdot \psi_{a'-a}$ is a *zs*-differentiant that satisfies condition (31) $[r=a'-a,\ \omega_r \equiv \widehat{yx}^{a'-a}$, and $S_r=(a'-a)!\,(2a'-\sigma)^{(a'-a)}]$ if $0 \leqq a \leqq a' \leqq \sigma$ and $\sigma - a \leqq a'$. Therefore, by (35), if $\frac{1}{2}\sigma \leqq a \leqq \sigma$,

$$N_{xy,\,zs}(a, \sigma - a) = (1 - \overline{xy}) \cdot N_{zs}(a, \sigma - a);$$

that is, on restoring w_x for a and w_y for $\sigma - a$, writing the weights in z and s, and expressing the denumerant of zs-differentiants by (37), we have, if $w_y \leqq w_x$ and $w_s \leqq w_z$,

$$(61) \qquad N_{xy,\,zs}(w_x,\, w_y,\, w_z,\, w_s) = (1 - \overline{xy})(1 - \overline{zs}) \cdot N(w_x,\, w_y,\, w_z,\, w_s).$$

Formula (61) shows that, if ϕ is the *most general* form of any type of which there can be xy- and zs-differentiants in accordance with theorem A of § 8, the linear equations between the multipliers of the terms of ϕ that are implied in the identities $\widehat{xy} \cdot \phi \equiv 0$ and $\widehat{zs} \cdot \phi \equiv 0$ are linearly independent in this sense that, when ϕ shall have been determined as the *most general* form of its type $(\cdots)$ that satisfies the identity $\widehat{zs} \cdot \phi \equiv 0$, so that the number of independent arbitrary multipliers of its terms is $N_{zs}(\cdots)$, the number of linearly independent linear equations between these multipliers that are implied in the identity $\widehat{xy} \cdot \phi \equiv 0$ is $\overline{xy} \cdot N_{zs}(\cdots)$.

It is evident that the method of this section can be employed to determine the denumerant of forms of a proper type, in accordance with theorem A of § 8, that have any multiple differentiant character that can be *compounded* of two or more simple or multiple characters of which *no two have a common variable* in their defining pairs when we know the general formulæ for the denumerants of forms having the *several* component characters. For example, if H and K represent two simple or multiple differentiant characters such that no variable in the pairs that define H occurs in the pairs that define K, if we know that $N_H(\cdots) = P \cdot N(\cdots)$ for *every* type $(\cdots)$ of which there can be H-differentiants and that $N_K(\cdots) = Q \cdot N(\cdots)$ for *every* type $(\cdots)$ of which there can be K-differentiants, in accordance with theorem A of § 8, where P and Q are certain type-operators, then, if $(\cdots)$ is any type of which there can be H- and K-differentiants, we shall have $N_{H,\,K}(\cdots) = PQ \cdot N(\cdots)$.

24. If $w_x = w_y = w_z = w$, we have, by (50) and (51),

$$(62) \begin{cases} N_{xyz}(w,\, w,\, w) = N(w,\, w,\, w) - 2N(w+1,\, w,\, w-1) \\ \quad + N(w+2, w-1, w-1) + N(w+1, w+1, w-2) - N(w+2, w, w-2). \end{cases}$$

This is the culminating formula for denumerants of double differentiants. It gives, if x, y, and z are the only variables, the number of linearly independent *invariants* or *covariants* (according as the order is or is not 0) of any possible type $(w,\, w,\, w)$ that belong to any system of *ternary* quantics; namely, as we have stated in § 8, every homogeneous and isobaric xyz-differentiant of equal weights in x and z and (therefore) y is a *complete* differentiant in x, y and z and, therefore, an *invariant* or *covariant* if x, y, and z are the only variables; and, conversely, every invariant or covariant in x, y, and z is a homogeneous isobaric xyz-differentiant of equal weights in x, y, and z. What Cayley's formula does for the invariants and covariants of a *binary* quantic and Sylvester's extension of it for the invariants and covariants of any system of binary quantics, this formula (62) does for the invariants and covariants of any system of *ternary* quantics.

