QUATERNION DEVELOPMENTS WITH APPLICATIONS*

BY

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Introduction.

This paper is an extension of the quaternion algebra along lines analogous to those of a preceding paper.† Certain functions of one or more quaternions are studied in detail, all of them independent of the orientation of the unit axes, $1, i, j, k$. The notation is functional and seems preferable to the use of brackets such as those used by Hamilton,‡ who, indeed, called his notation temporary. We seem to have here a natural method for treating several functions that occur frequently in analysis. For example, if we represent quaternion numbers by vectors in four-dimensional space, as was done by Stringham§ and Hathaway,|| these functions are expressions for geometrical relations which are independent of the particular system of rectangular axes used. These expressions are useful in electrodynamics and in relativity problems. They include the Hamiltonian $[ab]$, (abc), $[abc]$, $(abcd)$, and the equivalent forms used by Joly,¶ and Schrutka von Rechtenstamm.**

The development of these formulæ enable us also to formulate certain differential operators of frequent occurrence, which, from their nature, produce expressions, each invariant under orthogonal transformation in the sense that after the transformation has been performed on the various quaternions of which it is a function, the result is the transform of the initial function (§ 8). Certain of these forms are also invariant under any linear homogeneous transformation. Similar forms were used by MacMahon†† in the study of the ternary and quaternary invariants and covariants under orthogonal transformations. The study of differential parameters and their syzygies is made

* Presented to the Society (Chicago), April 8, 1910, and April 28, 1911.

† American Journal of Mathematics, vol. 19 (1897), pp. 193–216.

‡ *Elements of Quaternions*, 2d ed., vol. I, page 563.

§ These Transactions, vol. 2 (1901), pp. 183–214.

|| These Transactions, vol. 3 (1902), pp. 46–59.

¶ *Manual of Quaternions*, page 264.

** Wiener Sitzungsberichte, vol. 115 (1906), Abt. IIa, pp. 739–775.

†† Proceedings of the London Mathematical Society, ser. 2, vol. 1 (1904), pp. 210–229.

simpler by the use of these forms, and the symbolic invariants of MASCHKE * appear as a natural part of the development. Two differential operators are introduced which are the extensions of the quaternion ∇ to space of four dimensions and to curvilinear three-dimensional space.

Up to the present time, the greater part of the development of the quaternion algebra has been in terms of scalars and vectors rather than in terms of quaternions proper. This often introduces prolixity in the development. It is hoped that the present paper will present some fruitful extensions along the more general line. In the expansion of the forms appearing here it will be evident that they are of the nature of determinants and pfaffians. That other forms might be studied is of course self-evident, but these are the ones of most frequent occurrence in the functions of geometry and physics. However the developments below are on the formal side and valid irrespective of the interpretation of the quaternions.

PART I. FUNDAMENTAL FORMULÆ.

1. Throughout the paper italic letters will be used to represent quaternions, which may occasionally reduce to scalars or vectors. The conjugate of a quaternion q will usually be denoted by q', but the conjugate of an expression, as qrs, will be denoted by $\mathrm{K} \cdot qrs$.

2. We notice that the quaternions uv' and $v'u$ are important, particularly on account of the relations shown in the formulæ

$$uv' = \mathrm{S} \cdot uv' + \mathrm{V}u\mathrm{S}v - \mathrm{V}v\mathrm{S}u - \mathrm{V}\mathrm{V}u\mathrm{V}v,$$

which resolves uv' into three parts whose products have vanishing scalars; and if l and m are scalars,

$$uv'(lu + mv) = u(2l\mathrm{S}uv' + m\mathrm{T}^2v) - lv\mathrm{T}^2u = (lu + mv)v'u,$$

which shows that any quaternion of the form $lu + mv$ is converted into another of the same form by left-hand multiplication by uv' or by right-hand multiplication by $v'u$. Also uv' and $v'u$ differ only in their axes, having the same tensors and angles. The quaternions uv' and vu' however differ only in having opposite angles. If $r = lu + mv$, we have $uv'r - rv'u = 0$.

3. We define first the two functions

$$\mathrm{I} \cdot uv = \tfrac{1}{2}(uv' + vu'), \qquad \mathrm{A}_2 \cdot uv = \tfrac{1}{2}(uv' - vu').$$

The first of the two functions is obviously a scalar, since it is the sum of a quaternion and its conjugate, and the second is a vector. Indeed,

$$\mathrm{I} \cdot uv = \mathrm{S} \cdot uv' = \mathrm{T}^2v \cdot \mathrm{S} \cdot \frac{u}{v}, \qquad \mathrm{A}_2 \cdot uv = \mathrm{V}uv' = \mathrm{T}^2v \cdot \mathrm{V} \cdot \frac{u}{v}.$$

* These Transactions, vol. 1 (1900), pp. 197–204; vol. 7 (1906), pp. 69–80.

From the definitions we have at once

$$\mathrm{I} \cdot uv = \mathrm{I} \cdot vu = \mathrm{I} \cdot u' v' = \mathrm{I} \cdot v' u', \qquad \mathrm{A}_2 \cdot uv = - \mathrm{A}_2 \cdot vu.$$

The latter expression is alternating and therefore vanishes if u is a scalar multiple of v. If we write $\rho = \mathrm{UVV}u\mathrm{V}v$, we find with no difficulty that

$$\mathrm{A}_2 \cdot u' v' = \rho \cdot \mathrm{A}_2 \, uv \cdot \rho^{-1}.$$

Hence $\mathrm{A}_2 \cdot uv$ differs from $\mathrm{A}_2 \, u' v'$ only in direction. By expanding the two sides of the equations, we find that

$$u^{-1} \mathrm{A}_2 \cdot uv \cdot u = \mathrm{A}_2 \cdot v' u', \qquad v^{-1} \cdot \mathrm{A}_2 \cdot uv \cdot v = - \mathrm{A}_2 \cdot u' v'.$$

The two forms used by Joly are, in this notation, as follows:

$$(uv) = v\mathrm{S}u - u\mathrm{S}v = \tfrac{1}{2} (\mathrm{A}_2 \cdot u' v' - \mathrm{A}_2 \cdot uv),$$

$$[uv] = \mathrm{VV}u\mathrm{V}v = \tfrac{1}{2}(- \mathrm{A}_2 \cdot u' v' - \mathrm{A}_2 \cdot uv).$$

The single form given by Hamilton, $[uv]$, is twice the corresponding one given by Joly, that is, is $2 \, \mathrm{VV}u\mathrm{V}v$.

