APPLICABLE SURFACES WITH ASYMPTOTIC LINES OF ONE SURFACE CORRESPONDING TO A CONJUGATE SYSTEM OF ANOTHER.*

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Given a surface S and a surface S_1 applicable to it; the lines on S_1 corresponding to the asymptotic lines on S are called the *virtual asymptotic lines* of S_1 in its application upon S. It is our problem to determine the surfaces S which admit of one or more applicable surfaces S_1 with the virtual asymptotic lines forming a conjugate system. We find that in every case the surface S is an associate surface of a spherical surface, that is, of a surface whose gaussian curvature is a positive constant. Moreover, every surface associate to a spherical surface is of the kind sought.

In deriving the equations of condition to be satisfied by the fundamental quantities of a surface in order that it admit of the given deformation (we shall refer to such a surface as a *surface* S), we assume that the surface is referred to its asymptotic lines. Then the parametric lines on S_1 form a conjugate system. From the form of the equations of condition it is seen that, if there are more than two surfaces S_1 applicable to S with conjugate virtual asymptotic lines, there are an infinity. When there are only one or two surfaces S_1, they can be found by quadratures. But when there are an infinity of them, their determination requires the integration of a system of two linear partial differential equations of the first order in two unknowns. Certain of the equations of condition are satisfied identically when S is a quadric or a ruled surface, but all are satisfied only when S is a skew helicoid with plane director. In the latter case there are an infinity of surfaces S_1 — they are the catenoid and the surfaces of revolution applicable to it with lines of curvature in correspondence.

In § 4 we show that S is an associate of a spherical surface, Σ, and that when S_1 is known, Σ can be found by quadratures. Conversely, when Σ is

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known, S_1 can be found by quadratures. The relation between S and S_1 being reciprocal, S_1 also is an associate of a spherical surface, Σ_1, which can be found by quadratures. In § 5 we discover that Σ and Σ_1 are the Hazzidakis transforms* of one another. Hence we have a means of finding by quadratures the Hazzidakis transform of a spherical surface when a conjugate system with equal point or tangential invariants is known.†

It can be shown that when a spherical surface Σ has a conjugate system of lines with equal point and equal tangential invariants the corresponding conjugate system on the Hazzidakis transform is of the same kind. In this case these are two surfaces, S and S', associate to Σ which admit of applicable surfaces, S_1 and S_1', with conjugate virtual asymptotic lines. All four surfaces, S, S', S_1, S_1', and the surface Σ_1 associate to the last two are found by quadratures. These results are obtained in § 6 and applied to several particular cases in § 7.

In § 8 we study the spherical representation of the surfaces S_1. The equations to be satisfied by this representation are of such a form that when S admits of an infinity of applicable surfaces of the kind S_1, the representation of its asymptotic lines represents also the conjugate virtual asymptotic lines on another infinity of surfaces S. From this it follows that when a surface S is known, all the surfaces S_1 and the spherical associate surfaces Σ and Σ' can be found by quadratures.

§ 1. *General formulae.*

Let a surface S be referred to its asymptotic lines. The Codazzi equations‡ of condition reduce to

$$(1)\qquad \frac{\partial}{\partial u}\left(\frac{D'}{H}\right) = -2\left\{\begin{matrix}12\\2\end{matrix}\right\}\frac{D'}{H}, \qquad \frac{\partial}{\partial v}\left(\frac{D'}{H}\right) = -2\left\{\begin{matrix}12\\1\end{matrix}\right\}\frac{D'}{H},$$

where for the sake of brevity we have put

$$(2)\qquad H = \sqrt{EG - F^2}.$$

We seek the conditions which must be satisfied by the fundamental functions for S in order that upon one of the surfaces applicable to it the virtual asymptotic lines form a conjugate system. We assume that there is a surface applicable to S in this manner and denote it by S_1, indicating by a subscript 1 the functions pertaining to it.

* BIANCHI, *Lezioni*, II, p. 437. Also EISENHART, *Surfaces of constant curvature and their transformations*, these Transactions, vol. 6 (1905), pp. 472–485.

† L. c., where I have shown that the Hazzidakis transform can be found readily when one knows an isothermal-conjugate system.

‡ BIANCHI, *Lezioni*, I, p. 120.

Since the parametric lines on S are asymptotic, on S_1 they form a conjugate system; so that the Codazzi equations* for S_1 are

$$(3)\qquad \begin{cases} \dfrac{\partial}{\partial v}\left(\dfrac{D_1}{H}\right) + \left\{ {22 \atop 2} \right\} \dfrac{D_1}{H} + \left\{ {11 \atop 2} \right\} \dfrac{D_1''}{H} = 0, \\ \dfrac{\partial}{\partial u}\left(\dfrac{D_1''}{H}\right) + \left\{ {22 \atop 1} \right\} \dfrac{D_1}{H} + \left\{ {11 \atop 1} \right\} \dfrac{D_1''}{H} = 0. \end{cases}$$

In these equations and in equations (1) the Christoffel symbols $\left\{ {rs \atop t} \right\}$ are formed with respect to the common linear element of the two surfaces

$$(4)\qquad ds^2 = Edu^2 + 2Fdu\,dv + Gdv^2.$$

Since the gaussian curvatures of the two surfaces are equal, we have the equation

$$(4)\qquad -D'^2 = D_1 D_1'',$$

which we replace by

$$(5)\qquad D_1 = e^t D', \qquad D_1'' = -e^{-t} D',$$

where t is an auxiliary function. If these values of D_1 and D_1'' be substituted in (3) and reduction be made in accordance with equations (1), we obtain the following equations for the determination of t:

$$(6)\qquad \begin{cases} \dfrac{\partial t}{\partial v} + \left\{ {22 \atop 2} \right\} - 2\left\{ {12 \atop 1} \right\} - \left\{ {11 \atop 2} \right\} e^{-2t} = 0, \\ \dfrac{\partial t}{\partial u} - \left\{ {11 \atop 1} \right\} + 2\left\{ {12 \atop 2} \right\} + \left\{ {22 \atop 1} \right\} e^{2t} = 0. \end{cases}$$

The condition of integrability of these equations is

$$(7)\qquad \begin{aligned} & e^{2t}\left[\frac{\partial}{\partial v}\left\{ {22 \atop 1} \right\} - 2\left\{ {22 \atop 1} \right\}\left\{ {22 \atop 2} \right\} + 4\left\{ {22 \atop 1} \right\}\left\{ {12 \atop 1} \right\}\right] \\ & \quad + e^{-2t}\left[\frac{\partial}{\partial u}\left\{ {11 \atop 2} \right\} - 2\left\{ {11 \atop 2} \right\}\left\{ {11 \atop 1} \right\} + 4\left\{ {11 \atop 2} \right\}\left\{ {12 \atop 2} \right\}\right] \\ & \quad + \left[\frac{\partial}{\partial u}\left(2\left\{ {12 \atop 1} \right\} - \left\{ {22 \atop 2} \right\}\right) + \frac{\partial}{\partial v}\left(2\left\{ {12 \atop 2} \right\} - \left\{ {11 \atop 1} \right\}\right) + 4\left\{ {11 \atop 2} \right\}\left\{ {22 \atop 1} \right\}\right] = 0. \end{aligned}$$

This equation is quadratic in e^{2t}. From (5) it is seen that the two surfaces S_1 arising from a value of e^{2t} are symmetric with respect to the origin and consequently can be referred to as one surface. For convenience we write the above equation in the abbreviated form

$$(8)\qquad Ae^{2t} + Be^{-2t} + C = 0.$$

* BIANCHI, *Lezioni*, I, p. 120.