25. Just as (30) follows from (26) and (29) so from theorem D' and the equation preceding (45), together with (52) and (53), follows, if $w_z \leqq w_y \leqq w_x$ and $0 \leqq r \leqq w_z$,

$$N_{xy}^{\widehat{xz}^r}(w_x, w_y, w_z) = \sum_{k=0}^{w_x-w_y} N_{xy,\,xz}^{\widehat{yz}^k}(w_x + r, w_y, w_z - r)$$

$$= (1 - \overline{xz}) \cdot \sum_{k=0}^{w_x-w_y} (1 - \overline{yz}) \cdot \overline{yz}^k \cdot N_{xy}(w_x + r, w_y, w_z - r)$$

$$= (1 - \overline{xz}) \cdot [N_{xy}(w_x + r, w_y, w_z - r) - N_{xy}(w_x + r, w_z + 1, w_y + w_z - w_x - r - 1)]$$

$$(63) \quad = N_{xy,\,xz}(w_x + r, w_y, w_z - r) - N_{xy,\,xz}(w_x + r, w_x + 1, w_y + w_z - w_x - r - 1).$$

But the condition $\frac{1}{2}(\sigma - a') \leqq b'$ for the formula preceding (45) might have been replaced by $\sigma - a - b \leqq b'$, as was previously stated, so that the first expression given above for $N_{xy}^{\widehat{xz}^r}(w_x, w_y, w_z)$ holds also if $w_y \leqq w_z \leqq w_x$ and $0 \leqq r \leqq w_z$; excepting that,

if $w_y \leqq w_z \leqq w_x$ and $0 \leqq r \leqq w_z - w_y$, the lower limit of k is $w_z - w_y - r$ and the first term of the developed expression is $(1 - \overline{xz}) \cdot N_{xy}(w_x + r, w_z - r, w_y)$, which, by (37), is the same as $(1 - \overline{xz}) \cdot N_{xy}(w_x + r, w_y, w_z - r)$, as before, and

if $w_y \leqq w_x$, $w_z \leqq w_x$, and $w_y + w_z - w_x \leqq r \leqq w_z$, the upper limit of k is $w_z - r$ and the second term of the developed expression vanishes, together with the second term of (63). Therefore, (63) holds if $w_y \leqq w_x$, $w_z \leqq w_x$, and $0 \leqq r \leqq w_z$, that is, for *every* type of which there can be xy- and xz-differentiants in accordance with theorem A and for *every* possible rank quâ $\widehat{xz}$ of such differentiants.

Formula (63) shows that the number of linearly independent xy- and xz-differentiants of any type $(w_x + r, w_y, w_z - r)$ subordinate to (w_x, w_y, w_z) quâ $\widehat{xz}$ that must be annexed to those derivable quâ $\widehat{xz}$ from xy-differentiants of type (w_x, w_y, w_z) in order to produce a *complete* system is

$$N_{xy,\,xz}(w_x + r, w_x + 1, w_y + w_z - w_x - r - 1) \text{ if } w_y \leqq w_x \text{ and } w_z \leqq w_x.$$

If $w_y + w_z - w_x \leqq r \leqq w_z$, *every* xy- and xz-differentiant of type $(w_x + r, w_y, w_z - r)$ is derivable quâ $\widehat{xz}$ from the xy-differentiants of type (w_x, w_y, w_z), provided $w_y \leqq w_x$ and $w_z \leqq w_x$. In particular, we find (on putting $r = 1$ and replacing w_x by $w_x - 1$) that the number of linearly independent xy- and xz-differentiants that must be annexed to those derivable quâ $\widehat{xz}$ from the xy-differentiants of type $(w_x - 1, w_y, w_z + 1)$, where $w_y \leqq w_x - 1$ and $w_z \leqq w_x - 1$, in order to produce a *complete* system of xy- and xz-differentiants of type (w_x, w_y, w_z) is

$N_{xy,\,xz}(w_x,\, w_x,\, w_y + w_z - w_x)$, which is evidently $N_{xyz}(w_x,\, w_x,\, w_y + w_z - w_x)$, because, by theorems B and C of § 8, every xy- and xz-differentiant of equal weights in x and y is also a yx- and yz-differentiant and, therefore, an xyz-differentiant; but no such differentiants have to be annexed if $w_y + w_z < w_x$.

It is evident that the result of applying $\widehat{xz}$ to any xy-differentiant of type $(w_x - 1,\, w_y,\, w_z + 1)$, where $w_y \leqq w_x - 1$, is an xy-differentiant of type $(w_x,\, w_y,\, w_z)$, if not identically 0, and the number of xy-differentiants in a complete system of type $(w_x - 1,\, w_y,\, w_z + 1)$ *reduced* quâ $\widehat{xz}$ that are annihilated by $\widehat{xz}$ is $N_{xy,\,xz}(w_x - 1,\, w_y,\, w_z + 1)$, so that the number of linearly independent xy-differentiants of type $(w_x,\, w_y,\, w_z)$ that can be obtained by applying the operator $\widehat{xz}$ to xy-differentiants of type $(w_x - 1,\, w_y,\, w_z + 1)$ is

