IRREDUCIBLE HOMOGENEOUS LINEAR GROUPS IN AN ARBITRARY DOMAIN*

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I have given elsewhere a necessary and sufficient condition that certain categories of abstract groups of finite order be simply isomorphic with irreducible homogeneous linear groups in the domain of all real and complex numbers.† In the present paper I establish a two-fold generalization of this result by showing that the same condition applies to all groups of finite order and to an arbitrary domain.

The proof for the necessity of this condition rests upon a conclusion drawn from Theorem II of SCHUR's *Neue Begründung der Theorie der Gruppencharaktere*.‡

Suppose that we have a homogeneous linear group G of finite order whose coefficients belong to Ω (an arbitrary finite field or an arbitrary domain) and which is irreducible in Ω. If P is a given substitution on the same f variables as G with coefficients in Ω, the equation

$$|P - xE_f| = 0$$

is an equation in the unknown x with known coefficients. If this equation has a root in Ω and if P is commutative with every substitution of G, SCHUR's proof of his Theorem II applies, therefore P must be a similarity substitution.

If Ω is any finite field, say the $GF[p^m]$, then in the field F_p, which is the aggregate of all the fields § $GF[p^i]$ $(i = 1, 2, \cdots)$, G can be so transformed that in the matrix of every substitution of its central H all the elements below the main diagonal are zero. If Ω is a domain, the same result can be accom-

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† Transactions of the American Mathematical Society, vol. 7 (1906), p. 65; Bulletin of the American Mathematical Society, vol. 14 (1908), p. 327.

‡ Berliner Sitzungsberichte, 1905, I, p. 410.

§ Cf. DICKSON, Transactions of the American Mathematical Society, vol. 8 (1907), p. 389; Bulletin of the American Mathematical Society, vol. 13 (1907), p. 477.

plished by passing to the domain of all real and complex numbers.* In this transformed form of G no two substitutions S_1 and S_2 of H can change x_j into the same function, since, if this were possible, $S_1 S_2^{-1}$ would leave x_j unchanged —that is, one of the multipliers of $S_1 S_2^{-1}$ would be unity. But, since this substitution is invariant in the irreducible group G, our conclusion from SCHUR's Theorem shows that this is possible only in case $S_1 S_2^{-1}$ is a similarity substitution with its multipliers all equal to unity—that is, in case $S_1 S_2^{-1}$ is the identical substitution. But S_1 and S_2 were taken as distinct substitutions.

If now H is of order $p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$, where $p_1, p_2, \cdots, p_r$, are distinct primes, it is the direct product of groups $H_1, H_2, \cdots, H_r$, of order $p_1^{a_1}, p_2^{a_2}, \cdots, p_r^{a_r}$, respectively; and it follows from what has just been proved that $H_i\,(i=1, 2, \cdots, r)$, can contain only one subgroup of order p_i and is therefore cyclic.† *Hence H is cyclic.*

Suppose now that G is any group of finite order g with a cyclic central generated by the operation h of order $a\ (> 1)$. If we write G as a regular permutation group h will take the form

$$h \equiv (x_{11} x_{12} \cdots x_{1a})(x_{21} x_{22} \cdots x_{2a}) \cdots (x_{\gamma 1} x_{\gamma 2} \cdots x_{\gamma a}),$$

where $\gamma = g/a$. If now we write this permutation group as a homogeneous linear group and then transform it by the substitution

$$S: \qquad y_{ij} = \sum_{k=1}^{a} \omega^{(k-1)(j-1)} x_{ik},$$

where ω is a primitive ath root of unity, h will be transformed into its normal form. Moreover in this transformed form of G the variables

$$x_{1j},\ x_{2j},\ \cdots,\ x_{\gamma j} \qquad (j = 1, 2, \cdots, a)$$

are transformed among themselves in semi-canonical form.‡ We shall denote the group formed by the substitutions on these variables by G_j. If $j-1$ is relatively prime to a, G_j is simply isomorphic with G.

If G were not simply isomorphic with any irreducible group in C (the domain of all real and complex numbers), then when G_j ($j-1$ relatively prime to a) is completely reduced in C there must be some irreducible component such that when all the irreducible representations of G that are equivalent to this

* Cf. WEBER, *Algebra*, II (second edition), pp. 172, 173; 176–178. This discussion of WEBER's is with reference to the domain of all real and complex numbers, but the processes described are valid in the field F_p.

† BURNSIDE, *Theory of groups of finite order*, pp. 73, 75. If $p_i = 2$, H_i could not be the non-cyclic group with only one subgroup of order 2, since this contains more than one cyclic subgroup of order 4.

