LINEAR EQUATIONS WITH UNSYMMETRIC SYSTEMS OF COEFFICIENTS*

BY

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In two previous papers† we have considered cases of integral equations

$$\lambda\phi(s) = \int_a^b K(s,t)\phi(t)\,dt,$$

with unsymmetric kernel $K(s,t)$, which have real characteristic functions, in terms of which $\int_a^b\int_a^b K(s,t)f(s)g(t)\,ds\,dt$ may be expanded. In this paper we consider the corresponding theory for linear equations in infinitely many unknowns, which includes the theory of integral equations as a special case. The method is the same as that used by us for integral equations, reduction, by means of a biorthogonal system, to a system of linear equations with a limited symmetric matrix of coefficients. We find in Section 4 that, for a limited matrix A, a necessary and sufficient condition for the existence of real characteristic forms of the equations

$$\lambda l_i = \sum_k a_{ik}\, l_k, \qquad \int_\Delta \lambda dp_i = \sum_k a_{ik}\,\Delta p_k,$$

which form a complete system with limited linear forms $F_i(x)$ such that $AF_i(x) = 0$, or in terms of which $A(x,y)$ may be expanded, is the existence of a positive definite, symmetric, and limited matrix T such that AT is symmetric. The condition obtained by us on the kernel $K(s,t)$ for integral equations, is that there exist a functional transformation‡ T such that $T_s K(s,t)$ is symmetric. This functional transformation differs only slightly from the one, which by the Riesz-Fischer theorem corresponds to the matrix T, the additional restriction being that it transform every continuous function into a continuous function.

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* Presented to the Society, September, 1910.

† Bulletin of the American Mathematical Society, July, 1910, pp. 513–515, and these Transactions, vol. 12, pp. 165–180.

‡ J. Marty obtained necessary and sufficient conditions for the problem in integral equations, expressed in terms of a special functional transformation $T(f) = \int_a^b k(s,t)f(t)\,dt$, Comptes Rendus, April, 1910, and June, 1910.

Section 3 deals with properties of biorthogonal systems of linear forms and linear differential forms.

1. Notation

We confine our attention to real constants and real functions of real variables.

If the range of a variable is that of a sequence, finite or infinite, it is indicated by the subscripts i, j, k, l, m, α, and β. Similarly, constants and functions with these subscripts denote sequences, finite or infinite, of the constants and functions.

The coefficients in the linear form $F(x)$ of the variables $\{x_i\}$ are denoted by the corresponding small letters f_i, thus

$$F(x) = \sum_i f_i x_i .$$

The increment of a continuous function $f(\lambda)$ for an interval $\Delta = (\lambda_1, \lambda_2)$ is denoted by $\Delta f(\lambda) = f(\lambda_2) - f(\lambda_1)$. If Δ_1 and Δ_2 are two intervals of λ then $\Delta_{12} f(\lambda)$ is the increment of $f(\lambda)$ in the interval common to Δ_1 and Δ_2 if they overlap, and is zero if they do not overlap.

Matrices are denoted by capital letters. The elements of a matrix A are denoted by the corresponding small letters a_{ik} and we write $A = (a_{ik})$. The elements of the matrix A' are $a'_{ik} = a_{ki}$. The corresponding quadratic and bilinear forms in the variables $\{x_i\}$ and $\{y_i\}$ are denoted by $A(x, x)$ and $A(x, y)$ respectively.

The unit matrix is denoted by E

$$e_{ik} = \begin{cases} 1 & i = k, \\ 0 & i \neq k. \end{cases}$$

If $F(x)$ and $G(x)$ are two linear forms, the symbol (F, G) denotes the following operation

$$(F, G) = \sum_i f_i g_i .$$

For the matrices A and B we write

$$AB = \left(\sum_j a_{ij} b_{jk}\right).$$

The transformation of a linear form $F(x)$ by a matrix A is expressed by

$$AF(x) = \sum_i \left(\sum_j a_{ij} f_j\right) x_i .$$

2. Definitions and general properties of linear forms and matrices

Limited linear forms. If for a sequence $\{c_i\}$ the sum $\sum_i c_i^2$ converges, it is called the *norm* of $\{c_i\}$. and the sequence is of *finite norm.* A linear form $F(x)$ is *limited*, if there exists a quantity M such that for every sequence $\{x_i\}$ of finite norm, and for every n,

$$\left|\sum_{i=1}^{n} f_i x_i\right| \leqq M\sqrt{\sum_{i=1}^{n} x_i^2}.$$

(A) A linear form $F(x)$ is limited,* when and only when the coefficients $\{f_i\}$ are of finite norm, and for two limited linear forms $F(x)$ and $G(x)$, (F, G) converges.