$$(64)\qquad \begin{cases} N_{xy}(w_x - 1,\, w_y,\, w_z + 1) - N_{xy,\,xz}(w_x - 1,\, w_y,\, w_z + 1) \\ \qquad = N_{xy}(w_x,\, w_y,\, w_z) - N_{xy}(w_x,\, w_x,\, w_y + w_z - w_x), \end{cases}$$

by (53), if $w_y \leqq w_x - 1$ and $w_z \leqq w_x - 2$; that is, the number of linearly independent xy-differentiants of type $(w_x,\, w_y,\, w_z)$ that must be annexed to those obtained by applying the operator $\widehat{xz}$ to xy-differentiants of type $(w_x - 1,\, w_y,\, w_z + 1)$ in order to produce a *complete* system of xy-differentiants of type $(w_x,\, w_y,\, w_z)$ is $N_{xy}(w_x,\, w_x,\, w_y + w_z - w_x)$; but this number is 0 if $w_y + w_z < w_x$, and then *no* such differentiants have to be annexed in order to complete the system. If $w_x - 1 \leqq w_z$, there are *no* xz-differentiants of type $(w_x - 1,\, w_y,\, w_z + 1)$, and the forms obtained by applying $\widehat{xz}$ to a complete system of xy-differentiants of that type are *linearly independent* xy-differentiants of type $(w_x,\, w_y,\, w_z)$, which, however, do *not* generally constitute a complete system.

Similarly, from the equation preceding (55), together with (59) and (60), follows, if $w_z \leqq w_x$, $w_z \leqq w_y$, and $0 \leqq r \leqq w_z$,

$$N_{yz}^{\widehat{xz}^r}(w_x,\, w_y,\, w_z) = \sum_{k=0}^{w_y - w_z} N_{xz,\,yz}^{\widehat{xy}^k}(w_x + r,\, w_y,\, w_z - r)$$

$$= (1 - \overline{xz})\cdot \sum_{k=0}^{w_y - w_z} (1 - \overline{xy})\cdot \overline{xy}^k \cdot N_{yz}(w_x + r,\, w_y,\, w_z - r)$$

$$=(1-\overline{xz})\cdot [N_{yz}(w_x+r,\, w_y,\, w_z-r)-N_{yz}(w_x+w_y-w_z+r+1,\, w_z-1,\, w_z-r)]$$

$$(65)\quad = N_{xz,\,yz}(w_x+r,\, w_y,\, w_z-r)-N_{xz,\,yz}(w_x+w_y-w_z+r+1,\, w_z-1,\, w_z-r);$$

so that the number of linearly independent xz- and yz-differentiants of type $(w_x + r,\, w_y,\, w_z - r)$ that must be annexed to those derivable quâ $\widehat{xz}$ from yz-differentiants of type $(w_x,\, w_y,\, w_z)$ in order to produce a *complete* system is $N_{xz,\,yz}(w_x + w_y - w_z + r + 1,\, w_z - 1,\, w_z - r)$. In particular, we find (on putting $r = 1$ and replacing w_z by $w_z + 1$) that the number of linearly independent xz- and yz-differentiants of type (w_x, w_y, w_z) that must be annexed to those derivable quâ $\widehat{xz}$ from yz-differentiants of type $(w_x - 1, w_y, w_z + 1)$, where $w_z + 1 \leqq w_x$ and $w_z + 1 \leqq w_y$, in order to produce a *complete* system of xz- and yz-differentiants of type (w_x, w_y, w_z) is $N_{xz,\,yz}(w_x + w_y - w_z, w_z, w_z)$, which is also $N_{xyz}(w_x + w_y - w_z, w_z, w_z)$.

From (60) follows, if $w_z + 2 \leqq w_x$ and $w_z + 1 \leqq w_y$,

$$(66)\quad \left\{\begin{aligned} &N_{yz}(w_x - 1, w_y, w_z + 1) - N_{xz,\,yz}(w_x - 1, w_y, w_z + 1) \\ &\qquad\qquad = N_{yz}(w_x, w_y, w_z) - N_{yz}(w_x + w_y - w_z, w_z, w_z), \end{aligned}\right.$$

which shows that the number of linearly independent yz-differentiants of type (w_x, w_y, w_z) that must be annexed to those obtained by applying the operator $\widehat{xz}$ to yz-differentiants of type $(w_x - 1, w_y, w_z + 1)$ in order to produce a *complete* system of yz-differentiants of type (w_x, w_y, w_z) is $N_{yz}(w_x + w_y - w_z, w_z, w_z)$. If $w_x \leqq w_z + 1$, there are *no* xz-differentiants of type $(w_x - 1, w_y, w_z + 1)$ and the forms obtained by applying $\widehat{xz}$ to a complete system of yz-differentiants of that type are *linearly independent* yz-differentiants of type (w_x, w_y, w_z), which, however, do *not* generally constitute a complete system.