Hence unless

$$A = B = C = 0, \tag{9}$$

there are at most two surfaces S_1. In order to find the conditions to be satisfied, we solve equation (8) for e^{2t} and substitute the resulting value in equations (6). We shall find later that there are surfaces S for which the resulting equations of condition are satisfied.

When now equations (9) are satisfied, there are an infinity of surfaces applicable to S_1 with conjugate virtual asymptotic lines. It follows from (1) that

$$\frac{\partial}{\partial u}\left\{ {12 \atop 1} \right\} = \frac{\partial}{\partial v}\left\{ {12 \atop 2} \right\}, \tag{10}$$

so that in consequences of the identities

$$\frac{\partial}{\partial u}\log H = \left\{ {11 \atop 1} \right\} + \left\{ {12 \atop 2} \right\}, \qquad \frac{\partial}{\partial v}\log H = \left\{ {22 \atop 2} \right\} + \left\{ {12 \atop 1} \right\}, \tag{11}$$

we have also

$$\frac{\partial}{\partial u}\left\{ {22 \atop 2} \right\} = \frac{\partial}{\partial v}\left\{ {11 \atop 1} \right\}. \tag{12}$$

Hence the equations of condition (9) can be written thus:

$$\begin{aligned}
&\frac{\partial}{\partial v}\left\{ {22 \atop 1} \right\} - 2\left\{ {22 \atop 1} \right\}\left\{ {22 \atop 2} \right\} + 4\left\{ {22 \atop 1} \right\}\left\{ {12 \atop 1} \right\} = 0, \\
&\frac{\partial}{\partial u}\left\{ {11 \atop 2} \right\} - 2\left\{ {11 \atop 2} \right\}\left\{ {11 \atop 1} \right\} + 4\left\{ {11 \atop 2} \right\}\left\{ {12 \atop 2} \right\} = 0, \\
&\frac{\partial}{\partial u}\left[2\left\{ {12 \atop 1} \right\} - \left\{ {22 \atop 2} \right\}\right] = \frac{\partial}{\partial v}\left[2\left\{ {12 \atop 2} \right\} - \left\{ {11 \atop 1} \right\}\right] = -2\left\{ {11 \atop 2} \right\}\left\{ {22 \atop 1} \right\}.
\end{aligned} \tag{13}$$

When S and S_1 are referred to any parametric system whatever, the necessary and sufficient condition that to asymptotic lines on S correspond a conjugate system on S_1 is

$$DD_1'' + D''D_1 - 2D'D_1' = 0.$$

The symmetrical form of this equation enables us to state that the asymptotic lines on S_1 also correspond to a conjugate system on S. Hence, when a surface is referred to a conjugate system of lines, the necessary and sufficient condition that it be capable of a deformation in which the virtual asymptotic lines are conjugate is

$$\frac{\partial}{\partial u}\log\sqrt{-K} = -2\left\{ {12 \atop 2} \right\}, \qquad \frac{\partial}{\partial v}\log\sqrt{-K} = -2\left\{ {12 \atop 1} \right\}, \tag{14}$$

where K denotes the total curvature of the surface and the Christoffel symbols

are formed with respect to the linear element. For equations (14) are the condition that one of the surfaces with the linear element (4) have its asymptotic lines parametric.

§ 2. *Quadrics and ruled surfaces.*

We consider the particular case

$$\left\{ \begin{matrix} 11 \\ 2 \end{matrix} \right\} = \left\{ \begin{matrix} 22 \\ 1 \end{matrix} \right\} = 0; \tag{15}$$

that is, S is a quadric surface. Now equations (13) reduce to

$$\frac{\partial}{\partial u}\left[2\left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\} - \left\{ \begin{matrix} 22 \\ 2 \end{matrix} \right\}\right] = \frac{\partial}{\partial v}\left[2\left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\} - \left\{ \begin{matrix} 11 \\ 1 \end{matrix} \right\}\right] = 0. \tag{16}$$

Hence, if there is one surface S_1, there is an infinite number.

We consider first the case of central quadrics. The equations of such a surface can be put in the form

$$x = a\frac{1+uv}{u+v}, \qquad y = b\frac{u-v}{u+v}, \qquad z = c\frac{1-uv}{u+v}.$$

The constants a, b, c are real for the hyperboloid of one sheet. One finds readily

$$\left\{ \begin{matrix} 22 \\ 2 \end{matrix} \right\} = \frac{-2}{u+v},$$

$$\left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\} = \frac{a^2b^2(uv-1)(u^2+1) - 2a^2c^2(u-v)u + b^2c^2(uv+1)(u^2-1)}{(u+v)[a^2b^2(uv-1)^2 + a^2c^2(u-v)^2 + b^2c^2(vu+1)^2]},$$

which do not satisfy equation (16).

Again, the equations of the paraboloids are

$$x = a(u+v), \qquad y = b(u-v), \qquad z = uv.$$

Now

$$H = [4a^2b^2 + (a^2+b^2)(u+v) - 2(a^2-b^2)uv]^{\frac{1}{2}}$$

and

$$\left\{ \begin{matrix} 22 \\ 2 \end{matrix} \right\} = 0, \qquad \left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\} = \frac{\partial}{\partial v}\log H.$$

Hence equation (16) becomes

$$\frac{\partial^2}{\partial u \partial v}\log H = 0,$$

which evidently is not satisfied by the above value of H. Therefore equations (15) do not furnish a solution of the problem.

We consider next the case

$$\left\{ \begin{matrix} 11 \\ 2 \end{matrix} \right\} = 0.$$

Now the lines $v = \text{const.}$ are straight and consequently S is a ruled surface. The parameters of the asymptotic lines can be so chosen that the linear element takes the form *

$$ds^2 = du^2 + 2\cos\theta\, du\, dv + (M^2u^2 + 2Nu + 1)\, dv^2,$$

where $\cos\theta$, M and N are functions of v alone. Now

$$\left\{ {11 \atop 1} \right\} = 0, \qquad \left\{ {12 \atop 1} \right\} = -\cos\theta \frac{M^2u + N}{M^2u^2 + 2Nu + \sin^2\theta};$$

so that the first two of equations (13) are satisfied identically and the last reduces to

$$\frac{\partial}{\partial u}\left\{ {12 \atop 1} \right\} = \frac{\partial}{\partial u}\left\{ {22 \atop 2} \right\} = \frac{\partial}{\partial v}\left\{ {12 \atop 2} \right\} = 0.$$

But

$$\frac{\partial}{\partial u}\left\{ {12 \atop 1} \right\} = \frac{\cos\theta\,[M^4u^2 + 2M^2Nu + (2N^2 - M^2\sin\theta)]}{2(M^2u^2 + 2Nu + \sin^2\theta)^2}.$$

This can vanish only when $\cos\theta = 0$, or

$$M = N = 0.$$

In the latter case the total curvature of the surface is zero, so that it may be excluded; in the former case the ruled surface is the skew helicoid with plane director, which we know is applicable to an infinity of surfaces of revolution in such a way that the lines of curvature of the latter correspond to the asymptotic lines of the former. Hence:

The helicoid with plane director is the only ruled surface which can be deformed in an infinity of ways so that the asymptotic lines become a conjugate system.

