

THE SOUTHERLY DEVIATION OF FALLING BODIES*

BY

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INTRODUCTION

In the work of previous writers on the subject of this paper the potential function used for that of the earth's gravitational field of force has, in general, not been more than a first approximation, i. e., a development, in the neighborhood of the initial point of the falling body, which includes terms of only the first order in the independent variables. The circle of reference (parallel of latitude) used has also, in general, been different from that used in experiments for the determination of the same quantity.

From equations (1) and (2) which follow it appears that the southerly deviation is proportional to the square of the height h through which the body falls (at least for sufficiently small values of h) and that the constant of proportionality involves the first *and also the second derivatives of the potential function* f_1 .† Hence the truth of the following statement:

I. The potential function to be used for that of the earth's gravitational field of force should be a *second approximation*, i. e., a development in the neighborhood of the initial point of the falling body, which includes terms of at least the second order in the independent variables.

In experiments for the determination of the southerly deviation of falling bodies, a plumb-line P_0R is supported at the point P_0 (Fig. 5), from which the body falls, and the deviation is measured on the level (equipotential) surface which passes through the plumb-bob R , from the circle of latitude which passes through R . The direction of gravity at R is RP_0 , and that at P_0 is the limiting direction which is approached by the line RP_0 as the length P_0R approaches zero. Let us denote by t the limiting position of P_0R . It is easy to prove the following statement:

II. In a field of force in which the lines of force are not rectilinear, plumb-

* Presented to the Society (Southwestern Section), November 27, 1909, and November 26, 1910. See also footnote to formula (3).

† In this paper, as in the work of previous writers, a distribution of revolution is assumed, i. e., the earth's field of force is assumed to be the same in every plane through the axis of rotation.

lines* of different lengths which are supported at the same point (in particular a plumb-line P_0R and the line t) do not coincide.

In order to be able to compare the results of experiment with those of theory, the same circle of reference should be used in the computation as in the experiment. This, however, has not been done by previous writers. In their computations they used as their circle of reference on the level surface of R , not the circle of latitude of R , but that of the point T in which the line t pierces the level surface of R . That a great error has thus been made, appears in Assumption 4, below, in which it is shown that for a second approximation to the potential function the distance (measured along a meridian) between the circles of latitude of R and T (the part which has been neglected) is eighteen times as great as that between T and S (the part which has been computed), S being the point in which the body, which falls from P_0 , strikes the level surface of R . For a first approximation to the potential function, however, the distance between the circles of latitude of R and T may be small in comparison to that between T and S .

Now let us compare the assumptions and the results of previous writers, who disregarded either one or both of the facts set forth in I and II, with those of this paper, in which these facts are not disregarded. In order to do this it is desirable first to give a brief description of the method of this paper.

STATEMENT AND COMPARISON OF RESULTS.

After the falling body is released from its initial position, P_0 , of rest with respect to the rotating earth, it moves under the influence of a field of force F_1 which is fixed in space.† A plumb-line P_0R at rest is in equilibrium under the influence of a field of force F_2 which is fixed with respect to the rotating earth. If $U_1 = f_1(r, z)$ and $U_2 = f_2(r, z)$ (where r represents the distance of a general point from the earth's axis of rotation OZ , and z that from a plane π perpendicular to OZ at O , and where O is taken at the earth's center, although this is not necessary) are potential functions of the fields F_1 and F_2 respectively, there exists between them the relation

$$U_1 + \frac{1}{2}\omega^2 r^2 = U_2,$$

in which ω denotes the angular velocity of the earth's rotation.

The path of the falling body, which has received an initial velocity from the rotating earth, is a curve c in the fixed field F_1 . The curve c lies on a surface of revolution whose axis coincides with that of the earth. Let us denote by c_1

* It is assumed that the plumb-bob is a heavy particle and that the line is weightless and perfectly flexible.

† That is, fixed with respect to a set of axes whose origin is the center of the earth and whose directions are fixed directions in space.

that meridian curve of this surface of revolution which lies in the plane of OZ and P_0 . The locus of the plumb-bobs R of all plumb-lines which are supported at P_0 , is a curve c_2 which also lies in the plane of OZ and P_0 . It will be shown later that the curves c_1 and c_2 are tangent at P_0 . The line t defined above is their common tangent.

Let us now take the common tangent and the common normal to these curves as a pair of rectangular axes and refer the equations of curves c_1 and c_2 to them. The coördinate which is measured along the tangent from the point P_0 we will denote by ξ and regard as positive when measured in the direction of the earth's center. The other coördinate we will denote by η and regard as positive when measured in the direction of the north pole. (See Fig. 4.) The equations of c_1 and c_2 then become

$$(1) \quad \eta_1 = -\frac{1}{6} \frac{C + B_1 \omega^2 + A_1 \omega^4}{D^{\frac{3}{2}}} \xi^2 + \dots,$$

$$(2) \quad \eta_2 = -\frac{C + B_2 \omega^2 + A_2 \omega^4}{D^{\frac{3}{2}}} \xi^2 + \dots,$$

where

$$C = \left(\frac{\partial^2 f_1}{\partial r \partial z} \right)_0 \left[\left(\frac{\partial f_1}{\partial r} \right)_0^2 - \left(\frac{\partial f_1}{\partial z} \right)_0^2 \right] + \left[\left(\frac{\partial^2 f_1}{\partial z^2} \right)_0 - \left(\frac{\partial^2 f_1}{\partial r^2} \right)_0 \right] \left(\frac{\partial f_1}{\partial r} \right)_0 \left(\frac{\partial f_1}{\partial z} \right)_0,$$

$$B = 2 \left(\frac{\partial^2 f_1}{\partial r \partial z} \right)_0 \left(\frac{\partial f_1}{\partial r} \right)_0 r_0 + \left[\left(\frac{\partial^2 f_1}{\partial z^2} \right)_0 - \left(\frac{\partial^2 f_1}{\partial r^2} \right)_0 \right] \left(\frac{\partial f_1}{\partial z} \right)_0 r_0,$$

$$A = \left(\frac{\partial^2 f_1}{\partial r \partial z} \right)_0 r_0^2,$$

$$B_1 = B + 3 \left(\frac{\partial f_1}{\partial r} \right)_0 \left(\frac{\partial f_1}{\partial z} \right)_0, \quad B_2 = B - \left(\frac{\partial f_1}{\partial r} \right)_0 \left(\frac{\partial f_1}{\partial z} \right)_0,$$

$$A_1 = A + 3 \left(\frac{\partial f_1}{\partial z} \right)_0 r_0, \quad A_2 = A - \left(\frac{\partial f_1}{\partial z} \right)_0 r_0,$$

$$D = \left(\frac{\partial f_1}{\partial r} \right)_0^2 + \left(\frac{\partial f_1}{\partial z} \right)_0^2 + 2 \left(\frac{\partial f_1}{\partial r} \right)_0 r_0 \omega^2 + r_0^2 \omega^4,$$

the subscript (0) indicating that the derivatives of f_1 have been computed for the point $P_0 (r = r_0, z = z_0)$.

The southerly deviation is the difference $\eta_2 - \eta_1$, when we substitute for ξ the height $h = P_0 R$ through which the body falls, and for f_1 a sufficiently close approximation to the potential function of the earth's gravitational field of force.