If $(F, G) = 0$, the two linear forms $F(x)$ and $G(x)$ are *orthogonal.*

Limited matrices. A matrix A and the corresponding quadratic and bilinear forms are *limited*, if there exists a positive quantity M such that for all sequences $\{x_i\}$ and $\{y_i\}$ of finite norm, and for every n,

$$\left|\sum_{i,k=1}^{n} a_{ik} x_i y_k\right| \leqq M\sqrt{\sum_{i=1}^{n} x_i^2}\sqrt{\sum_{i=1}^{n} y_i^2}.$$

(B) If for every limited linear form $F(x)$, the linear form $AF(x)$ is also limited, the matrix A is limited.†

(C) If A is a limited matrix,‡ and the sequences $\{x_i\}$ and $\{y_i\}$ are of finite norm,

$$\sum_i\left(\sum_k a_{ik} x_i y_k\right) = \sum_k\left(\sum_i a_{ik} x_i y_k\right).$$

(*D*) If A and B are limited matrices,§ the matrix AB exists and is limited. A system of linear forms $\{L_i(x)\}$ forms an *orthogonal system* if $(L_i, L_k) = e_{ik}$.

(E) The matrix L corresponding to an orthogonal|| system of limited linear forms $L_i(x)$ is limited.

(F) If A_n are limited matrices¶ such that $|A_n(x, y)| \leqq M\sum_i x_i^2$, where M is independent of n, and if $\lim a_{ik}^{(n)} = a_{ik}$, then A is a limited matrix and $|A(x, x)| \leqq M\sum_i x_i^2$.

A matrix B is the *front reciprocal* of the matrix A if $AB = E$ and the *rear reciprocal* of A if $BA = E$.

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* Hilbert, *Grundzüge einer allgemeinen Theorie der Integralgleichungen*, p. 126.

† Hellinger and Toeplitz, *Theorie der unendlichen Matrizen*, § 10, Mathematische Annalen, vol. 69, 1910. See also E. Schmidt, *Ueber die Auflösung linearer Gleichungen mit unendlichvielen Unbekannten*, Rendiconti del Circolo Matematico di Palermo, t. 25, 1908, p. 54.

‡ Hilbert, l. c., p. 120.

§ Hilbert, l. c., pp. 128–129.

|| Hilbert, l. c., p. 130.

¶ Riesz, *Les systèmes d'équations linéaires à une infinité d'inconnues*, p. 106.

Stieltjes integral.* Let $f(x)$ be a continuous function of limited variation in the interval (a, b), and let the interval be divided into subintervals $\Delta_i = (\lambda_i, \lambda_{i+1})$ by the points $\lambda_0 = a, \lambda_1, \cdots, \lambda_{n-1}, \lambda_n = b$, in such a way that Δ_i approaches zero as n increases, and let u_i be any value of a continuous function $u(\lambda)$ in Δ_i, then the sum

$$\sum_{i=1}^{n} u_i \Delta_i f(\lambda)$$

has a limit, and it is denoted by the Stieltjes integral

$$\int_a^b u(\lambda)\, df(\lambda),$$

and this integral is a continuous function of the upper limit of integration.

Hellinger integrals. Let $f(x)$ be a continuous function, and $f_0(\lambda)$ be a continuous, monotonic non-decreasing function of λ in (a, b), and let $f(\lambda)$ be constant in every interval of (a, b) in which $f_0(\lambda)$ is constant. Divide (a, b) into n subintervals $\Delta_i = (\lambda_i, \lambda_{i+1})$ by the points, $\lambda_0 = a, \lambda_1, \cdots, \lambda_{n-1}, \lambda_n = b$, in such a way that Δ_i approaches zero, then the sum

$$\sum_{i=1}^{n} \frac{(\Delta_i f(\lambda))^2}{\Delta_i f_0(\lambda)}$$

has a limit, and it is denoted by the Hellinger integral

$$\int_a^b \frac{(df(\lambda))^2}{df_0(\lambda)}.$$

A sufficient condition that the Hellinger integral exist is that $(\Delta f)^2 \leqq \Delta h \Delta f_0$, where $h(\lambda)$ is a continuous, monotonic non-decreasing function in λ. If the integral with the upper limit λ is called $h(\lambda)$, then $h(\lambda)$ is a continuous, monotonic function, and

$$(\Delta f)^2 \leqq \Delta h\, \Delta f_0, \tag{1}$$

and hence $f(\lambda)$ is a function of limited variation.

The system of functions $\{f^{(\alpha)}\}$ is *integrable* $H(f_0^{(\alpha)})$ on an interval (a, b), if $f^{(\alpha)}$ are continuous functions, $f_0^{(\alpha)}$ are continuous, monotonic non-decreasing functions of λ in the interval (a, b), if the integrals $\int_a^b (df^{(\alpha)})^2/df_0^{(\alpha)}$ exist,† and the sum

$$\sum_{\alpha} \int_a^b \frac{(df^{(\alpha)})^2}{df_0^{(\alpha)}}$$

converges.

