A METHOD FOR CONSTRUCTING THE FUNDAMENTAL REGION OF A DISCONTINUOUS GROUP OF LINEAR TRANSFORMATIONS*

BY

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Two methods at present exist for the construction of the fundamental region of a discontinuous group.† The one applies only to groups which are capable of extension by reflection and consists in constructing first the region for the extended group (which can be done by a direct process) and then combining two such regions that are adjacent. The other method uses the process of the continuous reduction of a quadratic form. It is, however, tedious in application, and this fact appears to place a narrow restriction on its range of practical effectiveness. In the following pages I propose a much more simple and effective method, applicable to groups which leave a given Hermitian form invariant.

§ 1. *Groups of transformations of a single variable.*

Let the complex variable be denoted by z and its conjugate by $\bar{z}$. Also let the equation of the fixed circle for the given group be

$$F \equiv \alpha z\bar{z} + \gamma z + \bar{\gamma}\bar{z} + \beta = 0, \tag{1}$$

in which the discriminant $\delta = \gamma\bar{\gamma} - \alpha\beta$ is positive (α, β being real) and α is negative. The circle is then real and its interior will be defined by the inequality $F > 0$.

Let $F = 0$ be transformed into itself by the substitution

$$z' = \frac{az + b}{cz + d}, \qquad ad - bc = 1. \tag{2}$$

The conditions for this are

$$\begin{aligned} \alpha(a\bar{a} - 1) + \gamma a\bar{c} + \bar{\gamma}\bar{a}c + \beta c\bar{c} &= 0, \\ \alpha a\bar{b} + \gamma a\bar{d} + \bar{\gamma}c\bar{b} \times \beta c\bar{d} &= \gamma, \\ \alpha b\bar{b} + \gamma b\bar{d} + \bar{\gamma}\bar{b}d + \beta(d\bar{d} - 1) &= 0. \end{aligned} \tag{3}$$

Since $F = 0$ is also transformed into itself by the inverse of (2), the conditions

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† See FRICKE-KLEIN, *Automorphe Functionen*, I, p. 539.

(3) may be replaced by the equivalent ones,

$$
\begin{aligned}
&\alpha(d\bar{d}-1)-\gamma d\bar{c}-\bar{\gamma}\bar{d}c+\beta c\bar{c}=0,\\
(3')\qquad &-\alpha d\bar{b}+\gamma d\bar{a}+\bar{\gamma}c\bar{b}-\beta c\bar{a}=\gamma,\\
&\alpha b\bar{b}-\gamma b\bar{a}-\bar{\gamma}a\bar{b}+\beta(a\bar{a}-1)=0.
\end{aligned}
$$

The transformation (2) changes the function F into F' such that

$$F'=F/\Delta\bar{\Delta}, \tag{4}$$

$$\Delta=cz+d. \tag{5}$$

Suppose that for a given point z inside the circle (1) we have $\Delta\bar{\Delta}<1$; thence follows $F'>F$, and the effect of the substitution has been to increase the value of the positive function F. If there exists a second substitution changing the point z' into z'' so that $F''>F'$, we apply it to the preceding one and so proceed until F has reached its maximum value for the series of congruent points $z, z', z'', \cdots$. Such a maximum value will be attained after a finite number of steps provided that the substitutions employed all belong to a group G which is "properly discontinuous" in the z-plane. I will name the point so determined *the maximum point congruent to* z.

There may be several points congruent to z for which F' has a maximum value. The number of these is equal to the order* ν of the subgroup g of G defined by $c=0$, since under this condition it follows from (3′) that $\Delta\bar{\Delta}=1$. The totality of maximum congruent points obtained by varying z throughout the interior of F will form a region R bounded by certain circles, orthogonal to $F=0$, whose equations are

$$\Delta\bar{\Delta}=1. \tag{6}$$

The region R is divided by means of g into ν congruent regions $R_1, R_2, \cdots, R_\nu$. Any one of the regions R_i may then be taken as the fundamental region for the given group G.