§ 3. *Spherical representation of S.*

Between the Christoffel symbols $\left\{ {rs \atop t} \right\}'$ formed with respect to the linear element of the spherical representation of S and the symbols formed with respect to the linear element (4) the following relations obtain: †

$$(17)\qquad \begin{aligned} &\left\{ {11 \atop 1} \right\} = \left\{ {11 \atop 1} \right\}' - 2\left\{ {12 \atop 2} \right\}', \quad \left\{ {12 \atop 1} \right\} = -\left\{ {12 \atop 1} \right\}', \quad \left\{ {22 \atop 1} \right\} = -\left\{ {22 \atop 1} \right\}', \\ &\left\{ {11 \atop 2} \right\} = -\left\{ {11 \atop 2} \right\}', \quad \left\{ {12 \atop 2} \right\} = -\left\{ {12 \atop 2} \right\}', \quad \left\{ {22 \atop 2} \right\} = \left\{ {22 \atop 2} \right\}' - 2\left\{ {12 \atop 1} \right\}'. \end{aligned}$$

In consequence of these relations equations (6) can be given the form

$$(18)\qquad \begin{aligned} &\frac{\partial t}{\partial v} + \left\{ {22 \atop 2} \right\}' + \left\{ {11 \atop 2} \right\}' e^{-2t} = 0, \\ &\frac{\partial t}{\partial u} - \left\{ {11 \atop 1} \right\}' - \left\{ {22 \atop 1} \right\}' e^{2t} = 0. \end{aligned}$$

* BIANCHI, *Lezioni*, I, p. 254.

† BIANCHI, *Lezioni*, I, p. 157.

The condition of integrability takes the form

$$
\begin{aligned}
&\left[\frac{\partial}{\partial u}\left\{\begin{matrix}22\\2\end{matrix}\right\}' + \frac{\partial}{\partial v}\left\{\begin{matrix}11\\1\end{matrix}\right\}' - 4\left\{\begin{matrix}11\\2\end{matrix}\right\}'\left\{\begin{matrix}22\\1\end{matrix}\right\}'\right] \\
&\qquad + e^{2t}\left[\frac{\partial}{\partial v}\left\{\begin{matrix}22\\1\end{matrix}\right\}' - 2\left\{\begin{matrix}22\\1\end{matrix}\right\}'\left\{\begin{matrix}22\\2\end{matrix}\right\}'\right] \\
&\qquad\qquad + e^{-2t}\left[\frac{\partial}{\partial u}\left\{\begin{matrix}11\\2\end{matrix}\right\}' - 2\left\{\begin{matrix}11\\2\end{matrix}\right\}'\left\{\begin{matrix}11\\1\end{matrix}\right\}'\right] = 0.
\end{aligned}
\tag{19}
$$

We have seen that, unless S is a skew helicoid with plane director, $\left\{\begin{matrix}11\\2\end{matrix}\right\} \neq 0$, $\left\{\begin{matrix}22\\1\end{matrix}\right\} \neq 0$. Hence, excluding this exceptional case, we find, for the conditions that there be an infinity of surfaces S_1,

$$
\begin{aligned}
&\frac{\partial}{\partial u}\left\{\begin{matrix}22\\2\end{matrix}\right\}' = \frac{\partial}{\partial v}\left\{\begin{matrix}11\\1\end{matrix}\right\}' = 2\left\{\begin{matrix}11\\2\end{matrix}\right\}'\left\{\begin{matrix}22\\1\end{matrix}\right\}', \\
&\frac{\partial}{\partial u}\log\left\{\begin{matrix}11\\2\end{matrix}\right\}' = 2\left\{\begin{matrix}11\\1\end{matrix}\right\}', \qquad \frac{\partial}{\partial v}\log\left\{\begin{matrix}22\\1\end{matrix}\right\}' = 2\left\{\begin{matrix}22\\2\end{matrix}\right\}'.
\end{aligned}
\tag{20}
$$

§ 4. *Surfaces associate to S.*

We have shown elsewhere* that the coördinates, x', y', z', of an associate surface of the surface S are given by the quadratures

$$
\frac{\partial x'}{\partial u} = \mu\frac{\partial x}{\partial v}, \qquad \frac{\partial x'}{\partial v} = \sigma\frac{\partial x}{\partial u},
\tag{22}
$$

where μ and σ are a pair of solutions of the equations

$$
\begin{aligned}
&\frac{\partial \mu}{\partial v} + \mu\left\{\begin{matrix}22\\2\end{matrix}\right\} - \sigma\left\{\begin{matrix}11\\2\end{matrix}\right\} = 0, \\
&\frac{\partial \sigma}{\partial u} + \sigma\left\{\begin{matrix}11\\1\end{matrix}\right\} - \mu\left\{\begin{matrix}22\\1\end{matrix}\right\} = 0.
\end{aligned}
\tag{23}
$$

On comparing these equations with (3) we remark that an associate surface, Σ, is determined by the pair of solutions

$$
\mu = \frac{D_1}{H}, \qquad \sigma = -\frac{D_1''}{H}.
\tag{24}
$$

If the fundamental functions for Σ be denoted by E', F', G'; Δ, Δ', Δ'', it follows from (5) and (22) that

$$
\begin{aligned}
&E' = e^{2t}G\frac{D'^2}{H^2}, \qquad F' = \frac{D'^2}{H^2}F, \qquad G' = e^{-2t}E\frac{D'^2}{H^2}, \\
&\Delta = e^{t}\frac{D'^2}{H}, \qquad \Delta' = 0, \qquad \Delta'' = e^{-t}\frac{D'^2}{H}.
\end{aligned}
\tag{25}
$$

* *Associate surfaces*, Mathematische Annalen, vol. 68 (1906), p. 507.

From these values it follows that the total curvature of Σ is equal to unity. Hence Σ is a spherical surface.

Since S is referred to its asymptotic lines, the spherical representation of the parametric lines satisfies the condition

$$\frac{\partial}{\partial u}\begin{Bmatrix}12\\1\end{Bmatrix}' = \frac{\partial}{\partial v}\begin{Bmatrix}12\\2\end{Bmatrix}', \tag{26}$$

that is, the parametric system on Σ is a conjugate system with equal tangential invariants.

Conversely, given a system of lines on the sphere satisfying the condition (26), we seek the other conditions to be satisfied in order that it may represent a conjugate system on a spherical surface. Now

$$\frac{\Delta\Delta''}{\mathbf{EG}-\mathbf{F}^2} = 1,$$

where the linear element of the spherical representation of Σ is written

$$d\sigma^2 = \mathbf{E}du^2 + 2\mathbf{F}du\,dv + \mathbf{G}dv^2.$$

We replace the above equation by the two

$$\frac{\Delta}{\sqrt{\mathbf{EG}-\mathbf{F}^2}} = e^t, \qquad \frac{\Delta''}{\sqrt{\mathbf{EG}-\mathbf{F}^2}} = e^{-t}.$$

If these values be substituted in the Codazzi equations *

$$\frac{\partial}{\partial v}\left(\frac{\Delta}{\sqrt{\mathbf{EG}-\mathbf{F}^2}}\right) + \begin{Bmatrix}22\\2\end{Bmatrix}'\frac{\Delta}{\sqrt{\mathbf{EG}-\mathbf{F}^2}} + \begin{Bmatrix}11\\2\end{Bmatrix}'\frac{\Delta''}{\sqrt{\mathbf{EG}-\mathbf{F}^2}} = 0,$$

$$\frac{\partial}{\partial u}\left(\frac{\Delta''}{\sqrt{\mathbf{EG}-\mathbf{F}^2}}\right) + \begin{Bmatrix}22\\1\end{Bmatrix}'\frac{\Delta}{\sqrt{\mathbf{EG}-\mathbf{F}^2}} + \begin{Bmatrix}11\\1\end{Bmatrix}'\frac{\Delta''}{\sqrt{\mathbf{EG}-\mathbf{F}^2}} = 0,$$

we get equations (18). Hence, if we have a spherical surface referred to a conjugate system with equal tangential invariants, we have a solution of equations (18) and thus can get a surface S_1 applicable to the surface S associate to Σ and upon which the parametric lines are asymptotic. It is well known that the surface S is found by quadratures. Therefore the problem of finding surfaces capable of a deformation in which the asymptotic lines become conjugate on the new surface is equivalent to the problem of finding spherical surfaces referred to conjugate systems with equal tangential invariants. And we have the theorem:

The surfaces associate to a spherical surface can be deformed in such a way that on one of the deforms the virtual asymptotic lines form a conjugate system; and they are the only surfaces possessing this property.

