CERTAIN QUARTIC SURFACES BELONGING TO INFINITE DISCONTINUOUS CREMONIAN GROUPS*

BY

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1. **Statement of the problem.** The following paper has for its purpose the establishment of two theorems:

THEOREM I. *The quartic surface subjected to the single condition of passing through a non-hyperelliptic sextic curve of genus three is invariant under an infinite discontinuous group of birational transformations.*

THEOREM II. *The transformations of the infinite discontinuous group under which the most general quartic surface passing through a sextic curve of genus two remains invariant can be expressed in terms of cremonian transformations.*

In connection with the first theorem the equation of the surface is derived and the equations of the transformations are determined; it is shown that the transformations are cremonian and non-involutorial, and that no transformations exist other than those obtained. It is believed that this surface is the first illustration of one which possesses an infinite discontinuous group, but contains neither a pencil of elliptic curves, nor a net of hyperelliptic curves of genus two. The equation of the surface mentioned in the second theorem is found, and also the equations of two involutorial space transformations which generate the infinite group.

1. SURFACES THROUGH A SEXTIC OF GENUS THREE

2. **The cubic transformation.** Consider the birational transformations between the spaces (x) and (x') by means of the three bilinear equations†

$$(1)\qquad A = \sum a_{ik}\, x_i\, x'_k = 0, \qquad B = \sum b_{ik}\, x_i\, x'_k = 0, \qquad C = \sum c_{ik}\, x_i\, x'_k = 0 \qquad (a_{ik} \neq a_{ki},\ b_{ik} \neq b_{ki},\ c_{ik} \neq c_{ki}).$$

* Presented to the Society, January 1, 1915.

† This transformation is discussed analytically by Doehelemann, *Geometrische Transformationen*, vol. 2, pp. 286–296, and synthetically by Sturm, *Theorie der geometrischen Verwandtschaften*, vol. 3, pp. 484–486 and vol. 4, pp. 370–371.

By solving these equations for x'_k in terms of x_1, x_2, x_3, x_4 we obtain

$$x'_k = \phi_k(x) \qquad (k = 1, 2, 3, 4). \tag{2}$$

The cubic surfaces $\phi_k(x) = 0$ all pass through the fundamental curve of the transformation (2). This curve is a space sextic C_6 of genus three, the most general of its kind. The ruled surface R_8 of order eight formed by the trisecants of C_6 is the fundamental surface of the transformation. It contains C_6 as a three-fold curve and no other multiple curve. Similarly, by solving the system (1) for x_i we obtain

$$x_i = \psi_i(x') \qquad (i = 1, 2, 3, 4). \tag{3}$$

Let C'_6 and R'_8 be respectively the fundamental curve and surface for the space (x').

3. **Equation of the surface.** The equation of the most general quartic surface passing through C_6 may be written in the form

$$F_4 = \sum d_{ik}\, x_i\, \phi_k(x) = 0 \qquad (d_{ik} \neq d_{ki}). \tag{4}$$

By means of equations (2) and (3) we may write the equation of the transformed surface in the form

$$F'_4 = \sum d_{ik}\, \psi_i(x')\, x'_k = 0. \tag{5}$$

A general plane section of $F_4 = 0$ made by $\sum u_i x_i = 0$ goes into the residual section of $F'_4 = 0$ with a cubic surface $\sum u_i \psi_i(x') = 0$ passing through C'_6. This residual section is also a space sextic $\overline{C}'_6$ of genus three. Hence $F'_4 = 0$ contains ∞^3 coresidual space sextic curves $\overline{C}'_6$, images of the plane sections of $F_4 = 0$. Any two of the curves $\overline{C}'_6$ meet each other in four points. Similarly, the surface $F_4 = 0$ contains ∞^3 coresidual space sextic curves $\overline{C}_6$, images of the plane sections of $F'_4 = 0$ under the transformation (3). Any two of the curves $\overline{C}_6$ meet in four points.

The points of C_6 are transformed into the generators of R'_8. The intersection of F'_4 with R'_8 consists of C'_6 counted three times and of a space curve C'_{14} of genus three and order fourteen. Since C_6 and C_4 meet in six points, it follows that C'_{14} and $\overline{C}'_6$ meet in six points.