If $\mathrm{I} \cdot uv = 0$ we shall speak of u and v as orthogonal. The solution of the two equations which express that r is orthogonal to both u and v is

$$r = x \, (uv' - v' u) + y \, (uv' uv' - uv' v' u),$$

where x and y are arbitrary scalars; for, this value of r satisfies the two equations, and r can depend only upon two arbitrary parameters.

4. Passing now to three quaternions, we define the functions

$$\mathrm{A}_1 \cdot uvw = \tfrac{1}{2} \, (uv' w + wv' u), \qquad \mathrm{A}_3 \cdot uvw = \tfrac{1}{2} \, (uv' w - wv' u).$$

It is evident that $\mathrm{A}_1 \cdot uvw = \mathrm{A}_1 \cdot wvu$. Also

$$\mathrm{A}_1 \cdot uvw = \tfrac{1}{2} \, (vw' u + wv' u - vu' w - vw' u + uv' w + vu' w)$$

$$= u\mathrm{I} \cdot vw - v\mathrm{I} \cdot wu + w\mathrm{I} \cdot uv.$$

Again from the definition it is obvious that

$$\mathrm{A}_3 \cdot uvw = - \mathrm{A}_3 \cdot wvu.$$

Since we have identically $2u\mathrm{I} \cdot vw = 2\mathrm{I} \cdot vw \cdot u$, we have by expansion the identity

$$uv' w + uw' v = vw' u + wv' u,$$

from which by transposition, and by reapplying the result, we arrive at

$$uv' w - wv' u = vw' u - uw' v = wu' v - vu' w.$$

Therefore, returning to the form A_3, we have

$$A_3 \cdot uvw = A_3 \cdot vwu = A_3 \cdot wuv = - A_3 \cdot wvu = - A_3 \cdot uwv$$
$$= - A_3 \cdot vuw .$$

This function is therefore an alternating function of u, v, and w. It vanishes if they are linearly connected. Joly's form $[uvw]$ is in this notation $A_3 \cdot u' v' w'$. His form (uvw) is $SA_3 \cdot u' v' w'$. We may expand in the form

$$A_3 \cdot uvw = \tfrac{1}{6} [uv' w - uw' v + vw' u - vu' w + wu' v - wv' u]$$
$$= \tfrac{1}{3} [uA_2 \cdot v' w' + vA_2 \cdot w' u' + wA_2 \cdot u' v'] .$$

This form shows that the function is of the nature of a determinant.* Again,

$$I \cdot uA_3 \cdot uvw = 0 = I \cdot vA_3 \cdot uvw = I \cdot wA_3 \cdot uvw .$$

For, if we expand $I \cdot uA_3 \cdot uvw$ we have

$$\tfrac{1}{4} [u' uv' w - u' wv' u + w' vu' u - u' vw' u]$$
$$= \tfrac{1}{4} [u' uI \cdot vw - u' I \cdot vw \cdot u] = 0 .$$

A similar proof holds for v and w. Hence the solution of the three equations expressing the orthogonality of r to u, v, w, is $r = xA_3 \cdot uvw$, where x is an arbitrary scalar.

We see also from these equations and the expanded form of $A_1 \cdot uvw$ that we may resolve $uv' w$ into two orthogonal parts:

$$uv' w = A_1 \cdot uvw + A_3 \cdot uvw .$$

If $A_1 \cdot uvw = 0$, we have $uv' w + wv' u = 0$, whence

$$uv' ww' v + wv' uw' v = 0 = wv' uv' w + wv' wv' u .$$

Adding the two sides of this double equality we have

$$[wv' wv' + wv' (w' v + v' w) + vv' ww'] u = 0 .$$

This gives easily $wv' uI \cdot vw = 0$. Similarly we find that $uv' wI \cdot uw = 0$. Hence either we must have $I \cdot vw = 0 = I \cdot uw$ or else $uv' w = 0 = wv' u$. The latter cannot hold unless $T^2 u$, or $T^2 v$, or $T^2 w$ vanishes. Then if the quaternions are real we must have, when $A_1 \cdot uvw = 0$, the three equations $I \cdot uv = 0 = I \cdot uw = I \quad vw$, and the three quaternions form an orthogonal system.

We find that†

$$TA_1 \cdot uvw = \surd \, (IuuI^2 vw + IvvI^2 wu + IwwI^2 uv - 2IuvIvwIwu)$$
$$= TA_1 \cdot vwu = \cdots .$$

* Cf. JOLY: Transactions of the Royal Irish Academy, 1902, 32*A*, Part II, pp. 17–30.

† The period following I and A will be omitted when no confusion can result, just as it is after S and V in treatises on Quaternions.

