FUNCTIONAL DIFFERENTIAL GEOMETRY[*]

BY

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It is well known that a certain analogy exists between ordinary vectors or points and functions of a variable x in an interval $\alpha \le x \le \beta$. Such an analogy, for instance, may be observed in certain formulas of integral equations[†] as well as in recent papers dealing with identities connecting integrals.[‡] It is the purpose of this paper to give some details of this correspondence in the case of differential geometry of curves and surfaces.

In the ordinary theory a surface is defined by a vector whose projections on the axes are

$$y_i = f(i; u_1, u_2) \qquad (i = 1, 2, 3),$$

depending upon a parameter i. We consider here instead n-dimensional spaces defined by a function

$$f(x; u_1, u_2, \cdots, u_n) \qquad (\alpha \le x \le \beta),$$

depending upon a continuous parameter x; i. e., n-dimensional spaces in a space of infinitely many dimensions.

For *curves* in the space of infinitely many dimensions a sequence of directions are obtained which are generalizations of the tangent, principal normal, and binormal of ordinary curves; also a sequence of curvatures which correspond to the usual first curvature and torsion. For spaces of higher dimensions the usual tangent properties of ordinary surfaces are generalized, and formulas analogous to the formulas of the Grassmann theory are obtained which express relations among the tangents to subspaces. In the concluding sections the

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[*] This paper combines two papers: *Curves in a Function Space*, and *Surfaces in a Function Space*, both presented to the Society November 26, 1910.

[†] This analogy was emphasized by Professor E. H. Moore in lectures on integral equations at the University of Chicago, 1905–07; see his *Introduction to a Form of General Analysis* in the *New Haven Mathematical Colloquium*, Yale University Press, New Haven, 1910. The author owes to Professor Moore the suggestion that the ideas there expressed might be extended to differential relations.

[‡] Richardson and Hurwitz, *Note on determinants whose terms are certain integrals*, Bulletin of the American Mathematical Society, vol. 16 (1909), pp. 14–19; also Curtiss, *Relations between the Gramian, the Wronskian and a third determinant connected with the problem of linear dependence*, Bulletin of the American Mathematical Society, vol. 17 (1911), pp. 462–467.

two fundamental forms of surface theory are generalized and formulas are derived which are the extensions of the well-known relations of GAUSS and CODAZZI.

It is assumed that the functions defining the spaces have partial derivatives of all orders with respect to the parameters u, and these derivatives as well as the functions themselves are assumed to be continuous functions of x.

PART I.   CURVES IN A SPACE OF INFINITELY MANY DIMENSIONS.

§ 1.   *The notion of curve and the parameter s.*

As explained in the introduction, a function $f(x; u)$ is considered to be analogous to the defining vector* of a curve in ordinary space.   We shall say that this function defines a curve in space of infinitely many dimensions and shall refer to the curve as the curve f.   The expression $\mu \partial f / \partial x$, where μ is independent of x, will be called a tangent to f.

If the parameter u is replaced by a function of some other letter s, say $u = u(s)$, $f(x; u)$ is transformed into a new function $\varphi(x; s)$.   It is possible to choose the new parameter so that

$$(1) \qquad \int_a^\beta \left(\frac{\partial \varphi(x; s)}{\partial s} \right)^2 dx = 1 . \dagger$$

For

$$\frac{\partial \varphi(x; s)}{\partial s} = \frac{\partial f(x; u)}{\partial u} \frac{du}{ds},$$

and hence

$$\int_a^\beta \left(\frac{\partial \varphi(x; s)}{\partial s} \right)^2 dx = \left(\frac{du}{ds} \right)^2 \int_a^\beta \left(\frac{\partial f(x; u)}{\partial u|} \right)^2 dx.$$

Equation (1) is satisfied if

$$ds = \sqrt{\int_a^\beta \left(\frac{\partial f(x; u)}{\partial u} \right)^2 dx}\, du,$$

from which s may be found in terms of u.

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§ 2. *The sequence of normals and curvatures.*

It will be assumed from this point on that the parameter s is so chosen that equation (1) is satisfied. The derivative

$$(2) \qquad \frac{\partial f(x;\,s)}{\partial s}*$$

which is tangent to f, will be denoted by $t\,(x:s)$.

The following formulas define, in terms of $f(x;\,s)$ and its derivatives with respect to s, a set of functions which we shall call the *normals*. For uniformity $t\,(x;s)$ is included in the set and is denoted by $n_0\,(x;s)$. These formulas are

$$n_0\,(x;\,s) = t\,(x;\,s),$$

$$(3) \qquad \frac{n_1\,(x;\,s)}{r_1\,(s)} = \frac{\partial t\,(x;\,s)}{\partial s},$$

$$\frac{n_i\,(x;\,s)}{r_i\,(s)} = \frac{\partial n_{i-1}\,(x;\,s)}{\partial s} + \frac{n_{i-2}\,(x;\,s)}{r_{i-1}\,(s)},\dagger \qquad\qquad (i \geqq 2),$$

where the expressions $1/r_i\,(s)$ are called the curvatures $k_i\,(s)$ and are defined by the equations

$$k_1^2\,(s) = \left(\frac{1}{r_1\,(s)}\right)^2 = \int_a^\beta \left(\frac{\partial n_0\,(x;\,s)}{\partial s}\right)^2 dx,$$

$$k_i^2\,(s) = \left(\frac{1}{r_i\,(s)}\right)^2 = \int_a^\beta \left(\frac{\partial n_{i-1}\,(x;\,s)}{\partial s} + \frac{n_{i-2}\,(x;\,s)}{r_{i-1}\,(s)}\right)^2 dx.$$

It follows at once that

$$(4) \qquad \int_a^\beta n_i^2\,(x;\,s)\,dx = 1.$$

If the integrand vanishes identically when $i = n$, the integral which defines $k_i\,(s)$ vanishes and the series of equations (3) terminates. Otherwise there is an infinity of normals.

From the definitions (3) it is possible to write the normals as linear expressions

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* This is analogous to the tangent vector of an ordinary curve determined by the set of derivatives $df_i\,(s)\,/\,ds$, where .

$$x_1 = f_1\,(s), \qquad x_2 = f_2\,(s), \qquad x_3 = f_3\,(s)$$

are the rectangular coördinates in terms of length of arc at a point on the curve.

† These are the well-known Frenet formulas. They do not appear to have been used previously for the purpose of defining the normals. They have been obtained for a curve in function space by KOWALEWSKI in the paper, *Les Formules de Frenet dans l'espace fonctionnel*, Comptes Rendus, vol. 151 (1910), p. 1338.

in $t(x; s)$ and its derivatives with respect to s. For from (3) by differentiation we find

$$\frac{\partial n_1(x; s)}{\partial s} = \frac{dr_1(s)}{ds}\frac{\partial t(x; s)}{\partial s} + r_1(s)\frac{\partial^2 t(x; s)}{\partial s^2}.$$

After substituting in (3) we have

$$n_2(x; s) = \frac{r_2(s)}{r_1(s)}t + r_2(s)\frac{dr_1(s)}{ds}\frac{\partial t(x; s)}{\partial s} + r_1(s)\,r_2(s)\cdot t(x; s),$$

and continuing in this way we obtain:

$$
(5)\qquad
\begin{aligned}
n_i = A(s)\cdot t + A_1(s)\frac{\partial t}{\partial s} + \cdots + A_{i-1}(s)\frac{\partial^{i-1}t}{\partial s^{i-1}} \\
+ r_1(s)\cdot r_2(s)\cdots r_i(s)\frac{\partial^i t}{\partial s^i},
\end{aligned}
$$

where the coefficients A, $A_1 \cdots A_{i-1}$ are functions of s depending on the r's and their derivatives.

