TWO-DIMENSIONAL CHAINS AND THE ASSOCIATED COLLINEATIONS IN A COMPLEX PLANE*

BY

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Introduction.

The recognition of the abstract identity of geometry and analysis, which results from the notion of coördinates on the one hand and from the classical work of VON STAUDT † on the algebra of throws on the other, and which recent work on the foundations of geometry has fully established, has brought with it a broader conception of the content of geometry. It has meant not only the introduction of imaginary elements and the resulting conception of a complex space (of any number of dimensions), but it has also led to the consideration of geometries with respect to any number-system (finite or infinite), i. e., of spaces the elements of which may be determined by sets of numbers (coördinates) belonging to a given number-system. ‡ An important result of the recognition of the identity referred to is the emphasis it places on the possibility of using geometric or synthetic methods in the solution of analytic problems. It seems likely that the fact that such methods have received comparatively little attention hitherto has resulted in a loss of power. The present paper, it is hoped, will tend to substantiate this assertion.

We are here concerned with certain fundamental problems in the projective geometry on a *complex* plane, i. e., a plane the points of which are determined by sets of homogeneous coördinates (x_1, x_2, x_3), where the x_i are any ordinary complex numbers. not all zero. Though the investigation is essentially geometric and the results are susceptible of immediate application to problems of importance in geometry, these results are of even greater interest in the theory of functions of two complex variables.

To make this clear we shall glance briefly at the corresponding problems on a complex line, the results of which are well-known in the theory of functions

* Presented to the Society, April 25, 1908, under a slightly different title.

† VON STAUDT, *Beiträge zur Geometrie der Lage*, Nürnberg, 1856–60, Heft 2, p. 261.

‡ Cf., e. g., § 2 of the paper by Professor VEBLEN and myself cited below; and for the finite cases, O. VEBLEN and W. H. BUSSEY, *Finite projective geometries*, Transactions of the American Mathematical Society, vol. 7 (1906), pp. 241–59.

of one complex variable and which I have recently considered from the point of view of projective geometry.* The notion of a *linear* (or one-dimensional) *chain* on a complex line was introduced by VON STAUDT † in connection with his introduction of imaginary elements into geometry and has since been made fundamental in recent work on the foundations of geometry. ‡ Such a chain may be defined as any class of points on a line which is projective with the class of real points on a line. There is on a line one and only one chain containing any three distinct points of the line. § The place of this notion in the theory of functions of a complex variable becomes apparent, if we adopt the usual representation of complex numbers (i. e., of the points of our line) by the real points of a plane or sphere. The chains on the line are then represented by the real circles (and straight lines) of the plane or sphere, and the study of the projectivities on the line (i. e., of the linear fractional transformations on the complex variable) with reference to their behavior toward the chains on the line is fundamental in the projective geometry on the line as well as in the theory of functions. This study leads to the classification of the projectivities into hyperbolic, elliptic, parabolic and loxodromic and to the well-known systems of chains (circles) associated with the first three of these types of projectivities.

In the present paper we are concerned with the corresponding problems in the plane. Here the notion of a *planar* (or two-dimensional) *chain* is fundamental. A planar chain may be conveniently described as any class of points and lines in the plane which may be obtained from the class of real points and lines by a projective collineation. A planar chain is then a two-dimensional spread of points and lines within the four-dimensional spread of all points and lines in our complex plane.‖ It follows readily from the description given that there is one and only one planar chain through any four points of the plane, no three of which are collinear.¶ The notion of a planar chain was first introduced

* J. W. YOUNG, *The geometry of chains on a complex line*, Annals of Mathematics, series 2, vol. 11 (1909), No. 1, pp. 33–48. This paper is referred to in the sequel by the letter *A*.

† VON STAUDT, loc. cit., p. 137.

‡ Cf. PIERI, *Nuovi principii di geometria projettiva complessa*, Memorie della R. Accademia delle Scienze di Torino, series II, vol. 45 (1905), pp. 189–235, in which the chain is chosen as one of the primitive notions; also O. VEBLEN and J. W. YOUNG, *A set of assumptions for projective geometry*, American Journal of Mathematics, vol. 30 (1908), pp. 347–380, where the chain is defined in terms of order and continuity relations.

§ The coördinate system on a line is determined as soon as 0, 1, ∞ are assigned to three distinct points of the line, which are entirely arbitrary. The chain determined by the three points is then simply the class of all points on the line that have real coördinates.

‖ In this paper we use the word "dimension" throughout in the sense of *real* dimension, i. e., a spread of n dimensions is one whose elements may be made to depend on n independent *real* parameters. The plane is, of course, a spread of two *complex* dimensions, but we shall not use the word here in this sense.