* BIANCHI, *Lezioni*, I, p. 166.

It is evident from this theorem that when a surface can be deformed in the above manner in an infinity of ways it has an infinity of spherical surfaces associate to it.

§ 5. *Hazzidakis transformations.*

We proceed now to the consideration of the surface S_1. Since the parametric system is composed of virtual asymptotic lines, equation (10) is satisfied and consequently these lines form a conjugate system with equal point invariants. We have shown* that, when a surface is referred to such a family of lines, an associate surface is given by the quadratures

$$\frac{\partial \xi_1}{\partial u} = \lambda \frac{\partial x_1}{\partial u}, \qquad \frac{\partial \xi_1}{\partial v} = -\lambda \frac{\partial x_1}{\partial v}, \tag{27}$$

and similar quadratures for η_1, ζ_1, where (ξ_1, η_1, ζ_1) are the coordinates of the associate Σ_1, and the function λ is given by

$$\frac{\partial \log \lambda}{\partial v} = -2 \begin{Bmatrix} 12 \\ 1 \end{Bmatrix}_1, \qquad \frac{\partial \log \lambda}{\partial u} = -2 \begin{Bmatrix} 12 \\ 2 \end{Bmatrix}_1. \tag{28}$$

Since S_1 is applicable to S, it follows from (1) that†

$$\lambda = \frac{D'}{H}.$$

From these results it follows that the fundamental functions of the associate surface Σ_1, denoted by E'_1, F'_1, G'_1; Δ_1, Δ'_1, Δ''_1, have the values

$$\begin{aligned} E'_1 &= \frac{D'^2}{H^2} E_1, & F'_1 &= -\frac{D'^2}{H^2} F_1, & G'_1 &= \frac{D'^2}{H^2} G_1, \\ \Delta_1 &= \frac{D'^2}{H} e^t, & \Delta'_1 &= 0, & \Delta''_1 &= \frac{D'^2}{H} e^{-t}. \end{aligned} \tag{29}$$

Since S and S_1 are applicable, it follows that the expression for the total curvature of Σ_1 reduces to unity, that is Σ_1 is a spherical surface.

Incidentally, we remark from (29) that if S admits of more than one applicable surface of the kind S_1 all of the corresponding associate surfaces Σ_1 will be applicable to one another with their parametric lines in correspondence.

From the form of equations (27) it follows that the parametric system on Σ_1 is conjugate and that the point invariants are equal. Moreover, if the Christoffel symbols $\left\{ {r s \atop t} \right\}_{\Sigma_1}$ are formed with respect to the linear element of Σ_1, it follows that we must have

$$\frac{\partial \log \lambda}{\partial u} = 2 \begin{Bmatrix} 12 \\ 2 \end{Bmatrix}_{\Sigma_1}, \qquad \frac{\partial \log \lambda}{\partial v} = 2 \begin{Bmatrix} 12 \\ 1 \end{Bmatrix}_{\Sigma_1}. \tag{30}$$

* l. c., p. 531.

† Another surface is given by $\lambda = -D'/H$, but it is symmetric to this one with respect to the origin.

Conversely, given a surface of unit total curvature referred to a conjugate system with equal point invariants, and let ξ_1, η_1, ζ_1 be its rectangular coördinates, then the surface of coördinates x_1, y_1, z_1 determined by the quadratures (27) where λ is given by (30) is an associate of Σ_1. Between the gaussian curvatures of two surfaces associate to one another by means of equations of the form (27) the following relation exists: *

$$K_1 = -\frac{K_2}{\lambda^2}.$$

For the present case it is

$$K_1 = -\frac{1}{\lambda^2}.$$

This result and equations (28), which are the same as (30), show that the parametric system on S_1 is virtual asymptotic and from (27) it follows that it is conjugate.

We gather together the foregoing results into the following theorems:

When a surface admits of an applicable surface upon which the virtual asymptotic lines form a conjugate system, and either surface is referred to its asymptotic lines, an associate of each surface can be found by quadratures; these associates are spherical surfaces and have the same second quadratic forms.

When a spherical surface is referred to a conjugate system with equal point or tangential invariants, a surface associate to it can be found which admits of a deformation such that the virtual asymptotic lines are conjugate.

The general equation of the lines of curvature of a surface is †

$$(ED' - FD)du^2 + (ED'' - GD)\,du\,dv + (FD'' - GD')dv^2 = 0.$$

For Σ this reduces to

$$e^t F du^2 + (Ee^{-t} - Ge^t)\,du\,dv - e^{-t} F dv^2 = 0, \tag{31}$$

where E, F, G are the first coefficients of S referred to its asymptotic lines. From (29) it follows that the equation for the lines of curvature of Σ_1 is likewise (31). Since Σ and Σ_1 have their lines of curvature in correspondence and the same second quadratic forms, they are the Hazzidakis transforms of one another.‡ Since each spherical surface has a unique Hazzidakis transform, it follows that the surface Σ_1 is the same no matter which conjugate system with equal point or tangential invariants is taken for the parametric system. Hence we have the theorems:

* Loc. cit., p. 506.

† BIANCHI, *Lezioni*, I, p. 128.

‡ *Surfaces of constant curvature*, etc., these Transactions, vol. 6 (1905), p. 478.

When a spherical surface is referred to a conjugate system with equal point or tangential invariants, its Hazzidakis transform can be found by quadratures.

The surfaces S_1 into which the associates S of a spherical surface Σ are deformed so that virtual asymptotic lines are conjugate are the associate surfaces of the Hazzidakis transform of Σ.

From the general expressions for the coefficients of the spherical representation of a surface referred to a conjugate system* one deduces by means of (25) and (29) that the coefficients of the spherical representation of Σ and Σ_1 are

$$\mathbf{E} = \frac{D'^2}{H^2} E, \qquad \mathbf{F} = -\frac{D'^2}{H^2} F, \qquad \mathbf{G} = \frac{D'^2}{H^2} G,$$

$$\mathbf{E}_1 = e^{2t} \frac{D'^2}{H^2} G, \quad \mathbf{F}_1 = \frac{D'^2}{H^2} F, \quad \mathbf{G}_1 = e^{-2t} \frac{D'^2}{H^2} E.$$

Comparing these expressions with (25) and (29), we see that *each of the surfaces Σ and Σ_1 is applicable to the spherical representation of the other.*

§ 6. *Conjugate systems with equal point and tangential invariants.*

In consequence of the foregoing result, if the conjugate system on Σ corresponding to the asymptotic lines on S has equal point invariants, the corresponding system on Σ_1 has equal tangential invariants. Hence:

When there is known a spherical surface referred to a conjugate system with equal point and tangential invariants, another spherical surface can be found upon which the corresponding lines form a conjugate system of the same kind; it is the Hazzidakis transform of the given surface.