‡ Cf. Transactions of the American Mathematical Society, vol. 8 (1907), p. 108.

one (in number equal to the degree of each one*) are taken together, there is an r_1, r_2 isomorphism between the group formed by these representations and the group in the remaining variables, and $r_1 > 1$, $r_2 > 1$; since otherwise G would be simply isomorphic with some irreducible group. Call these two components L_1 and L_2 respectively and let their orders be denoted by l_1 and l_2 respectively. If H_i $(i = 1, 2)$ is the subgroup of L_i that corresponds to identity in the other component, H_i can contain no invariant substitution of L_i, except identity, since every invariant substitution G_j is a similarity substitution. Now each irreducible component of L_1 is of degree at most $\sqrt{l_1/a}$, since each such component is of order l_1 and contains at least a invariant substitutions.† But by virtue of the manner in which G_j is formed from L_1 and L_2 we must have $g = r_1 r_2 \cdot l_1/r_1$ and $l_1 = g/r_2$. Then the number of variables in L_1 is at most $g/(ar_2)$.

Any substitution of H_1, such as t of order b, when combined with identity of H_2 gives a substitution of G_j that has at least $g/a - g/(ar_2)$ multipliers equal to unity. Now there are $\phi(a)$ values of j such that $j - 1$ is relatively prime to a and there is a similar substitution in each G_j corresponding to these values of j. These similar substitutions combine with one another and with a certain substitution from every G $(j - 1$ not relatively prime to $a)$ to form a certain substitution of G of order b.

Since every G_j is in semi-canonical form, in any substitution of G_j the variables are permuted among themselves (except for certain coefficients). If now $j - 1$ is not relatively prime to a, the cycles of the permutation in a substitution of G_j that corresponds to a non-invariant substitution of G must all be of the same order, since otherwise a certain power of this substitution that is not invariant in G would change the variables of at least one of the sets of variables into themselves multiplied by certain constants. This means that in the regular permutation group there would be a non-invariant operation that leaves some of the systems of intransitivity of H unchanged. But this is impossible in a regular group. Hence in G_j $(j - 1$ not relatively prime to $a)$ the variables of t_j (the substitution that corresponds to t) are permuted among themselves (except for certain coefficients) in cycles of order b. The characteristic determinant D_j of t_j is then the product of g/ab determinants each of order b:

$$D_j \equiv \{(-1)^b \lambda^b - (-1)^b\}^{g/ab}.$$

Hence t_j has g/ab multipliers equal to unity. The total number of multipliers that equal unity in the substitution of G that corresponds to t would then be at least

$$\left(\frac{g}{a} - \frac{g}{ar_2}\right)\phi(a) + \frac{g}{ab}[a - \phi(a)].$$

* BURNSIDE, Acta Mathematica, vol. 28 (1904), p. 383.

† Transactions of the American Mathematical Society, vol. 7 (1906), p. 67.

But this number is greater than the actual number of multipliers of this substitution that equal unity, namely* g/h, provided $r_2 > 2$. However, if $r_2 = 2$, H_2 would contain only one substitution besides identity. This substitution would be invariant in L_2 and would combine with identity of L_1 to form an invariant substitution of G_j that is not a similarity substitution.

Hence G must be simply isomorphic with an irreducible group in C.

If G has no invariant operation except identity, we can form the direct product $\overline{G}$ of G and an operation h of order $a\,(>1)$. It follows from the preceding discussion that $\overline{G}$ is simply isomorphic with an irreducible group in C. The subgroup of this irreducible group that corresponds to G must also be irreducible, since otherwise, in view of the fact that the substitution corresponding to h is a similarity substitution, the group that is simply isomorphic with $\overline{G}$ would be reducible.

Hence any group of finite order with a cyclic central is simply isomorphic with a homogeneous linear group that is irreducible in C. But every irreducible representation of G (in C) is equivalent to one of the irreducible components of the regular permutation group that is simply isomorphic with G when this is written as a homogeneous linear group.† If now no irreducible component of this latter group in any domain Ω were simply isomorphic with G, then no irreducible component in C could be simply isomorphic with G.

We can formulate the results of the preceding discussion into the

THEOREM. *A necessary and sufficient condition that any group of finite order be simply isomorphic with an irreducible group in any domain is that its central be cyclic.*

It should be remarked that the sufficiency of this condition is established only for an arbitrary domain, whereas the necessity of it is established also for any finite field.

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* Transactions of the American Mathematical Society, vol. 8 (1907), p. 108.
† BURNSIDE, loc. cit., p. 387.