Since $\mathrm{T}^2\, uv'\, w = 1$ when u, v, and w are unit quaternions, and

$$\mathrm{T}^2\, uv'\, w = \mathrm{T}^2\, \mathrm{A}_1 \cdot uvw + \mathrm{T}^2\, \mathrm{A}_3 \cdot uvw,$$

we might call $\mathrm{TA}_1 \cdot uvw$ the cosine of the triple u, v, w and $\mathrm{TA}_3 \cdot uvw$ the sine. The latter for geometric reasons could also be called the Staudtian. Moreover,

$$\mathrm{T}^2\, \mathrm{A}_3 \cdot uvw = \mathrm{I} \cdot \mathrm{A}_3 \cdot uvw\mathrm{A}_3 \cdot uvw = \begin{vmatrix} \mathrm{I}uu & \mathrm{I}uv & \mathrm{I}uw \\ \mathrm{I}vu & \mathrm{I}vv & \mathrm{I}vw \\ \mathrm{I}wu & \mathrm{I}wv & \mathrm{I}ww \end{vmatrix}.$$

5. If we write out each quaternion in terms of $1, i, j, k$ we find that

$$\mathrm{A}_3 \cdot uvw = \begin{vmatrix} 1 & i & j & k \\ u_0 & u_1 & u_2 & u_3 \\ v_0 & v_1 & v_2 & v_3 \\ w_0 & w_1 & w_2 & w_3 \end{vmatrix}.$$

From this we have at once

$$\mathrm{I} \cdot t\mathrm{A}_3 \cdot uvw = |\, t_0\, u_1\, v_2\, w_3 \,|.$$

Also from the properties of determinant multiplication we have the important formula

$$\mathrm{I} \cdot \mathrm{A}_3 \cdot abc\mathrm{A}_3 \cdot uvw = \begin{vmatrix} \mathrm{I}au & \mathrm{I}av & \mathrm{I}aw \\ \mathrm{I}bu & \mathrm{I}bv & \mathrm{I}bw \\ \mathrm{I}cu & \mathrm{I}cv & \mathrm{I}cw \end{vmatrix}.$$

We may also easily verify that

$$\mathrm{A}_3 \cdot bc\mathrm{A}_3 \cdot uvw = \begin{vmatrix} -u & -v & -w \\ \mathrm{I}bu & \mathrm{I}bv & \mathrm{I}bw \\ \mathrm{I}cu & \mathrm{I}cv & \mathrm{I}cw \end{vmatrix}.$$

Hence we have at once

$$\mathrm{I} \cdot a\mathrm{A}_3\, bc\mathrm{A}_3\, uvw = -\, \mathrm{IA}_3\, abc\mathrm{A}_3\, uvw.$$

6. Some useful formulæ are

$$\mathrm{A}_2 \cdot \mathrm{A}_2\, uv\mathrm{A}_2\, wx = -\, \mathrm{A}_2\, uw\mathrm{I}vx + \mathrm{A}_2\, ux\mathrm{I}vw + \mathrm{A}_2\, vw\mathrm{I}ux - \mathrm{A}_2\, vx\mathrm{I}uw,$$

$$\mathrm{A}_2 \cdot u\mathrm{A}_3\, vwx = -\, \mathrm{A}_2\, vw\mathrm{I}ux - \mathrm{A}_2\, wx\mathrm{I}uv - \mathrm{A}_2\, xv\mathrm{I}uw,$$

$$\mathrm{A}_3\, uvw \cdot u' = u\mathrm{A}_3\, u'\, v'\, w',$$

$$\mathrm{A}_3 \cdot \mathrm{A}_2\, uv\mathrm{A}_2\, vw\mathrm{A}_2\, wu = \mathrm{T}^2\, \mathrm{A}_3\, uvw.$$

7. For four quaternions we define the three functions

$$\mathrm{I} \cdot uvwx = \mathrm{I}u\mathrm{A}_1 vwx = \mathrm{I}uv\mathrm{I}wx - \mathrm{I}uw\mathrm{I}vx + \mathrm{I}ux\mathrm{I}vw$$

$$= \tfrac{1}{4}(uv'wx' + ux'wv' + vw'xu' + vu'xw') = \tfrac{1}{2}(\mathrm{S} \cdot uv'wx' + \mathrm{S} \cdot vw'xu').$$

$$\mathrm{A}_4 \cdot uvwx = -\mathrm{I}u\mathrm{A}_3 vwx = \tfrac{1}{4}(uv'wx' - ux'wv' + xw'vu' - vw'xu')$$

$$= \tfrac{1}{2}(\mathrm{S} \cdot uv'wx' - \mathrm{S} \cdot vw'xu').$$

$$\mathrm{A}_2 \cdot uvwx = \tfrac{1}{2}(uv'wx' - xw'vu') = \mathrm{V} \cdot uv'wx'.$$

Evidently in $\mathrm{I}uvwx$ we may permute the four quaternions cyclically, or may reverse the order. A_4 is an alternating function. In A_2 the order may be reversed by changing the sign. $\mathrm{I}uvwx$ is a pfaffian, for it is the square root of a skew symmetric determinant of even order; indeed we have by actually squaring the expanded form,

$$(\mathrm{I}uvwx)^2 = \begin{vmatrix} 0 & \mathrm{I}uv & \mathrm{I}uw & \mathrm{I}ux \\ -\mathrm{I}vu & 0 & \mathrm{I}vw & \mathrm{I}vx \\ -\mathrm{I}wu & -\mathrm{I}wv & 0 & \mathrm{I}wx \\ -\mathrm{I}xu & -\mathrm{I}xv & -\mathrm{I}xw & 0 \end{vmatrix}.$$

8. If now we define a function $\mathrm{A}_5 \cdot uvwxy$ in the same manner as above it will be alternating and vanish, since any five quaternions are linearly connected. Indeed the vanishing of this expression gives us the important identity

$$u\mathrm{A}_4 vwxy - v\mathrm{A}_4 uwxy + w\mathrm{A}_4 uvxy - x\mathrm{A}_4 uvwy + y\mathrm{A}_4 uvwx = 0.$$