The southerly deviation can also be expressed in the following very simple form*

* This formula was presented to the Society (Chicago Section) April 29, 1911.

$$(3) \quad \frac{4\omega^2 \sin \phi \cos \phi + 5 \left(\frac{\partial g}{\partial \eta} \right)_0 h^2}{\cdot 6g_0}$$

in which g_0 and ϕ denote the acceleration and the astronomical latitude, respectively, at the point P_0 . For,

$$g_0 = \sqrt{\left(\frac{\partial f_2}{\partial z} \right)_0^2 + \left(\frac{\partial f_2}{\partial r} \right)_0^2} = \sqrt{D}, \quad g_0 \sin \phi = \left(\frac{\partial f_2}{\partial z} \right)_0 = \left(\frac{\partial f_1}{\partial z} \right)_0,$$

$$g_0 \cos \phi = \left(\frac{\partial f_2}{\partial r} \right)_0 = \left(\frac{\partial f_1}{\partial r} \right)_0 + \omega^2 r_0.$$

By the equation of transformation in § 7,

$$U_1 = f_1(r, z) = f_1(-\xi \cos \phi - \eta \sin \phi + r_0, -\xi \sin \phi + \eta \cos \phi + z_0),$$

and therefore

$$\frac{\partial^2 U_1}{\partial \eta \partial \xi} = (\sin^2 \phi - \cos^2 \phi) \frac{\partial^2 f_1}{\partial r \partial z} + \sin \phi \cos \phi \left[\frac{\partial^2 f_1}{\partial r^2} - \frac{\partial^2 f_1}{\partial z^2} \right].$$

On the other hand,

$$\frac{\partial^2 U_1}{\partial \eta \partial \xi} = \frac{\partial^2 U_2}{\partial \eta \partial \xi} - \omega^2 \sin \phi \cos \phi.$$

Hence equations (1) and (2) assume the following forms:

$$\eta_1 = \frac{1}{6g_0} \left[\left(\frac{\partial^2 U_2}{\partial \eta \partial \xi} \right)_0 - 4\omega^2 \sin \phi \cos \phi \right] \xi^2, \quad \eta_2 = \frac{1}{g_0} \left(\frac{\partial^2 U_2}{\partial \eta \partial \xi} \right)_0 \xi^2,$$

and

$$\eta_2 - \eta_1 = \frac{1}{6g_0} \left[4\omega^2 \sin \phi \cos \phi + 5 \left(\frac{\partial^2 U_2}{\partial \eta \partial \xi} \right)_0 \right] \xi^2.$$

But

$$\left(\frac{\partial^2 U_2}{\partial \eta \partial \xi} \right)_0 = \left[\frac{\partial}{\partial \eta} \left(\frac{\partial U_2}{\partial \xi} \right) \right]_0 = \left(\frac{\partial g}{\partial \eta} \right)_0,$$

and hence formula (3).

It will be observed that the coefficient of ξ^2 in each of the equations (1) and (2) contains the second as well as the first derivatives of f_1 . It is for this reason that the potential function must be developed to terms of the second order inclusive in the neighborhood of $P_0(r_0, z_0)$.

In the discussion which follows we shall call η_1 and η_2 the parts of the southerly deviation which are contributed by the trajectory and the plumb-bob locus respectively. It is the part η_2 which has been neglected, and the part η_1 which has been computed, by previous writers. We shall compare these for each of several forms of f_1 , we shall also express the results corresponding to different forms of f_1 in terms of the same unit.

1. *The Assumption of Gauss.*

Gauss assumed that the force to which a body, at rest with respect to the rotating earth, is subjected is constant in magnitude and direction. That is,*

$$f_2(r, z) = -g \cos \phi \cdot r - g \sin \phi \cdot z,$$

and therefore

$$f_1(r, z) = f_2(r, z) - \frac{1}{2}\omega^2 r^2,$$

in which ϕ and g are the astronomical latitude and the acceleration at P_0 . For this potential function equations (1) and (2) become †

$$\eta_1 = -\frac{2}{3} \sin \phi \cos \phi \cdot \frac{\omega^2}{g} \cdot \xi^2, \quad \eta_2 = 0 \cdot \xi^2,$$

and

$$\eta_2 - \eta_1 = -\eta_1 = \frac{2}{3} \sin \phi \cos \phi \cdot \frac{\omega^2}{g} \cdot \xi^2.$$

Thus we see that in this case the *whole southerly deviation is due to the trajectory*, since the plumb-bob locus coincides with the tangent t . (See Fig. 1.) The curve c_1 lies to the south of the tangent t .

The potential function used by Gauss is a first approximation to that of the earth's field. By statement I this is not a sufficiently close approximation.

Before making the next assumption let us express the result just obtained in a form in which it may be compared with the results which follow.

With sufficient approximation

$$\sin \phi = \frac{z_0}{\rho_0}, \quad \cos \phi = \frac{r_0}{\rho_0}, \quad \frac{\omega^2}{g} = \frac{\sigma}{\rho_0},$$

where ‡

$$\sigma = \frac{\omega^2 \rho_0}{M/\rho_0^2}, \quad \rho_0 = \sqrt{r_0^2 + z_0^2},$$

and M represents the mass of the earth. Then

*Strictly speaking this function does not quite represent Gauss' assumption. Under Gauss' assumption the equipotential surfaces of the field F_2 are parallel planes, while the equipotential surfaces corresponding to the function f_2 here assumed are parallel cones of revolution which have as a common axis that of the earth.

†The formula given by GAUSS is

$$\frac{1}{3} \sin \phi \cos \phi g \omega^2 t^4.$$

But since

$$\xi = \frac{1}{2} g t^2 + \dots,$$

we see that the result is the same as the one just found. See GAUSS' *Werke*, vol. 5 (1867), p. 502. GAUSS' formula can also be obtained by putting $(\partial g / \partial \eta)_0 = 0$ in formula (3).

‡The value of σ is about $\frac{1}{2} \frac{1}{3} g$.

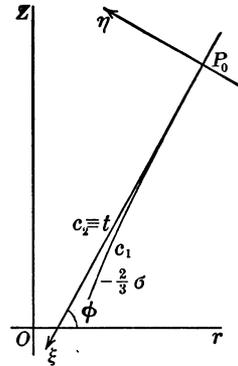


FIG. 1.

$$\eta_2 - \eta_1 = -\eta_1 = \frac{r_0 z_0}{\rho_0^2} \cdot \frac{1}{\rho_0} \cdot \frac{2}{3} \sigma \cdot \xi^2.$$

2. A Second Assumption.

Let us now assume that the field of force F_1 is central, the center being at the earth's center and the law of attraction that of Newton. Then*

$$f_1(r, z) = \frac{M}{\rho}, \quad \rho = \sqrt{r^2 + z^2},$$

where M represents the mass of the earth. For this potential function equations (1) and (2) become †

$$\eta_1 = \frac{r_0 z_0}{\rho_0^2} \cdot \frac{1}{\rho_0} \left[0 + 0 + \frac{1}{2} \frac{z_0^2}{\rho_0^2} \sigma^2 + \dots \right] \xi^2,$$

$$\eta_2 = \frac{r_0 z_0}{\rho_0} \cdot \frac{1}{\rho_0} \left[0 + 4\sigma + \frac{8r_0^2 - z_0^2}{\rho_0^2} \sigma^2 + \dots \right] \xi^2,$$

and

$$\eta_2 - \eta_1 = \frac{r_0 z_0}{\rho_0^2} \cdot \frac{1}{\rho_0} \left[0 + 4\sigma + \frac{16r_0^2 - 3z_0^2}{2\rho_0^2} \sigma^2 + \dots \right] \xi^2 + \dots$$

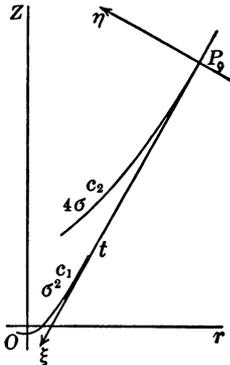


FIG. 2.