4. **A second cubic transformation.** Equation (4) may be written in the form

$$\begin{vmatrix} \sum a_{i1}\, x_i & \sum a_{i2}\, x_i & \sum a_{i3}\, x_i & \sum a_{i4}\, x_i \\ \sum b_{i1}\, x_i & \sum b_{i2}\, x_i & \sum b_{i3}\, x_i & \sum b_{i4}\, x_i \\ \sum c_{i1}\, x_i & \sum c_{i2}\, x_i & \sum c_{i3}\, x_i & \sum c_{i4}\, x_i \\ \sum d_{i1}\, x_i & \sum d_{i2}\, x_i & \sum d_{i3}\, x_i & \sum d_{i4}\, x_i \end{vmatrix} = 0. \tag{6}$$

If we consider the fourth bilinear equation $D = 0$ we see from (6) that $F_4 = 0$ is also transformed into $F'_4 = 0$ by the transformation defined by $A = 0$, $B = 0$, $D = 0$, and by two other similar ones. These transformations are distinct for an arbitrary point of space but, from a known property of determinants,* they are identical for points of $F_4 = 0$.

From the form of equation (6) it is seen that $F_4 = 0$ is also transformed into $F'_4 = 0$ by means of any three of the four bilinear equations

$$(7) \quad (\textstyle\sum_i a_{ik} x_i) x'_1 + (\sum_i b_{ik} x_i) x'_2 + (\sum_i c_{ik} x_i) x'_3 + (\sum_i d_{ik} x_i) x'_4 = 0 \qquad (k = 1, 2, 3, 4).$$

The four transformations thus obtained are distinct for an arbitrary point of space, but are identical for points of $F_4 = 0$.

We shall designate the transformation defined by (2) by T_1, and its inverse (3) by T_1^{-1}. Similarly, that defined by (7) by T_2, and its inverse by T_2^{-1}. By means of T_2 the planes of (x) are transformed into cubic surfaces of (x') which have a fundamental curve of order six and of genus three in common. The surface $\psi_1(x') = 0$ is the image of $x_1 = 0$ by both transformations, which have different fundamental curves, namely, the two sextics which together form the complete intersection of $\psi_1(x') = 0$ with $F'_4 = 0$. Hence we see that C'_6 belongs to a triply infinite system of coresidual curves, any two of which have four points in common. Similarly for the systems on $F_4 = 0$.

5. **An infinite discontinuous group.** The product of the two transformations T_1, T_2^{-1} leaves $F_4 = 0$ invariant. By T_1 a plane section is transformed into a space sextic of (x'), which by T_2^{-1} is transformed into a C_{14} of (x). Since the two transformations do not have common fundamental elements, the product $T_1 T_2^{-1}$ is not periodic and therefore generates an infinite group.

6. **Linear systems of curves on $F_4 = 0$.** If we apply the method used by Severi in his study of the quartic surface through a sextic curve of genus two† we may choose the plane sections C_4 and one system of sextics C_6 for a minimum basis. The other system of sextics has the symbol $3C_4 - C_6$. Any system of curves on the surface is expressible in the form $\lambda C_4 + \mu C_6$, in which λ and μ are integers. If the curve is of grade n and of genus π, we have the formula

$$n = 2\pi - 2 = 4\lambda^2 + 12\lambda\mu + 4\mu^2.$$

The surface therefore contains no system of curves of even genus, nor pencil of elliptic curves.

7. **Possible birational transformations.** Suppose a birational transforma-

* See, *e. g.*, Burnside and Panton, *Theory of Equations*, 2d edition, p. 266.