Operating by $\mathrm{I} \cdot z(\ )$, where z is any quaternion, we find the important identity

$$z\mathrm{A}_4 vwxy = -\mathrm{A}_3 wxy\mathrm{I}vz + \mathrm{A}_3 vxy\mathrm{I}wz - \mathrm{A}_3 vwy\mathrm{I}xz + \mathrm{A}_3 vwx\mathrm{I}yz.$$

We define further

$$\mathrm{A}_1 \cdot uvwxy = \tfrac{1}{2}(uv'wx'y + yx'wv'u),$$

$$\mathrm{A}_3 \cdot uvwxy = \tfrac{1}{2}(uv'wx'y - yx'wv'u).$$

Functions like these and the even numbered functions I, A_2, A_4 may be defined for any number of quaternions, and are useful in reductions.

As previously stated, all these forms are invariant or pseudo-invariant under the substitution of aqb for q, where $\mathrm{T}a = \mathrm{T}b = 1$. This is evident by mere substitution in the definitions. For example,

$$\mathrm{A}_2 uv = a'\mathrm{A}_2(aub)(avb) \cdot a, \qquad \mathrm{A}_1 uvw = a^{-1} \cdot \mathrm{A}_1(aub)(avb)(awb) \cdot b^{-1},$$

$$\mathrm{A}_3 \cdot uvw = a' \cdot \mathrm{A}_3(aub)(avb)(awb) \cdot b',$$

while $\mathrm{I} \cdot uv$, $\mathrm{I} \cdot uvwx$, $\mathrm{A}_4 \cdot uvwx$ are invariant.

The functions $\mathrm{A}_3\, uvw$ and $\mathrm{A}_4\, uvwx$ are invariant under any linear homogeneous substitution, in the sense that such a substitution merely multiplies the function by the determinant of the substitution.

PART II. DIFFERENTIAL OPERATORS.

9. Let us suppose that the coördinates of a point in space of four dimensions are w, x, y, z, corresponding to the real or scalar axis and the vector axes i, j, k. We then define the differentiating operator * D as follows:

$$\mathrm{D} = \frac{\partial}{\partial w} + i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}.$$

In physical applications it would be desirable to define this operator directly in physical terms, and in [illegible]etry in geometrical terms, but for the present formal developments this definition seems simpler. It is obvious that if the quaternion q becomes $q + dq$, where

$$dq = dw + idx + jdy + kdz,$$

then the operator

$$\mathrm{I} \cdot dq\mathrm{D} = dw\frac{\partial}{\partial w} + dx\frac{\partial}{\partial x} + dy\frac{\partial}{\partial y} + dz\frac{\partial}{\partial z}$$

gives the differential change in Q, any function of the position q, due to the differential change dq. If Q is a scalar function of position and its levels given by $Q = c$, then the change in Q which is greatest will be normal to these levels and will be given by DQ itself. Let

$$Q = W + iX + jY + kZ.$$

Then

$$\mathrm{I} \cdot \mathrm{D}Q = \frac{\partial W}{\partial w} + \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}.$$

We might call this the *divergence* of the function Q. Its vanishing gives an equation well known in hydrodynamics, the equation of continuity. In this case we take $Q = \rho\,(1 + \sigma)$ where σ is the velocity, ρ the density, w the time.

If we combine D with itself we have

$$\mathrm{I} \cdot \mathrm{DD} = \frac{\partial^2}{\partial w^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{dz^2}, \qquad \mathrm{I} \cdot \mathrm{DD}' = \frac{\partial^2}{\partial w^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}.$$

The first is the extension of the Laplacian to four dimensions and leads to what may be called four-dimensional potential functions. We have an example of the vanishing of these differential forms in the differential equation of wave motion, which if we set $w = at$ may be written

$$\mathrm{I} \cdot \mathrm{DD} \cdot u = 0, \qquad \text{or} \qquad \mathrm{I} \cdot \mathrm{DD}' \cdot u = 0,$$

according as the term containing the differential of the time is negative or positive.

* This is the operator *lor* of MINKOWSKI, Mathematische Annalen, vol. 68 (1909–10), pp. 472–525.

The first partial derivatives of a scalar function Q as to the coördinates, are the coördinates of the quaternions $1, i, j, k$ in $\mathrm{D}Q$. The second partial derivatives would be the coefficients of the dyads in $\mathrm{DID}Q$. In fact, the scalar invariants† of the linear quaternion differential operator $\mathrm{DI} \cdot \mathrm{D}\,(\,)$, namely M_1, M_2, M_3, M_4, lead not only to the Laplacian, but to other forms of operators, all of which are invariant under orthogonal transformations of the axes $1, i, j, k$.

We have

$$\mathrm{A}_2\,\mathrm{D}Q = i\left(\frac{\partial W}{\partial x} - \frac{\partial X}{\partial w} + \frac{\partial Y}{\partial z} - \frac{\partial Z}{\partial y}\right) + j\left(\frac{\partial W}{\partial y} - \frac{\partial Y}{\partial w} + \frac{\partial Z}{\partial x} - \frac{\partial X}{\partial z}\right)$$
$$+ k\left(\frac{\partial W}{\partial z} - \frac{\partial Z}{\partial w} + \frac{\partial X}{\partial y} - \frac{\partial Y}{\partial x}\right),$$
$$\mathrm{A}_2\,\mathrm{D}'Q' = i\left(-\frac{\partial W}{\partial x} + \frac{\partial X}{\partial w} + \frac{\partial Y}{\partial z} - \frac{\partial Z}{\partial y}\right) + j\left(-\frac{\partial W}{\partial y} + \frac{\partial Y}{\partial w} + \frac{\partial Z}{\partial x} - \frac{\partial X}{\partial z}\right)$$
$$+ k\left(-\frac{\partial W}{\partial z} + \frac{\partial Z}{\partial w} + \frac{\partial X}{\partial y} - \frac{\partial Y}{\partial x}\right).$$