It only remains to show how to select from among the circles (6) those which bound the region R. For this purpose we observe that when F increases, the point z approaches the center C of the circle $F=0$. Accordingly the boundaries of R are to be selected from those Δ-circles (6) which pass nearest to C. The point of $\Delta\bar{\Delta}=1$ nearest to C is

$$z=\frac{d\lambda+\sqrt{\lambda\bar{\lambda}}}{-c\lambda}, \tag{7}$$

in which

$$\lambda=\gamma\bar{c}-\alpha\bar{d}. \tag{8}$$

*This order may be infinite as in the case of the modular group when it has the real axis for its fixed circle.

The value of F at this point is

$$F = 2(\alpha + \sqrt{\lambda\bar{\lambda}})/c\bar{c}. \tag{9}$$

This is evidently the greatest value attained by F on the given Δ-circle, and it may be found by the usual process for determining the maximum value of a function.

From (8) we deduce

$$\gamma\bar{\gamma}c\bar{c} = \alpha\gamma\bar{c}d + \alpha\bar{\gamma}c\bar{d} - \alpha^2 d\bar{d} + \lambda\bar{\lambda};$$

also, from the first equation (3′),

$$\alpha\beta c\bar{c} = \alpha\gamma d\bar{c} + \alpha\bar{\gamma}c\bar{d} - \alpha^2 d\bar{d} + \alpha^2,$$

whence, by subtraction,

$$\delta c\bar{c} = \lambda\bar{\lambda} - \alpha^2.$$

Using this result in (9) we have for the maximum value of F on (6),

$$F = \frac{2\delta}{\sqrt{\lambda\bar{\lambda}} - \alpha}.$$

From this it follows that the greatest values attained by F at the points defined by (7) are those for which the absolute values of λ are the least.

It is evident, then, that the boundary circles for the region R occur among those determined by the smallest values of $|\lambda|$. There does not appear to be a sufficiently simple and general criterion by which to select out the required circles from among those found in this way, but a graphical construction will quickly indicate which ones are to be used.

The coefficients a, b, d are expressible in terms of λ, c in the form

$$a = \frac{\lambda + \bar{\gamma}c}{-\alpha}, \qquad d = \frac{\bar{\gamma}c - \bar{\lambda}}{\alpha}, \qquad b = \frac{\bar{\gamma}(\bar{\lambda} - \bar{\gamma}c) + \delta\bar{c} - \lambda\bar{\gamma}}{\alpha^2}, \tag{10}$$

the numbers λ, c being connected by the relation

$$\lambda\bar{\lambda} = \alpha^2 + \delta c\bar{c} \tag{11}$$

which is the condition that the determinant shall equal 1. Equation (11) is also the condition that the circle $\Delta\bar{\Delta} = 1$ shall be orthogonal to $F = 0$.

To give a clearer idea of how the preceding method works out in practise, we consider the following examples in which a, b, c, d are restricted to complex integers.

Example I. $\alpha = 1$, $\gamma = 0$, $\beta = -5$.

For $c = 0$ a substitution of period 2 is obtained having the origin as fixed point. Hence the region R consists of two congruent regions R_1, R_2. For $|c| > 0$ the minimum value of $\lambda\bar{\lambda}$ is 26, and hence $\lambda = \pm 5 \pm i$ or $\pm 1 \pm 5i$;

also $c\bar{c} = 5$ and therefore $c = \pm 2 \pm i$ or $\pm 1 \pm 2i$. The equation of the Δ-circle is $(cz - \bar{\lambda})(\bar{c}\bar{z} - \lambda) = 1$, which may be written

$$\left(z - \frac{\bar{\lambda}}{c}\right)\left(\bar{z} - \frac{\lambda}{\bar{c}}\right) = \frac{1}{c\bar{c}},$$

and hence its center is the point $z = \bar{\lambda}/c$ and its radius is $1/\sqrt{c\bar{c}}$. With the values just obtained for λ, c we construct the corresponding circles $1, 1, \cdots$, Fig. 1, using only those which occur in the upper half plane in which we take