* The statements concerning Stieltjes and Hellinger integrals are results taken from Hellinger, *Neue Begründung der Theorie quadratischer Formen von unendlichvielen Veränderlichen*, Journal für Mathematik, vol. 136, pp. 234–242, or are easily derived from them.

† The argument in the integral is omitted whenever there is no ambiguity.

If $f(\lambda)$ and $f_1(\lambda)$ are both integrable $H(f_0)$ on the interval (a, b), and u_i is the value of a continuous function $u(\lambda)$ at some point in Δ_i, the sum

$$\sum_{i=1}^{n} u_i \frac{\Delta_i f \Delta_i f_1}{\Delta_i f_0}$$

has a limit, and it is denoted by

$$\int_a^b u \frac{df\, df_1}{df_0}.$$

Linear differential forms. If $f_i(\lambda)$ are continuous functions in the interval (a, b), the system of linear form $\Delta F(\lambda;\, x) = \sum_i \Delta f_i(\lambda) x_i$ for all possible subintervals Δ of (a, b), forms a system of linear differential forms, for which we use the symbol $dF(\lambda;\, x) = \sum_i df_i(\lambda) x_i$, and denote the matrix of coefficients $\Delta f_i(\lambda)$ by dF.

The linear differential forms $dF(\lambda;\, x)$ are *limited,* if the functions are continuous in the interval (a, b), and if the norm $f_0(\lambda)$ is continuous in (a, b).

Given a system of limited linear differential forms $\{dF^{(a)}(\lambda;\, x)\}$ and an interval (a, b); $\{dF^{(a)}\}$ is *limited with respect to* $\{f_0^{(a)}\}$, if $f_0^{(a)}$ are continuous, monotonic non-decreasing functions in (a, b), if $\{dF^{(a)}(\lambda;\, x)\}$ is integrable $H(f_0^{(a)})$, and if there exists a quantity M such that for every sequence $\{x_i\}$ of finite norm

$$\sum_a \int_a^b \frac{(dF^{(a)}(\lambda;\, x))^2}{df_0^{(a)}(\lambda)} \leqq M \sum_i x_i^2.$$

(G) If $\{f^{(a)}\}$ is integrable $H(f_0^{(a)})$ and if $\{dF^{(a)}\}$ is limited with respect to $\{f_0^{(a)}\}$ for an interval (a, b), then

$$\sum_a \int_a^b \frac{dF^{(a)}(\lambda;\, x)\, df^{(a)}(\lambda)}{df_0^{(a)}(\lambda)} = \sum_i x_i \left(\sum_a \int_a^b \frac{df_i^{(a)}\, df^{(a)}}{df_0^{(a)}} \right), \tag{2}$$

and is a limited linear form. If $\{dG^{(a)}(\lambda;\, x)\}$ is also a system of differential forms, limited with respect to $\{f_0^{(a)}\}$, and $u(\lambda)$ is a continuous function,

$$\sum_a \int_a^b u(\lambda) \frac{dF^{(a)}(\lambda;\, x)\, dG^{(a)}(\lambda;\, y)}{df_0^{(a)}(\lambda)} = \sum_{i,k} x_i y_k \left(\int_a^b u \cdot \frac{df_i^{(a)}\, df_k^{(a)}}{df_0^{(a)}} \right),$$

and is a limited bilinear form.

A system consisting* of limited linear forms $\{L_i(x)\}$, and limited linear differential forms $\{dP^{(a)}(\lambda;\, x)\}$, for an interval (a, b), forms a normalized *orthogonal system with basis functions* $\{p_0^{(a)}(\lambda)\}$, if

$$(L_i, L_k) = e_{ik};$$

* It is to be understood throughout this paper, that in a system consisting of linear forms $\{L_i(x)\}$, and linear differential forms $\{dP^{(a)}(\lambda;\, x)\}$, either all the $L_i(x)$ or all the $dP^{(a)}(\lambda;\, x)$ may be zero.

if for any two intervals Δ_1 and Δ_2 of (a, b)

$$(3) \qquad (\Delta_1 P^{(\alpha)}, \Delta_2 P^{(\beta)}) = \begin{cases} \Delta_{12}\, p_0^{(\alpha)}(\lambda) & \alpha = \beta, \\ 0 & \alpha \neq \beta; \end{cases}$$

and if for any interval Δ of (a, b)

$$(L_i, \Delta P^{(\alpha)}) = 0.$$

Since the relation (3) involves only the increments of $p_i^{(\alpha)}(\lambda)$ we assume for an orthogonal system that $p_i^{(\alpha)}(\lambda_0) = 0$ for some point λ_0 of (a, b). It follows that