We may say that if $\mathrm{A}_2\,\mathrm{D}Q = 0$, Q is *leftward irrotational,* if $\mathrm{A}_2\,\mathrm{D}'Q' = 0$, then Q is *rightward irrotational.* The expressions A_2 we may speak of as *left curl* and *right curl.* If both vanish we may speak of Q as *irrotational.* The rotations here are of course planar and not axial. As an example of the use of these forms, let $Q = \varphi - A$ be a quaternion potential function, φ being a scalar potential, and A a vector potential. Then

$$\mathrm{A}_2\,\mathrm{D}Q = \nabla\varphi + \partial A / \partial w + \mathrm{V}\nabla A = -x + y,$$
$$\mathrm{A}_2\,\mathrm{D}'Q' = -\nabla\varphi - \partial A / \partial w + \mathrm{V}\nabla A = x + y.$$

Thence

$$2x = \mathrm{A}_2\,\mathrm{D}'Q' - \mathrm{A}_2 \cdot \mathrm{D}Q, \qquad 2y = \mathrm{A}_2\,\mathrm{D}'Q' + \mathrm{A}_2\,\mathrm{D}Q.$$

From the identity $\mathrm{A}_3\,\mathrm{D}Q\mathrm{D} = 0$ we have, taking scalars, $\mathrm{S} \cdot \mathrm{VDV}Q\mathrm{VD} = 0$, or

$$\mathrm{S}\nabla y = 0.$$

Taking vectors we have

$$\mathrm{V}\,(\mathrm{A}_2\,\mathrm{D}Q \cdot \mathrm{D} + \mathrm{DA}_2\,\mathrm{D}'Q') = 0, \qquad \mathrm{VD}'\,(-x + y) + \mathrm{VD}\,(x + y) = 0,$$

that is,

$$\mathrm{V}\nabla x + \frac{\partial y}{\partial w} = 0.$$

Both equations are thus included in $\mathrm{A}_3\mathrm{D}Q\mathrm{D} = 0$, an identity.

Setting $2\mathrm{DA}_2 \cdot Q'\mathrm{D}' = \mathrm{A}_2\,\mathrm{D}Q \cdot \mathrm{D} - \mathrm{DA}_2\,\mathrm{D}'Q' = 2q$, we have, by taking scalars,

$$\mathrm{S}\nabla x = -Sq.$$

† These invariants are coefficients of the characteristic equation; cf. JOLY, *Manual of Quaternions*.

Taking vectors we have

$$\mathrm{V}\nabla y + \frac{\partial x}{\partial w} = -\mathrm{V}q.$$

These two equations arise thus from a single equation.

Now if we set in these formulæ $w = \sqrt{-1}\cdot ct$, and in the first place set $x = -\sqrt{-1}\cdot E, y = H, Sq = \sqrt{-1}\cdot\rho$, we have the common equations of the electro-magnetic field.* If we set in the next place $x = -\sqrt{-1}\cdot E$, $y = B$, or $x = H$, $y = -\sqrt{-1}\cdot D$, respectively, we arrive at the vectors called by MINKOWSKI † electric force of rest, and magnetic force of rest,

$$R = \tfrac{1}{2}(\mathrm{A}_2\,\mathrm{D}Q\cdot r - r\mathrm{A}_2\,\mathrm{D}'Q'), \qquad S = \mathrm{A}_3\,\mathrm{D}Qr,$$

provided r is the velocity given by

$$\sqrt{-1}\cdot r = (1 - \sqrt{-1}\cdot q)/\sqrt{(1-q^2)}.$$

If we start with such equations as the familiar ones

$$\mathrm{V}\nabla H - \frac{\partial E}{\partial t} = \rho\sigma, \qquad \mathrm{S}\nabla E = -\rho, \qquad \mathrm{V}\nabla E + \frac{\partial H}{\partial t} = 0, \qquad \mathrm{S}\nabla H = 0,$$

we may reduce them to the present forms. We have first

$$\nabla H - \frac{\partial E}{\partial t} = \rho\sigma, \qquad -E\nabla + \frac{\partial H}{\partial t} = \rho.$$

Adding these, we have the single equivalent equation $\mathrm{D}H - E\mathrm{D} = q$. If $H = \mathrm{V}\nabla A$, and $E = \nabla\varphi + \partial A/\partial t$ this reduces with little trouble to

$$\mathrm{D}'\,\mathrm{I}\mathrm{D}'\,Q - \mathrm{I}\mathrm{D}\mathrm{D}'\cdot Q' = 2q.$$

Formulæ for the extensions of Stokes' theorem and Green's theorem will be found in the reference to Joly below.‡

When we operate on $\mathrm{T}q^{-2}$ or $(qq')^{-1}$ the operator $\mathrm{I}a\mathrm{D}$ will give the hyperspherical harmonics, and the results are susceptible of interpretations similar to those for three dimensions. The introduction of doublets follows analogous lines. These applications can only be mentioned.§

* Cf. LEWIS, Proceedings of the American Academy of Arts and Sciences, vol. 46 (1910), pp. 165–181; M. ABRAHAM, Rendiconti del Circolo Matematico di Palermo, vol. 30 (1900), pp. 33–46. A. SOMMERFELD, Annalen der Physik und Chemie, ser. 4, vol. 32 (1910), pp. 749–776; vol. 33, pp. 649–689.