In this case *practically the whole southerly deviation is due to the plumb-bob locus*, since in the equation of the curve c_1 the coefficient of ξ^2 is an infinitesimal of second order with respect to the coefficient of ξ^2 in the equation of c_2 . (See Fig. 2.) Both of the curves c_1 and c_2 lie to the north of the tangent t .

Since the curve c_1 lies to the north of t , it follows that those computers who neglect the part η_2 are forced to say that, for this potential function, the falling body has a *northerly*, and not a southerly, deviation.

3. A Third Assumption.

Let us assume, as above, that the field of force F_1 is central, the center being at the earth's center. Here we will assume, not the law of Newton, but that the

* For this potential function the level (equipotential) surface of the field F_2 which passes through the plumb-bob R , is an ellipsoid of oblateness $\frac{1}{2}\sigma$. [Put $\epsilon = 0$ in the assumption 4.]

† In this case the curve c_1 is a conic of which $O(r=0, z=0)$ is a focus. This follows from the fact that the curve c is an ellipse with focus at O .

magnitude of the force varies directly as the distance from the center. Then *

$$f_1(r, z) = -\frac{K}{2} \rho^2, \quad \rho^2 = r^2 + z^2,$$

where K is a positive constant. For this potential function equations (1) and (2) become †

$$\begin{aligned} \eta_1 &= -\frac{1}{2} \frac{r_0 z_0}{\rho_0^2} \cdot \frac{1}{\rho_0} \left[0 + \frac{\omega^2}{K} + \frac{2r_0^2 - z_0^2}{\rho_0^2} \frac{\omega^4}{K^2} \dots \right] \xi^2 + \dots, \\ \eta_2 &= -2\eta_1 \quad \text{and} \quad \eta_2 - \eta_1 = -3\eta_1. \end{aligned}$$

In this case c_1 falls to the south, and c_2 to the north, of t . Hence the trajectory contributes $\frac{1}{3}$, and the plumb-bob locus $\frac{2}{3}$, of the southerly deviation. (See Fig. 3.)

If, at the initial point, the forces due to assumptions 2 and 3 be the same, $K\rho_0 = M/\rho_0^2$ and therefore $\omega^2/K = \sigma$.

The General Differential Equation of Gauss.

Before making the supposition stated in Assumption 1 above, Gauss found the differential equations of motion (relative to a set of axes fixed with respect to the rotating earth) of a falling body, on the supposition that the field of force F_1 is central, ‡ the center being the point in which the earth's axis is pierced by the vector which represents the force of the field F_1 at the initial point of the falling body. Gauss does not solve these differential equations, in which he denotes by p the magnitude of the force of the field F_1 and by M a constant due to air resistance. If we put this M equal to zero (i. e., neglect air resistance) and assume that p varies

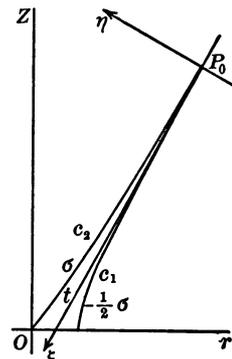


FIG. 3.

* For this potential function all of the level (equipotential) surfaces of the field F_2 are ellipsoids of oblateness $\frac{1}{2}\sigma$. Their equations are

$$K(r^2 + z^2) - \omega^2 r^2 = C.$$

Their oblateness is

$$1 - \frac{\rho_p}{\rho_e} = 1 - \frac{\frac{\sqrt{C}}{\sqrt{K}}}{\frac{\sqrt{C}}{\sqrt{K - \omega^2}}} = 1 - \sqrt{1 - \frac{\omega^2}{K}} = \frac{1}{2} \frac{\omega^2}{K} = \frac{1}{2} \sigma.$$

† In this case the curve c_1 is an hyperbola whose center is at O and whose foci are on the r -axis; the curve c_2 is an equilateral hyperbola which passes through O and whose asymptotes are parallel to the axes of r and z . The truth of the first part of this statement follows from the fact that the curve c is an ellipse whose center is at O . The curve c_2 is of the form mentioned since it is the locus of the feet of perpendiculars dropped from a point to a family of similar and concentric ellipses.

‡ At least if g is constant in his equations [3].

inversely as the square of the distance (or directly as the distance) from the center ($X = 0$, $Y = 0$, $Z = q$) of Gauss' field F_1 , we obtain a set of differential equations, the solution of which should yield for the x of Gauss the same value as that obtained by replacing z_0 by $z_0 + \alpha$ in the expression given for η_1 under Assumption 2 (or 3), α being the distance of the center of Gauss' field F_1 to the south from the earth's center O .

4. A Fourth Assumption.

Now let us assume that the potential function f_1 has the form* :

$$f_1(r, z) = \frac{M}{\rho} + \frac{\epsilon}{3} M \rho_1^2 \frac{r^2 - 2z^2}{\rho^5},$$

in which ρ and M have the same meaning as in assumption 2. The symbol ρ_1 represents the mean radius of the Standard Spheroid, and

$$\epsilon = e_1 - \frac{1}{2}\sigma_1,$$

where e_1 represents the ellipticity (or oblateness) of the Standard Spheroid, and

$$\sigma_1 = \frac{\omega^2 \rho_1}{M/\rho_1^2}.$$

The values of the constants are †

$$\sigma_1 = \frac{1}{288.38} = .003468 \quad \text{and} \quad e_1 = \frac{1}{293.5} = .003407,$$

$$\epsilon = .00167.$$

For this form of f_1 , equations (1) and (2) become

$$\eta_1 = -\frac{1}{6} \frac{r_0 z_0}{\rho_0^2} \cdot \frac{1}{\rho_0} \left[2 \left(\frac{\rho_1}{\rho_0} \right)^2 \epsilon + \left(\frac{\rho_1}{\rho_0} \right)^4 \frac{16z_0^2 - 10r_0^2}{\rho_0^2} \epsilon^2 - 2 \left(\frac{\rho_1}{\rho_0} \right)^2 \frac{r_0^2}{\rho_0^2} \epsilon \sigma - 3 \frac{z_0^2}{\rho_0^2} \sigma^2 \dots \right] \xi^2,$$