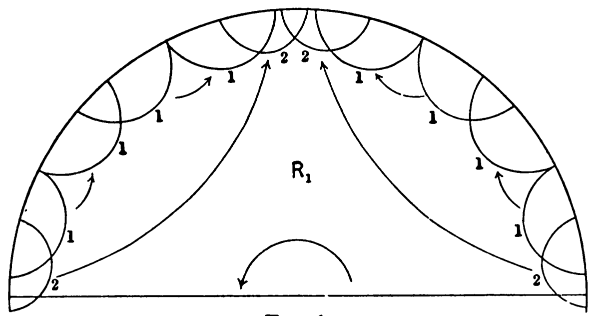

FIG. 1.

R_1. These do not form a closed region and we accordingly take the next higher value of $\lambda\bar{\lambda}$ which is 41, whence $c\bar{c} = 8$. The circles $2, 2, \cdots$ are determined by these values, and now the region above the real axis being closed we have in R_1 thus determined the fundamental region for the given group.*

Example II. $\alpha = -3$, $\beta = \gamma = 1$.

It is useful to observe from the relation $\lambda + \bar{\lambda} = -\alpha(a + d)$ that $\lambda + \bar{\lambda}$ is divisible by α. For $c = 0$ the only substitution of the group is identity. The values of λ and c first to be considered are those for which $\lambda\bar{\lambda} = 13$, $c\bar{c} = 1$. Among these the only ones which give different substitutions with complex integral values for a, b, d are $\lambda = 3 + 2\epsilon i$, $c = \epsilon i$, $\epsilon = \pm 1$. These determine the two circles 1, 2, Fig. 2. The next higher values of λ and c are derived from $\lambda\bar{\lambda} = 25$, $c\bar{c} = 4$, whence we obtain the circles 3, 4, 5. We exclude 4, 5 since they lie inside 1, 2. As the region about the center C is not yet closed we take the next case $\lambda\bar{\lambda} = 45$, $c\bar{c} = 9$, which determines the circles 6, 7, 8, 9, the last two of which are to be excluded. As the region R is now closed the operation is completed.

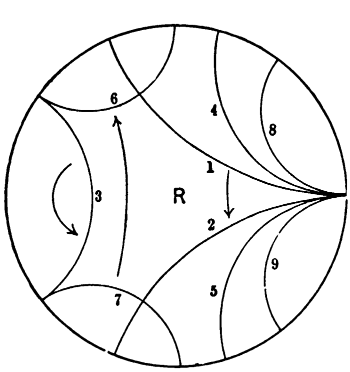

FIG. 2.

* Cf. FRICKE-KLEIN, *Automorphe Functionen*, I, p. 479.

§ 2. *Transformations on two variables.*

It is evident that the general method explained above is at once applicable to groups of substitutions on any number of variables, the region R being then defined by certain inequalities of the form $\Delta\bar{\Delta} > 1$. For the sake of simplicity I will confine myself in the present paper to the case of a group G of linear transformations on two complex variables x, y, the general substitution being written in the form

$$x' = (ax + by + c)/\Delta, \qquad y' = (dx + ey + f)/\Delta \tag{12}$$

in which $\Delta = px + qy + r$. I will further suppose that the group G leaves the hypersphere

$$F \equiv 1 - x\bar{x} - y\bar{y} = 0 \tag{13}$$

invariant. The conditions on the coefficients are

$$\begin{aligned} a\bar{a} + d\bar{d} - p\bar{p} &= 1, & a\bar{b} + d\bar{e} - p\bar{q} &= 0, \\ b\bar{b} + e\bar{e} - q\bar{q} &= 1, & a\bar{c} + d\bar{f} - p\bar{r} &= 0, \\ c\bar{c} + f\bar{f} - r\bar{r} &= -1, & b\bar{c} + e\bar{f} - q\bar{r} &= 0. \end{aligned} \tag{14}$$