† MINKOWSKI, Göttinger Nachrichten, 1908, pp. 53, 111. The imaginary is not really necessary.

‡ Proceedings of the Royal Irish Academy, ser. 3, vol. 8 (1902), pp. 6–20.

§ Cf. JOLY, Appendix to HAMILTON's *Elements of Quaternions*, 2d ed., Appendix 13. TAIT, *Scientific Papers*, vol. 1, 37–42, 134–135, 136–150, 176–193, 234–236, 280–281; vol. 2, p. 312; Quarterly Journal of Mathematics, vol. 6 (1864), pp. 279–301. MCAULAY, Messenger of Mathematics, vol. 14 (1884), pp. 26–37; Proceedings of the Royal Society of Edinburgh, vol. 18 (1890), pp. 98–123.

10. We pass now to the development of certain differential formulæ for three-dimensional curvilinear space, or what is the same thing, quaternions depending upon three parameters. This is equivalent to considering the three-dimensional space in a flat four-dimensional space, so that the problem becomes that of the differential geometry of curved three-spreads in four-dimensional space.* Let us suppose, then, that

$$q = f(u_1, u_2, u_3).$$

This equation will limit q to a certain three spread. We shall write

$$q_1 = \frac{\partial f}{\partial u_1}, \qquad q_2 = \frac{\partial f}{\partial u_2}, \qquad q_3 = \frac{\partial f}{\partial u_3}, \qquad dq = q_1\, du_1 + q_2\, du_2 + q_3\, du_3.$$

The quaternions q_1, q_2, q_3 are independent, as the parameters are essential. Let the quaternion $\mathrm{A}_3\, q_1 q_2 q_3 = n$. Then n is the normal at q to the three-flat tangent to the three-spread, since $\mathrm{I}nq_1 = 0$, $\mathrm{I}nq_2 = 0$, $\mathrm{I}nq_3 = 0$. The differential dq lies in the tangent three-flat.

11. We shall indicate the quaternions

$$\frac{-\mathrm{A}_3\, nq_2 q_3}{\mathrm{I}nn}, \qquad \frac{-\mathrm{A}_3\, nq_3 q_1}{\mathrm{I}nn}, \qquad \frac{-\mathrm{A}_3\, nq_1 q_2}{\mathrm{I}nn}$$

respectively by $\bar{q}_1$, $\bar{q}_2$, and $\bar{q}_3$. These give at once

$$\mathrm{I}q_i\, \bar{q}_i = 1, \qquad \mathrm{I}q_i\, \bar{q}_j = 0 \quad (i \neq j), \qquad \mathrm{I}n\bar{q}_i = 0.$$

Any quaternion in the three-dimensional space may be expanded in the forms

$$r = q_1\, \mathrm{I}\bar{q}_1\, r + q_2\, \mathrm{I}\bar{q}_2\, r + q_3\, \mathrm{I}\bar{q}_3\, r = q_1\, \mathrm{I}q_1\, r + \bar{q}_2\, \mathrm{I}q_2\, r + \bar{q}_3\, \mathrm{I}q_3\, r.$$

We now define the operator

$$\Delta = \bar{q}_1 \frac{\partial}{\partial u_1} + \bar{q}_2 \frac{\partial}{\partial u_2} + \bar{q}_3 \frac{\partial}{\partial u_3}.$$

We have by substitution of values the operator

$$\mathrm{I}dq\Delta = du_1 \frac{\partial}{\partial u_1} + du_2 \frac{\partial}{\partial u_2} + du_3 \frac{\partial}{\partial u_3}.$$

That is to say, the operator $\mathrm{I}a\Delta$ gives the rate of variation of any operand due to a variation in the three-spread of q in the direction a, where a is a unit quaternion. It is evident again that if Q is a point function defined for the three-spread, a scalar let us say at first, then it will have certain level surfaces. The quaternion ΔQ gives us then the rate of maximum change, which will be the normal to a level surface. The operator Δ is thus the extension to curved

* Thus, if q is to terminate on a hypersphere, we might write

$$q = u_1 i + u_2 j + u_3 k + \sqrt{(a^2 - u_1^2 - u_2^2 - u_3^2)}.$$

space of the operator ∇ for flat three-dimensional space. We proceed to show that it is invariant under transformations of the parameters. To do this let

$$u_1 = \varphi_1(v_1, v_2, v_3), \qquad u_2 = \varphi_2(v_1, v_2, v_3), \qquad u_3 = \varphi_3(v_1, v_2, v_3),$$

where the functions φ are differentiable and without singularities at the points considered. Then

$$\mathrm{A}_2\, q_2\, q_3 = \Sigma \mathrm{A}_2\, q_{1v}\, q_{2v} \cdot \frac{\partial(v_1, v_2)}{\partial(u_2, u_3)}, \cdots \qquad \left(q_{1v} = \frac{\partial q}{\partial v_1}, \cdots\right).$$

Thence building up Δ_v we have $\Delta_v = \Delta_u$. Also we have

$$\mathrm{U}n_u = \mathrm{UA}_3\, q_1\, q_2\, q_3 = \mathrm{UA}_3\, (q_1)_v\, (q_2)_v\, (q_3)_v = \mathrm{U}n_v.$$

The normal is thus an invariant under transformation of the parameters. It follows naturally that such linear operators as $\mathrm{I}(\ )\Delta \cdot \mathrm{U}n$, $\mathrm{I}(\ )\Delta \cdot \Delta x$ and powers and combinations of these are also invariant. As an example, we see that the function $\mathrm{I}a\Delta \cdot Q$ gives the rate of variation of Q at the point in the direction a. The axes of this function are the directions of extremal variation and the roots are the rates of such change, at least if the function is self-transverse. If we lay off in each direction the value of the function (when Q is a scalar) the points so determined will have a hyper-quadric for their locus. The method of treatment differs little from the corresponding case for three dimensions.