$$\eta_2 = -\frac{r_0 z_0}{\rho_0^2} \cdot \frac{1}{\rho_0} \left[2 \left(\frac{\rho_1}{\rho_0} \right)^2 \epsilon - 4\sigma + \left(\frac{\rho_1}{\rho_0} \right)^4 \frac{16z_0^2 - 10r_0^2}{\rho_0^2} \epsilon^2 - 6 \left(\frac{\rho_1}{\rho_0} \right)^2 \frac{r_0^2}{\rho_0^2} \epsilon \sigma + \frac{z_0^2 - 8r_0^2}{\rho_0^2} \sigma^2 \dots \right] \xi^2,$$

and

$$\eta_2 - \eta_1 = \frac{r_0 z_0}{\rho_0^2} \cdot \frac{1}{\rho_0} \left[-\frac{5}{6} \left(\frac{\rho_1}{\rho_0} \right)^2 2\epsilon + 4\sigma - \frac{5}{6} \left(\frac{\rho_1}{\rho_0} \right)^4 \frac{16z_0^2 - 10r_0^2}{\rho_0^2} \epsilon^2 \right. \\ \left. + \frac{17}{3} \left(\frac{\rho_1}{\rho_0} \right)^2 \frac{r_0^2}{\rho_0^2} \epsilon \sigma + \frac{16r_0^2 - 3z_0^2}{2\rho_0^2} \sigma^2 \dots \right] \xi^2.$$

The ratio ρ_1/ρ_0 is nearly equal to unity and $2\epsilon (= .00334)$ is nearly equal to $\sigma (= .00345)$. Hence, approximately, the above equations become

* This function may be regarded as a sufficiently close approximation to the potential function for which the Standard Spheroid is an equipotential surface.

† For the derivation of this function and the values of e_1 and σ_1 see POINCARÉ, *Figures d'Equilibre d'une Masse Fluide* (1902), Chap. V.

$$\eta_1 = \frac{r_0 z_0}{\rho_0^2} \cdot \frac{1}{\rho_0} \left[-\frac{1}{6} \sigma \right] \xi^2, \quad \eta_2 = \frac{r_0 z_0}{\rho_0^2} \cdot \frac{1}{\rho_0} \left[3\sigma \right] \xi^2,$$

and

$$\eta_2 - \eta_1 = \frac{r_0 z_0}{\rho_0^2} \frac{1}{\rho_0} \left[\frac{19}{6} \sigma \right] \xi^2.$$

In this case c_1 falls to the south, and c_2 to the north, of t . The trajectory contributes $\frac{1}{19}$, and the plumb-bob locus $\frac{18}{19}$, to the southerly deviation. (See Fig. 4.)

It will be observed that the potential function and the results under the second assumption may be obtained from those under this assumption by setting ϵ equal to zero.

The preceding results and figures 1, 2, 3 and 4 clearly show that the curves c_1 (or c_2) which correspond to the different assumptions are not even approximations to one another. The four potential functions assumed above may be regarded as having equal first derivatives* at $P_0(r_0, z_0)$. But in order that the curves c_1 (or c_2) which correspond to two potential functions f_1 , shall osculate at P_0 , both the first and second derivatives of these two potential functions must be equal at P_0 . This follows from the fact that the coefficient of ξ^2 in each of equations (1) and (2) involves both the first and second derivatives of f_1 .

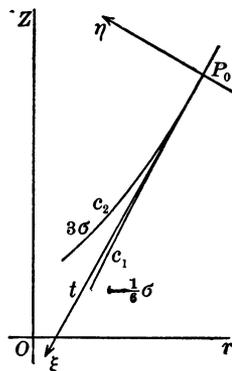


FIG. 4.

The Work of M. le Comte de Sparre.†

M. le Comte de Sparre computes the portion of the southerly deviation which is denoted by η_1 in this paper. In his first paper he assumes potential functions which are practically identical with those of Assumptions 2 and 3. His results, under these assumptions, are

$$x = -\frac{1}{2} \frac{R\omega^4 h^2}{g^2} \sin^3 \lambda \cos \lambda \quad \text{and} \quad x = \frac{1}{2} \frac{\omega^2 h^2}{g} \sin \lambda \cos \lambda,$$

respectively, and those of this paper are

$$\eta_1 = \frac{1}{2} \frac{r_0 z_0}{\rho_0^2} \cdot \frac{z_0^2}{\rho_0^2} \cdot \frac{1}{\rho_0} \sigma^2 \xi^2 \quad \text{and} \quad \eta_1 = -\frac{1}{2} \frac{r_0 z_0}{\rho_0^2} \frac{1}{\rho_0} \left[\sigma + () \sigma^2 \right] \xi^2,$$

respectively. Since x is positive for southerly, and η_1 is positive for northerly deviations, and since

*That is, the vector which represents the force of the field F_1 at the initial point of the falling body in F_1 , may be taken to be the same for all the assumptions. As a consequence of this the line t will be the same for all the assumptions.

†Bulletin de la Société Mathématique de France, vol. 33 (1905). First paper, page 65, second paper, page 146.

$$R = \rho_0, \quad \sin \lambda = \frac{z_0}{\rho_0}, \quad \cos \lambda = \frac{r_0}{\rho_0}, \quad h^2 = \xi^2, \quad R \frac{\omega^2}{g} = \sigma,$$

it follows that the two sets of formulas are identical.

In the second paper a potential function is chosen which is practically equivalent to that of Assumption 4. The result of M. le Comte de Sparre is

$$x = \frac{\omega^2}{2g} \sin \lambda \cos \lambda g t^4$$

and that of this paper is

$$\eta_1 = \frac{r_0 z_0}{\rho_0^2} \frac{1}{\rho_0} \left(-\frac{1}{6} \sigma \dots \right) \xi^2.$$

Since $h = \xi = \frac{1}{2} g t^2 + \dots$, and by the above substitution, these formulas may be written in the following forms:

$$x = \frac{4}{2g} \sin \lambda \cos \lambda \frac{\sigma}{R} h^2 \quad \text{and} \quad -\eta_1 = +\frac{1}{6} \sin \lambda \cos \lambda \frac{\sigma}{R} h^2.$$

Therefore these expressions are nearly the same.

Conclusion.

Of the facts set forth in statements I and II of the introduction, the assumption of Gauss disregarded the first but not the second, and the assumption of the second paper of M. le Comte de Sparre disregarded the second but not the first, while the two assumptions of the first paper of M. le Comte de Sparre disregarded both the first and the second. On the other hand the difference $\eta_2 - \eta_1$ for assumption 4 was obtained with due regard for the facts set forth in Statements I and II. This difference yields results for the southerly deviation which are several times as great as those obtained from the formulas of Gauss and M. le Comte de Sparre. It is not necessary to retain all the terms which are included in the expression for this difference, since σ^2 , ϵ^2 and $\epsilon\sigma$ are small in comparison with σ and ϵ . Since the ratio ρ_1/ρ_0 is nearly equal to unity, and therefore

$$\sigma \left(= \frac{\omega^2 \rho_0}{M/\rho_0^2} \right) \quad \text{and} \quad \sigma_1 \left(= \frac{\omega^2 \rho_1}{M/\rho_1^2} \right)$$

are nearly equal, the expression for the southerly deviation of a falling body assumes the form:

$$\frac{1}{2} \sin 2\phi_0 \left(4\sigma_1 - \frac{5}{3}\epsilon \right) \frac{h^2}{\rho_0},$$

where ϕ_0 denotes the geocentric latitude of the initial point P_0 , ρ_0 the distance from the earth's center to P_0 , h the height through which the body falls, and σ_1 and ϵ the numerical constants given in Assumption 4.