It is easily seen that the first normal $n_1(x; s)$ is orthogonal to the tangent $t(x; s)$. For from the equation

$$\int_a^\beta n_0^2(x; s)\,dx = 1$$

it follows by differentiation that

$$\int_a^\beta n_0\frac{\partial n_0}{\partial s}\,dx = \frac{1}{r_1(s)}\int_a^\beta n_0\,n_1\,dx = 0.$$

Hence n_0 and n_1 are orthogonal.

It will now be shown that all the normals are mutually orthogonal. Assume that $n_0(x; s)$, $n_1(x; s)$, $\cdots$, $n_h(x; s)$ form an orthogonal system. The following considerations show that the system $n_0(x; s)$, $n_1(x; s)$, $\cdots$, $n_h(x; s)$, $n_{h+1}(x; s)$ is also orthogonal. We have immediately the equations:

$$
(6)\qquad
\int_a^\beta n_i\,n_j\,dx = \begin{cases} 0 \text{ if } i \neq j \\ 1 \text{ if } i = j \end{cases}\quad (i, j = 0, 1, 2, \cdots, h),
$$

and from (6) by differentiation:

$$
(7)\qquad
\int_a^\beta n_i\frac{\partial n_j}{\partial s}\,dx + \int_a^\beta n_j\frac{\partial n_i}{\partial s}\,dx = 0 \quad (i, j = 0, 1, 2, \cdots, h).
$$

In (7) substitute for $\partial n_i / \partial s$ its value from (3). This gives

$$\int_a^\beta \frac{n_j\, n_{i+1}}{r_{i+1}}\, dx - \int_a^\beta \frac{n_j\, n_{i-1}}{r_i}\, dx + \int_a^\beta n_i \frac{\partial\, n_j}{\partial s}\, dx = 0 \quad (i,j=0,1,2,\cdots,h).^*$$

Hence by (6)

(8) $$\int_a^\beta n_i \frac{\partial\, n_j}{\partial s}\, dx = 0 \quad (i = 0,1,2,\cdots,h-1; j = 0,1,2,\cdots,h; i + j \pm 1),$$

and by (5) and (6)

(9) $$\int_a^\beta n_{j+1} \frac{\partial\, n_j}{\partial s}\, dx = \frac{1}{r_{j+1}} \quad (j = 0,1,2,\cdots,h-1),$$

$$\int_a^\beta n_{j-1} \frac{\partial\, n_j}{\partial s}\, dx = -\frac{1}{r_j} \quad (j = 0,1,2,\cdots,h).$$

By using these results $n_{h+1}(x; s)$ is seen to be orthogonal to all the previous n's; for from (3) by multiplying by n_j and integrating

$$\int_a^\beta \frac{n_{h+1}\, n_j}{r_{h+1}}\, dx = \int_a^\beta \frac{\partial\, n_h}{\partial s} n_j\, dx + \int_a^\beta \frac{n_{h-1}\, n_j}{r_h}\, dx \quad (j = 0,1,2,\cdots,h).$$

In this equation if $j < h - 1$ the two integrals on the right vanish by (6) and (8). If $j = h - 1$ they cancel each other by (9) and (4), and if $j = h$ they vanish by (6) and (7). Thus the induction is complete.

§ 3.   *On the vanishing of the j-th curvature.*

In the ordinary theory of curves it is shown that in case the second curvature (torsion) vanishes identically the curve is a plane curve. A corresponding result holds for curves in a function space. In order to obtain this result, it is convenient to have another expression for the j-th curvature which we proceed to develop. Let $x, x_1, \cdots, x_j$, be $j + 1$ independent variables on the interval $\alpha \leq x_i \leq \beta$, and consider the following integral:

$$\int_a^\beta \frac{1}{(j+1)!} \begin{vmatrix} n_0(x; s) & n_0(x_1; s) & \cdots & n_0(x_j; s) \\ n_1(x; s) & n_1(x_1; s) & \cdots & n_1(x_j; s) \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ n_j(x; s) & n_j(x_1; s) & \cdots & n_j(x_j; s) \end{vmatrix}^2 dx\, dx_1 \cdots dx_j.$$

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* If $i = 0$ it is to be understood that the second term of this formula does not appear. This is the same as defining the curvature k_0 to be 0.

For convenience, we write this in the form

$$\int_\alpha^\beta [n_0, n_1, \cdots, n_j]^2 \, dx \, dx_1 \cdots dx_j,$$

where the bracket stands for the above determinant divided by the square root of $(j+1)!$ and is analogous to the Grassmann outer product of $j+1$ vectors.

To determine the value of this integral, consider that the determinant expanded consists of $(j+1)!$ terms which are squares, together with certain cross products. The general squared term contributes the value

$$\frac{1}{(j+1)!} \int_\alpha^\beta n_{m_0}^2(x; s) \, n_{m_1}^2(x_1; s) \cdots n_{m_j}^2(x_j; s) \, dx \, dx_1, \cdots, dx_j = \frac{1}{(j+1)!}$$

since by equation (4)

$$\int_\alpha^\beta n_{m_i}^2(x; s) \, dx = 1.$$

Hence all of these terms together yield unity.

The integral of each cross product is zero since each such term must contain as a factor at least one combination like $n_l(x_i; s) \cdot n_h(x_i; s)$, where h and l are different, and the integral of this product is zero because n_l and n_h are orthogonal. The entire integral, therefore, reduces to unity. The square root of the integral of the square of a function $F(x, x_1, \cdots, x_j)$ taken over the region $\alpha \leq x_i \leq \beta$ is called the *norm* of F with reference to the variables $x, x_1, \cdots, x_j$. We have, therefore,

(10) $$\text{Norm}\, [n_0, n_1, \cdots, n_j] = 1.$$

We may, therefore, write the j-th curvature k_j in the form

$$k_j = \frac{1}{r_j} = \frac{1}{r_j} \text{norm}\, [n_0, n_1, \cdots, n_j],$$

and substituting for n_j its value from equation (5), we may write the result as the norm of a linear combination of bracket terms

$$\left[n_0, n_1, \cdots, n_{j-1}, \frac{\partial^i t}{\partial s^i} \right] \qquad (i = 0, 1, 2, \cdots, j).$$

These terms all vanish except the one for which $i = j$. Hence

$$k_j = \frac{r_1 r_2 \cdots r_j}{r_j} \text{norm}\, \left[n_0, n_1, \cdots, n_{j-1}, \frac{\partial^j t}{\partial s^j} \right].$$

Substituting in like manner for $n_{j-1}, n_{j-2}, \cdots, n_0$, we have finally

(11) $$k_j = r_1^j r_2^{j-1} \cdots r_{j-1}^2 \, \text{norm}\, \left[t, \frac{\partial t}{\partial s}, \frac{\partial^2 t}{\partial s^2}, \cdots, \frac{d^j t}{\partial s^j} \right].$$

We now prove the following theorem.

The necessary and sufficient condition that there exist a finite series for $t(x; s)$ of the form

$$(12) \quad t(x; s) = a_1(s) e_1(x) + a_2(s) e_2(x) + \cdots + a_j(s) e_j(x),$$

is that the j-th curvature be the first of the curvatures to vanish identically.

For, if $k_j = 0$, we have

$$\int_a^\beta \left(\frac{\partial n_j}{\partial s} + \frac{n_{j-2}}{r_{j-1}} \right)^2 dx = 0$$

and hence

$$\frac{\partial n_{j-1}}{\partial s} = -\frac{1}{r_{j-1}} n_{j-2}.$$

If the values of n_{j-2} and $\partial n_{j-1} / \partial s$ obtained from (5) are substituted in this equation, it becomes an ordinary linear differential equation of order j for $t(x; s)$.