Again consider the function $\mathrm{N} = \mathrm{I}(\ )\Delta \cdot \mathrm{U}n$. This is evidently the rate of change of the unit normal at the point. From the character of Δ it is self-transverse. Hence the axes are orthogonal and are the directions of the curvatures (reciprocals of the radii) at the point. The roots are real and are the curvatures themselves for real quaternions q_1, q_2, q_3. One root is zero, the corresponding axis being the normal. The usual scalar invariants of N are therefore the mean curvature, the second mean curvature, and the total or Kronecker-Gaussian curvature. The mean curvature (as in the similar three-dimensional case) is $\mathrm{I}\Delta\mathrm{U}n$. The second mean curvature and the total curvature are respectively

$$\mathrm{I} \cdot \mathrm{A}_3\, n\Delta'\, \Delta''\, \mathrm{A}_3\, n^{-1}\, \mathrm{U}n'\, \mathrm{U}n'', \qquad \mathrm{IA}_3\, \Delta'\, \Delta''\, \Delta'''\, \mathrm{A}_3\, \mathrm{U}n'\, \mathrm{U}n''\, \mathrm{U}n''',$$

where the accents are dropped after the expansion.

12. Since Δu, Δv, Δw, $\cdots$ are invariant, also $\mathrm{T}\Delta u$, $\mathrm{T}\Delta v$, $\cdots$ are invariant. Indeed it is evident that the differential parameters of the first order for the functions u, v, $\cdots$ are, in the usual notation for differential parameters,

$$\Delta_1 u = \mathrm{I}\Delta u \Delta u, \qquad \Delta_1(u, v) = \mathrm{I}\Delta u \Delta v.$$

In addition to these we have the following extensions of the Darboux $\Theta(u, v)$

for two dimensions to three dimensions:

$$A_2\,\Delta u\Delta v, \qquad A_1\,\Delta u\Delta v\Delta w, \qquad A_3\,\Delta u\Delta v\Delta w,$$

giving the respective parameters

$$IA_2\,\Delta u\Delta v A_2\,\Delta u\Delta v, \qquad I\cdot A_1\,\Delta u\Delta v\Delta w A_1\,\Delta u\Delta v\Delta w, \qquad IA_3\,\Delta u\Delta v\Delta w A_3\,\Delta u\Delta v\Delta w,$$

$$I\Delta u\Delta v\Delta w\Delta x, \qquad IA_2\,\Delta u\Delta v\Delta w\Delta x A_2\,\Delta u\Delta v\Delta w\Delta x.$$

The form $A_4\,\Delta u\Delta v\Delta w\Delta x$ vanishes identically.

The formulæ of part I give various syzygies connecting these, for example

$$I\cdot\Delta u\Delta v\Delta w\Delta x = \Delta_1(u,\ v)\,\Delta_1(w,\ x) - \Delta_1(u,\ w)\,\Delta_1(v,\ x) + \Delta_1(u,x)\,\Delta_1(v,w).$$

Since Δu is the normal in the three-spread for the levels of the scalar function u, the parameter of first order of u, $\Delta_1 u$, is the square of the length of this normal. The differential parameter of first order of u and v is the projection of the normal to the level of either, on the other, multiplied by the length of the latter. If it vanishes the levels are orthogonal. The interpretation of the other forms follows the exactly analogous forms for ∇ in a flat three-dimensional space.

13. Any parameter of the first order is itself subject to operation and we arrive thus at compound forms such as $I\Delta u\Delta\,(\Delta_1 u)$. We are led thus to consider operators such as $\Phi = I(\ )\Delta\cdot\Delta u$. The first scalar invariant of this linear quaternion operator is $I\Delta\Delta u$, which is the well-known differential parameter $\Delta_2\,u$. It vanishes if u is a solution of the generalized Laplacian equation. Indeed if the space becomes flat and the position is determined by ρ instead of q, the second differential parameter becomes $-\nabla^2$. If the lines u_1, u_2, u_3 are isothermal, we have a simplification which is not difficult to follow. The second invariant of Φ is

$$IA_3\,n\Delta'\,\Delta''\,A_3\,n^{-1}\,(\Delta u)'\,(\Delta u)''.$$

This can be expanded in the usual manner for the coefficients of the characteristic equation of a matrix or linear quaternion operator, in the form

$$\tfrac{1}{2}\,(I\Delta\Delta u)^2 - I\Delta'\,\Delta''\,uI\Delta'\,\Delta''\,u.$$

It is the well-known parameter $\Delta_{22}\,u$. Finally the third invariant is

$$-\tfrac{1}{6}\,IA_3\,\Delta'\,\Delta''\,\Delta'''\,A_3\,(\Delta u)'\,(\Delta u)''\,(\Delta u)'''.$$

This might be called the parameter $\Delta_{222}\,u$. The fourth invariant vanishes identically.

The function Φ is the rate of change in any direction of the vector rate of change of the function which is itself the normal to a level of the function.

14. If we consider only the direction of the normal of the level, we have the function $\Xi = \mathrm{I}\,(\;)\,\Delta \cdot \mathrm{U}\Delta u$, which gives the rate of deviation of the normal direction. There are now only two invariants that do not vanish identically, and these determine what may be called geodetic curvatures. The first invariant is $\mathrm{I}\Delta\mathrm{U}\Delta u$, the geodetic mean curvature. The second is the expression

$$\mathrm{IA}_3\, n\Delta'\,\Delta''\,\mathrm{A}_3\, n^{-1}\,\mathrm{U}\Delta u'\,\mathrm{U}\Delta u''.$$

These are the extensions of the usual geodetic curvature for two dimensions.