$$p_0^{(\alpha)}(\lambda) = \begin{cases} \sum_i (p_i^{(\alpha)}(\lambda)^2) & \lambda - \lambda_0 > 0, \\ -\sum_i (p_i^{(\alpha)}(\lambda)^2) & \lambda - \lambda_0 < 0, \end{cases}$$

and the $p_0^{(\alpha)}(\lambda)$ are continuous, monotonic non-decreasing functions of λ. The orthogonal relation (3) is equivalent* to

$$(4) \qquad \sum_i \int_a^b \frac{df^{(\alpha)}\, dp_i^{(\alpha)}}{dp_0^{(\alpha)}} \int_a^b \frac{df_1^{(\beta)}\, dp_i^{(\beta)}}{dp_0^{(\beta)}} = \begin{cases} \int_a^b \frac{df^{(\alpha)}\, df_1^{(\alpha)}}{dp_0^{(\alpha)}} & \alpha = \beta, \\ 0 & \alpha \neq \beta, \end{cases}$$

where $\{f^{(\alpha)}\}$ and $\{f_1^{(\alpha)}\}$ are any functions integrable $H(p_0^{(\alpha)})$.

(H) An orthogonal† system of linear differential forms $\{dP^{(\alpha)}(\lambda; x)\}$ with basis functions $\{p_0^{(\alpha)}\}$ for an interval (a, b), is limited with respect to $\{p_0^{(\alpha)}\}$.

In conformity with the notation for the composition of two matrices, both of whose arguments have the range of a sequence, we adopt the following notation, when there is no ambiguity about the basis functions $\{f_0^{(\alpha)}\}$,

$$\sum_\alpha \int_a^b \frac{df_i^{(\alpha)}\, dg_k^{(\alpha)}}{df_0^{(\alpha)}} = F'\, G_a^b = F'\, G,$$

according as there is, or is not any ambiguity about the limits of integration. The product of the matrix above with a matrix $A = (a_{ik})$ is indicated by

$$\left(\sum_j a_{ij}\left(\sum_\alpha \int_a^b \frac{df_j^{(\alpha)}\, dg_k^{(\alpha)}}{df_0^{(\alpha)}}\right)\right) = AF'\, G_a^b = AF'\, G,$$

and

$$\left(\sum_j \left(\sum_\alpha \int_a^b \frac{df_i^{(\alpha)}\, dg_j^{(\alpha)}}{df_0^{(\alpha)}}\right) a_{jk}\right) = F'\, G_a^b A = F'\, GA.$$

A system of linear forms is *complete* if there exists no limited linear form orthogonal to all the forms of the system.

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* Hellinger, l. c., p. 250.

† Hellinger, l. c., pp. 248 and 251.

If the orthogonal system consisting of the linear forms $\{ L_i(x) \}$, and the linear differential forms $\{ dP^{(\alpha)}(\lambda, x) \}$ with the basis functions $\{ p_0^{(\alpha)}(\lambda) \}$ for the interval (a, b), is complete, then

$$L'L + P'P = E.$$

Linear equations with symmetric systems of coefficients.

A *characteristic linear form* of the system of equations

$$\lambda l_i = \sum_k a_{ik} l_k, \tag{5}$$

where the matrix A of coefficients is limited, is a limited linear form $L(x)$ not identically zero, whose coefficients l_i for some value of λ, called *characteristic number,* satisfy the equations (5).

The characteristic numbers form the *point spectrum.*

A system of *characteristic linear differential forms* for an interval (a, b) of the equations

$$\int_\Delta \lambda dp_i(\lambda) = \sum_k a_{ik} \Delta p_k(\lambda), \tag{6}$$

where the matrix A of coefficients is limited, is a system of linear differential forms $\{ dP(\lambda\,;\, x) \}$ not identically zero, limited with respect to a continuous, monotonic non-decreasing function $p_0(\lambda)$, and such that the equations (6) are satisfied for every subinterval Δ of (a, b).

The intervals of λ, for which characteristic linear differential forms exist, form the *continuous spectrum.*

(I) If A is a limited symmetric* matrix, there exists a spectrum associated with it, which lies within a finite interval of the λ-axis. The characteristic linear forms $\{ L_i(x) \}$, corresponding to the characteristic numbers $\{ \lambda_i \}$, and the characteristic linear differential forms $\{ dP^{(\alpha)}(\lambda\,;\, x) \}$ corresponding to the continuous spectrum (a, b), form an orthogonal† system for (a, b), and the basis functions are given by

$$p_0^{(\alpha)}(\lambda) = \begin{cases} \sum_i (p_i^{(\alpha)}(\lambda)^2) & \lambda > 0, \\ -\sum_i (p_i^{(\alpha)}(\lambda)^2) & \lambda < 0 \end{cases}$$

under the assumption that $p_i^{(\alpha)}(0) = 0$. For sequences $\{ x_i \}$ and $\{ y_i \}$ of finite norm,

$$A(x, y) = \sum_i \lambda_i L_i(x) L_i(y) + \sum_\alpha \int_a^b \lambda \cdot \frac{dP^{(\alpha)}(\lambda; x)\, dP^{(\alpha)}(\lambda; y)}{dp_0^{(\alpha)}(\lambda)}. \tag{7}$$

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* Hilbert, l. c., Kap. 11, and Hellinger, l. c., pp. 210–271.