† F. Severi, *Complementi alla teoria della base per la totalità delle curve di una superficie algebrica*, Rendiconti del Circolo Matematico di Palermo, vol. 30 (1911), pp. 265–288.

tion exists which leaves $F_4 = 0$ invariant. Let it change C_4 into $\alpha C_4 + \beta C_6$ and C_6 into $\gamma C_4 + \delta C_6$, where $\alpha\delta - \beta\gamma = \pm 1$. Since $[C_4, C_4] = 4$, $[C_4, C_6] = 6$, $[C_6, C_6] = 4$, we must have the relations

$$\alpha^2 + 3\alpha\beta + \beta^2 = 1, \qquad \gamma^2 + 3\gamma\delta + \delta^2 = 1,$$

$$2\alpha\gamma + 3(\alpha\delta + \beta\gamma) + 2\beta\delta = 3.$$

If $\alpha\delta - \beta\gamma = 1$, we may write

$$2\alpha = t + 3u, \qquad \beta = -u, \qquad \gamma = u, \qquad 2\delta = t - 3u,$$

so that

$$t^2 - 5u^2 = 4.$$

Put $u = 1$, so that $t = 3$. In this case we have

$$\begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} = \begin{vmatrix} 3 & -1 \\ 1 & 0 \end{vmatrix} = \tau.$$

It is seen that any power of τ may be expressed by the recurring formula

$$\tau^p = \begin{vmatrix} g_p & -g_{p-1} \\ g_{p-1} & -g_{p-2} \end{vmatrix},$$

in which $g_p = 3g_{p-1} - g_{p-2}$, with $g_1 = 3$, $g_0 = 1$, $g_{-1} = 0$.

If

$$\alpha\delta - \beta\gamma = -1,$$

one solution is $\alpha = 0$, $\beta = 1$, $\gamma = 1$, $\delta = 0$. Set

$$\sigma = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}.$$

The product $\tau^p \sigma$ is also an involution, and all transformations of determinant -1 are of the form $\tau^p \sigma$. Hence we have the theorem:

THEOREM. *All the birational transformations which exist and leave the surface $F_4 = 0$ invariant are of the form τ^p or $\tau^p \sigma$. The former are not periodic and the latter are involutorial.*

The transformation $T_2 T_1^{-1}$ is readily seen to be τ^2.

8. **Non-existence of involutions on $F_4 = 0$.** If σ or $\tau^2 \sigma$ exists on $F_4 = 0$, there must be a transformation L which transforms $F_4 = 0$ into $F_4' = 0$ such that

$$T_2 L^{-1} = \tau^2 \sigma, \qquad T_1 L^{-1} = \sigma.$$

From the definition of T_1 and T_2 it is easily seen that L must be linear. But a simple illustration shows that $F_4 = 0$ and $F_4' = 0$ are not linearly equivalent.

If

$$A = x_1 x_2' + x_2 x_3' = 0, \qquad B = -x_1 x_3' + x_2 x_4' - x_3 x_2' = 0,$$

$$C = x_2 x_1' - x_4 x_4' = 0, \qquad D = -x_1 x_3' + x_3 x_1' = 0,$$

the equation of $F_4 = 0$ is

$$x_1^2 x_2^2 - x_1^2 x_3 x_4 + x_2 x_3^2 x_4 = 0$$

and of $F_4' = 0$ is

$$x_4' (x_1' x_2' x_4' - x_2' x_3'^2 + x_1' x_3'^2) = 0.$$

These surfaces are evidently not linearly equivalent.

The transformation σ, which should interchange the plane sections and the sextics of one family, is thus seen not to exist. Similarly, the operation τ, which should change the plane sections into one system of sextics and the other system of sextics into the plane sections, does not exist. Hence the only effective transformations are powers of τ^2.

2. THE GENERAL QUARTIC SURFACE THROUGH A SEXTIC CURVE OF GENUS TWO*

9. **Cubic surfaces through a sextic curve of genus two.** A space sextic curve C_6 of genus two has one quadrisecant s_4. A plane through s_4 cuts C_6 in two residual points. The locus of the lines joining these residual points as the plane turns about s_4 is a cubic ruled surface $f_3 = 0$ having s_4 for double directrix. Through C_6 pass two other linearly independent cubic surfaces $f_1 = 0$ and $f_2 = 0$. Since s_4 lies on $f_1 = 0$ and $f_2 = 0$, their residual intersection is a conic. The equations of the conic may be taken as

$$x_3 = 0, \qquad Q_3(x_1, x_2, x_4) = 0,$$

in which $Q_3 = 0$ is a quadric cone with vertex at $(0, 0, 1, 0)$. If s_4 has the equations $x_1 = 0, x_2 = 0$, and if $Q_1 = 0, Q_2 = 0$ are two quadrics through s_4, we may write