Let $a_1(s)$, $a_2(s)$, $\cdots a_j(s)$ be j independent solutions of the differential equation satisfied by $t(x; s)$. Then every solution which is a function of both x and s can be written in the form (12), and hence t can be written in this form. Conversely, if t has the form (12), substitution in (11) shows that $k_j \equiv 0$. The functions $e_1(x)$, $e_2(x)$, $\cdots$, $e_j(x)$ are linearly independent. For if not, suppose that all can be expressed linearly in terms of g of them $(g < j)$. Then, from (12) t can be expressed linearly in terms of g of the e's, and hence by (11)

$$k_g \equiv 0 \ (g < j),$$

but this is contrary to the assumption made above that k_j is the *first* of the curvatures to vanish identically.

PART II. SPACES OF n DIMENSIONS IN SPACE OF AN INFINITE NUMBER OF DIMENSIONS.

§ 4. *The notion of space vector.*

We go at once from one parameter u to any number n of independent parameters and consider a function $f(x; u_1, u_2, \cdots, u_n)$ of x and n parameters $u_1 \cdots u_n$ Such a function will be said to characterize a space of locus of n dimensions which will be called the space f. The different functions of x obtained from f by giving fixed values to the parameters u, may be regarded as the points of the space in question, or they may also be thought of as vectors from the origin to points of the space f.

It is assumed that no linear relation connects the partial derivatives $\partial f(x; u_1, u_2 \cdots u_n) / \partial u_i$. It follows that the function f, for arbitrary values

of $u_1, u_2, \cdots u_n$, cannot be written as a function of x and fewer parameters. In case only such values of the parameters u are considered as can be expressed in terms of k parameters $v_1, v_2, \cdots, v_k$ ($k < n$), the function f becomes a function of the form

$$f(x; u_1, u_2, \cdots, u_n) = \varphi(x; v_1, v_2, \cdots, v_k).$$

The function φ then characterizes a space of k dimensions. We shall say that the space φ lies in the space f. In the special case $k = 1$, φ is a *curve lying in* f.

The curves defined by

$$u_1, u_2, \cdots, u_{i-1}, u_{i+1}, \cdots, u_n = \text{const.} \quad (i = 1, 2, \cdots, n).$$

will be called the parametric curves corresponding to the parameters u_i.

The space

$$\varphi = u_1 e_1(x) + u_2 e_2(x) + \cdots + u_n e_n(x),$$

where the u's are independent parameters and the e's are linearly independent functions of x alone, will be called a *linear* space of n dimensions.

It is known from the theory of orthogonal functions that there exists a set of normed and mutually orthogonal functions

$$\epsilon_1(x), \epsilon_2(x), \cdots, \epsilon_n(x)$$

in terms of which the e's can be expressed linearly. We may therefore write

$$e_i(x) = \sum_{j=1}^{n} a_{ij} \, \epsilon_j(x).$$

If these values are substituted in the expression for the linear space above, it becomes

$$\varphi = \sum_{i=1}^{n} u_i \sum_{j=1}^{n} a_{ij} \, \epsilon_j(x) = \sum_{j=1}^{n} \left(\sum_{i=1}^{n'} a_{ij} u_i \right) \epsilon_j(x) = \sum_{j=1}^{n} v_j \, \epsilon_j(x),$$

where the coefficients

$$v_j = \sum_{i=1}^{n} a_{ij} u_i$$

are independent parameters. Thus the function φ characterizing a linear space can be expressed in terms of a normed orthogonal set of functions obtained from the e's.

From the above definitions it is seen that if

$$f(x; u_1, u_2, \cdots, u_n) = a_1 e_1 + a_2 e_2 + \cdots + a_r e_r \quad (n < r),$$

where the a's are functions of $u_1, u_2, \cdots, u_n$, and the e's are functions of x alone, the space f lies in a linear space of r dimensions.

§ 5.   The first fundamental form.

If the total differential of the function $f(x; u_1, u_2, \cdots, u_n)$ with respect to parameters u_i, be squared and integrated with respect to x from α to β there results a quadratic differential form in the variables u_i;

$$E = \int_\alpha^\beta (df)^2\, dx = \int_\alpha^\beta \left(\sum_{i=1}^n \frac{\partial f}{\partial u_i} du_i \right)^2 dx = \sum_{i,j=1}^n \left(\int_\alpha^\beta \frac{\partial f}{\partial u_i} \frac{\partial f}{\partial u_j} dx \right) du_i\, du_j$$

$$= \sum_{i,j=1}^n E_{ij}\, du_i\, du_j .$$

The form E is called the first fundamental quadratic differential form associated with the function f.   It plays the same rôle as the differential form giving length of arc in the ordinary theory.   The coefficients E_{ij} are called the first fundamental quantities.   The discriminant of E is denoted by

$$\frac{1}{\lambda^2} = |E_{ij}| \qquad\qquad (i, j = 1, 2, \cdots, n),$$

and λ always denotes the positive square root.

In the symbolic theory of the invariants of differential forms* E is represented symbolically as the square of a linear form.   It has been shown that the expression

$$\int_\alpha^\beta \left(\sum_{i=1}^n \frac{\partial f}{\partial u_i} du_i \right)^2 dx$$

may be used as such a symbolic representation of E† and that all of the identities of the symbolic differential invariant theory may be interpreted as identities involving ordinary functions and their integrals.

If $v_1, v_2, \cdots, v_n$ are any n functions of the parameters u_i we may use the notation

$$(v_1, v_2, \cdots, v_n) = \lambda \frac{\partial (v_1, v_2, \cdots, v_n)}{\partial (u_1, u_2, \cdots, u_n)} .$$

In case two sets of functions a, b are used, we may write

$$(a_1, a_2, \cdots, a_k, b_1, b_2, \cdots, b_{n-k}) = (a, b) = (a, b; k, n-k),$$

with similar notations for the Jacobian of n functions made up of three or more sets.

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* See MASCHKE, *A symbolic treatment of the theory of invariants of quadratic differential quantics in n variables.*   Transactions of the American Mathematical Society, vol. 4 (1903), p. 448.

† See the author's paper, *Note on identities connecting certain integrals*, Bulletin of the American Mathematical Society, vol. 17 (1911), pp. 184–189.

§ 6. *Tangents and normals.*

The function $\mu\partial\psi\,(x;s)\,/\,\partial s$, where μ is independent of x, has been called the tangent to the curve $\psi\,(x;s)$. The tangent to any curve *lying in* a space $f\,(x;u_1,\cdots,u_n)$ will be called a tangent to f. All of the tangents to f at a given point $\bar{u}$ (i. e., for given values of the parameters, say $\bar{u}_1,\bar{u}_2,\cdots,\bar{u}_n$) lie in the linear space

$$\varphi = v_1\overline{\frac{\partial f}{\partial u_1}} + v_2\overline{\frac{\partial f}{\partial u_2}} + \cdots + v_n\overline{\frac{\partial f}{\partial u_n}},$$

where $\overline{\partial f}/\partial u_i$ represents the value of $\partial f\,/\,\partial u_i$ at the point $\bar{u}$. For if the parameters are all functions of s so that

$$f\,(x;u_1,u_2,\cdots,u_n) = \psi\,(x;s),$$

then

$$\frac{\partial\psi}{\partial s} = \frac{\partial f}{\partial u_1}\frac{du_1}{ds} + \frac{df}{\partial u_2}\frac{du_2}{ds} + \cdots + \frac{\partial f}{\partial u_n}\frac{du_n}{ds}$$

which is of the form of φ above. Conversely, any linear function of $\partial f\,/\,\partial u_1$, $\partial f\,/\,\partial u_2,\cdots,\partial f\,/\,\partial u_n$ is a tangent to some curve lying in f and is therefore a tangent to f.