We inquire under what conditions a conjugate system is of this kind. The Codazzi equations for Σ can be written in the form

$$(32) \qquad \frac{\partial \log \Delta}{\partial v} - \begin{Bmatrix} 12 \\ 1 \end{Bmatrix}_\Sigma + \begin{Bmatrix} 11 \\ 2 \end{Bmatrix}_\Sigma \frac{\Delta''}{\Delta} = 0, \quad \frac{\partial \log \Delta''}{\partial u} - \begin{Bmatrix} 12 \\ 2 \end{Bmatrix}_\Sigma + \begin{Bmatrix} 22 \\ 1 \end{Bmatrix}_\Sigma \frac{\Delta}{\Delta''} = 0.$$

In terms of the functions of Σ equation (26) can be put in the form

$$\frac{\partial}{\partial u}\left(\frac{\Delta''}{\Delta} \begin{Bmatrix} 11 \\ 2 \end{Bmatrix}_\Sigma \right) = \frac{\partial}{\partial v}\left(\frac{\Delta}{\Delta''} \begin{Bmatrix} 22 \\ 1 \end{Bmatrix}_\Sigma \right).$$

Hence it follows from (32) that the necessary and sufficient condition that the point invariants be equal is

$$\frac{\partial^2}{\partial u \, \partial v} \log \frac{\Delta}{\Delta''} = 0.$$

By a suitable choice of parameters this can be replaced by

$$\Delta = \Delta''.$$

* BIANCHI, *Lezioni*, I, p. 150.

But the necessary and sufficient conditions that a conjugate system on a spherical surface be isothermal-conjugate are *

$$(33)\qquad \begin{Bmatrix} 11 \\ 1 \end{Bmatrix}' + \begin{Bmatrix} 22 \\ 1 \end{Bmatrix}' = 0, \qquad \begin{Bmatrix} 22 \\ 2 \end{Bmatrix}' + \begin{Bmatrix} 11 \\ 2 \end{Bmatrix}' = 0,$$

which are equivalent to

$$(34)\qquad \frac{\partial \mathbf{E}}{\partial u} - \frac{\partial \mathbf{G}}{\partial u} + 2\frac{\partial \mathbf{F}}{\partial v} = 0, \qquad \frac{\partial \mathbf{E}}{\partial v} - \frac{\partial \mathbf{G}}{\partial v} - 2\frac{\partial \mathbf{F}}{\partial u} = 0.$$

Eliminating $\mathbf{E} - \mathbf{G}$ from these equations, we get

$$\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right)\mathbf{F} = 0,$$

so that

$$\mathbf{F} = \Phi(u + iv) + \Phi_0(u - iv),$$

where Φ is an arbitrary function and Φ_0 is its conjugate, as we are confining ourselves to the consideration of real surfaces, and

$$\mathbf{E} - \mathbf{G} = -i\left[\Phi(u + iv) - \Phi_0(u - iv)\right].$$

In order that the conditions of the problem be satisfied, the function Φ must be such that these values of $\mathbf{E}, \mathbf{F}, \mathbf{G}$ satisfy the Gauss equation for the sphere and the equation (26).

We assume that the representation of S is of this kind. Equation (19) reduces to

$$(e^t - 1)^2\left[\frac{\partial}{\partial u}\begin{Bmatrix} 11 \\ 2 \end{Bmatrix}' + 2\begin{Bmatrix} 11 \\ 2 \end{Bmatrix}'\begin{Bmatrix} 22 \\ 1 \end{Bmatrix}'\right] = 0.$$

This is satisfied by $e^t = 1$, which is a solution of equations (18). From this it is seen that if there are more than one surface of the kind there are an infinity of them and the condition of the latter case is

$$\frac{\partial}{\partial u}\begin{Bmatrix} 11 \\ 2 \end{Bmatrix}' = -2\begin{Bmatrix} 11 \\ 2 \end{Bmatrix}'\begin{Bmatrix} 22 \\ 1 \end{Bmatrix}'.$$

Since the conjugate virtual asymptotic lines on S_1 have equal tangential invariants, one finds the coördinates of an associate S_1' of S_1 by the quadratures of the form

$$\frac{\partial x_1'}{\partial u} = \mu'\frac{\partial x_1}{\partial v}, \qquad \frac{\partial x_1'}{\partial v} = \sigma'\frac{\partial x_1}{\partial u},$$

where

$$(35)\qquad \begin{aligned} \mu' &= \rho_1\frac{D_1}{H_1}, & \sigma' &= \rho_1\frac{D_1''}{H_1}, \\ \frac{\partial \log \rho_1}{\partial u} &= -2\begin{Bmatrix} 12 \\ 2 \end{Bmatrix}_1', & \frac{\partial \log \rho_1}{\partial v} &= -2\begin{Bmatrix} 12 \\ 1 \end{Bmatrix}_1'. \end{aligned}$$

* BIANCHI, *Lezioni*, I, p. 168.

Moreover, we have shown* that these functions μ' and σ' are solutions of

$$\frac{\partial \mu'}{\partial v} + \mu' \left\{ \begin{matrix} 22 \\ 2 \end{matrix} \right\}_1 - \sigma' \left\{ \begin{matrix} 11 \\ 2 \end{matrix} \right\}_1 = 0, \qquad \frac{\partial \sigma'}{\partial u} + \sigma' \left\{ \begin{matrix} 11 \\ 1 \end{matrix} \right\}_1 - \mu' \left\{ \begin{matrix} 22 \\ 1 \end{matrix} \right\}_1 = 0,$$

where the Christoffel symbols $\left\{ \begin{matrix} rs \\ t \end{matrix} \right\}_1$ are formed with respect to the linear element of S_1 and consequently of S. Hence these equations are the same as equations (23), so that we can find a surface S' associate to S by means of the quadratures of the form

$$\frac{\partial x'}{\partial u} = \mu' \frac{\partial x}{\partial v}, \qquad \frac{\partial x'}{\partial v} = \sigma' \frac{\partial x}{\partial u}, \tag{36}$$

where μ' and σ' have the values (35). We see at once that the two surfaces S' and S_1' are applicable, for they have the same linear element

$$\mu'^2 G du^2 + 2\mu'\sigma' F du\, dv + \sigma'^2 E dv^2,$$

and the parametric lines are conjugate on S' and asymptotic on S_1'.

Taking $e^t = 1$, we get from (5), (25), and (29),

$$\Delta = \Delta'', \qquad D_1 = -D_1'', \qquad \Delta_1 = \Delta_1''. \tag{37}$$

Moreover, the linear elements of Σ and Σ_1 reduce to

$$\frac{D'^2}{H^2}(G du^2 + 2F du\, dv + E dv^2), \qquad \frac{D'^2}{H^2}(E du^2 - 2F du\, dv + G dv^2),$$

showing that the surfaces are the Hazzidakis transforms of one another.*

If we denote by (D), (D'), (D'') the second fundamental functions of S_1', it follows from (36) that

$$(D) = \mu' D', \qquad (D') = 0, \qquad (D'') = \sigma' D'.$$

But $\mu' = -\sigma'$, hence

$$(D)\Delta'' + (D'')\Delta = 0.$$

Therefore S' and S_1' are associate surfaces of Σ and Σ_1 respectively; consequently S' can be found from Σ by making use of the fact that the conjugate parametric system has equal point invariants, and S_1' from Σ_1 from the standpoint that the conjugate system has equal tangential invariants. In both cases the determination requires only quadratures. Hence we have the theorem:

When a spherical surface has an isothermal-conjugate system with equal tangential invariants and this system is parametric, one can find by quadratures two surfaces which can be deformed in such a way that the asymptotic lines become conjugate; and these new surfaces are associate surfaces of the Hazzidakis transform of the given spherical surface.

* *Surfaces of constant curvature, etc.*, Transactions, vol. 6 (1905), p. 478.