$$f_1 \equiv x_2 Q_3 - x_3 Q_2 = 0, \qquad f_2 \equiv x_3 Q_1 - x_1 Q_3 = 0.$$

The equation of the cubic ruled surface then has the form

$$f_3 \equiv x_1 Q_2 - x_2 Q_1 = 0.$$

Now consider the conic $\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 = 0, \lambda_1 Q_1 + \lambda_2 Q_2 + \lambda_3 Q_3 = 0$. Through it pass the pencil of cubic surfaces $a_1 f_1 + a_2 f_2 + a_3 f_3 = 0$, where $a_1 \lambda_1 + a_2 \lambda_2 + a_3 \lambda_3 = 0$. Hence in each plane through $(0, 0, 0, 1)$ lies a conic through which pass a pencil of cubic surfaces, the residual intersection being C_6 and s_4.

* G. Fano, *Sopra alcune superficie del quarto ordine rappresentabili sul piano doppio*, Rendiconti del Reale Istituto Lombardo, vol. 39 (1906), pp. 1071–1086.

Let the equations of the simple directrix of $f_3 = 0$ be $x_3 = 0$, $x_4 = 0$; let $x_4 = 0$ meet s_4 in a point of C_6, and let the two generators of $f_3 = 0$ in this plane be $x_1 = 0$ and $x_2 = 0$. The equation of $f_3 = 0$ now has the form

$$f_3 = (x_1^2 + kx_1 x_2 + x_2^2) x_4 - x_1 x_2 x_3 = 0. \tag{8}$$

The projecting cone K_5 of C_6 from $(0, 0, 1, 0)$ is of order five, has s_4 for triple line, one generator, say $x_2 = 0$, $x_4 = 0$ for double line, and the other, $x_1 = 0$, $x_4 = 0$ for simple line. Its equation may be written in the form

$$\begin{aligned} & x_4^2\{(Px_1 + Sx_2)(x_1^2 + kx_1 x_2 + x_2^2) + Rx_1 x_2^2\} \\ & + x_4 x_2\{(px_1 + sx_2)(x_1^2 + kx_1 x_2 + x_2^2) + (qx_1 + rx_2) x_1 x_2\} \\ & \qquad + x_1 x_2^2 (ax_1^2 + bx_1 x_2 + cx_2^2) = 0. \end{aligned} \tag{9}$$

Since C_6 and s_4 lie on $K_5 = 0$ and on $f_3 = 0$, by combining (8) and (9) we have for the equations of the other cubic surfaces,

$$\begin{aligned} f_1 &= x_3\{x_4(Px_2 + Sx_2) + x_2(px_1 + sx_2)\} \\ &\quad + x_2\{Rx_4^2 + x_4(qx_1 + rx_2) + x_2(ax_1^2 + bx_1 x_2 + cx_2^2)\} = 0, \\ f_2 &= x_3\{(S - kP) x_1 x_4 - Px_2 x_4 + Px_1 x_3 + x_1(px_1 + sx_2)\} \\ &\quad - x_1\{Rx_4^2 + x_4(qx_1 + rx_2) + x_2(ax_1^2 + bx_1 x_2 + cx_2^2)\} = 0. \end{aligned} \tag{10}$$

10. **Quartic surfaces through** C_6. The equation $\sum m_{ik} x_i f_k = 0$ represents a quartic surface passing through c_6, but it also passes through s_4. The equation contains but eleven independent homogeneous constants, since $x_1 f_1 + x_2 f_2 + x_3 f_3$ vanishes identically. If $F = 0$ is the equation of a quartic surface through C_6 but not through s_4, the equation

$$\phi_4 = F + \sum m_{ik} x_i f_k = 0$$

represents a surface passing through C_6 and containing eleven non-homogeneous constants; it is therefore the most general quartic surface through C_6.