If the coefficients v_i in the expression for φ are independent parameters, φ is called the tangent space to the space f at the point $\bar{u}$.

In case f is the linear space

$$(13)\qquad f\,(x;u_1,u_2,\cdots,u_n) = u_1f_1 + u_2f_2 + \cdots + u_nf_n,$$

where $f_1,f_2,\cdots,f_n$ are linearly independent functions of x alone, the tangent space φ coincides with f.

Suppose that we wish to determine the tangents at a given point of a subspace of $n-1$ dimensions determined for the space $f\,(x;u_1,u_2,\cdots,u_n)$ by an equation of the form

$$v_1\,(u_1,u_2,\cdots,u_n) = \text{const.}$$

We may select $n-1$ other functions $v_2,\cdots,v_n$ so that the n functions $v_1,v_2,\cdots,v_n$ are independent and transform the coördinates in f from the u's to the v's. Then

$$f\,(x;u_1,u_2,\cdots,u_n) = \psi\,(x;v_1,v_2,\cdots,v_n),$$

$$\frac{\partial f}{\partial u_i} = \sum_{j=1}^{n}\frac{\partial\psi}{\partial v_j}\frac{\partial v_j}{\partial u_i}\qquad (i=1,2,\cdots,n),$$

and the solutions of these equations are

$$\frac{\partial\psi}{\partial v_j} = \frac{(v_1,v_2,\cdots,v_{j-1},f,v_{j+1},\cdots,v_n)}{(v_1,v_2,\cdots,v_n)}.$$

Any curve in the subspace of $n - 1$ dimensions can be determined by properly selecting $v_2, v_3, \cdots, v_n$ and letting all but one, say v_n, remain constant. The tangent to the curve so determined is $\partial \psi / \partial v_n$, or what is the same thing, the vector

$$\varphi = (w_1, v_2, v_3, \cdots, v_{n-1}, f),$$

where v_1 is replaced by w_1 to show that it is the locus $w_1 = $ const. which we are studying. The totality of tangents to this locus is found by using arbitrary functions in place of $v_2, \cdots, v_{n-1}$.

By a similar argument it is seen that the totality of tangents to a subspace of $n - k$ dimensions,

$$w_1 = \text{const.}, \quad w_2 = \text{const.}, \quad \cdots, \quad w_k = \text{const.},$$

is given by the formula

(14) $$(w_1, w_2, \cdots, w_k, v_{k+1}, \cdots, v_{n-1}, f),$$

where $v_{k+1}, \cdots, v_{n-1}$ are arbitrary functions.

Consider now the case in which f is a linear space as in equation (13), with the functions $f_1, \cdots, f_n$ normed and orthogonal.

The expression

(15) $$W = \sum_{i=1}^{n} \frac{\partial f}{\partial u_i} \frac{\partial w}{\partial u_i}$$

is called the normal to the space $w = $ const. in the space f, since it is orthogonal to every one of the tangents.

To prove this let φ be expanded in the form

$$\varphi = \sum_{i=1}^{n} V_i \frac{\partial f}{\partial u_i},$$

where V_i is the co-factor of $\partial f / du_i$ in the determinant representing φ. Then

$$\int_{\alpha}^{\beta} W \varphi \, dx = \sum_{i=1}^{n} V_i \frac{\partial w}{\partial u_i} = 0,$$

since the second member is the determinant for φ with f replaced by w.

§ 7. *Auxiliary notions.*

Let $\varphi_1(x), \varphi_2(x), \cdots, \varphi_k(x)$ represent any k functions of x and any other parameters. Then we may use the notation

$$[\varphi_1, \varphi_2, \cdots, \varphi_k] = \frac{1}{\sqrt{k!}} \begin{vmatrix} \varphi_1(x_1) & \varphi_1(x_2) & \cdots & \varphi_1(x_k) \\ \varphi_2(x_1) & \varphi_2(x_2) & \cdots & \varphi_2(x_k) \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \varphi_k(x_1) & \varphi_k(x_2) & \cdots & \varphi_k(x_k) \end{vmatrix},$$

where x_1, x_2, $\cdots$, x_k are independent variables in the region $\alpha \leq x_i \leq \beta$. This expression is analogous to a Grassmann outer product of k vectors, called by him a *simple element* of the k-th order, or of k dimensions. A linear combination of simple elements which cannot be reduced to a single simple element is called a *compound element*.

We shall assume for the present that the functions $f_1(x), f_2(x), \cdots, f_n(x)$ occurring in the function f in equation (13) are normed and mutually orthogonal. The factor λ defined in §5 is then unity. The determinant $(f(x_1), f(x_2), \cdots, f(x_n))$ defined at the end of §5 and all its minors are clearly simple elements except for constant factors.

The simple elements corresponding to minors of the k-th order will be called the *fundamental simple elements of the k-th order* with reference to the functions $f_1, f_2, \cdots, f_n$.

The following notations will be found convenient.

$$[f(x), f(y); k, n - k] = \left(f(x_1), \cdots, f(x_k), f(y_1), \cdots, f(y_{n-k}) \right),$$

$$[f(x), a; k, n - k] = \left(f(x_1), \cdots, f(x_k), a_1, \cdots, a_{n-k} \right),$$

$$[f(x); k] = [f(x), u_{c'}; k, n - k] = \lambda \frac{\partial \left(f(x_1), f(x_2), \cdots, f(x_k) \right)}{\partial (u_{c_1}, u_{c_2}, \cdots, u_{c_k})}$$

$$= \sqrt{k!} \left[\frac{\partial f(x)}{\partial u_{c_1}}, \frac{\partial f(x)}{\partial u_{c_2}}, \cdots, \frac{\partial f(x)}{\partial u_{c_k}} \right],$$

$$(a_c; k) = (a_c, u_{c'}; k, n - k) = \frac{\partial (a_1, a_2, \cdots, a_k)}{\partial (u_{c_1}, u_{c_2}, \cdots, u_{c_k})}.$$

In these formulas $c = (c_1, c_2, \cdots, c_k)$ represents any combination of k of the integers $1, 2, \cdots, n$, and $c' = (c'_1, c'_2, \cdots, c'_{n-k})$ is the complementary set arranged so that the permutation $(c_1, c_2, \cdots, c_k, c'_1, c'_2, \cdots, c'_{n-k})$ can be obtained from the permutation $(1, 2, \cdots, n)$ by an even number of transpositions.

The elements so defined satisfy the following relations:

$$(16) \qquad \int_a^\beta [f_c(x), u_{c'}; k, n - k]^2 \, dx_1 \, dx_2 \cdots dx_k = \frac{1}{k!},$$

$$(17) \quad \int_a^\beta [f_c(x), u_{c'}; k, n - k][f_d(x), u_{d'}; k, n - k] \, dx_1 \, dx_2 \cdots dx_k = 0,$$

where c and d represent different combinations.* The proof is the same as for formula (10).