A Numerical Example.

For the data of Benzenberg's experiment in St. Michael's Tower, namely $\phi = 53^\circ 33'$, $h = 235$ feet, the formula of Gauss ($\frac{2}{3} \sin \phi \cos \phi \omega^2 h^2 / g$) yields the value .00046 lines*, and the formula of this paper [$\frac{1}{2} \sin 2\phi_0 (4\sigma_1 - \frac{5}{8}\epsilon) h^2 / \rho_0$, where ϕ_0 denotes the geocentric latitude of P_0 and equals $53^\circ 22'$ if the astronomical latitude is $\phi = 53^\circ 33'$] yields the value .00202 lines. The latter result is about four and one-half times as great as the former, as we knew it should be from a comparison of the expressions for $\eta_2 - \eta_1$ under assumptions 1 and 4.

The Results of Experiments.

In experiments for the determination of the southerly deviation of falling bodies, results are obtained which are many times greater than those accounted for by any of the formulas given above. †

Since the greater part of the southerly deviation is due to the plumb-bob locus, it occurs to the author that experiments for the determination of the deviation between a very long plumb-line and a short plumb-line, which are supported at the same point, might give results of greater value than those for the determination of the southerly deviation of a falling body. The annoying factor of air resistance occurs in the latter, but not in the former. By enclosing the plumb-lines in an air-tight tube and making the observations through glass windows in the tube, air currents could also be eliminated.

THE ESTABLISHMENT OF GENERAL EQUATIONS (1) AND (2).

§ 1. *The curve c.* The curve c has already been defined as the path of the falling body in the fixed field F_1 . In order to get its equations let us choose a set of rectangular axes in F_1 and denote by x_1, y_1, z_1 the coördinates of a general point referred to these axes, the origin of which is taken at the earth's center O . The axis of z_1 is taken coincident with the earth's axis of rotation and is regarded as positive in the direction of the earth's north pole. The axes of x_1 and y_1 then lie in the earth's equatorial plane. The direction of the axis of x_1 is so chosen that the plane $x_1 z_1$ contains the position \bar{P}_0 which the falling body occupied at the instant when its connection with the rotating earth was severed, and its sense is such that the coördinate of this initial point is positive. The axis of y_1 , which is perpendicular to that of x_1 , is positive in the direction in which the earth rotates at the initial point. (See Fig. 5.)

* 144 lines = 1 foot.

† See the following papers: Professor FLORIAN CAJORI; *The Unexplained Southerly Deviation of Falling Bodies*, Science, Vol. 14 (1901), p. 853. Professor EDWIN H. HALL, *Do Falling Bodies Move South?* Physical Review, Vol. 17 (1903), Nos. 3, 4.

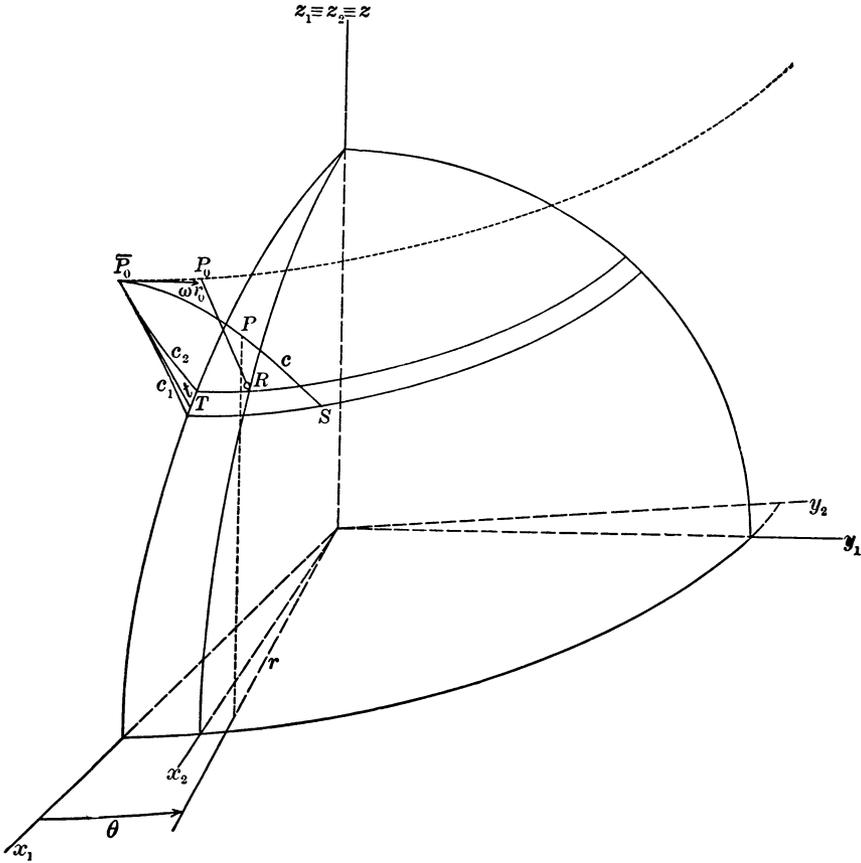


FIG. 5.

If, as above,

$$U_1(x_1, y_1, z_1) = f_1(r, z)$$

represents the potential function of the field F_1 , and ω the angular velocity of the earth's rotation, then the curve c is that integral curve of the differential equations

$$\frac{d^2 x_1}{dt^2} = \frac{\partial U_1}{\partial x_1}, \quad \frac{d^2 y_1}{dt^2} = \frac{\partial U_1}{\partial y_1}, \quad \frac{d^2 z_1}{dt^2} = \frac{\partial U_1}{\partial z_1},$$

which, at $t = 0$, is subject to the conditions

$$\begin{aligned} x_1 &= r_0, & y_1 &= 0, & z_1 &= z_0, \\ \frac{dx_1}{dt} &= 0, & \frac{dy_1}{dt} &= \omega r_0, & \frac{dz_1}{dt} &= 0. \end{aligned}$$

§ 2. *The curve c_1 .* The curve c_1 has already been defined as the meridian curve of the surface of revolution, of axis z_1 , on which the curve c lies. It may also be defined as the path of the falling body in the moving plane which is determined by the body and the axis of z_1 . In order to get its equation let us make the following transformation of coördinates :

$$x_1 = r \cos \theta, \quad y_1 = r \sin \theta, \quad z_1 = z,$$

in which θ denotes the angle which the plane determined by the falling body and the axis of z_1 makes with the plane $x_1 z_1$. (See Fig. 5.) The preceding differential equations are then transformed into the following differential equations :

$$\begin{aligned} r'' - r\theta'^2 &= \frac{\partial f_1}{\partial r}, \\ r\theta'' + 2r'\theta' &= 0, \\ z'' &= \frac{\partial f_1}{\partial z}, \end{aligned}$$

where the primes (') and seconds (") denote the first and second derivatives, respectively, with regard to the time t . The second of these equations on integration becomes

$$\theta' = \frac{k}{r^2},$$

where k is a constant of integration, and hence reduces the first to

$$r'' = \frac{k^2}{r^3} + \frac{\partial f_1}{\partial r}.$$

The initial conditions assume the following form : when $t = 0$,

$$\begin{aligned} r &= r_0, & \theta &= 0, & z &= z_0, \\ r' &= 0, & \theta' &= \omega, & z' &= 0. \end{aligned}$$