† The argument given by Hellinger, l. c., pp. 244–246, for the orthogonality covers only the case in which the spectrum is positive; it can however be extended to cover the general case.

A limited symmetric matrix A is *positive* if for every sequence of finite norm

$$\sum_{i,k} a_{ik}\, x_i\, x_k \geqq 0\,,$$

the equality sign holding only for $\sum_k a_{ik}\, x_k = 0\,,\ i = 1, 2, \cdots;$ and *positive definite,* if the equality sign holds only for $x_i = 0\,, i = 1, 2, \cdots.$

(J) For a positive symmetric limited matrix of coefficients the spectrum contains no points to the left of the origin.

A function $F(x)$ of the variables $\{x_i\}$ of finite norm is *completely continuous* if

$$\lim_{n=\infty} F(x_i + \epsilon_i^{(n)}) = F(x_i)\,,$$

whenever

$$\lim_{n=\infty} \epsilon_i^{(n)} = 0 \qquad (i = 1, 2, \cdots).$$

(K) For a completely continuous* quadratic form $A(x, x)$, there exist no characteristic linear differential forms of the equations (6), and if for a value of λ there exists a characteristic linear form of (5), then there exists a characteristic linear form for the same value of λ of the adjoint system of equations

$$\lambda l_i = \sum_k a_{ki}\, l_k\,.$$

(L) A quadratic form $A(x, x)$ is completely continuous† if $\sum_{i,k} a_{ik}^2$ converges; also if $A = BC$, where $B(x, x)$ is limited and $C(x, x)$ is completely continuous.

3. Biorthogonal Systems

Two finite or infinite systems of limited linear forms $\{U_i(x)\}$ and $\{V_i(x)\}$ form a *biorthogonal system* if

$$(U_i, V_k) = e_{ik}\,.$$

This condition could be written

$$UV' = E\,,$$

which expresses that U is the rear reciprocal of V'. In a biorthogonal system of limited linear forms $\{U_i(x)\}$ and $\{V_i(x)\}$ each system is linearly independent, and the rows of the matrices U and V are of finite norm (A § 2), although the columns may not be. The matrices may both be limited, or both unlimited, or one may be limited and the other unlimited.

Theorem I. *If T is a limited, positive, symmetric matrix, not identically zero, there exists a biorthogonal system, finite or infinite, of limited linear forms $\{U_i(x)\}$ and $\{V_i(x)\}$, for which the matrix V is limited, and*

$$V_i(x) = TU_i(x)\,, \qquad T = VV'\,.$$

* Hilbert, l. c., pp. 165–170.

† Hilbert, l. c., pp. 150 and 176.

Let $\{f_i(x)\}$ be a system of linearly independent limited linear forms, such that $TF_i \neq 0$ and either the system is complete or forms a complete system together with limited linear forms $\{\overline{F}_i(x)\}$, such that $T\overline{F}_i(x) = 0$ and which form a linearly independent system together with the $\{F_i(x)\}$. For example let $F_i(x) = T_i(x)$, omitting those $T_i(x)$ which are linearly dependent on a finite number of the preceding. Then the system

$$G_i(x) = TF_i(x)$$

is linearly independent. The biorthogonal system is constructed as follows

$$U_i(x) = \frac{\begin{vmatrix} F_1(x) & F_2(x) & \cdots & F_i(x) \\ (F_1, G_1) & (F_2, G_1) & \cdots & (F_i, G_1) \\ \cdot & \cdot & \cdots & \cdot \\ (F_1, G_{i-1}) & (F_2, G_{i-1}) & \cdots & (F_i, G_{i-1}) \end{vmatrix}}{\sqrt{K_{i-1}K_i}},$$

$$V_i(x) = \frac{\begin{vmatrix} G_1(x) & G_2(x) & \cdots & G_i(x) \\ (G_1, F_1) & (G_2, F_1) & \cdots & (G_i, F_1) \\ \cdot & \cdot & \cdots & \cdot \\ (G_1, F_{i-1}) & (G_2, F_{i-1}) & \cdots & (G_i, F_{i-1}) \end{vmatrix}}{\sqrt{K_{i-1}K_i}},$$

where

$$K_i = \begin{vmatrix} (F_1, G_1) & (F_2, G_1) & \cdots & (F_i, G_1) \\ (F_1, G_2) & (F_2, G_2) & \cdots & (F_i, G_2) \\ \cdot & \cdot & \cdots & \cdot \\ (F_1, G_i) & (F_2, G_i) & \cdots & (F_i, G_i) \end{vmatrix}.$$