_____

* These are special cases of the formula

$$(M) \quad \int_a^\beta [\varphi_1(x_1), \varphi_2(x_2), \cdots, \varphi_k(x_k)][\psi_1(x_1), \psi_2(x_2), \cdots, \psi_k(x_k)] dx_1 \cdots dx_k$$
$$= \left| \int_a^\beta \varphi_i(x) \psi_j(x) \, dz \right|$$

The normal in the space f to the locus $a_i(u_1, u_2, \cdots, u_n) = \text{const.}$ will be denoted by the corresponding german letter $\mathfrak{a}$. In the element

$$[\mathfrak{a}_1, \mathfrak{a}_2, \cdots, \mathfrak{a}_k]$$

let $\mathfrak{a}_i$ be replaced by its value from formula (15). Then

$$\mathfrak{A}^{(k)} = [\mathfrak{a}_1, \mathfrak{a}_2, \cdots, \mathfrak{a}_k] = \frac{1}{\sqrt{k!}} \sum_c (a_c; k)[f_c(x); k].$$

This is analogous to the expansion of a k-dimensional vector in terms of the k-dimensional units. But it follows by use of formulas (16) and (17) that

$$\frac{1}{(n-k)!\sqrt{k!}} \int_a^\beta [f(y), f(x); n-k, k][f(y), a; n-k, k] dy_1\, dy_2 \cdots dy_{n-k}$$
$$= \frac{1}{\sqrt{k!}} \sum_c (a_c; k)[f_c(x); k].$$

Hence

(I) $\quad \mathfrak{A}^{(k)} = \dfrac{1}{(n-k)!\sqrt{k!}} \displaystyle\int_a^\beta [f(y), f(x); n-k, k]$
$$\times [f(y), a; n-k, k] dy_1, \cdots, dy_{n-k}.^*$$

In particular

(II) $\quad \mathfrak{a} = \dfrac{1}{(n-1)!} \displaystyle\int_a^\beta [f(y), f(x); n-1, 1][f(y), a; n-1, 1] dy_1 \cdots dy_{n-1}.$

The expression

(III) $\quad |\mathfrak{A}^{(k)} = \dfrac{1}{\sqrt{k!(n-k)!}} \displaystyle\int_a^\beta [f(y), f(x); k, n-k]\vdash$
$$\times [a(y_1), \cdots, a(y_k)] dy_1 \cdots dy_k$$

is called the complement of $\mathfrak{A}^{(k)}$. If we expand the element

$$[f(y), f(x); k, n-k]$$

in terms of the elements $[f_c(x); k]$ and write $\mathfrak{A}^{(k)}$ also in terms of them, we

where the determinant on the right is of the kth order, i and j ranging from 1 to k, and where $\varphi_1, \varphi_2, \cdots, \varphi_k$ and $\psi_1 \psi_2, \cdots, \psi_k$ are any functions of x which are such that the products $\varphi_i \psi_j$ are integrable. This result was first published by RICHARDSON and HURWITZ in the paper already referred to, and later by LANDSBERG in the paper, *Theorie der Elementarteiler linearer Integralgleichungen*, M a t h e m a t i s c h e A n n a l e n, vol. 69 (1910), pp. 227–265. The formula is a special case of a formula proved by Professor MOORE in a course on Determinants at the University of Chicago, summer, 1907.

* The formulas of this and the following section are closely analogous to formulas given by Grassmann. See the *Ausdehnungslehre* (1862), in Grassmann's collected works edited by F. ENGEL.

have by means of (16) and (17)

(IV) $|\mathfrak{A}^{(k)} = \dfrac{[a, f(x); \ k, \ n-k]}{\sqrt{(n-k)!}} = \dfrac{(-1)^{k(n-k)}}{\sqrt{(n-k)!}} [f(x), a; \ n-k, k].$

Taking complements again, we find

(V) $||\mathfrak{A}^{(k)} = \dfrac{(-1)^{k(n-k)}}{(n-k)! \sqrt{k!}} \displaystyle\int_a^\beta [f(y), f(x); \ n-k, k]$

$\times [f(y), a; \ n-k, k] dy_1 \cdots dy_{n-k} = (-1)^{k(n-k)} \mathfrak{A}^{(k)}.$*

§ 8. *Properties of the elements.*

The three following formulas, (VI), (VII), (VIII), are frequently useful in making reductions:

(VI) $\displaystyle\int_a^\beta \mathfrak{A}^{(k)} \mathfrak{B}^{(k)} dx_1, \cdots, dx_k = \int_a^\beta |\mathfrak{A}^{(k)}| \mathfrak{B}^{(k)} dy_1, \cdots, dy_{n-k}$

$= \dfrac{1}{(n-k)!} \displaystyle\int_a^\beta [f(x), a; \ n-k, k][f(x), b; \ n-k, k] dx_1 \cdots dx_{n-k},$

(VII) $\dfrac{1}{(n-k)!} \displaystyle\int_a^\beta [f(x), a; \ n-k, k][f(x), b; \ n-k, k] dx_1 \cdots dx_{n-k}$

$= \left| \displaystyle\int_a^\beta [f(x), a_i; \ n-1, 1][f(x), b_j; \ n-1, 1] dx_1 \cdots dx_{n-1} \right|,$†

$(i, j = 1, 2, \cdots, k).$

(VIII) $\displaystyle\int_a^\beta [f(y), f(x); \ n-k, k][f(y), a; \ n-k, k][f(x), b; \ k, n-k]$

$\times dy_1 \cdots dy_{n-k} dx_1 \cdots dx_k = k! (n-k)! (a_1, \cdots, a_k, b_1, \cdots, b_{n-k}).$

Formula (VI) is proved by showing each expression to be equal to

$$\sum_c (a_c; k)(b_c; k).$$

Formula (VII) can be proved by the use of (VI) and formula (M) of the foot-note p. (330). For by (VI) the left hand side is the left hand side of (M), and also by (VI) the general term in the determinant on the right is equal to

$$(n-1)! \int_a^\beta a_i b_j dx.$$

Hence the determinant reduces to the right hand side of (M) and the formula is proved. By expanding both sides of formula (VIII) in terms of $(a_c; k)$,

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* See GRASSMAN, l. c., No. 92.

† Ibid., No. 175.

$(b_c; k)$, they are reduced by formulas (16) and (17) to the same expression

$$k!(n-k)!\sum_c (a_c; k)(b_c; k).$$

The preceding formulas enable us to prove that the elements $[f_c(x); k]$ are linearly independent. For suppose there is a linear relation connecting them, say

$$\sum_c p_c[f(x), u_{c'}; k, n-k] = 0,$$

then

$$\sum_c p_c \int_a^\beta [f(x), f(y); k, n-k][f(x), u_{c'}; k, n-k][f(y), u_h; n-k, k]$$

$$\times dx_1 \cdots dx_k dy_1 \cdots dy_{n-k} = \sum_c p_c k!(n-k)!(u_{c_1'}, \cdots, u_{c_{n-k}'}, u_{h_1}, \cdots, u_{h_k}) = 0,$$

by (VIII). But every term of this vanishes except the term for which the combinations c and h are the same, and this reduces to $k!(n-k)!p_h$. Hence $p_h = 0$ for every h.

In the proofs of the remaining formulas of this section, the following theorem on determinants is needed. If $|\alpha_1, \cdots, \alpha_n|$ and $|\beta_1, \cdots, \beta_n|$ denote two determinants of the nth order, then

$$(D) \quad |\alpha_1, \cdots, \alpha_n||\beta_1, \cdots, \beta_n| = \sum_{i=1}^n |\beta_i, \alpha_2, \cdots, \alpha_n||\beta_1, \cdots, \beta_{i-1}, \alpha_1, \beta_{i+1}, \cdots, \beta_n|.*$$

The theorem can be applied to each term of the sum on the right by putting α_2 successively in place of β_1, β_2, etc.; and clearly this process can be repeated until $\alpha_1, \cdots, \alpha_k$ replace in all possible ways a like number of columns of the second determinant.