¶ We may indeed establish a coördinate system in the plane by choosing any three points by JUEL and SEGRE* in connection with their study of anti-projective transformations.† These planar chains are of value in those problems in the plane in which the distinction is maintained between real and imaginary elements. The projective point of view, however, emphasizes the fact that this distinction is relative, since any planar chain may be taken as the real planar chain; the points and lines not belonging to this chain are then "imaginary" with reference to the points and lines of the chain.

In § 1 below we define a planar chain geometrically in terms of the notion of a linear chain and derive certain properties of these chains which are needed in the later developments. In § 2 we determine necessary and sufficient conditions that a collineation in the plane leave a planar chain invariant; this is equivalent to the determination of the conditions that a collineation may be represented (transformed into one) with real coefficients. We are thus led to certain characteristic *systems of planar chains* which form a generalization of the systems of circles referred to above. In the Conclusion these systems are briefly considered and reference is made to certain immediate applications of the results obtained. Here also is outlined the important problem of the order relations in a complex plane, to which I expect to return on a future occasion.

§ 1. *Definition and fundamental properties of planar chains.*

We assume the theorems of alignment in the ordinary complex projective geometry of the plane.‡ All points and lines considered are coplanar. We assume further the notion and fundamental properties of (linear) chains on a line. We then define a planar chain as follows:

(which are not collinear) as the vertices of the triangle of reference and assigning to any fourth point which is not on a side of this triangle the coördinates (1, 1, 1). The planar chain determined by these four points then consists of all points and lines the ratios of whose coördinates are real numbers.

* JUEL, *Über einige Grundgebilde der projectiven Geometrie*, Acta Mathematica, vol. 14 (1890), pp. 1–30; SEGRE, *Un nuovo campo di ricerche geometriche*, Atti della R. Accademia delle Scienze di Torino, vol. 25 (1890), pp. 276–301, 430–457; vol. 26 (1890), pp. 35–71, 592–612. Though closely related in subject matter, the present paper has little direct contact with either of the two papers cited.

† The term "anti-projective" is due to SEGRE and seems to have been generally adopted. JUEL calls the anti-projective transformations in the plane "Symmetralitäten." He describes a planar chain in his introduction substantially as we have described it above (though in his description the planar chain consists of a class of points only). SEGRE on the other hand obtains them as the class of invariant elements of certain anti-projective transformations. It may be noted in passing that the property of a planar chain to the effect that the point of intersection of two lines joining two pairs of points of the chain is a point of the chain, which JUEL apparently takes as his definition of a planar chain (loc. cit., p. 3) is not sufficient to characterize it. The property is characteristic of any *planar field*, i. e., of any set of points and lines in a plane whose coördinates are numbers of a given field. The notion of a general field in a geometry of any number of dimensions I defined in a paper presented to the Society (Chicago section), January 2, 1909.

‡ Our developments are subject, e. g., to Assumptions A, E, H, C and I of the paper by Professor VEBLEN and myself quoted above.

DEFINITION. Given two linear chains $\mathfrak{C}_1$, $\mathfrak{C}_2$, on distinct lines and having a point O in common, a point is said to be *linearly related* to $\mathfrak{C}_1$, $\mathfrak{C}_2$, if it is the intersection of two lines, each of which joins a point of $\mathfrak{C}_1$ to a distinct point of $\mathfrak{C}_2$. A line is said to be *linearly related* to $\mathfrak{C}_1$, $\mathfrak{C}_2$, if it joins two points linearly related to $\mathfrak{C}_1$, $\mathfrak{C}_2$. The class of all points and lines linearly related to $\mathfrak{C}_1$, $\mathfrak{C}_2$ is called the *two dimensional chain* or the *planar chain* determined by $\mathfrak{C}_1$, $\mathfrak{C}_2$.

If a scale * be established on the line of $\mathfrak{C}_1$ by choosing the point O as 0 and any other two distinct points of $\mathfrak{C}_1$ as 1 and ∞ respectively, the points of $\mathfrak{C}_1$ correspond to the set of all ordinary real numbers (and ∞) and only these. If then a scale be established in a similar manner on the line of $\mathfrak{C}_2$, the coördinate system in the plane determined by these two scales will assign to every point (or line) of the planar chain $\mathfrak{C}^2$ determined by $\mathfrak{C}_1$, $\mathfrak{C}_2$ a set of real coördinates; and, conversely, to every set of real coördinates will correspond a point (or line) of $\mathfrak{C}^2$. Since a projective collineation in the plane is equivalent to a change in the system of coördinates, this justifies the analytic definition suggested in the introduction:

Any class of points and lines which can be obtained from the real points and lines of the plane by a projective collineation is a planar chain.