That the matrix V is limited, follows from the substitution of

$$F(x) = H(x) - \sum_{j=1}^{n} U_j(x)(V_j, H),$$

where $H(x)$ is any limited linear form, in $(F, TF) \geqq 0$. We obtain

$$\sum_{k=1}^{n} (V_k, H)^2 \leqq (H, TH).$$

Since T is limited it follows from B § 2 that V is limited. It can be seen directly from the construction that $V_i(x) = TU_i(x)$, and hence $(V_i, \overline{F}) = 0$. To prove the last part of the theorem, let $V'V - T = R$. Then $RU_i(x) = 0$, and also $R\overline{F}_i(x) = 0$. Since $\{u_i(x)\}$ and $\{\overline{F}_i(x)\}$ form a complete system if $\{F_i(x)\}$ and $\{\overline{F}_i(x)\}$ do, $R = 0$ and $V'V = T$.

The limited linear forms $\{U_i(x)\}$ and $\{V_i(x)\}$, and the limited linear differential forms $\{dQ^{(\alpha)}(\lambda; x)\}$ and $\{dR^{(\alpha)}(\lambda; x)\}$ form a *biorthogonal system with the basis functions* $\{r_0^{(\alpha)}\}$ for an interval (a, b) if the functions $\{r_0^{(\alpha)}\}$ are continuous, monotonic non-decreasing in (a, b); if

$$(U_i , V_k) = e_{ik} ;$$

if for any two intervals Δ_1 and Δ_2 of (a, b)

$$(\Delta_1 Q^{(\alpha)}, \Delta_2 R^{(\beta)}) = \begin{cases} \Delta_{12} r_0^{(\alpha)} & \alpha = \beta, \\ 0 & \alpha \neq \beta; \end{cases}$$

and if for any interval Δ of (a, b)

$$(\Delta Q^{(\alpha)}, V_i) = (\Delta R^{(\alpha)}, U_i) = 0.$$

THEOREM II. *If for the biorthogonal system* $(U_i(x), V_i(x), dQ^{(\alpha)}(\lambda; x), dR^{(\alpha)}(\lambda; x))$ *with basis functions* $\{ r_0^{(\alpha)} \}$ *for an interval* (a, b), *the matrix* V *is limited, and the matrix* $\{ dR^{(\alpha)} \}$ *is limited with respect to* $\{ r_0^{(\alpha)} \}$ *for the interval* (a, b), *there exists a positive, symmetric, and limited matrix* T *such that*

$$V_i(x) = TU_i(x), \qquad dR^{(\alpha)}(\lambda; x) = TdQ^{(\alpha)}(\lambda; x).$$

Such a matrix T is given by

$$T = V'V + R'R.$$

By D and G § 2 it is limited; it is obviously positive and symmetric.

THEOREM III. *If* T *is a positive, symmetric, and limited matrix, there exists a biorthogonal system* $(U_i(x), V_i(x), dQ^{(\alpha)}(\lambda; x), dR^{(\alpha)}(\lambda; x))$ *with basis functions* $\{ r_0^{(\alpha)} \}$ *for an interval* (a, b), *such that the matrix* V *is limited, the matrix* $\{ dR^{(\alpha)} \}$ *is limited with respect to the functions* $\{ r_0^{(\alpha)} \}$ *for the interval* (a, b), *and*

$$V_i(x) = TU_i(x), \qquad dR^{(\alpha)}(\lambda; x) = TdQ^{(\alpha)}(\lambda; x), \qquad T = V'V + R'R.$$

Let $\{ L_i(x) \}$ be the characteristic linear forms associated with the matrix T and corresponding to the characteristic numbers λ_i, and $\{dP^{(\alpha)}(\lambda; x)\}$ the characteristic linear differential forms for the continuous spectrum (a, b), which form an orthogonal system with basis functions $\{p_0^{(\alpha)}\}$ for (a, b) (I § 2).