In this form the theorem will be used to prove the following formula:

$$\int_a^\beta [f(x), f(y); n-k, k][f(z), a; l, n-l]$$

$$\times [f(y), f(z), b; k, l, r] dy_1 dy_2 \cdots dy_k dz_1 dz_2 \cdots dz_l$$

$$(IX)$$

$$= \frac{k!l!}{(k+l)!}\sum_c [f(x), a_c; n-k, k] \int_a^\beta [f(y), a_{c'}; n-r, r]$$

$$\times [f(y), b; n-r, r] dy_1 dy_2 \cdots dy_{n-r},$$

where $k + l = n - r$ and a_c denotes any combination of k of the a's and a_c' denotes the combination of the remaining r of the a's.† To prove formula (IX)

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* See MASCHKE, *Differential parameters of the first order*, these Transactions, vol. 7 (1906), p. 70, equation (1).

† If $r = 0$ there is only one combination a which consists of all of the a's. The right hand side reduces to a single term and the integral factor of this term reduces to $n! = (k+l)!$; whence it is seen that (IX) reduces to (VIII) in this case.

apply the theorem (D) to the first two factors under the integral sign on the left and exchange in all possible ways $f(y_1), \cdots, f(y_k)$ of the first factor for a like number of elements of the second. Adding the results and denoting the left side of (IX) by P, we have

$$P = \sum_h \int_a^\beta [f(x), f(z_t); n-k, k][f(y), f(z_{t'}), a; k, l-k, n-l]$$
$$\times [f(y), f(z), b; k, l, r] dy_1 \cdots dy_k dz_1 \cdots dz_l$$
$$+ \sum_{h, i} \int_a^\beta [f(x), f(z_h), a_i; n-k, k-1, 1]$$
$$\times [f(y), f(z_{h'}), a_{i'}; k, l-k+1, n-l-1]$$
$$\times [f(y), f(z), b; k, l, r] dy_1 \cdots dy_k dz_1 \cdots dz_l$$
$$+ \sum_{m, j} \int_a^\beta [f(x), f(z_m), a_j; n-k, k-2, 2]$$
$$\times [f(y), f(z_{m'}), a_{j'}; k, l-k+2, n-l-2]$$
$$\times [f(y), f(z), b; k, l, r] dy_1 \cdots dy_k dz_1 \cdots dz_l$$
$$+ \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$+ \sum_s \int_a^\beta [f(x), a_s; n-k, k][f(y), f(x), a_{s'}; k, l, r]$$
$$\times [f(y), f(z), b; k, l, r] dy_1 \cdots dy_k dz_1 \cdots dz_l,$$

where $f(z_t)$ is any combination of k of the quantities $f(z_1) \cdots f(z_l)$, and $f(z_{t'})$ is the combination of the remaining ones. This series of sums is for the case in which k is at most equal to l. If $k = l - g$ the first g sums will not appear. Each term of the first sum is equal to P, except perhaps for sign. By applying theorem (D) to the first two factors of each term of any sum except the first, interchanging in all possible ways the $f(x)$ of the first factor for a like number of terms of the second, each sum is reduced to a series of sums of succeeding types, and by successive reductions they are all reduced to a sum of terms of the type appearing in the last sum, each term being multiplied by a numerical factor. Thus we have

$$P = \sum_c K_c [f(x), a_c: n-k, k] \int_a^\beta [f(y), a_{c'}; n-r, r]$$
$$\times [f(y), b; n-r, r] dy_1 \cdots dy_{n-r},$$

where the coefficients K_c depend only on the integers k, l, n, r, and not at all on the functions a, b.

Now let

$$\{ b_1, b_2, \cdots, b_r \} = \{ u_{i_1}, u_{i_2}, \cdots, u_{i_r} \} = \{ a_{h_1'}, a_{h_2'}, \cdots, a_{h_r'} \},$$
$$\{ a_{h_1}, \cdots, a_{h_k} \} = \{ u_{\theta_1}, \cdots, u_{\theta_k} \}.$$

Then on the right all terms vanish except the one for which the combinations c and h are the same, and the term which remains has the value

$$K_h (l + k)! [f_{\bar{h}} (x); n - k] / \sqrt{(n - k)!}.$$

But the expression for P becomes $k! l! [f_s (x); n - k] / \sqrt{(n - k)!}$. Since all of the functions $[f_c (x); n - k]$ are independent, $[f_s (x); n - k]$ and $[f_{\bar{h}} (x); n - k]$ are the same and we have finally

$$K_h = \frac{k! l!}{(k + l)!} *$$

for every h, and formula (IX) is proved.

From the formulas given, many others can be obtained by multiplying both sides by a factor involving the function f and integrating. For example, taking complements of both sides of (IX), we have

$$\int_a^\beta [f (x), f (y), b; k, l, r] [f (y), a; l, n - l] dy_1 \cdots dy_l$$

(X)
$$= \frac{l!}{(k + l)! (n - k)!} \sum_c \int_a^\beta [f (y), a_c, n - r, r] [f (y), b; n - r, r]$$

$$\times dy_1 \cdots dy_{n-r} \int_a^\beta [f (z), f (x); n - k, k] [f (z), a_{c'}; n - k, k] dz_1 \cdots dz_{n-k}.\dagger$$

The proofs of formulas in this and the previous sections are for the case in which f is a linear space whose coefficients f_i are normed and orthogonal. The formulas hold, however, when f is general. Formula (VIII)$\ddagger$ for the special cases $k = 1$ and $k = 2$ and formula (VII)§ have been proved by Maschke in symbolic notations. His argument applies equally well to functions. The proofs will therefore not be repeated here.

Formula (VIII) can be shown by induction to hold for any value of k. Denoting the left member by M_k and using the determinant theorem (D),

---

* This represents the number of terms of the type of those on the right side of (IX) which for a given h are found in the form to which P reduces by the above process. The same expression could be obtained for P for a *general space f* and the same reductions would lead to the same number of terms of the above type. Hence K has the same value for the general case.

† See GRASSMANN, l. c., No. 173.

‡ MASCHKE, *A Symbolic Treatment* etc., loc. cit., formulas (34) and (39).

§ MASCHKE, *Differential parameters of the first order*, these T r a n s a c t i o n s, vol. 7 (1906), p. 74, equation (8).

we find

$$M_k = \int_a^\beta [f(y), f(x), n-k, k]$$

$$\times \left\{ \sum_{i=1}^{n-k} \left(f(y_1), \cdots, f(y_{i-1}), b_1, f(y_{i+1}), \cdots, f(y_{n-k}) \right) \right.$$

$$\times \left(f(x_1), \cdots, f(x_k), f(y_i), b_2, \cdots, b_{n-k} \right)$$

$$+ \sum_{i=1}^{k} \left(f(y_1), \cdots, f(y_{n-k}), a_1, \cdots, a_{i-1}, b_1, a_{i+1}, \cdots, a_k \right)$$

$$\left. \times \left(f(x_1), \cdots, f(x_k), a_i, b_2, \cdots, b_{n-k} \right) \right\} dy_1 \cdots dy_{n-k} dx_1 \cdots dx_k.$$

By interchanging equivalent variables $x_1 \cdots x_k \, y_1 \cdots y_{n-k}$ each term of the first summation is seen to be equal to $- M_k$. Hence

$$M_k = (n - k) M_{k+1} - k M_k,$$

$$M_{k+1} = \frac{k+1}{n-k} M_k = (k+1)! (n-k-1)! (a_1, \cdots, a_k, b_1, \cdots, b_{n-k}).$$

Consequently the formula holds in general, since it holds for $k = 1$ and $k = 2$. It is clear that the same proof is valid in case any of the quantities a_i, b_i are functions of x or an equivalent variable distinct from the variables of integration.