In this connection we have been considering the case where a surface S has only one surface S_1 applicable to it with its virtual asymptotic lines conjugate. In case there are more than one, for each surface S_1 there is a spherical surface Σ associate to S.

From (37) it is seen that if we put

$$u = u_1 - v_1, \qquad v = u_1 + v_1,$$

the curves $u_1 =$ const. and $v_1 =$ const. on S_1 are asymptotic and on S conjugate. And the equation of the asymptotic lines on S is of the form

$$\lambda(du_1^2 - dv_1^2) = 0.$$

Recalling the preceding results, we see that the function t_1, by means of which S is formed from S_1 when the latter is referred to its asymptotic lines, is zero also. Consequently *when the asymptotic lines on S satisfy the conditions* (33), *a similar set of equations are satisfied by the spherical representation of the asymptotic lines of S_1*. And in all respects it can be shown that the relation between the surfaces S and S_1 and the surfaces adjoined to them is perfectly reciprocal.

§ 7. *Minimal surfaces.*

As an example of the foregoing results, we consider the case where S is a minimal surface. By a suitable choice of parameters we can put

$$\mathbf{E} = \mathbf{G} = \frac{1}{\rho}, \qquad \mathbf{F} = 0, \qquad \frac{1}{\rho^2} = -K, \tag{38}$$

where K denotes the Gaussian curvature. Equations (34) are evidently satisfied, as is also equation (26). The linear element of S is

$$ds^2 = \rho(du^2 + dv^2) \tag{39}$$

and $D' = 1$. Hence for the value $e^t = 1$,

$$D_1 = -D_1'' = 1,$$

so that S_1 is the minimal surface adjoint to S. From (25) and (29) it is seen that Σ and Σ_1 are each the unit sphere and corresponding points coincide. And from (35) and (36) it follows that S' coincides with S_1 and S_1' with S.

We inquire whether there are any other surfaces S_1 applicable in the desired way to a minimal surface. Now equations (18) reduce to

$$\frac{\partial t}{\partial v} - \frac{\partial}{\partial v}\log\sqrt{\rho}(1 - e^{-2t}) = 0, \qquad \frac{\partial t}{\partial u} + \frac{\partial}{\partial u}\log\sqrt{\rho}(1 - e^{2t}) = 0, \tag{40}$$

and the equation (19) becomes

$$(e^t - e^{-t})^2 \frac{\partial^2 \rho}{\partial u \partial v} = 0.$$

Neglecting the solutions $e^t = \pm 1$, which we considered above, we see that equations (40) are integrable when

$$\rho = \phi(u) + \psi(v).$$

But the functions $\phi(u)$ and $\psi(v)$ are not arbitrary, for there remains to be satisfied the Gauss equation of the sphere,* which is now

$$\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right)\log \rho = \frac{2}{\rho}.$$

This is satisfied by the above value of ρ only in case ϕ or ψ is equal to a constant.† Hence all the surfaces S_1 are surfaces of revolution referred to their meridians and parallels, and the surface S_1 corresponding to the solution $t = 0$ is the catenoid. Hence S is the skew helicoid with plane director.

By a suitable choice of parameters the linear element of the skew helicoid with plane director can be reduced to the form

$$ds^2 = \frac{(e^u + a^2 e^{-u})^2}{4}(du^2 + dv^2). \tag{41}$$

Now the general integral of equations (40) is

$$e^t = \frac{2}{\sqrt{(e^u + a^2 e^{-u})^2 b + 4}}, \tag{42}$$

where b is the constant of integration. When $b = 0$, S_1 is the catenoid and for the other values of b the surfaces S_1 are the surfaces of revolution into which the catenoid can be deformed with preservation of the lines of curvature.‡

From equations (25), (41) and (42), we find the following expressions for the fundamental functions of the spherical surface Σ associate to S:

$$E' = \frac{16}{(e^u + a^2 e^{-u})^4 b + 4(e^u + a^2 e^{-u})^2}, \quad F' = 0, \quad G' = \frac{(e^u + a^2 e^{-u})^2 b + 4}{(e^u + a^2 e^{-u})^2}, \tag{43}$$

$$\Delta = \frac{8}{\sqrt{(e^u + a^2 e^{-u})^2 b + 4(e^u + a^2 e^{-u})^2}}, \quad \Delta' = 0, \quad \Delta'' = \frac{2\sqrt{(e^u + a^2 e^{-u})^2 b + 4}}{(e^u + a^2 e^{-u})^2}. \tag{44}$$

From these values it is found that Σ is a sphere only in case $b = 0$. We have seen that all the spherical surfaces Σ_1 associate to S_1 are applicable to one another and to the spherical representation of Σ, that is, their linear element is

$$d\sigma^2 = \frac{4}{(e^u + a^2 e^{-u})^2}(du^2 + dv^2) \tag{45}$$

and their second fundamental quantities are given by (44).

* BIANCHI, *Lezioni*, I, p. 93.

† Ibid., II, p. 45.

‡ Ibid., I, p. 232.

We apply now to this case the results of the preceding section for the determination of the surfaces S' and S_1'. Since the surface Σ is applicable to the spherical representation of S_1, the last two of equations (35) can be replaced by

$$\frac{\partial \log \rho_1}{\partial u} = -2 \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\}_\Sigma, \qquad \frac{\partial \log \rho_1}{\partial v} = -2 \left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\}_\Sigma,$$

of which the solution, without loss of generality, is

$$\rho_1 = \frac{1}{G'},$$

in consequence of (43). Hence

$$\mu' = \frac{8}{[(e^u + a^2 e^{-u})^2 b + 4]^{\frac{3}{2}}}, \qquad \sigma' = -\frac{2}{[(e^u + a^2 e^{-u})^2 b + 4]^{\frac{1}{2}}},$$

so that the linear element of the surface S' and S_1' is

$$ds'^2 = \frac{(e^u + a^2 e^{-u})^2}{(e^u + a^2 e^{-u})^2 b + 4} \left[\frac{16 du^2}{[(e^u + a^2 e^{-u})^2 b + 4]^2} + dv^2 \right],$$

and the respective second fundamental quantities are

$$(D) = \mu', \qquad (D') = 0, \qquad (D'') = \sigma',$$
$$(D_1) = 0, \qquad (D_1') = \sqrt{-\mu'\sigma'}, \qquad (D_1'') = 0.$$

From these results it is seen that S' is minimal only when $b = 0$, in which case it is the catenoid; moreover, S_1' is a minimal surface for all values of b and is applicable upon a surface of revolution.

Another solution of equations (34) is $\mathbf{E} = \mathbf{G}$; $\mathbf{F} = \text{const}$. For the sake of convenience we put

$$\mathbf{E} = \mathbf{G} = \cosh \theta, \qquad \mathbf{F} = 1, \qquad \mathbf{EG} - \mathbf{F}^2 = \sinh^2 \theta.$$

Equation (26) becomes

$$\frac{\partial}{\partial u}\left(\frac{1}{\sinh \theta} \frac{\partial \theta}{\partial u}\right) - \frac{\partial}{\partial v}\left(\frac{1}{\sinh \theta} \frac{\partial \theta}{\partial v}\right) = 0,$$

of which the general integral is

$$\tanh \frac{\theta}{2} = \phi(u + v) \cdot \psi(u - v),$$

where ϕ and ψ are arbitrary functions. The Gauss equation* for the sphere is

$$\frac{\partial^2 \theta}{\partial u^2} + \frac{\partial^2 \theta}{\partial v^2} + 2 \sinh \theta = 0.$$

* BIANCHI, *Lezioni*, I, p. 93.