Since a projective transformation transforms linear chains into linear chains, it follows at once from the definition given above (and also from the analytic characterization just obtained), that

I. *Any class of points and lines of a plane which is projective with a planar chain is a planar chain.* †

The following fundamental *internal* properties of a planar chain $\mathfrak{C}^2$ may be deduced readily (they are simply properties of the points and lines of a *real* projective plane):

II. *There is one and only one planar chain containing four points (lines) of a plane no three of which are collinear (concurrent).*

III. *Any two lines of a planar chain meet in a point of the chain.*

IV. *Any line of a planar chain has the points of a linear chain in common with the planar chain, and only these.*

We should note also the *principle of duality* in a planar chain, where the elements "point" and "linear chain" are dual elements.

The following *external* properties are of fundamental importance:

VI. *Every line of the plane which is not a line of a planar chain $\mathfrak{C}^2$ in the plane has one and only one point in common with $\mathfrak{C}^2$.*

This theorem is most readily proved analytically. Let a system of homo-

* VEBLEN and YOUNG, loc. cit., § 2, p. 353.

† In this theorem the projective transformation may be either a collineation or a reciprocity. The theorem is still true, moreover, if the transformation in question is *anti-projective* (cf. SEGRE, *Un nuovo campo etc.*, loc. cit.).

geneous coördinates be established in such a way that the points of $\mathfrak{C}^2$ correspond to *real* ratios $x_1 : x_2 : x_3$ of the coördinates. The equation of any line in the plane which is not a line of $\mathfrak{C}^2$ is then $u_1 x_1 + u_2 x_2 + u_3 x_3 = 0$, where the ratios $u_1 : u_2 : u_3$ are not all real. Placing $u_j = u_j' + iu_j''$ ($j = 1, 2, 3$; u_j', u_j'' real) the equations

$$u_1' x_1 + u_2' x_2 + u_3' x_3 = 0, \qquad u_1'' x_1 + u_2'' x_2 + u_3'' x_3 = 0,$$

determine a real point $x_1 : x_2 : x_3$ of the line. The line cannot have more than one point in common with $\mathfrak{C}^2$, for otherwise it would be a line of $\mathfrak{C}^2$.

The argument dual to this gives at once:

VII. *Through every point of the plane which is not a point of a planar chain $\mathfrak{C}^2$ in the plane passes one and only one line of $\mathfrak{C}^2$.*

These theorems lead at once to the notion of pairs of points (lines) *inverse* or *conjugate* with respect to a planar chain. Let P be any point not on a given planar chain $\mathfrak{C}^2$ and let l be the line of $\mathfrak{C}^2$ through P (VII). This line l has a linear chain in common with $\mathfrak{C}^2$ (IV); the point $\bar{P}$ which is the inverse of P with respect to this linear chain is called the *inverse* or *conjugate* of P with respect to the planar chain $\mathfrak{C}^2$; P, $\bar{P}$ form a pair of *conjugate points* with respect to $\mathfrak{C}^2$. The latter phrase is justified by the evident fact that either of two such points is the conjugate of the other. Analytically two conjugate points with respect to the planar chain all of whose elements have real coördinates are determined by coördinates of which one set consists of the conjugate complex numbers of those forming the other set. Every point of a planar chain $\mathfrak{C}^2$ is, by definition, its own conjugate with respect to $\mathfrak{C}^2$.*

The process dual to the one described gives rise to the notion of pairs of *conjugate lines* with respect to a planar chain. It follows further that *the line joining two points conjugate with respect to a planar chain $\mathfrak{C}^2$ is a line of $\mathfrak{C}^2$; and the point of intersection of two lines conjugate with respect to $\mathfrak{C}^2$ is a point of $\mathfrak{C}^2$.* It is then at once evident analytically that

VIII. *The line joining two points A, B is conjugate with respect to a planar chain with the line joining the two points $\bar{A}$, $\bar{B}$ which are conjugate with A, B respectively; and every point on a line l is conjugate with a point on the line conjugate with l.*

Theorems I–VIII just given are sufficient to derive the fundamental properties of the geometry of chains in a plane. In preparation for the application of these properties to the classification of collineations in the plane with reference to their behavior toward the planar chains of the plane and the derivation of the resulting characteristic systems of planar chains, we call attention to a few general results of importance.

* The transformation obtained by replacing each point of the plane by its conjugate with respect to a planar chain (wherein the points of the planar chain are self-conjugate) is called by SEGRE (loc. cit.) an *antinvolution* with double elements. Cf. the footnote, p. 282.