We may take for $F = 0$ a quartic having $x_3 = 0$, $x_4 = 0$ for double line. The intersection of $F = 0$ and $f_3 = 0$ is C_6 and a residual sextic $\bar{C}_6$ of genus two, having s_4 for quadrisecant, and meeting C_6 in twelve other points. Since any quartic surface through C_6 but not through s_4 will suffice, we may think of the residual $\bar{C}_6$ as composite and as projected from $(0, 0, 1, 0)$ by the composite quintic cone

$$x_4^2\{(Sx_1 + Px_2)(x_1^2 + kx_1 x_2 + x_2^2) + Rx_1^2 x_2\} = 0. \tag{11}$$

To obtain the equation of $F = 0$ we write the general equation of a quartic passing through $x_3 = 0$, $x_4 = 0$ twice, and determine the coefficients from the condition that, if x_3 is eliminated by means of (8), the resultant shall be composite, having for factors the first members of (2) and (11). Hence we find

$$F = R^2 x_4^4 + R(2S - kP) x_3 x_4^3 + (PR + P^2 - kPS + S^2) x_3^2 x_4^2$$
$$+ R(qx_1 + rx_2) x_4^3 + \{R(px_1 + sx_2) + S(qx_1 + rx_2) - Prx_1$$
$$+ P(q - kr) x_2\} x_3 x_4^2 + \{S(px_1 + sx_2) - Psx_1$$
$$+ (Pr + Pp - ksP) x_2\} x_3^2 x_4 + Psx_2 x_3^3 + R(ax_1^2 + bx_1 x_2 + cx_2^2) x_4^2$$
$$+ \{S(ax_1^2 + bx_1 x_2 + cx_2^2) + P(a - c) x_1 x_2 + P(b - kc) x_2^2\} x_3 x_4$$
$$+ Pcx_2^2 x_3^2 = 0.$$

Through the composite $\overline{C}_6$ determined by (8) and (11) pass the cubic surfaces

$$f_1' \equiv x_2 Q_3' - x_3 Q_2' = x_2 Rx_4^2 - x_3\{(S - kP) x_2 x_4 - Px_1 x_4 + Px_2 x_3\} = 0,$$
$$f_2' \equiv x_3 Q_1' - x_1 Q_3' = - x_1 Rx_4^2 - x_3\{Sx_1 x_4 + Px_2 x_4\} = 0.$$

The curves C_6 and $\overline{C}_6$ meet the plane quartic curve

$$F = 0, \qquad \sum_{i=1}^{3} \lambda_i x_i \equiv (\lambda x) = 0,$$

each in six points, lying respectively on the conics $(\lambda x) = 0$, $(\lambda Q) = 0$, and $(\lambda x) = 0$, $(\lambda Q') = 0$. This set of twelve points is the complete intersection of C_4 and $f_3 = 0$. The curve of intersection of $F = 0$ and $(\lambda x) = 0$ therefore lies on the quartic surface

$$(\lambda Q)(\lambda Q') + f_3 \sum_{i=1}^{4} A_i x_i = 0.$$

By making this equation simultaneous with $(\lambda x) = 0$, we find

$$A_1 = \lambda_1 \lambda_3 a, \qquad A_2 = \lambda_2 \lambda_3 b - \lambda_1 \lambda_3 c, \qquad A_3 = \lambda_1 \lambda_3 p - \lambda_1 \lambda_3 s - \lambda_1 \lambda_2 P,$$
$$A_4 = \lambda_2 \lambda_3 q - \lambda_1 \lambda_3 r - (\lambda_1^2 - k\lambda_1 \lambda_2 + \lambda_2^2) P.$$

For the general quartic surface $\phi_4 = F + \sum m_{ik} x_i f_k = 0$ we make use of the fact that the intersection of $\sum m_{ik} x_i f_k = 0$ and $(\lambda x) = 0$ lies on the quartic surface

$$(\lambda Q)(x_2 \sum m_{ik} x_i - x_1 \sum m_{i2} x_i) + f_3 (\lambda_1 \sum m_{i1} x_i + \lambda_2 \sum m_{i2} x_i$$
$$+ \lambda_3 \sum m_{i3} x_i) = 0.$$

We shall say that $(\lambda Q')$ becomes $(\lambda Q'')$ and that $\sum A_i x_i$ goes into $\sum A_i' x_i$.

11. **Involutorial transformations of** $\phi_4 = 0$. In every plane through $(0, 0, 0, 1)$ the quartic surface determines a line which meets each of two conics in two points. These four points lie on the given quartic surface.