Hence the constant k is given by the relation

$$\omega = k/r_0^2.$$

Therefore the curve c_1 is that integral curve of the differential equations :

$$r'' = \frac{\omega^2 r_0^4}{r^3} + \frac{\partial f_1}{\partial r}, \quad z'' = \frac{\partial f_1}{\partial z},$$

which, at $t = 0$, is subject to the conditions

$$r = r_0, \quad z = z_0, \quad r' = 0, \quad z' = 0.$$

If we put

$$V(r, z) = -\frac{\omega^2 r_0^4}{2r^2} + f_1(r, z),$$

the last differential equations may be written in the following form:

$$r'' = \frac{\partial V}{\partial r}, \quad z'' = \frac{\partial V}{\partial z}.$$

§ 3. *The curve c_2 .* The curve c_2 has already been defined as the locus of plumb-bobs of all plumb-lines which are supported at the initial point, P_0 , of the falling body in the field F_2 . It is therefore the locus of the points of tangency of tangent lines which are drawn from P_0 to the lines of force of the field F_2 . In order to get its equation let us choose a set of rectangular axes in F_2 and denote by x_2, y_2, z_2 the coördinates of a general point referred to these axes, the origin of which is also taken at the earth's center O . The axis of z_2 is taken, just as that of z_1 , coincident with the earth's axis of rotation and is regarded as positive in the direction of the earth's north pole. The axes of x_2 and y_2 lie in the earth's equatorial plane, and when $t = 0$, they coincide with the axes of x_1 and y_1 , respectively. (See Fig. 5.) The point P_0 lies in the plane $x_2 z_2$. If, as above,

$$U_2(x_2, y_2, z_2) = f_2(r, z),$$

where

$$r = \sqrt{x_2^2 + y_2^2} \quad \text{and} \quad z = z_2,$$

represents the potential function of the field F_2 , the equations of c_2 are

$$\frac{x_2 - r_0}{\frac{\partial U_2}{\partial x_2}} = \frac{y_2 - 0}{\frac{\partial U_2}{\partial y_2}} = \frac{z_2 - z_0}{\frac{\partial U_2}{\partial z_2}}.$$

But since

$$\frac{\partial U_2}{\partial x_2} = \frac{\partial f_2}{\partial r} \cdot \frac{x_2}{r}, \quad \frac{\partial U_2}{\partial y_2} = \frac{\partial f_2}{\partial r} \cdot \frac{y_2}{r}, \quad \frac{\partial U_2}{\partial z_2} = \frac{\partial f_2}{\partial z},$$

the curve lies in the plane $y_2 = 0$, in which it has the equation

$$\frac{r - r_0}{\frac{\partial f_2}{\partial r}} = \frac{z - z_0}{\frac{\partial f_2}{\partial z}}.$$

§ 4. *The equation of curve c_1 in powers of $r - r_0$.*

That solution of the differential equations

$$r'' = \partial V / \partial r, \quad z'' = \partial V / \partial z$$

which, at $t = 0$, is subject to the conditions $r = r_0, z = z_0, r' = 0, z' = 0$, is

of the form

$$r - r_0 = \alpha_2 t^2 + \alpha_4 t^4 + \dots, \quad z - z_0 = \beta_2 t^2 + \beta_4 t^4 + \dots,$$

where

$$\alpha_2 = \frac{1}{2} \left(\frac{\partial V}{\partial r} \right)_0, \quad \beta_2 = \frac{1}{2} \left(\frac{\partial V}{\partial z} \right)_0,$$

$$\alpha_4 = \frac{1}{4!} \left\{ \left(\frac{\partial^2 V}{\partial r^2} \right)_0 \left(\frac{\partial V}{\partial r} \right)_0 + \left(\frac{\partial^2 V}{\partial r \partial z} \right)_0 \left(\frac{\partial V}{\partial z} \right)_0 \right\},$$

$$\beta_4 = \frac{1}{4!} \left\{ \left(\frac{\partial^2 V}{\partial r \partial z} \right)_0 \left(\frac{\partial V}{\partial r} \right)_0 + \left(\frac{\partial^2 V}{\partial z^2} \right)_0 \left(\frac{\partial V}{\partial z} \right)_0 \right\},$$

the subscript $(_0)$ indicating that the derivatives of V have been computed for the point $P_0(r_0, z_0)$. By the elimination of t from these equations, we obtain the equation

$$z - z_0 = a_1(r - r_0) + a_2(r - r_0)^2 + \dots,$$

where

$$a_1 = \frac{\left(\frac{\partial V}{\partial z} \right)_0}{\left(\frac{\partial V}{\partial r} \right)_0},$$

$$a_2 = \frac{1}{6} \frac{\left(\frac{\partial^2 V}{\partial r \partial z} \right)_0 \left\{ \left(\frac{\partial V}{\partial r} \right)_0^2 - \left(\frac{\partial V}{\partial z} \right)_0^2 \right\} + \left\{ \left(\frac{\partial^2 V}{\partial z^2} \right)_0 - \left(\frac{\partial^2 V}{\partial r^2} \right)_0 \right\} \left(\frac{\partial V}{\partial r} \right)_0 \left(\frac{\partial V}{\partial z} \right)_0}{\left(\frac{\partial V}{\partial r} \right)_0^3}.$$

§ 5. *The equation of curve c_2 in powers of $r - r_0$.*

The equation of curve c_2 , whose finite equation has already been found in § 3, may also be written in the following form:

$$z - z_0 = b_1(r - r_0) + b_2(r - r_0)^2 + \dots,$$

where

$$b_1 = \left(\frac{z}{dr} \right)_0, \quad b_2 = \frac{1}{2} \left(\frac{d^2z}{dr^2} \right)_0.$$

In order to do this let us first write its equation in the form

$$F(r, z) = 0,$$

where

$$F(r, z) = \frac{\partial f_2}{\partial z}(r - r_0) - \frac{\partial f_2}{\partial r}(z - z_0).$$