We note first, that a triangle in the plane determines an infinite system of chains containing the vertices and sides of this triangle; there is, in fact, through every point of the plane which is not on a side of the triangle one and only one chain of this system, and one and only one through every line which does not meet one of the vertices (II). In general, two planar chains of this system have only the vertices and sides of the triangle in common; certain pairs of chains, however, have more than these elements in common, as the following considerations will show. A linear chain $\mathfrak{C}$ of points and a point not on $\mathfrak{C}$ determine an infinite system of planar chains each of which contains P and the points of $\mathfrak{C}$; viz., there is one such chain through any point (distinct from P and not in $\mathfrak{C}$) of a line joining P to any point of $\mathfrak{C}$. For there are thus determined two linear chains ($\mathfrak{C}$ and one on the line through P and containing P) which determine a planar chain (Definition). *Two distinct planar chains, however, can not have more points in common than a linear chain* $\mathfrak{C}$ *of points and a point* P *not on* $\mathfrak{C}$. For, let Q be another common point; there exist then under the hypothesis two points A, B of $\mathfrak{C}$ such that no three of the points A, B, P, Q are collinear. Hence, the two planar chains would coincide (II).* In other words *two planar chains which have the points of a linear chain and two other points in common coincide.*

It should be noted, however, that a linear chain of points and two points not on this chain do not in general determine a planar chain in which they lie. They will determine a planar chain only if the line joining the two points meets the linear chain in question (IV).

Another important system of chains is determined as follows:

Definition. Any planar chain is said to be *about* two points B, C, if B, C are conjugate points with respect to the planar chain.

Now let A, B, C be the vertices of a triangle. The class of all planar chains through A and about B, C is the system in question. There is one and only one chain of this system through any point not on a side of the triangle ABC. For, let P be such a point; the line AP meets the line BC in a point Q. Let $\mathfrak{C}$ be the linear chain through Q and about B, C. The chain $\mathfrak{C}$ and the two points P, A determine a unique planar chain with the specified property. Two chains of this system have in general only the point A and the line BC in common; certain pairs of these chains have, however, also a linear chain of points on BC and a linear chain of lines through A in common.

* Two distinct chains through the points of a linear chain C and a point P not on C have also a linear chain of lines through P in common (the linear chains of points on these lines are, of course, different for the two planar chains).

§ 2. *Collineations with invariant planar chains.*

We now seek necessary and sufficient conditions that a collineation in the plane leave a planar chain invariant.* Suppose π is a collineation leaving a planar chain $\mathfrak{C}^2$ invariant, and let a system of homogeneous coördinates be determined so that the elements of $\mathfrak{C}^2$ are determined by real ratios of the coördinates. It is then clear that a necessary and sufficient condition that π leave $\mathfrak{C}^2$ invariant is that when π is represented analytically by equations

$$x'_i = \Sigma a_{ij} x_j \qquad (i, j = 1, 2, 3),$$

the coefficients a_{ij} shall be real numbers (or at least that they may be rendered real by multiplication with a suitably chosen factor of proportionality). We seek, however, a simpler and geometric condition. The five well-known types of collineations in the plane will be treated successively.

Type I. *Three (and only three) double points.* If the collineation be represented analytically as described above, the characteristic equation will have real coefficients and three distinct roots. There are then two cases to consider:

h) The roots of the characteristic equation are all real;

e) One of the roots is real and the other two conjugate complex.†

Type Ih. If the roots of the characteristic equation are all real the double points are all real. In other words, any invariant planar chain $\mathfrak{C}^2$ contains all three double points, A, B, C. Each of the sides of the triangle ABC then has a linear chain in common with $\mathfrak{C}^2$ (IV) and, since these lines are invariant, these linear chains are invariant under the projectivities on the invariant lines. Since these linear chains contain the (distinct) double points of the projectivities mentioned, *the projectivities on the sides of the invariant triangle must all be hyperbolic.*‡ This necessary condition is also sufficient. For let π be any collineation of Type I such that the projectivities on two sides AB and BC of the invariant triangle are hyperbolic, and let $\mathfrak{C}^2$ be *any* planar chain through A, B, C. The linear chains $\mathfrak{C}_1$, $\mathfrak{C}_2$ on AB, BC respectively are then invariant, since a hyperbolic projectivity on a line leaves every chain through the double points invariant (A, Theorem 14). Since $\mathfrak{C}^2$ is determined by $\mathfrak{C}_1$, $\mathfrak{C}_2$, it follows that $\mathfrak{C}^2$ is invariant. That the linear chain of $\mathfrak{C}^2$ on CA is likewise invariant then follows almost immediately. Hence,

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* That such collineations exist is obvious. Let A, B, C, D be the vertices of any quadrangle, and let $\mathfrak{C}^2$ be the planar chain determined by A, B, C, D. Let A', B', C', D' be any other four points of $\mathfrak{C}^2$ no three of which are collinear. The collineation determined by the homologous pairs A, A'; B, B'; C, C'; D, D' then clearly leaves $\mathfrak{C}^2$ invariant, since it transforms $\mathfrak{C}^2$ into the planar chain determined by A', B', C', D', which is identical with $\mathfrak{C}^2$ (II).