A biorthogonal system satisfying the conditions in the theorem is constructed as follows

$$(8) \quad \begin{gathered} U_i(x) = \frac{L_i(x)}{\sqrt{\lambda_i}}, \qquad V_i(x) = \sqrt{\lambda_i}\, L_i(x), \qquad Q^{(\alpha)}(\lambda; x) = P^{(\alpha)}(\lambda; x); \\ R^{(\alpha)}(\lambda; x) = \int_a^\lambda \lambda dP^{(\alpha)}(\lambda; x), \qquad r_0^{(\alpha)}(\lambda) = \int_a^\lambda \lambda dp_0^{(\alpha)}(\lambda). \end{gathered}$$

The spectrum is positive (J § 2), and hence $U_i(x)$ and $V_i(x)$ are real. The matrix V is limited, since L is limited (E § 2) and the characteristic numbers are finite (I § 2). The functions $r_0^{(\alpha)}$ are continuous, monotonic non-decreasing functions of λ in (a, b). If $a \neq 0$, and $\{dF^{(\alpha)}\}$ is limited with respect to $\{p_0^{(a)}\}$, it is limited with respect to $\{r_0^{(a)}\}$. The matrix $\{dR^{(a)}\}$ is limited with respect to $\{r_0^{(a)}\}$ in any case, for, from the inequality

$$\left(\sum_{j=1}^{n}\lambda_j\,\delta_j\,P^{(a)}\right)^2 \leqq \sum_{j=1}^{n}\lambda_j\frac{(\delta_j P^{(a)})^2}{\delta_j p_0^{(a)}}\sum_{j=1}^{n}\lambda_j\,\delta_j\,p_0^{(a)},$$

if follows that

$$\left(\int_\Delta \lambda dP^{(a)}\right)^2 \leqq \int_\Delta \lambda\cdot\frac{(dP^{(a)})^2}{dp_0^{(a)}}\int_\Delta \lambda dp_0^{(a)},$$

and

$$\int_a^b \frac{(dR^{(a)})^2}{dr_0^{(a)}} \leqq \int_a^b \lambda\frac{(dP^{(a)})^2}{dp_0^{(a)}}.$$

From $\int_{\Delta_i}\lambda df = \lambda_i\,\Delta_i f + \Delta_i\lambda\,(f(\lambda_{i+1}) - f(\bar\lambda))$ where $\lambda_i \leqq \bar\lambda \leqq \lambda_{i+1}$, and from $\int_{\Delta_i}\lambda dp_0^{(a)} = \lambda_i^*\,\Delta_i\,p_0^{(a)}$, we obtain for $a \neq 0$ and for $\{f^{(a)}\}$ integrable $\mathrm{H}(p_0^{(a)})$

$$\int_a^b \frac{dr_i^{(a)}\,df^{(a)}}{dr_0^{(a)}} = \int_a^b \frac{dp_i^{(a)}\,df^{(a)}}{dp_0^{(a)}}.$$

For the case $a = 0$ and $b > \epsilon > 0$, we define

$$\int_0^b \frac{dr_i^{(a)}\,df^{(a)}}{dr_0^{(a)}} = \lim_{\epsilon \doteq 0}\int_\epsilon^b \frac{dr_i^{(a)}\,df^{(a)}}{dr_0^{(a)}} = \int_0^b \frac{dp_i^{(a)}\,df^{(a)}}{dp_0^{(a)}}.$$

We have also

$$\int_a^b \frac{(dR^{(a)})^2}{dr_0^{(a)}} = \int_a^b \lambda\frac{(dP^{(a)})^2}{dp_0^{(a)}}.$$

The remainder of the theorem follows from (5), (6), and (7).

If the matrix T is positive definite, the system $\{U_i(x),\ dQ^{(a)}(\lambda;\ x)\}$ is complete.

THEOREM IV. *If* $(U_i(x),\ V_i(x),\ dQ^{(a)}(\lambda;\ x),\ dR^{(a)}(\lambda;\ x))$ *is a biorthogonal system with basis functions* $\{r_0^{(a)}\}$ *for an interval* (a, b), *such that the matrix* V *is limited, the matrix* $\{dR^{(a)}\}$ *limited with respect to* $\{r_0^{(a)}\}$ *on* (a, b), *and the system* $(U_i(x),\ dQ^{(a)}(\lambda;\ x))$ *complete, then for any sequence* $\{x_i\}$ *of finite norm, and for any sequence* $\{y_i\}$ *such that* $\{U_i(y)\}$ *is of finite norm and* $\{Q^{(a)}(\lambda;\ y)\}$ *is integrable* $H(r_0^{(a)})$ *on* (a, b),

$$(11)\qquad \sum_i x_i\,y_i = \sum_i V_i(x)\,U_i(y) + \sum_a\int_a^b \frac{dR^{(a)}(\lambda;\ x)\,dQ^{(a)}(\lambda;\ y)}{dr_0^{(a)}(\lambda)}.$$

Call the difference between the two sides of (11) $F(x)$, and set x_i equal to U_{ij} and also to $\Delta q_i^{(a)}$. Then by C § 2 and (2), $(F,\ U_i) = 0 = (F,\ \Delta Q^{(a)})$ and therefore $F(x) \equiv 0$.

The relation (11) may hold for other values of $\{y_i\}$ than those stated in the theorem; for example, for the biorthogonal system in Theorem III the relation (11) is true for all sequences $\{y_i\}$ of finite norm, by virtue of (8) and (10).