Multiply the two equations (12) by $\bar{a}$, $\bar{d}$ respectively and add; then subtract $\bar{p}$ from both members of the resulting equation and use (14). This gives

$$\bar{a}x' + \bar{d}y' - \bar{p} = x/\Delta;$$

similarly,

$$\bar{b}x' + \bar{e}y' - \bar{q} = y/\Delta \qquad \text{and} \qquad \bar{c}x' + \bar{f}y' - \bar{r} = -1/\Delta.$$

Hence the inverse of (12) may be written

$$x = (\bar{a}x' + \bar{d}y' - \bar{p})/\Delta', \qquad y = (\bar{b}x' + \bar{e}y' - \bar{q})/\Delta', \tag{15}$$

$$\Delta' = -\bar{c}x' - \bar{f}y' + \bar{r}.$$

As the inverse substitution also leaves the hypersphere invariant we have the conditions

$$\begin{aligned} a\bar{a} + b\bar{b} - c\bar{c} &= 1, & a\bar{d} + b\bar{e} - c\bar{f} &= 0, \\ d\bar{d} + e\bar{e} - f\bar{f} &= 1, & a\bar{p} + b\bar{q} - c\bar{r} &= 0, \\ p\bar{p} + q\bar{q} - r\bar{r} &= -1, & d\bar{p} + e\bar{q} - f\bar{r} &= 0. \end{aligned} \tag{14'}$$

From the fourth and fifth of these equations we deduce

$$a = k(\bar{f}\bar{q} - \bar{e}\bar{r}), \qquad b = k(\bar{d}\bar{r} - \bar{f}\bar{p}), \qquad c = k(\bar{d}\bar{q} - \bar{e}\bar{p}).$$

In order to evaluate the unknown k, we substitute these expressions for a, b, c

in the determinant

$$\begin{vmatrix} a & b & c \\ d & e & f \\ p & q & r \end{vmatrix} = N$$

and obtain

$$k(-A\bar{A} - B\bar{B} + C\bar{C}) = N,$$

in which A, B, C are the minors corresponding to a, b, c. But since the inverse of the substitution (12) is

$$x = \frac{Ax' + Dy' + P}{Cx' + Fy' + R}, \qquad y = \frac{Bx' + Ey' + Q}{Cx' + Fy' + R},$$

it follows that $A\bar{A} + B\bar{B} - C\bar{C} = 1$ and therefore $k = -N$. Hence,

$$\begin{aligned} a &= N(\bar{e}\bar{r} - \bar{f}\bar{q}) = N\bar{A}, \\ b &= N(\bar{f}\bar{p} - \bar{d}\bar{r}) = N\bar{B}, \\ c &= N(\bar{e}\bar{p} - \bar{d}\bar{q}) = -N\bar{C}, \end{aligned} \tag{16}$$

in which $d, e, \cdots$ satisfy the three relations

$$\begin{aligned} p\bar{p} + q\bar{q} - r\bar{r} &= -1, \\ d\bar{d} + e\bar{e} - f\bar{f} &= 1, \\ d\bar{p} + e\bar{q} - f\bar{r} &= 0. \end{aligned} \tag{17}$$

We next proceed to determine the maximum point of the *spread*

$$\Delta\bar{\Delta} = 1, \qquad \Delta = px + qy + r, \tag{18}$$

that is, the point at which the $F = 1 - x\bar{x} - y\bar{y}$ takes its maximum value when x, y are restricted to the locus (18). Regarding $\bar{y}$ as a dependent variable on account of (18) we differentiate F with respect to $x, \bar{x}, y$ and obtain as conditions for a maximum

$$\bar{x} + y\frac{\partial \bar{y}}{\partial x} = 0, \qquad x + y\frac{\partial \bar{y}}{\partial \bar{x}} = 0, \qquad \bar{y} + y\frac{\partial \bar{y}}{\partial y} = 0.$$