When the above value of θ is substituted, this equation becomes

$$\frac{\phi''}{\phi}+\frac{\psi''}{\psi}+2\frac{\phi'^2\psi^2+\psi'^2\phi^2}{1-\phi^2\psi^2}+1=0, \tag{46}$$

where the primes denote differentiation with respect to the argument. If we differentiate this equation consecutively with respect to $u+v$ and $u-v$, we get

$$\frac{\phi\phi'\psi\psi'}{1-\phi^2\psi^2}\left(\frac{\phi''}{\phi}+\frac{\psi''}{\psi}+2\frac{\phi'^2\psi^2+\psi'^2\phi^2}{1-\phi^2\psi^2}\right)=0.$$

Hence either ϕ' or ψ' must be zero. Let it be the latter and take $\psi=1$; then the integral of (46) is given by the quadrature

$$\int\frac{d\phi}{\sqrt{c(\phi^2-1)^2+(\phi^2-1)}}=u+v,$$

where c is a constant of integration, the additive constant being taken equal to zero without loss of generality.

In terms of this function ϕ the coefficients of the linear element of the spherical representation of S are given by

$$\mathbf{E}=\mathbf{G}=\frac{1+\phi^2}{1-\phi^2},\qquad \mathbf{F}=1.$$

From the equations

$$\frac{\partial\log\rho}{\partial u}=-2\left\{\begin{matrix}12\\2\end{matrix}\right\}',\qquad \frac{\partial\log\rho}{\partial v}=-2\left\{\begin{matrix}12\\1\end{matrix}\right\}',$$

we find

$$\rho=1-\phi^2,$$

and consequently

$$E=G=\rho^2\mathbf{E}=1-\phi^4,\qquad F=-\rho^2\mathbf{F}=(1-\phi^2)^2,\qquad D'=\rho\sqrt{\mathbf{EG}-\mathbf{F}^2}=2\phi.$$

For the coefficients of the linear element of the spherical representation we have

$$\mathbf{E}_1=\mathbf{G}_1=\frac{1+\phi^2}{1-\phi^2},\qquad \mathbf{F}_1=-1,$$

and from (35)

$$\rho_1=\frac{\phi^2-1}{\phi^2},\qquad \mu'=-\sigma'=\frac{1}{\phi^2}.$$

Therefore the linear element of the surfaces S' and S_1' is

$$ds'^2=\frac{1}{\phi^4}[(1-\phi^4)du^2+2(1-\phi^2)^2dudv+(1-\phi^4)dv^2], \tag{47}$$

and the coefficients of the second quadratic forms of these surfaces are respectively

$$\frac{2}{\phi},\ 0,\ -\frac{2}{\phi};\qquad 0,\ \frac{2}{\phi},\ 0.$$

The parameters of the lines of curvature on S are given by

$$u+v=u_1,\qquad u-v=v_1.$$

In terms of these parameters the two quadratic forms of S are

$$ds^2=(1-\phi^2)\left[\phi^2 du_1^2+dv_1^2\right],$$
$$\phi(du_1^2-dv_1^2).$$

Hence S is a surface of revolution. Moreover, these are the parameters of the asymptotic lines on S_1, hence S_1 is a minimal surface. In like manner (47) becomes

$$ds'^2=\frac{1-\phi^2}{\phi^4}(du_1^2+\phi^2 dv_1^2),$$

and the curves $u_1=$ const., $v_1=$ const. are the asymptotic lines on S'. Moreover, the curves $v_1=$ const. are geodesics; consequently S' is the skew helicoid with plane director. Hence the surfaces S which we have just been considering are the surfaces S' corresponding to the case where S is the skew helicoid.

§ 8. *Spherical representation of the surfaces S_1.*

We return to the consideration of the general problem and write the linear element of the spherical representation of S_1, referred to its conjugate virtual asymptotic lines, in the form

$$d\sigma_1^2=\mathbf{E}_1 du^2+2\mathbf{F}_1 du\,dv+\mathbf{G}_1 dv^2. \tag{48}$$

Between the Christoffel symbols $\left\{ {r s \atop t} \right\}'$, formed with respect to (48) and those with respect to the linear element of S and S_1 the following relations obtain: *

$$\left\{\begin{array}{lll} \left\{ {11 \atop 1} \right\}=\dfrac{\partial \log D_1}{\partial u}-\left\{ {11 \atop 1} \right\}_1', & \left\{ {12 \atop 1} \right\}=-\dfrac{D_1''}{D_1}\left\{ {11 \atop 2} \right\}_1', & \left\{ {22 \atop 1} \right\}=-\dfrac{D_1''}{D_1}\left\{ {12 \atop 2} \right\}_1', \\ \left\{ {22 \atop 2} \right\}=\dfrac{\partial \log D_1''}{\partial v}-\left\{ {22 \atop 2} \right\}_1', & \left\{ {12 \atop 2} \right\}=-\dfrac{D_1}{D_1''}\left\{ {22 \atop 1} \right\}_1', & \left\{ {11 \atop 2} \right\}=-\dfrac{D_1}{D_1''}\left\{ {12 \atop 1} \right\}_1'. \end{array}\right. \tag{49}$$

From (3), (5) and (6) we get

$$\left\{\begin{array}{l} \dfrac{\partial \log D_1}{\partial u}=2\left\{ {11 \atop 1} \right\}-3\left\{ {12 \atop 2} \right\}-\left\{ {22 \atop 1} \right\}e^{2t}, \\ \dfrac{\partial \log D_1''}{\partial v}=2\left\{ {22 \atop 2} \right\}-3\left\{ {12 \atop 1} \right\}-\left\{ {11 \atop 2} \right\}e^{-2t}. \end{array}\right. \tag{50}$$

* BIANCHI, *Lezioni*, I, p. 167.

By means of (49), (50), and

$$\frac{D_1}{D_1''} = -e^{2t}, \tag{51}$$

equations (6) can be put in the form

$$\begin{aligned} \frac{\partial t}{\partial v} &= -e^{-2t}\left\{ {11 \atop 2} \right\}_1' - \left\{ {22 \atop 2} \right\}_1', \\ \frac{\partial t}{\partial u} &= e^{2t}\left\{ {22 \atop 1} \right\}_1' + \left\{ {11 \atop 1} \right\}_1'. \end{aligned} \tag{52}$$

The condition of integrability of these equations reduces to the form

$$A_1 e^{2t} + B_1 e^{-2t} + C_1 = 0, \tag{53}$$

where for the sake of brevity we have put

$$\begin{aligned} A_1 &= \frac{\partial}{\partial v}\left\{ {22 \atop 1} \right\}_1' - 2\left\{ {22 \atop 2} \right\}_1'\left\{ {22 \atop 1} \right\}_1', \\ B_1 &= \frac{\partial}{\partial u}\left\{ {11 \atop 2} \right\}_1' - 2\left\{ {11 \atop 1} \right\}_1'\left\{ {11 \atop 2} \right\}_1', \\ C_1 &= \frac{\partial}{\partial v}\left\{ {11 \atop 1} \right\}_1' + \frac{\partial}{\partial u}\left\{ {22 \atop 2} \right\}_1' - 4\left\{ {11 \atop 2} \right\}_1'\left\{ {22 \atop 1} \right\}_1'. \end{aligned} \tag{54}$$