26. Because we have always excluded denumerants of such types that their values as given by our formulæ might come out negative, these formulæ may serve to determine the *relative* values of some denumerants involved in the expressions of others, just as (37) determined the relative numbers of terms in the general forms of certain types.

It follows from (45) that, if $w_z \leqq w_y \leqq w_x$,

$$N_{xy}(w_x, w_y + 1, w_z - 1) \leqq N_{xy}(w_x, w_y, w_z);$$

that is, of two types of which there can be xyz-differentiants in accordance with theorem A of § 8 and of which the one is subordinate to the other quâ $\overline{yz}$, that one has the more linearly independent xy-differentiants for which the weights in y and z are the more nearly equal, unless they have the same number.

It follows from (55) that, if $w_z \leqq w_y \leqq w_x$,

$$N_{yz}(w_x + 1, w_y - 1, w_z) \leqq N_{yz}(w_x, w_y, w_z);$$

or, because the sequence of variables is immaterial, if $w_y \leqq w_x \leqq w_z$,

$$N_{xy}(w_x - 1, w_y, w_z + 1) \leqq N_{xy}(w_x, w_y, w_z);$$

that is, of two types of which there can be zxy-differentiants in accordance with theorem A and of which the one is subordinate to the other quâ $\widehat{zx}$ (or $\widehat{xz}$), that one has the more linearly independent xy-differentiants for which the weights in x and z are the more nearly equal, unless they have the same number.

It follows from (61) and (37) that, if $w_y \leqq w_x$ and $w_s \leqq w_z$,

$$N_{xy,\,zs}(w_x, w_y, w_z, w_s) = (1 - \overline{zs}) \cdot N_{xy}(w_x, w_y, w_z, w_s)$$

and, therefore,

$$N_{xy}(w_x, w_y, w_z+1, w_s-1) \leqq N_{xy}(w_x, w_y, w_z, w_s).$$

Similarly, if $w_y \leqq w_x$ and $w_z \leqq w_s$,

$$N_{xy}(w_x, w_y, w_z-1, w_s+1) \leqq N_{xy}(w_x, w_y, w_z, w_s);$$

that is, of two types of which there can be xy-differentiants in accordance with theorem A and of which the one is subordinate to the other relatively to some pair of *other* variables than x and y, that one has the more linearly independent xy-differentiants for which the weights in the two other variables are the more nearly equal, unless they have the same number.

It follows from (39) that, if $w_z \leqq w_y \leqq w_x$,

$$N_{xz,\,yz}(w_x+1, w_y-1, w_z) \leqq N_{xz,\,yz}(w_x, w_y, w_z)$$

and, if $w_z \leqq w_x \leqq w_y$,

$$N_{xz,\,yz}(w_x-1, w_y+1, w_z) \leqq N_{xz,\,yz}(w_x, w_y, w_z);$$

that is, of two types of which there can be xz- and yz-differentiants and of which the one is subordinate to the other quâ xy, that one has the more linearly independent xz- and yz-differentiants for which the weights in x and y are the more nearly equal, unless they have the same number.

It follows from (42) that if $w_z \leqq w_y \leqq w_x$,

$$N_{xy,\,xz}(w_x, w_y+1, w_z-1) \leqq N_{xy,\,xz}(w_x, w_y, w_z)$$

and, if $w_y \leqq w_z \leqq w_x$,

$$N_{xy,\,xz}(w_x, w_y-1, w_z+1) \leqq N_{xy,\,xz}(w_x, w_y, w_z);$$

that is, of two types of which there can be xy- and xz-differentiants and of which the one is subordinate to the other quâ $\widehat{yz}$, that one has the more linearly independent xy- and xz-differentiants for which the weights in y and z are the more nearly equal, unless they have the same number.

For the actual numerical calculation of the denumerants considered in this paper nothing is wanting but a formula for the *number of terms* in the *general form* of any given type, that is, a formula for the denumerant $N(w_x, w_y, w_z, \cdots)$ for *any* type $(w_x, w_y, w_z, \cdots)$.

In conclusion, I wish to call attention to the fact that the real basis of this whole investigation is SYLVESTER's method, described in his *Proof of the hitherto undemonstrated Fundamental Theorem of Invariants* in the Philosophical Magazine for March, 1878, by which we pass from (29) through (26) to (30).

To facilitate reference I append here a brief table of contents.

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