By differentiating $\Delta\bar{\Delta} = 1$ and substituting the expressions thus obtained for the derivatives, we deduce

$$x = \frac{\bar{p}}{\bar{q}}y, \qquad \bar{x} = \frac{p\bar{\Delta}}{\bar{q}\Delta}y, \qquad \bar{y} = \frac{q\bar{\Delta}}{\bar{q}\Delta}y. \tag{19}$$

By substitution of these results in the second equation (18) we obtain

$$\Delta = (r\bar{r} - 1)\bar{y}/\bar{q},$$

and by substituting in (10),

$$\bar{y} = \frac{q\bar{r}}{\bar{q}r} y, \qquad \bar{x} = \frac{p\bar{r}}{\bar{q}r} y, \qquad x = \frac{\bar{p}}{\bar{q}} y.$$

Substitute now in $\Delta\bar{\Delta} = 1$ and solve for y. This gives

$$y = \frac{\bar{q}(-\bar{r}r \pm \sqrt{r\bar{r}})}{\bar{r}(r\bar{r} - 1)}.$$

Therefore

$$F = \frac{-2 \pm 2\sqrt{r\bar{r}})}{r\bar{r} - 1}.$$

From the first equation (17) it is evident that $r\bar{r}$ cannot be less than 1. In order that the point determined may be inside the hypersphere it is necessary to take the lower sign. Hence, *the point*

$$x = \frac{\bar{p}(\sqrt{r\bar{r}} - r\bar{r})}{\bar{r}(r\bar{r} - 1)}, \qquad y = \frac{\bar{q}(\sqrt{r\bar{r}} - r\bar{r})}{\bar{r}(r\bar{r} - 1)}$$

is the maximum point for the spread $\Delta\bar{\Delta} = 1$ *and gives to* F *the value*

(20) $$F = \frac{2}{\sqrt{r\bar{r}} + 1}.$$

A glance at (20) shows that the largest maximum values are attained by F when $r\bar{r}$ has its smallest values. The corresponding Δ-spreads are those of (18) which pass nearest to the center of $F = 0$, and the boundaries for the region R are to be selected from among these.

§ 3. *Klein's group of* $(n+1)!$ *collineations.*

The working out of the fundamental region for an infinite discontinuous group of the class just treated appears to involve a considerable amount of necessary detail and for that reason I reserve for a future paper the discussion of particular groups. In the present article I will only notice, by way of illustration, the fundamental region R for KLEIN's group of $(n+1)!$ collineations defined by the homogeneous equations of transformation on the variables y_i,

(21) $$y'_i = y_{a_i} \qquad (i = 1, 2, \cdots, n+1),$$

in which $a_1, a_2, \cdots, a_{n+1}$ is any permutation of the numbers $1, 2, \cdots, n+1$. We first make the group non-homogeneous by introducing the variables $x_k = y_k / y_{n+1}$ $(k = 1, 2, \cdots, n)$. Equations (21) then become

(22) $$x'_k = x_{a_k} / x_i \qquad (k = 1, 2, \cdots, n),$$

in which x_{a_k} of the numerator is to be replaced by 1 when $a_k = i$. We have, then, $\Delta = x_i$ and the inequality conditions $\Delta\bar{\Delta} > 1$ reduce to

$$x_i\bar{x}_i - 1 > 0 \qquad (i=1, 2, \cdots, n). \tag{23}$$

Apply now to (23) the different transformations of the group. We obtain additional inequalities of the form

$$x_i\bar{x}_i - x_j\bar{x}_j > 0 \qquad (i, j=1, 2, \cdots, n). \tag{24}$$

A complete set of consistent inequalities of this type will be obtained by subjecting i and j to the condition $i > j$.