In terms of the symbols $\left\{ {r s \atop t} \right\}_1'$ equations (3) and (50) have the forms

$$\frac{\partial}{\partial v}\log D_1 = \left\{ {12 \atop 1} \right\}_1' + \left\{ {11 \atop 2} \right\}_1' e^{-2t}, \quad \frac{\partial}{\partial u}\log D_1'' = \left\{ {12 \atop 2} \right\}_1' + \left\{ {22 \atop 1} \right\}_1' e^{2t}, \tag{55}$$

and

$$\begin{aligned} \frac{\partial \log D_1}{\partial u} &= \left\{ {12 \atop 2} \right\}_1' + 2\left\{ {11 \atop 1} \right\}_1' + 3\left\{ {22 \atop 1} \right\}_1' e^{2t}, \\ \frac{\partial \log D_1''}{\partial v} &= \left\{ {12 \atop 1} \right\}_1' + 2\left\{ {22 \atop 2} \right\}_1' + 3\left\{ {11 \atop 2} \right\}_1' e^{-2t}. \end{aligned} \tag{56}$$

The conditions of integrability of these expressions are reducible to

$$\begin{aligned} 3e^{2t}A_1 - e^{-2t}B_1 + 2\left(\frac{\partial}{\partial v}\left\{ {11 \atop 1} \right\}_1' - 2\left\{ {22 \atop 1} \right\}_1'\left\{ {11 \atop 2} \right\}_1' \right) &= 0, \\ -e^{2t}A_1 + 3e^{-2t}B_1 + 2\left(\frac{\partial}{\partial u}\left\{ {22 \atop 2} \right\}_1' - 2\left\{ {22 \atop 1} \right\}_1'\left\{ {11 \atop 2} \right\}_1' \right) &= 0, \end{aligned} \tag{57}$$

which, when added together, give equation (53).

It is evident that there is at most one value of e^{2t} which satisfies equations (53) and (57). Suppose that we have a common solution of them and that it satisfies equations (52). From (55) and (56) we find D_1, D_1'' by quadratures

and the corresponding surface S_1 is capable of a deformation in which the parametric lines become asymptotic. For the total curvature of S_1 is given by

$$K = -\frac{1}{\rho^2} = \frac{\mathbf{E}_1\mathbf{G}_1 - \mathbf{F}_1^2}{D_1 D_1''} = -\frac{\mathbf{E}_1\mathbf{G}_1 - \mathbf{F}_1^2}{D_1^2 e^{-2t}},$$

which is such that

$$\frac{\partial \log \rho}{\partial u} = 2e^{2t}\left\{ \begin{matrix} 22 \\ 1 \end{matrix} \right\}_1' = 2\left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\}, \qquad \frac{\partial \log \rho}{\partial v} = 2e^{-2t}\left\{ \begin{matrix} 11 \\ 2 \end{matrix} \right\}_1' = 2\left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\}.$$

But this is the condition that there be a surface S with the same linear element as S_1 and with asymptotic lines parametric.

By means of equations (49) and (56) the first two of equations (13) can be put in the form

$$\frac{\partial}{\partial u}\left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\}_1' = \frac{\partial}{\partial v}\left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\}_1' = 2\left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\}_1' \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\}_1'.$$

But these are the necessary and sufficient conditions that the curves upon the sphere represent the asymptotic lines upon a surface whose total curvature is of the form

$$K' = -\frac{1}{[\phi(u) + \psi(v)]^2}. \tag{58}$$

Hence *when a surface S is applicable upon an infinity of surfaces S_1 with the virtual asymptotic lines conjugate on the latter, each of the surfaces S_1 is associate to a surface whose total curvature is of the form* (58), *when the asymptotic lines of this surface are parametric.*

From (53) and (57) it is seen that in order that there be more than one surface S_1 with the given spherical representation, it is necessary and sufficient that

$$A_1 = B_1 = C_1 = 0, \qquad \frac{\partial}{\partial v}\left\{ \begin{matrix} 11 \\ 1 \end{matrix} \right\}_1' = \frac{\partial}{\partial u}\left\{ \begin{matrix} 22 \\ 2 \end{matrix} \right\}_1'.$$

But these are equations of the same form as equations (20). Hence when a system of lines on the sphere represents the asymptotic lines on a surface S applicable to an infinity of surfaces S_1 with conjugate virtual asymptotic lines, these lines represent the conjugate lines upon an infinity of surfaces S' each of which is applicable to a surface S_1' upon which the asymptotic lines are parametric.

We consider this case and fix our attention upon one of the surfaces S'. We know that there is a spherical surface Σ' associate to it, which is referred to a conjugate system with equal point invariants. But the tangential invariants also are equal. Hence the conditions (33) must be satisfied; consequently equations (20) reduce to

$$(59)\qquad \frac{\partial}{\partial u}\log\left\{\begin{matrix}11\\2\end{matrix}\right\}' = -2\left\{\begin{matrix}22\\1\end{matrix}\right\}', \qquad \frac{\partial}{\partial v}\log\left\{\begin{matrix}22\\1\end{matrix}\right\}' = -2\left\{\begin{matrix}11\\2\end{matrix}\right\}'.$$

From these equations we find

$$\frac{\partial^2 \log}{\partial u\,\partial v}\left\{\begin{matrix}11\\2\end{matrix}\right\}' = \frac{\partial^2 \log}{\partial u\,\partial v}\left\{\begin{matrix}22\\1\end{matrix}\right\}',$$

so that

$$\left\{\begin{matrix}11\\2\end{matrix}\right\}' = \lambda V', \qquad \left\{\begin{matrix}22\\1\end{matrix}\right\}' = \lambda U',$$

where λ is an auxiliary function, U and V are functions of u and v alone and the primes indicate differentiation. Substituting in (59) and integrating, we get

$$\lambda = \frac{1}{2(U+V)}.$$

Hence equations (20) can be replaced by

$$(60)\qquad \begin{aligned} \left\{\begin{matrix}11\\1\end{matrix}\right\}' &= -\left\{\begin{matrix}22\\1\end{matrix}\right\}' = -\frac{U'}{2(U+V)},\\ \left\{\begin{matrix}22\\2\end{matrix}\right\}' &= -\left\{\begin{matrix}11\\2\end{matrix}\right\}' = -\frac{V'}{2(U+V)}.\end{aligned}$$

Equations (18) reduce to

$$(61)\qquad \begin{aligned} e^{-2t}\frac{\partial t}{\partial u} &= -(e^{-2t}-1)\frac{U'}{2(U+V)},\\ e^{2t}\frac{\partial t}{\partial v} &= (e^{2t}-1)\frac{V'}{2(U+V)}.\end{aligned}$$

Neglecting for the moment the solution

$$(62)\qquad e^{2t} = 1,$$

the integrals of these respective equations are

$$(63)\qquad e^{-2t}-1 = (U+V)\psi(v), \qquad e^{2t}-1 = (U+V)\phi(u),$$

where $\psi(v)$ and $\phi(u)$ are arbitrary functions of v and u. Since equations (61) are simultaneous, we get on dividing the above equations

$$e^{2t} = -\frac{\phi(u)}{\psi(v)}.$$

Substituting in (63), we have

$$\phi + \psi = -(U+V)\phi\psi.$$

Consequently

$$\phi = \frac{1}{U - c}, \qquad \psi = \frac{1}{V + c},$$

where c is an arbitrary constant. Hence the general integral of equations (61) is

$$e^{2t} = \frac{c + V}{c - U}. \tag{64}$$

From (5), (22), (24) and (25) we obtain the following theorem:

When a surface S is applicable to an infinity of surfaces S_1 with the virtual asymptotic lines of the latter conjugate, the fundamental coefficients of the surfaces S_1 are given without quadrature; the surface S is associate to an infinity of spherical surfaces Σ whose rectangular coördinates are given by quadratures; and the integration of the equations of the asymptotic lines on the surfaces Σ reduces to quadratures.

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