Formula (VIII) may now be used to show the linear independence of the expressions $[f(x), u_c; k, n-k]$ for the general, non-linear, case precisely as at the beginning of this section for a linear space. The proofs given for formulas (IX) and (X) hold for the general case.*

There remain formulas (I), (II), $\cdots$, (VI). Formulas (II) and (III) may be taken as definitions of a and $[a_1, a_2, \cdots, a_k]$ respectively for the more general space. Let the expressions for $a_1, \cdots, a_k$ from (II) be substituted in the determinant $\mathfrak{A}$ in (I). The result is

$$\frac{1}{\sqrt{k!}} \frac{1}{\{(n-1)!\}^k} \left| \int_a^\beta [f(y), f(x_i); n-1, 1] \right.$$

$$\left. \times [f(y), a_j; n-1, 1] dy_1 \cdots dy_{n-1} \right|,$$

where the determinant is of the kth order, i and j ranging from 1 to k. This expression by (VII) can be written

$$\frac{1}{\sqrt{k!}} \frac{1}{(n-k)!} \int_a^\beta [f(y), f(x); n-k, k][f(y), a; n-k, k] dy_1 \cdots dy_{n-k},$$

_____

* See the footnote to the proof of formula (IX).

the right-hand side of formula (I), which is therefore proved. Using this formula and formula (VIII) with $f(x_1), \cdots, f(x_{n-k})$ in place of $b_1, \cdots, b_{n-k}$, we have

$$
|\mathfrak{A}^{(k)} = \frac{1}{\sqrt{k!(n-k)!}} \int_\alpha^\beta [f(y), f(x); k, n-k] \frac{1}{(n-k)!\sqrt{k!}}
$$

$$
\times \int_\alpha^\beta [f(z), f(y); n-k, k][f(z), a; n-k, k] dz_1 \cdots dz_{n-k} dy_1 \cdots dy_k
$$

$$
= \frac{1}{k!(n-k)!\sqrt{(n-k)!}} \int_\alpha^\beta [f(z), f(y); n-k, k]
$$

$$
\times [f(z), a; n-k, k][f(y), f(x); k, n-k] dz_1 \cdots dz_{n-k} dy_1 \cdots dy_k
$$

$$
= \frac{1}{\sqrt{(n-k)!}} (a_1, a_2, \cdots, a_k, f(x_1), f(x_2), \cdots, f(x_{n-k})),
$$

and this proves formula (IV). In a similar manner formulas (V) and (VI) can be proved.

§ 9. *Functions orthogonal to all tangents of a subspace.*

Given a subspace R_k lying in f, it is possible to find functions tangent to f but orthogonal to all tangents to R_k. In fact, if f is of n dimensions there should be precisely $n - k$ *independent* functions satisfying this condition. These functions are given in the following theorem:

Every function

$$
\psi(x) = \int_\alpha^\beta [f(y), f(x), U; k, 1, n-k-1][f(y), a; k, n-k] dy_1 \cdots dy_k,
$$

where the U's are arbitrary functions of $u_1, u_2, \cdots, u_n$, *is orthogonal to all tangents of the subspace* R_k *in f defined by the equations*

$$
a_1 = \text{const.}, \qquad a_2 = \text{const.}, \qquad \cdots, \qquad a_{n-k} = \text{const.}
$$

By formula (14), § 6, all tangents to R_k may be written in the form

$$
\varphi(x) = [f(x), V, a; 1, k-1, n-k],
$$

where the V's are arbitrary functions of $u_1, u_2, \cdots, u_n$. It is to be shown, then, that

$$
\int_\alpha^\beta \varphi(x) \psi(x) dx = \int_\alpha^\beta [f(y), f(x), U; k, 1, n-k-1][f(y), u; k, n-k]
$$

$$
\times [f(x), V, u; 1, k-1, n-k] dx \, dy_1 \cdots dy_k = 0.
$$

By formula (X)

$$\psi(x) = \sum_{i=1}^{n} \int_{a}^{\beta} [f(y), u_i; k+1, n-k-1][f(y), U; k+1, n-k-1]$$

$$\times \int_{a}^{\beta} [f(z), f(x); n-1, 1][f(z), a_i; n-1, 1] dy_1 \cdots dy_{k+1} dz_1 \cdots dz_{n-1}.$$

Hence, by (VIII)

$$\int_{a}^{\beta} \varphi(x) \psi(x) dx = \sum_{i=1}^{n} M_i \int_{a}^{\beta} [f(z), f_i(x); n-1, 1][f(z), a_i; n-1, 1]$$

$$\times [f(x), V, a; 1, k-1, n-k] dx \, dz_1 \cdots dz_{n-1}$$

$$= (n-1)! \sum_{i=1}^{n} M_i(a_i, V, a; 1, k-1, n-k).$$

But each term of this sum contains a determinant having two columns identical. Hence the sum vanishes, and this proves the theorem.

§ 10.   *Normal properties; Christoffel symbols.*

If all of the parameters u_i are functions of the same parameter s, the function f determines a curve $f(x; s)$.   The tangent to this curve is given by formula (2) of § 2.   Let s be so chosen that the condition (1) holds.   Then the first normal n to the curve is defined by (3) to be

$$\frac{1}{r} n = \frac{\partial^2 f(x; s)}{\partial s^2} = \sum_{i,j=1}^{n} \frac{\partial^2 f(x; s)}{\partial u_i \, \partial u_j} \frac{\partial u_i}{\partial s} \frac{du_j}{ds} + \sum_{i=1}^{n} \frac{\partial f}{\partial u_i} \frac{d^2 u_i}{ds^2},$$

where $1/r$ is the first curvature and is so chosen that

$$\int_{a}^{\beta} n^2 dx = 1.$$

Thus *the first normals to all curves lying in f are linearly expressible in terms of the tangents $\partial f / \partial u_i$ and the second partial derivatives $\partial^2 f / \partial u_i \, \partial u_j$.*

A function which is linearly expressible in terms of the derivatives $\partial f / \partial u_i$, and $\partial^2 f / \partial u_i \, \partial u_j$, and which is orthogonal to all of the tangents, $\partial f / \partial u_i$, we shall call *a first normal to the space f.*

If the second derivatives $\partial^2 f / \partial u_i \, \partial u_j$ are linearly independent there are $n(n+1)/2$ linearly independent first normals to f in terms of which all others can be expressed.   We write

$$(18) \qquad N_{ij}(u_1, u_2, \cdots, u_n) = \frac{\partial^2 f}{\partial u_i \, \partial u_j} - \sum_{k=1}^{n} \begin{Bmatrix} i \ j \\ k \end{Bmatrix} \frac{\partial f}{\partial u_k},$$

where the coefficients $\begin{Bmatrix} i \ j \\ k \end{Bmatrix}$ are to be determined as functions of the para-

meters u_i from the condition that the functions N_{ij} are orthogonal to the functions $\partial f / \partial u_i$. To calculate the $\{ {}^i_k {}^j \}$, multiply both sides of equation (18) by $\partial f / \partial u_l$ and integrate, remembering equation (6). This gives

$$\int_\alpha^\beta \frac{\partial^2 f}{\partial u_i \, \partial u_j} \frac{\partial f}{\partial u_l} \, dx = \sum_{k=1}^n \left\{ \begin{matrix} i \ j \\ k \end{matrix} \right\} E_{kl}.$$