† These two cases are also readily obtained by a purely synthetic argument. Cf. footnote on next page.

‡ In accordance with the definition given in A, p. 42, the involutions are to be considered as both hyperbolic and elliptic.

THEOREM 1. *A necessary and sufficient condition that a collineation of Type I leave a planar chain through the double points invariant is that the projectivities on two sides of the invariant triangle be hyperbolic. The projectivity on the third side is then likewise hyperbolic, and the collineation leaves every planar chain through the double points invariant.*

A collineation satisfying the conditions of this theorem we will call *hyperbolic of Type I*, or *of Type Ih.*

Moreover, a hyperbolic collineation of Type I leaves invariant no other planar chains than those mentioned in the theorem. For, if A is a double point, there is at least one line through A which meets a given invariant planar chain in a linear chain $\mathfrak{C}_1$ (VII). If this linear chain is invariant, A must be a point of C_1 and hence of the planar chain, since the projectivity in this chain is hyperbolic. If $\mathfrak{C}_1$ is not invariant, it is transformed into another linear chain of the invariant planar chain, and the lines of these two linear chains determine A as a point of the invariant planar chain. Hence,

THEOREM 2. *The system of invariant planar chains of a hyperbolic collineation of Type I consists of all the planar chains through the double points and only these. There is one chain of the system, and only one, through every point of the plane which is not on a side of the invariant triangle.*

Type Ie. If one of the roots of the characteristic equation is real and the other two are conjugate complex, any invariant planar chain $\mathfrak{C}^2$ contains one double point, say A, and the other two double points B, C are conjugate with respect to the planar chain.* The line BC is then a line of the planar chain (it also contains one invariant line, therefore; so that the invariant figure is self-dual within the planar chain) and has an invariant linear chain $\mathfrak{C}_1$ in common with the invariant planar chain (IV). Since $\mathfrak{C}_1$ does not contain the double points of the projectivity on BC, this projectivity must be elliptic.

This condition is not, however, sufficient. A necessary and sufficient condition is readily obtained by considering the projectivities on the two conjugate lines AB and AC. If P is any point of $\mathfrak{C}^2$ not on a side of the invariant triangle, and the line PC meets AB in a point M, the line PB meets AC in a point N which is the conjugate of M with respect to $\mathfrak{C}^2$; for the line conjugate with MC passes through B (the conjugate of C) and P, and must contain the conjugate of M which is on AC (VIII). Let P' be the point homologous with P and let the points M', N' on AB, AC be constructed. If $\mathfrak{C}^2$ is invariant,

* That any invariant planar chain of a collineation of Type I either passes through all the double points or passes through one and about the other two may be seen synthetically as follows. Suppose an invariant planar chain $\mathfrak{C}^2$ does not contain the double point B. Through B passes a line l of $\mathfrak{C}^2$. The line l must then be invariant; for otherwise B would be the intersection of two lines of $\mathfrak{C}^2$ and would hence be a point of $\mathfrak{C}^2$. The invariant linear chain of l is therefore about the two double points B and C (say) on l. No other line of $\mathfrak{C}^2$ can pass through B or C. Through the third double point A, however, passes a line of $\mathfrak{C}^2$, and since (as just noted) this line cannot be invariant, A is a point of $\mathfrak{C}^2$.

the projectivities on AB and AC must then be so related that if a pair of points are homologous under the projectivity on AB, their conjugates with respect to $\mathfrak{C}^2$ must be homologous under the projectivity on AC. Two projectivities which are related in this way with respect to a planar chain we will call *conjugate.* A necessary condition that $\mathfrak{C}^2$ be invariant under a collineation with double points A, B, C is then that the projectivities on AB and AC be conjugate with respect to $\mathfrak{C}^2$. But this condition is at once seen to be also sufficient. Hence,

THEOREM 3. *A necessary and sufficient condition that a collineation of Type I leave a planar chain through one double point A and about the other two B, C invariant is that the projectivities on the two conjugate invariant lines be conjugate. The projectivity on the third side of the invariant triangle is then elliptic. The collineation leaves every planar chain through A and about B, C invariant.**