The derivatives dz/dr and d^2z/dr^2 are expressible in terms of the derivatives of F by means of the relations:

$$\frac{\partial F}{\partial r} + \frac{\partial F}{\partial z} \cdot \frac{dz}{dr} = 0,$$

$$\frac{\partial^2 F}{\partial r^2} + 2 \frac{\partial^2 F}{\partial r \partial z} \cdot \frac{dz}{dr} + \frac{\partial^2 F}{\partial z^2} \left(\frac{dz}{dr} \right)^2 + \frac{\partial F}{\partial z} \cdot \frac{d^2 z}{dr^2} = 0,$$

and the derivatives of F , in terms of those of f_2 , by means of the relations :

$$\frac{\partial F}{\partial r} = \frac{\partial^2 f_2}{\partial r \partial z} (r - r_0) - \frac{\partial^2 f_2}{\partial r^2} (z - z_0) + \frac{\partial f_2}{\partial z},$$

$$\frac{\partial F}{\partial z} = \frac{\partial^2 f_2}{\partial z^2} (r - r_0) - \frac{\partial^2 f_2}{\partial r \partial z} (z - z_0) - \frac{\partial f_2}{\partial r},$$

$$\frac{\partial^2 F}{\partial r^2} = \frac{\partial^3 f_2}{\partial r^2 \partial z} (r - r_0) - \frac{\partial^3 f_2}{\partial r^3} (z - z_0) + 2 \frac{\partial^2 f_2}{\partial r \partial z},$$

$$\frac{\partial^2 F}{\partial r \partial z} = \frac{\partial^3 f_2}{\partial r \partial z^2} (r - r_0) - \frac{\partial^3 f_2}{\partial r^2 \partial z} (z - z_0) + \frac{\partial^2 f_2}{\partial z^2} - \frac{\partial^2 f_2}{\partial r^2},$$

$$\frac{\partial^2 F}{\partial z^2} = \frac{\partial^3 f_2}{\partial z^3} (r - r_0) - \frac{\partial^3 f_2}{\partial r \partial z^2} (z - z_0) - 2 \frac{\partial^2 f_2}{\partial r \partial z},$$

and therefore

$$\left(\frac{\partial F}{\partial r} \right)_0 = \left(\frac{\partial f_2}{\partial z} \right)_0,$$

$$\left(\frac{\partial F}{\partial z} \right)_0 = - \left(\frac{\partial f_2}{\partial r} \right)_0,$$

$$\left(\frac{\partial^2 F}{\partial r^2} \right)_0 = 2 \left(\frac{\partial^2 f_2}{\partial r \partial z} \right)_0,$$

$$\left(\frac{\partial^2 F}{\partial r \partial z} \right)_0 = \left(\frac{\partial^2 f_2}{\partial z^2} \right)_0 - \left(\frac{\partial^2 f_2}{\partial r^2} \right)_0,$$

$$\left(\frac{\partial^2 F}{\partial z^2} \right)_0 = -2 \left(\frac{\partial^2 f_2}{\partial r \partial z} \right)_0.$$

Hence we find

$$b_1 = \left(\frac{dz}{dr} \right)_0 = - \frac{\left(\frac{\partial F}{\partial r} \right)_0}{\left(\frac{\partial F}{\partial z} \right)_0} = \frac{\left(\frac{\partial f_2}{\partial z} \right)_0}{\left(\frac{\partial f_2}{\partial r} \right)_0},$$

$$b_2 = \frac{1}{2} \left(\frac{d^2 z}{dr^2} \right)_0 = \frac{1}{2} \frac{\left(\frac{\partial^2 F}{\partial r^2} \right)_0 + 2 \left(\frac{\partial^2 F}{\partial r \partial z} \right)_0 \left(\frac{dz}{dr} \right)_0 + \left(\frac{\partial^2 F}{\partial z^2} \right)_0 \left(\frac{dz}{dr} \right)_0^2}{- \left(\frac{\partial F}{\partial z} \right)_0}$$

$$= \frac{\left(\frac{\partial^2 f_2}{\partial r \partial z}\right)_0 \left\{ \left(\frac{\partial f_2}{\partial r}\right)_0^2 - \left(\frac{\partial f_2}{\partial z}\right)_0^2 \right\} + \left\{ \left(\frac{\partial^2 f_2}{\partial z^2}\right)_0 - \left(\frac{\partial^2 f_2}{\partial r^2}\right)_0 \right\} \left(\frac{\partial f_2}{\partial r}\right)_0 \left(\frac{\partial f_2}{\partial z}\right)_0}{\left(\frac{\partial f_2}{\partial r}\right)_0^3}.$$

§ 6. *The relation between the curves c_1 and c_2 .* In order to find a relation between the curves c_1 and c_2 let us express the coefficients a_1, a_2, b_1, b_2 in terms of the derivatives of f_1 . We already have the relations :

$$V = f_1 - \frac{\omega^2 r_0^4}{2r^2} \quad \text{and} \quad f_2 = f_1 + \frac{\omega^2}{2} r^2,$$

from which the following relations are found :

$$\frac{\partial V}{\partial r} = \frac{\partial f_1}{\partial r} + \frac{\omega^2 r_0^4}{r^3}, \quad \frac{\partial f_2}{\partial r} = \frac{\partial f_1}{\partial r} + \omega^2 r,$$

$$\frac{\partial V}{\partial z} = \frac{\partial f_1}{\partial z}, \quad \frac{\partial f_2}{\partial z} = \frac{\partial f_1}{\partial z},$$

$$\frac{\partial^2 V}{\partial r^2} = \frac{\partial^2 f_1}{\partial r^2} - \frac{3\omega^2 r_0^4}{r^4}, \quad \frac{\partial^2 f_2}{\partial r^2} = \frac{\partial^2 f_1}{\partial r^2} + \omega^2,$$

$$\frac{\partial^2 V}{\partial r \partial z} = \frac{\partial^2 f_1}{\partial r \partial z}, \quad \frac{\partial^2 f_2}{\partial r \partial z} = \frac{\partial^2 f_1}{\partial r \partial z},$$

$$\frac{\partial^2 V}{\partial z^2} = \frac{\partial^2 f_1}{\partial z^2}, \quad \frac{\partial^2 f_2}{\partial z^2} = \frac{\partial^2 f_1}{\partial z^2}.$$

Therefore :

$$\begin{aligned} \left(\frac{\partial V}{\partial r}\right)_0 &= \left(\frac{\partial f_2}{\partial r}\right)_0 = \left(\frac{\partial f_1}{\partial r}\right)_0 + \omega^2 r_0, & \left(\frac{\partial V}{\partial z}\right)_0 &= \left(\frac{\partial f_2}{\partial z}\right)_0 = \left(\frac{\partial f_1}{\partial z}\right)_0, \\ \left(\frac{\partial^2 V}{\partial r^2}\right)_0 &= \left(\frac{\partial^2 f_1}{\partial r^2}\right)_0 - 3\omega^2, & \left(\frac{\partial^2 f_2}{\partial r^2}\right)_0 &= \left(\frac{\partial^2 f_1}{\partial r^2}\right)_0 + \omega^2, \\ \left(\frac{\partial^2 V}{\partial r \partial z}\right)_0 &= \left(\frac{\partial^2 f_2}{\partial r \partial z}\right)_0 = \left(\frac{\partial^2 f_1}{\partial r \partial z}\right)_0, & \left(\frac{\partial^2 V}{\partial z^2}\right)_0 &= \left(\frac{\partial^2 f_2}{\partial z^2}\right)_0 = \left(\frac{\partial^2 f_1}{\partial z^2}\right)_0. \end{aligned}$$