Given a point P on $\phi_4 = 0$. By substituting its coördinates in the equations $(\lambda x) = 0$, $(\lambda Q) = 0$, values of $\lambda_1 : \lambda_2 : \lambda_3$ may be determined. These values are to be substituted in the equation $\sum A'_i\, x_i = 0$. The two equations $(\lambda x) = 0$, $\sum A'_i\, x_i = 0$ determine the line. It meets $(\lambda Q) = 0$ in two points, one of which is P; hence the coördinates of the other are rational functions of those of P. This operation defines an involutorial transformation I_1 under which the surface $\varphi_4 = 0$ is invariant. If we substitute the coördinates of P in $(\lambda Q'') = 0$ instead of in $(\lambda Q) = 0$, and proceed in the same way, we obtain a second involution I_2.

12. **Depiction on a double plane.** If $\lambda_1, \lambda_2, \lambda_3$ are taken as the coördinates of a point K in a plane then K corresponds to both P and to the image P_1 in I_1, hence the surface $\varphi_4 = 0$ is depicted on a double plane. When P and P_1 coincide, the line $(\lambda x) = 0$, $\sum A'_i\, x_i = 0$ is tangent to the quadric $(\lambda Q) = 0$. If we eliminate the coördinates x_i from these equations and the equation which expresses the condition for tangency, we obtain the equation of the sextic curve of branch points on the double plane.*

13. **Involutions expressed by cremonian transformations.** When the point $P \equiv (y_1, y_2, y_3, y_4)$ is not on the surface $\varphi_4 = 0$, consider the pencil of quartic surfaces $l_1\, \phi_4 + l_2 f_3 \sum u_i\, x_i = 0$, and choose $l_1 : l_2$ so that the surface passes through P. The equation $\sum A'_i\, x_i = 0$ changes to

$$l_1 \sum A_i\, x_i + l_2\, \lambda_3^2 \sum u_i\, x_i = 0,$$

while Q_i, Q''_i remain unchanged.

If $P_1 \equiv (y'_1, y'_2, y'_3, y'_4)$ we have

$$\frac{\sum A'_i\, y'_i}{\sum u_i\, y'_i} = \frac{\sum A'_i\, y_i}{\sum u_i\, y_i} = \frac{-\, l_2}{l_1},$$

$$(\lambda y) = 0, \qquad (\lambda Q''(y)) = 0, \qquad (\lambda y') = 0, \qquad (\lambda Q''(y')) = 0.$$

If x_3 and x_4 are eliminated from $(\lambda x) = 0$, $l_1 \sum A'_i\, x_i + \lambda_3^2\, l_2 \sum u_i\, x_i = 0$, and $(\lambda Q) = 0$, we obtain an equation of the form

$$H_{11}\, x_1^2 + 2H_{12}\, x_1\, x_2 + H_{22}\, x_2^2 = 0,$$

in which each coefficient is a rational function of $\lambda_1, \lambda_2, \lambda_3, l_1, l_2$. But

$$y_1\, y'_1 / y_2\, y'_2 = H_{22}/H_{11},$$

and hence

$$y_1 = \sigma y'_2\, \lambda_3\, H_{22}, \qquad y_2 = \sigma y'_1\, \lambda_3\, H_{11},$$
$$y_3 = -\, \sigma\, (\lambda_1\, H_{22}\, y'_2 + \lambda_2\, H_{21}\, y'_1), \tag{12}$$

from which y_4 can be found in terms of (y') by means of

$$l_1 \sum A_i\, y_i + l_2\, \lambda_3^2 \sum u_i\, y_i = 0.$$

* See Fano, loc. cit., p. 1071.

Since l_1, l_2, λ_1, λ_2, λ_3 have exactly the same values when expressed in terms of (y) as of (y'), therefore equations (12) have also the same form whether solved for y_i in terms of (y'), or for y'_i in terms of (y), and are applicable for every point of space. Similarly for the involution I_2. Hence we have the theorem:

THEOREM. *The involutions I_1, I_2, belonging to the quartic surface through a sextic curve of genus two, can be expressed in terms of cremonian transformations which are birational and involutorial for all space.*

CORNELL UNIVERSITY,
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