Returning now to the homogeneous variables, the conditions (23) and (24) become

$$y_i\bar{y}_i - y_j\bar{y}_j > 0, \qquad i > j \qquad (i, j=1, 2, \cdots, n+1). \tag{25}$$

These inequalities (some of which are superfluous) define a fundamental region R for the given group. For, since any substitution of the group produces a permutation of the y_i, it follows that at least one of the inequalities (25) is reversed. The only case in which this is not immediately apparent is that of the substitution which produces the cyclic permutation $(y_1, y_2, \cdots, y_{n+1})$. The inequalities (25) are permuted except that in place of $y_2\bar{y}_2 - y_1\bar{y}_1 > 0$ we now have $y_1\bar{y}_1 - y_{n+1}\bar{y}_{n+1} > 0$. But if we add this last to the $n-1$ inequalities $y_{i+1}\bar{y}_{i+1} - y_i\bar{y}_i > 0\,(i = 2, 3, \cdots, n)$, we obtain $y_1\bar{y}_1 - y_2\bar{y}_2 > 0$, which is a reversal of the first inequality (25). Hence every point within R is transformed into a point outside R by each substitution of the group. On the other hand, every point outside R can be transformed into a point within. For, the coördinates of an arbitrarily fixed point can evidently satisfy a set of conditions of the form

$$y_{a_i}\bar{y}_{a_i} - y_{a_j}\bar{y}_{a_j} > 0, \qquad i > j \qquad (i, j=1, 2, \cdots, n+1). \tag{26}$$

By choosing the a_i as all possible permutations of $1, 2, \cdots, n+1$ we obtain $(n+1)!$ different sets of conditions (26), including (25). It is also clear that no other set of conditions except those enumerated would be a consistent one. For if in the $n(n+1)$ inequalities (26) no variable occurred in the first term in more than n of the formulas, it would be necessary for two of the inequalities to have the form $y_k\bar{y}_k - y_l\bar{y}_l > 0$, $y_l\bar{y}_l - y_k\bar{y}_k > 0$, which are contradictory. Suppose, then, that y_{a_1} occurs in the first term of $n+1$ of the inequalities (26). A similar argument will show that some other variable, say y_{a_2}, must occur in the first term of n of the remaining inequalities, and so on. We accordingly have the arrangement of (26). But since there are $(n+1)!$ substitutions in the group and no substitution, except 1, leaves a particular set unaltered, it follows that any given set (26) can be transformed into (25) by a certain substitution of the group, and therefore any point outside R has a congruent point inside.

After excluding superfluous conditions it is evident that the $n(n+1)$ inequalities (25) reduce to *

$$y_{i+1}\bar{y}_{i+1} - y_i\bar{y}_i > 0 \qquad (i=1, 2, \cdots, n). \tag{27}$$

The boundary

$$y_i\bar{y}_i - y_{i+1}\bar{y}_{i+1} = 0 \tag{28}$$

is divided by the flat space $y_i = y_{i+1}$ into two parts which are interchanged by the substitution

$$y'_i = y_{i+1}, \qquad y'_{i+1} = y_i, \qquad y'_k = y_k \quad (k=1, 2, \cdots, i-1, i+2, \cdots, n+1),$$

while the $n-1$ remaining inequalities (27) are unaltered. Hence each of the boundaries (28) is paired with itself.

* The fundamental region for this group, when the variables y_i are restricted to real values, is given by E. H. MOORE, *Concerning Klein's group of* $(n+1)!$ *n-ary collineations*, American Journal of Mathematics, vol. 22 (1900), p. 336. When the variables are real the conditions (27) become $y_{i+1}^2 - y_i^2 > 0$. For positive values of the y's these reduce to exactly the conditions given by Professor Moore, p. 338; for negative values of the variables the two sets of conditions do not coincide, but they are equivalent and one may be reduced to the other by an "allowable change" of the fundamental region.

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