A collineation satisfying the condition of Theorem 3 we will call *elliptic of Type I* or *of Type Ie.* The double points on the side of the invariant triangle on which the projectivity is elliptic we will call the *conjugate* double points. We then have:

THEOREM 4. *The system of invariant planar chains of an elliptic collineation of Type I consists of all the planar chains about the conjugate double points and through the remaining double point, and only these. There is one and only one chain of this system through every point which lies in the plane but not on a side of the invariant triangle.*

Type II. *Two and only two double points.* The invariant figure consists of two points A, B, the line $l = AB$, and a line l' through A. The collineation is completely determined by the projectivity on the line l and the parabolic projectivity on l'. We note first that the characteristic equation of any collineation of this type has a double root, and that hence all the roots of this equation must be real. It then follows at once that *every invariant planar chain of such a collineation must contain the two points A and B and a linear chain of l'.*†

Let $\mathfrak{C}^2$ be any invariant planar chain. The linear chain of $\mathfrak{C}^2$ on AB is

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* The theorem that *if a collineation of Type I leaves one planar chain* $\begin{smallmatrix}\textit{through the double points } A,\ B,\ C\\ \textit{through one double point } A \textit{ and}\end{smallmatrix}$ *about the other double points B, C invariant, it leaves every chain* $\begin{smallmatrix}\textit{through } A,\ B,\ C\\ \textit{through } A \textit{ and about } B,\ C\end{smallmatrix}$ *invariant*, may also be proved directly as follows: Let π be the given collineation and $\mathfrak{C}^2$ a planar chain which it leaves invariant. Let P be any point of $\mathfrak{C}^2$ which is not on a side of the invariant triangle of π and let P_1 be any other point in the plane not on a side of this invariant triangle. Let π_1 be the collineation which has the same invariant triangle as π and which transforms P into P_1. π_1 then clearly transforms $\mathfrak{C}^2$ into a planar chain $\mathfrak{C}_1^2$ through P_1. Moreover, if $\mathfrak{C}^2$ contains the double points A, B, C of π, so also will $\mathfrak{C}_1^2$; and if $\mathfrak{C}^2$ is through A and about B, C, so also is $\mathfrak{C}_1^2$. Now the transformation $\pi_1\pi\pi_1^{-1}$ clearly leaves $\mathfrak{C}_1^2$ invariant. But since π, π_1 have the same invariant triangle, they are commutative and we have $\pi_1\pi\pi_1^{-1} = \pi$; i. e., π leaves $\mathfrak{C}_1^2$ invariant.

† This result, as well as the corresponding one under Type III below, may also be established readily by a purely synthetic argument.

invariant under the projectivity on AB; hence this projectivity must be *hyperbolic*. Now let P be any point not on l or l'. We show that one and only one invariant planar chain of the collineation contains P. Let the line BP meet l' in a point Q and let Q' be the point homologous with Q under the parabolic projectivity on l'. Any invariant planar chain through P must contain Q (III) and hence Q'. The points P, B, Q', A determine uniquely a planar chain; hence, there is no more than one invariant planar chain through P. That the planar chain thus determined is indeed invariant follows from the fact that it is uniquely determined by its two linear chains on the line $l = AB$ and l' (Def.) and each of these linear chains is invariant; the one on l because it contains the two double points of a hyperbolic projectivity (A, Theorem 14), the one on l' because it contains the double point and one pair of homologous points of a parabolic projectivity (A, Theorem 15). We have then the following:

THEOREM 5. *A necessary and sufficient condition that a collineation of Type II leave a planar chain invariant is that the projectivity on one of the invariant lines be hyperbolic. The collineation then leaves every planar chain through the point B and an invariant linear chain on l' invariant.*

THEOREM 6. *The system of invariant planar chains is determined by the point B and the system of invariant chains of the parabolic projectivity on l'. There is one and only one chain of this system through any point not on l or l'.*

A collineation of Type II satisfying the condition of Theorem 5 may be called *hyperbolic of Type II* or *of Type IIh*.