Now the expressions on the left may be shown to be the Christoffel triple index symbols* of the first kind belonging to the quadratic differential form (E), and can be expressed in terms of the functions E_{ij} and their derivatives. Denoting them by $[{}^i_l {}^j]$ we have the system of equations

$$\sum_{k=1}^n E_{kl} \left\{ \begin{matrix} i \ j \\ k \end{matrix} \right\} = \left[\begin{matrix} i \ j \\ l \end{matrix} \right] \qquad\qquad (l = 1, 2, \cdots n),$$

from which we obtain

$$\left\{ \begin{matrix} i \ j \\ k \end{matrix} \right\} = \lambda^2 \sum_{l=1}^n \mathfrak{E}_{kl} \left[\begin{matrix} i \ j \\ l \end{matrix} \right],$$

where $\mathfrak{E}_{kl}$ denotes the co-factor of E_{kl} in the determinant $| E_{ij} |$. The expressions $\{ {}^i_k {}^j \}$ are the Christoffel triple index symbols of the second kind belonging to the differential form (E).†

§ 11.   *The second fundamental form.*

For the study of properties depending upon the second derivatives of the function f certain other fundamental quantities are important. These may be taken to be the quantities of either of the sets

$$\int_\alpha^\beta \frac{\partial^2 f}{\partial u_i \, \partial u_j} \frac{\partial^2 f}{\partial u_r \, \partial u_s} \, dx, \qquad \int_\alpha^\beta N_{ij} N_{rs} \, dx,$$

or perhaps still others which can be expressed in terms of either of these sets and the first fundamental quantities E. Under certain restrictions the number of these quantities is reduced. Thus in the ordinary theory of spaces of n dimensions lying in a Euclidean space of $n + 1$ dimensions, the number is reduced to $n (n + 1) / 2$, the fundamental functions being the coefficients of the *second* quadratic differential form. Here, however, for the corresponding developments it is not assumed that f lies in a Euclidean space of $n + 1$ dimensions, but only that all *first* normals to f have the same direction; i. e., that *any two of the normals N_{ij} differ from each other at most by a factor which is independent of* x,‡ so that we may write

$$N_{ij} = L_{ij} N,$$

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* See MASCHKE, *A Symbolic Treatment*, etc., loc. cit., p. 455.

† See MASCHKE, *A Symbolic Treatment*, etc., loc. cit., p. 456, equation (67).

‡ The further assumption, that the third and higher derivatives are linearly expressible in terms of the first and second derivatives of f leads to the ordinary case mentioned above.

where L_{ij} is independent of x and where

$$\int_{\alpha}^{\beta} N^2 dx = 1.$$

Equation (18) may now be written

(19) $$L_{ij} N = \frac{\partial^2 f}{\partial u_i \partial u_j} - \sum_{k=1}^{n} \left\{ \begin{matrix} i & j \\ k \end{matrix} \right\} \frac{\partial f}{\partial u_k}.$$

Multiplying both sides of this equation by N and integrating, we have, since N is orthogonal to $\partial f / \partial u_k$,

$$L_{ij} = \int_{\alpha}^{\beta} N \frac{\partial^2 f}{\partial u_i \partial u_j} dx.$$

The L_{ij} are the coefficients of a quadratic differential form obtained in the following manner. We have

$$d^2 f = \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial u_i \partial u_j} du_i du_j + \sum_{i=1}^{n} \frac{\partial f}{\partial u_i} d^2 u_i.$$

If we multiply by N and integrate, we find

$$\int_{\alpha}^{\beta} (d^2 f) N dx = \sum_{i,j=1}^{n} \left\{ N \frac{\partial^2 f}{\partial u_1 \partial u_j} dx \right\} du_i du_j = \sum_{i,j=1}^{n} L_{ij} du_i du_j.$$

This is called the second fundamental differential form and the coefficients L_{ij} are called the second fundamental quantities associated with the function f.

The following relations are easily obtained by differentiation under the integral sign. From

$$\int_{\alpha}^{\beta} N \frac{\partial f}{\partial u_i} dx = 0, \qquad \int_{\alpha}^{\beta} N \frac{\partial f}{\partial u_j} dx = 0, \qquad \int_{\alpha}^{\beta} N^2 dx = 1$$

we have

(20) $$\int_{\alpha}^{\beta} N \frac{\partial^2 f}{\partial u_i \partial u_j} dx = - \int_{\alpha}^{\beta} \frac{\partial N}{\partial u_j} \frac{\partial f}{\partial u_i} \partial x = - \int_{\alpha}^{\beta} \frac{\partial N}{\partial u_i} \frac{\partial f}{\partial u_j} dx = L_{ij},$$

(21) $$\int_{\alpha}^{\beta} N \frac{\partial N}{\partial u_i} dx = 0.$$

§ 12. *The relations of Gauss and Codazzi.*

We now apply the formulas just obtained in the proof of the extension of the well-known Gauss and Codazzi relations. From equation (19) we have

$$\frac{\partial^2 f}{\partial u_i \, \partial u_j} = \sum_{r=1}^{n} \begin{Bmatrix} i \; j \\ r \end{Bmatrix} \frac{\partial f}{\partial u_r} + L_{ij}\, N \,,$$

$$\frac{\partial^2 f}{\partial u_i \, \partial u_k} = \sum_{r=1}^{n} \begin{Bmatrix} i \; k \\ r \end{Bmatrix} \frac{\partial f}{\partial u_r} + L_{ik}\, N \,.$$

Differentiating the first with respect to u_k and the second with respect to u_j and equating the results, we find

$$\frac{\partial^3 f}{\partial u_i \, \partial u_j \, \partial u_k} = \sum_{r=1}^{n} \begin{Bmatrix} i \; j \\ r \end{Bmatrix} \frac{\partial^2 f}{\partial u_r \, \partial u_k} + \sum_{r=1}^{n} \frac{\partial \begin{Bmatrix} i \; j \\ r \end{Bmatrix}}{\partial u_k} \frac{\partial f}{\partial u_r} + L_{ij} \frac{\partial N}{\partial u_k} + \frac{\partial L_{ij}}{\partial u_k} N$$

$$= \sum_{r=1}^{n} \begin{Bmatrix} i \; k \\ r \end{Bmatrix} \frac{\partial^2 f}{\partial u_r \, \partial u_j} + \sum_{r=1}^{n} \frac{\partial \begin{Bmatrix} i \; k \\ r \end{Bmatrix}}{\partial u_j} \frac{\partial f}{\partial u_r} + L_{ik} \frac{\partial N}{\partial u_j} + \frac{\partial L_{ik}}{\partial u_j} N \,.$$

Multiplying in turn by N and $\partial f / \partial u_m$ and integrating, we obtain, by the use of (20) and (21), after transposing,

$$(22) \qquad \sum_{r=1}^{n} \begin{Bmatrix} i \; j \\ r \end{Bmatrix} L_{rk} - \sum_{r=1}^{n} \begin{Bmatrix} i \; k \\ r \end{Bmatrix} L_{rj} + \frac{\partial L_{ij}}{\partial u_k} - \frac{\partial L_{ik}}{\partial u_j} = 0 \,,$$

$$\sum_{r=1}^{n} \begin{Bmatrix} i \; j \\ r \end{Bmatrix} \begin{bmatrix} r \; k \\ m \end{bmatrix} - \sum_{r=1}^{n} \begin{Bmatrix} i \; k \\ r \end{Bmatrix} \begin{bmatrix} r \; j \\ m \end{bmatrix} + \sum_{r=1}^{n} \frac{\partial \begin{Bmatrix} i \; j \\ r \end{Bmatrix}}{\partial u_k} E_{rm}$$

$$(23)$$

$$- \sum_{r=1}^{n} \frac{\partial \begin{Bmatrix} i \; k \\ r \end{Bmatrix}}{\partial u_j} E_{rm} + L_{ik}\, L_{jm} - L_{ij}\, L_{km} = 0 \,.$$

The last relation contains the result that the expression $L_{ik}\, L_{jm} - L_{ij}\, L_{km}$ can be expressed in terms of the E_{ij} and their derivatives.

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