Hence the relations

$$a_1 = b_1 = \frac{\left(\frac{\partial f_1}{\partial z}\right)_0}{\left(\frac{\partial f_1}{\partial r}\right)_0 + \omega^2 r_0},$$

$$a_2 = \frac{1}{6} \frac{C + B_1 \omega^2 + A_1 \omega^4}{\left\{ \left(\frac{\partial f_1}{\partial r}\right)_0 + \omega^2 r_0 \right\}^3}, \quad b_2 = \frac{C + B_2 \omega^2 + A_2 \omega^4}{\left\{ \left(\frac{\partial f_1}{\partial r}\right)_0 + \omega^2 r_0 \right\}^3},$$

where

$$\begin{aligned}
 C &= \left(\frac{\partial^2 f_1}{\partial r \partial z}\right)_0 \left\{ \left(\frac{\partial f_1}{\partial r}\right)_0^2 - \left(\frac{\partial f_1}{\partial z}\right)_0^2 \right\} + \left\{ \left(\frac{\partial^2 f_1}{\partial z^2}\right)_0 - \left(\frac{\partial^2 f_1}{\partial r^2}\right)_0 \right\} \left(\frac{\partial f_1}{\partial r}\right)_0 \left(\frac{\partial f_1}{\partial z}\right)_0, \\
 B &= 2 \left(\frac{\partial^2 f_1}{\partial r \partial z}\right)_0 \left(\frac{\partial f_1}{\partial r}\right)_0 r_0 + \left\{ \left(\frac{\partial^2 f_1}{\partial z^2}\right)_0 - \left(\frac{\partial^2 f_1}{\partial r^2}\right)_0 \right\} \left(\frac{\partial f_1}{\partial z}\right)_0 r_0, \\
 B_1 &= B + 3 \left(\frac{\partial f_1}{\partial r}\right)_0 \left(\frac{\partial f_1}{\partial z}\right)_0, \quad B_2 = B - \left(\frac{\partial f_1}{\partial r}\right)_0 \left(\frac{\partial f_1}{\partial z}\right)_0, \\
 A &= \left(\frac{\partial^2 f_1}{\partial r \partial z}\right)_0 r_0^2, \quad A_1 = A + 3 \left(\frac{\partial f_1}{\partial z}\right)_0 r_0, \quad A_2 = A - \left(\frac{\partial f_1}{\partial z}\right)_0 r_0.
 \end{aligned}$$

Since $a_1 = b_1$, it follows that the curves c_1 and c_2 , which lie in the meridian plane of $P_0(r_0, z_0)$ (i. e., in the plane $y_2 = 0$), have a common tangent* and a common normal at P_0 .

§ 7. *The General Equations (1) and (2). The Southerly Deviation.*

Let us now take the common tangent and the common normal to the curves c_1 and c_2 as a pair of rectangular axes, and obtain the equations of c_1 and c_2 referred to them. The coördinate which is measured along the tangent from P_0 we shall denote by ξ , and regard as positive when measured in the direction of the earth's center. The other coördinate we shall denote by η and regard as positive when measured in the direction of the north pole. The equations of transformation from the axes of r and z to those of ξ and η are

$$r - r_0 = -\cos \phi \cdot \xi - \sin \phi \cdot \eta, \quad z - z_0 = -\sin \phi \cdot \xi + \cos \phi \cdot \eta,$$

where

$$\sin \phi = \frac{b_1}{\sqrt{1 + b_1^2}} = \frac{a_1}{\sqrt{1 + a_1^2}}, \quad \cos \phi = \frac{1}{\sqrt{1 + b_1^2}} = \frac{1}{\sqrt{1 + a_1^2}}.$$

By this transformation the equation

$$z - z_0 = a_1(r - r_0) + a_2(r - r_0)^2 + \dots$$

* This common tangent is also tangent at P_0 to the line of force of the field F_2 which passes through P_0 . For the differential equation of the lines of force of F_2 which line in the plane $y_2 = 0$ is

$$\frac{dz}{dr} = \frac{\partial f_2}{\partial z} / \frac{\partial f_2}{\partial r}$$

and therefore the slope at P_0 of the line of force which passes through P_0 is

$$\left(\frac{\partial f_2}{\partial z}\right)_0 = \frac{\left(\frac{\partial f_1}{\partial z}\right)_0}{\left(\frac{\partial f_1}{\partial r}\right)_0 + \omega^2 r_0} = a_1 = b_1.$$

of the curve c_1 assumes the form

$$+ \frac{a_2}{1 + a_1^2} \xi^2 + \frac{2a_1 a_2}{1 + a_1^2} \xi \eta + \frac{a_1^2 a_2}{1 + a_1^2} \eta^2 - \sqrt{1 + a_1^2} \eta = 0,$$

which when solved for η , becomes* :

$$\eta = \frac{a_2}{[1 + a_1^2]^{\frac{3}{2}}} \xi^2 + \dots$$

By the same transformation the equation

$$z - z_0 = b_1(r - r_0) + b_2(r - r_0)^2 + \dots,$$

of the curve c_2 , becomes

$$\eta = \frac{b_2}{[1 + b_1^2]^{\frac{3}{2}}} \xi^2 + \dots$$

From the relations given in § 6 we find †

$$\frac{1}{[1 + a_1^2]^{\frac{3}{2}}} = \frac{1}{[1 + b_1^2]^{\frac{3}{2}}} = - \frac{\left[\left(\frac{\partial f_1}{\partial r} \right)_0 + \omega^2 r_0 \right]^3}{D^{\frac{3}{2}}},$$

where

$$D = \left[\left(\frac{\partial f_1}{\partial r} \right)_0 + \omega^2 r_0 \right]^2 + \left(\frac{\partial f_1}{\partial z} \right)_0^2 = \left(\frac{\partial f_1}{\partial r} \right)_0^2 + \left(\frac{\partial f_1}{\partial z} \right)_0^2 + 2 \left(\frac{\partial f_1}{\partial r} \right)_0 r_0 \omega + r_0^2 \omega^2.$$

Hence, if we denote the η 's of the curves c_1 and c_2 by η_1 and η_2 , respectively, we obtain the equations

$$\eta_1 = - \frac{1}{6} \frac{C + B_1 \omega^2 + A_1 \omega^4}{D^{\frac{3}{2}}} \xi^2 + \dots, \quad \eta_2 = - \frac{C + B_2 \omega^2 + A_2 \omega^4}{D^{\frac{3}{2}}} \xi^2 + \dots,$$

which are the equations (1) and (2) already given. As has been stated, the southerly deviation of a falling body is the difference $\eta_2 - \eta_1$, when ξ is replaced by the height h ‡ through which the body falls, and f_1 is a sufficiently close approximation to the potential function of the earth's gravitational field of force.

* The equation

$$0 = Ey + Ax^2 + Bxy + Cy^2 + Lx^3 + Mx^2y + Nxy^2 + \dots$$

when solved for y becomes

$$y = - \frac{A}{E} x^2 + \dots$$

† The negative sign is taken because $[1 + b_1^2]^{\frac{3}{2}}$ and $D^{\frac{3}{2}}$ are positive and $(\partial f_1 / \partial r)_0 + \omega^2 r_0$ is negative.

‡ As already remarked, the circle of reference must be determined by a plumb-line of length h supported at the initial point of the falling body.