Type III. *One and only one double point.* The invariant figure consists of a point A and a line l through A. The collineation is completely determined when the parabolic projectivity (with double point A) on l and one pair of homologous points (not on l) are given. As before, by reference to the characteristic equation which in this case has a triple root, we see that every invariant planar chain of such a collineation must contain A and a linear chain on l. Now let P, P' (not on l) be any pair of points homologous under the collineation in question. The line PP' then meets l in a point Q distinct from A. Let Q' be the point homologous with Q. Let $\mathfrak{C}^2$ be the planar chain determined by the linear chain $\mathfrak{C}$ through A, Q, Q' and the points P, P'. We show that $\mathfrak{C}^2$ is invariant under the collineation. We note first that the chain $\mathfrak{C}$ is invariant (A, Theorem 15). Similarly, the linear chain of lines at A determined by the lines l, AP, AP' is invariant, since it contains the double line and a pair of homologous lines of the parabolic projectivity in the pencil of lines at A. The line AP' is therefore transformed by the collineation into a line AM where M is a point of $\mathfrak{C}^2$ (III). The point P'' which is homologous with P' is now determined as the intersection of the lines $P'Q'$, AM, and is therefore a point of $\mathfrak{C}^2$ (III). Hence $\mathfrak{C}^2$ is invariant. Clearly also $\mathfrak{C}^2$ is the only invariant planar chain through P. We have then

THEOREM 6. *Every collineation of Type III leaves invariant one and only one planar chain through each point not on l. Every invariant planar chain contains an invariant linear chain of the parabolic projectivity on the double line; through every invariant linear chain of this parabolic projectivity there pass an infinite number of invariant planar chains of the collineation.*

A collineation of Type III may conveniently be called a *parabolic* collineation.

Type IV. *A point O and every point of a line l not through O is invariant.* A collineation of this type is called a *homology*, the point O and the line l being the *center* and *axis* respectively of the homology. A homology is completely determined when the center, axis, and one pair of homologous points (collinear with O) are given. Any invariant planar chain of a homology contains O and a linear chain on l. For let P, P' and Q, Q' be any two pairs of homologous points on any invariant planar chain (P, P', Q not collinear). The lines PP', QQ' meet in O which is therefore a point of the planar chain (III). The dual argument shows similarly that l contains two points and hence a linear chain of any invariant planar chain. Let $\mathfrak{C}^2$ be any planar chain through O and a linear chain $\mathfrak{C}$ on l. Any line joining O to a point of $\mathfrak{C}$ has a linear chain in common with $\mathfrak{C}^2$, and this linear chain must be invariant if $\mathfrak{C}^2$ is invariant. Hence, if a homology leaves a planar chain invariant, the projectivities along the invariant lines through O are *hyperbolic.* This necessary condition is also clearly sufficient, since the projectivities on all lines through O are hyperbolic, if one is, and a hyperbolic projectivity leaves every linear chain through its double points invariant. We have then

THEOREM 7. *A necessary and sufficient condition that a homology leave a planar chain invariant is that the projectivity on an invariant line through the center be hyperbolic. If this condition is satisfied, the homology leaves every planar chain through the center and any linear chain on the axis invariant.*

A homology satisfying the condition of this theorem we call a *hyperbolic homology* or a collineation of *Type IVh.*

THEOREM 8. *The system of invariant chains of a hyperbolic homology consists of all planar chains through the center and any linear chain on the axis, and of these only. Through any point not on the axis and distinct from the center there pass an infinite number of chains of this system. Through any two such points not collinear with the center passes one and only one chain of this system.**

*On every line through O there is a projectivity which has the center O and a point on the axis l for double points. It is well known, moreover, that any two projectivities on such lines are equivalent, i. e., can be transformed into each other (or have the same characteristic cross ratio). We may therefore classify homologies with reference to these projectivities into *hyperbolic, elliptic*, and *loxodromic.* The hyperbolic homologies alone leave a planar chain invariant. The elliptic homologies, however, leave invariant every linear chain about the double points on any line through A. The loxodromic homologies leave no planar chain invariant and no linear chain on any line except l. The elliptic and loxodromic projectivities leave invariant, however, every *system* of planar chains through A and a linear chain on l.

Type V. *All points of a line l and all lines through a point O on l are invariant.* A collineation of this type is called an *elation*, the line l and point O being the *axis* and *center* respectively. An elation is completely determined when the center, axis, and one pair of homologous points are given. If P, P' and Q, Q' are any two pairs of homologous points (not all collinear), the lines PP' and QQ' meet in O and the lines PQ and $P'Q'$ meet in a point of l. It follows at once that any invariant planar chain must contain O and a linear chain on l (III). The construction for the elation shows also at once that every planar chain containing a pair of homologous points and a linear chain on l through O is invariant. We have then

THEOREM 9. *Every elation leaves invariant every planar chain containing a pair of homologous points and a linear chain on the axis and through the center; and only these. Through any point not on the axis pass an infinite number of invariant planar chains; through two such points not collinear with the center passes one and only one.*

We note finally the following theorem which is an immediate consequence of the results of the preceding discussion:

THEOREM 10. *If a collineation leaves invariant more than one planar chain through a point which is not on a double line it is a perspective collineation, i. e., either a homology or an elation.*

§ 3. *Conclusion.*

Our discussion has led to the definition of six fundamental systems of planar chains in the plane, which form a natural generalization of the three well-known systems of linear chains on a line; i. e., of the three systems of circles (in the Argand representation of a complex variable) associated with the hyperbolic, elliptic, and parabolic transformations (linear fractional) on one complex variable. Each of the first four of these six systems, i. e., those associated with a hyperbolic collineation of Type I, an elliptic collineation of Type I, a hyperbolic collineation of Type II, or a parabolic collineation (Type III), consists of ∞^2* planar chains, one and only one through each point which is not a double point nor on a double line. Each of the last two systems, i. e., those associated with a hyperbolic homology or an elation, consists of ∞^4 planar chains, one and only one passing through two points neither of which is a double point and both of which are not on the same invariant line. In view of their importance, we shall describe these systems independently of the collineations by means of which they were obtained. They are as follows:

System Ih. Such a system is defined by any three points which are not collinear, and consists of all the planar chains through these points. Associated

* By ∞^2 is meant that the system may be made to depend on two independent *real* parameters; similarly for the ∞^4 in the next sentence, and similar expressions following.

with any such system is a two-parameter group of hyperbolic collineations having the three points as double points, each collineation of which leaves every chain of the system invariant.

System Ie. Three such systems are defined by any three points which are not collinear; each of these systems consists of all the chains through one of the points and about the other two. Associated with any system of this type is a two-parameter group of elliptic collineations having the three points as double points; each collineation of the group leaves every chain of the associated system invariant.

System II. Such a system consists of all the planar chains through two given points A, B and meeting a given line l' through A in a system of linear chains all mutually tangent at A. Associated with any such system is a two-parameter group of hyperbolic collineations, each collineation of which leaves every chain of the system invariant.

System III. Such a system meets a given line in a system of linear chains all mutually tangent at a given point A of the line, and meets the pencil of lines at A in a system of linear chains (of lines) all mutually tangent at the given line. Associated with any such system is a three-parameter group of parabolic collineations, each collineation of which leaves every chain of the system invariant.

System IV. Such a system consists of all the planar chains through a given line and a given point not on that line. Associated with any such system is a one-parameter group of hyperbolic homologies, each homology of which leaves every chain of the system invariant.

System V. Such a system consists of all the planar chains through a given line and through a given point not on that line. Associated with any such system is a one-parameter group of elations, each elation of which leaves every chain of the system invariant.

The results of this paper may be applied at once to the enumeration and geometric definition of *all* the continuous groups of collineations in the plane which leave a planar chain invariant. This is equivalent to the enumeration of all continuous groups of collineations in the plane which can be represented with real coefficients. No complete list of the groups of *real* collineations in a plane, in which the groups are characterized geometrically or in which the finite equations of the collineations are given, has ever been published, so far as the writer is aware.* The enumeration in both of these forms by the methods developed above is very simple, and offers a good example of a problem apparently analytic which yields readily to synthetic treatment.†

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* LIE has given the infinitesimal equations of these groups. Cf. LIE-ENGEL, *Theorie der Transformationgruppen*, Leipzig (1893), vol. 2, pp. 78–109 and pp. 380–384.

† The results of this enumeration were presented by the writer to the Society September 11, 1908, and will be published in the near future.

We close with a word regarding the relation of our developments to the theory of functions of two complex variables. In the theory of functions of one complex variable the circles in the Argand plane owe their importance in part to the fact that the necessary and sufficient condition that a linear fractional transformation on the variable may be transformed into one with real coefficients is that it leave a circle invariant. In all analogous problems in the theory of functions of two complex variables the planar chains will play a role similar to that of the circles (linear chains) in the one-dimensional case. On the other hand, the circles owe their importance in the theory of functions of one complex variable also in part to the fact that they divide the points of the Argand plane into two regions. This aspect of the linear chain does not find its generalization in the planar chain, since a two-dimensional spread of points can not divide the four-dimensional spread of points in the complex plane into two regions. The generalization here in question may be found by combining a system of ∞^1 planar chains (or a system of ∞^2 linear chains) into a spread of three dimensions with the desired property. A three-dimensional spread of points dividing the complex plane into two regions is obtained very simply, moreover, as the class of all the points on the lines of a linear chain of lines. To this aspect of the problem and in general to the question of the order relations in a complex plane, I shall return on a future occasion. To this problem the present paper forms a preliminary investigation.

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