15. We notice next the fundamental forms or covariants of different orders. The first is

$$\mathrm{I}dqdq = \Sigma \mathrm{I}q_1\, q_1\, du_1\, du_1 + 2\Sigma \mathrm{I}q_1\, q_2\, du_1\, du_2.$$

The coefficients $\mathrm{I}q_i\, q_j$ correspond to the E, F, G of the geometry of surfaces. They are expressible as parameters, for example,

$$\mathrm{I}q_1\, q_2 = \mathrm{IA}_3\, n\Delta u_2\, \Delta u_3\, \mathrm{A}_3\, n\Delta u_3\, \Delta u_1.$$

Other covariant forms are

$$\mathrm{I}dqdr,\quad \mathrm{A}_2\, dqdr,\quad \mathrm{A}_1\, dqdrds,\quad \mathrm{A}_3\, dqdrds,\quad \mathrm{I}dqdrdsdt,\quad \mathrm{A}_2\, dqdrdsdt,$$

$$\mathrm{A}_2\, ndq,\quad \mathrm{A}_1\, ndqdr,\quad \mathrm{A}_3\, ndqdr,\quad \mathrm{A}_2\, ndqdrds,\quad \mathrm{A}_4\, ndqdrds \equiv 0.$$

Denoting differentiation as to u_i by the subscript i we have, from $\mathrm{IU}n\mathrm{U}n = 1$,

$$\mathrm{IU}n\,(\mathrm{U}n)_i = 0,$$

$$\mathrm{IU}n\,(\mathrm{U}n)_i = 0, \qquad \mathrm{IU}n\,(\mathrm{U}n)_{ij} = -\,\mathrm{I}\,(\mathrm{U}n)_i\,(\mathrm{U}n)_j \quad (i, j = 1, 2, 3).$$

Since $\mathrm{I}nq_i = 0$, therefore

$$\mathrm{I}nq_{ij} = -\,\mathrm{I}n_iq_j = -\,\mathrm{I}n_jq_i.$$

In this we might also have written $\mathrm{U}n$ instead of n. We arrive thus at the extensions of the D, D′, D″, of two dimensions. The Christoffel symbols are

$$\begin{bmatrix} kl \\ i \end{bmatrix} = \mathrm{I}q_iq_{kl}, \qquad \begin{Bmatrix} kl \\ i \end{Bmatrix} = \mathrm{I}q_{kl}\Delta u_i = \mathrm{I}q_{kl}\bar{q}_i.$$

$$(ikrs) = \mathrm{IA}_3\, n^{-1}\,\mathrm{U}n_i\mathrm{U}n_k\mathrm{A}_3nq_rq_s.$$

Differentiating these and remembering that also Δ is subject to differentiation, we arrive at the symbols of higher order. This is the covariantive differentiation of Maschke.*

The second fundamental form is

$$\mathrm{I}dq d\mathrm{U}n = \mathrm{I}dq\mathrm{N}dq = \Sigma \mathrm{I}q_1\,(\mathrm{U}n)_1\, du_1\, du_1 + 2\Sigma \mathrm{I}q_i\,(\mathrm{U}n)_j\, du_i\, du_j.$$

* MASCHKE, loc. cit. Also these Transactions, vol. 4 (1903), pp. 457.

This form vanishes for the asymptotic directions, or the principal tangent lines of the three-spread. It is worth noting that the differential equation of the lines of curvature is $A_2\, dq N dq = 0$. This corresponds to the similar form for surfaces $V d\rho \varphi d\rho = 0$.

16. The symbolic invariants of Maschke are easily expressed in these forms by noticing the following equivalences * for his symbols

$$f_i = I\zeta q_i, \qquad f_i f_k = I\zeta q_i I\zeta q_k = I q_i q_k, \qquad f_{kl} = I\zeta q_{kl}.$$

$$(f) = A_4\, Un\zeta_1\, \zeta_2\, \zeta_3,$$

whence

$$(f)^2 = 3\,!\, IUnUn = 3\,!, \qquad (uf) = A_4\; n\Delta u \zeta_1\, \zeta_2.$$

$$(fu) = A_4\, Un\Delta u'\; \Delta u''\; \zeta_1,$$

whence

$$(uf)^2 = 2I\Delta u \Delta u = 2\Delta_1\, u.$$

$$((uf), f) = A_4\, Un\Delta\, (A_4\, Un\Delta u \zeta_1\, \zeta_2)\, \zeta_1\, \zeta_2 = 2IUnUnI\Delta\Delta u = 2\Delta_2\, u.$$

Not to elaborate too far, we need only to remember that each ( ) is an A_4, that u, v, w, x, etc., are functions of q such as we have been using, that for every f or like symbol we substitute $I\zeta q$, for subscripts we differentiate as to the corresponding u. All the Maschke symbols for four dimensions become at once interpretable in this system. The vectors of INGOLD † are the quaternions appearing in these forms, his formulæ being analogues of Maschke's, not equivalents. The ordinary formulæ hold of course for these quaternions, but the Maschke forms are all scalars and commutative, and are the scalar forms with the ζ-pairs introduced.

* The bilinear function $Q(\xi, \zeta)$ is an abbreviation for $Q(q_1, \bar{q}_1) + Q(q_2, \bar{q}_2) + Q(q_3, \bar{q}_3)$. We may always write for a bilinear function $Q(\zeta, \zeta)$ the form $Q(\Delta, q)$ or equally $Q(q, \Delta)$. See MCAULAY, *Utility of Quaternions in Physics*; SHAW, *Synopsis of Linear Associative Algebra.*

† These Transactions, vol. 11 (1910), pp. 449–474.