If the columns of the matrix U are of finite norm, and if the system of functions $\{ q_i^{(\alpha)}(\lambda) \}$ is integrable $H(r_0^{(\alpha)})$ on (a, b), we may take $y_i = e_{ik}$ and (11) becomes

$$V'U + R'Q = E. \tag{11'}$$

This relation is true for an orthogonal system which is complete, also for the biorthogonal system (8). If the system consists only of linear forms, (11') expresses that U is the front reciprocal of V'. Toeplitz* proved (11') for the special case of two matrices which are both limited. That the relation (11') is not true for every biorthogonal system of limited linear forms is shown by the following example

$$U_i(x) = L_{i+1}(x) - L_1(x), \qquad V_i(x) = L_{i+1}(x),$$

where $\{ L_i(x) \}$ is a complete orthogonal system of linear forms.

THEOREM V. *Let T be a positive definite, symmetric, limited matrix; let $(\bar{U}_i(x), \bar{V}_i(x))$ form a biorthogonal system of limited linear forms with the matrix $\bar{V}$ limited, the system $\{ \bar{U}_i(x) \}$ complete, and $\bar{V}_i(x) = T\bar{U}_i(x)$;* let *$(U_i(x), V_i(x), dQ^{(\alpha)}(\lambda; x), dR^{(\alpha)}(\lambda; x))$ form a biorthogonal system of limited linear forms and linear differential forms with the basis functions $\{ r_0^{(\alpha)} \}$ for the interval (a, b), V limited, $\{ dR^{(\alpha)} \}$ limited with respect to $\{ r_0^{(\alpha)} \}$ on (a, b), $V_i(x) = TU_i(x)$, $dR^{(\alpha)}(\lambda; x) = TdQ^{(\alpha)}(\lambda; x)$ and the system of $\{ U_i(x) \}$ and $\{ dQ^{(\alpha)}(\lambda; x) \}$ complete, then $\sum_j \bar{u}_{ij} v_{kj} = \sum_j \bar{v}_{ij} u_{kj}$* and *$\sum_j \bar{u}_{ij} r_j^{(\alpha)}(\lambda) = \sum_j \bar{v}_{ij} q_j^{(\alpha)}(\lambda)$ are the coefficients of a complete orthogonal system with the basis functions $\{ r_0^{(\alpha)} \}$ for the interval (a, b).*

The orthogonal property follows from Theorem 4, since

$$\sum_j (U, \bar{V}_j)(\bar{V}_j, U_k) = \sum_j (U_i, \bar{V}_j)(\bar{U}_j, V_k) = (U_i, V_k) = e_{ik},$$

$$\sum_j (\Delta_1 Q^{\alpha)}, \bar{V}_j)(\bar{V}_j, \Delta_2 Q^{(\beta)}) = \sum_j (\Delta_1 Q^{(\alpha)}, \bar{V}_j)(\bar{U}_j, \Delta_2 R^{(\beta)})$$

$$= (\Delta_1 Q^{(\alpha)}, \Delta_2 R^{(\beta)}) = \begin{cases} \Delta_{12} r_0^{(\alpha)} & \alpha = \beta, \\ 0 & \alpha \neq \beta, \end{cases}$$

$$\sum_j (U_i, \bar{V}_j)(\bar{V}_j, \Delta Q^{(\alpha)}) = \sum_j (U_i, \bar{V}_j)(\bar{U}_j, \Delta R^{(\alpha)}) = (U_i, \Delta R^{(\alpha)}) = 0.$$

The system is complete, for if $C(x)$ were orthogonal to $\bar{V}U_i(x)$ and $\bar{V}Q^{(\alpha)}(\lambda; x)$, then $\bar{V}'C(x)$ would be orthogonal to $U_i(x)$ and $dQ^{(\alpha)}(\lambda; x)$.

* *Die Jacobische Transformation der quadratischen Formen von unendlichvielen Veränderlichen*, Nachrichten der Kgl. Gesellschaft der Wissenschaften zu Göttingen, Math-Phys. Kl., 1907, p. 101.

4. Linear equations with unsymmetric systems of coefficients

Theorem VI. *If, for the limited matrix A, there exist for real values of λ real characteristic linear forms and linear differential forms of the equations* (5) *and* (6) *which form a complete system together with limited linear forms $F_i(x)$ such that $AF_i(x) = 0$, then there exists a positive definite, symmetric, and limited matrix T such that AT is symmetric.*

Let $\{ V_i(x) \}$ be the characteristic linear forms for which the matrix V may be assumed to be limited, and $\{ dR^{(\alpha)}(\lambda;\, x) \}$ the system of characteristic linear differential forms for which $\{ dR^{(\alpha)} \}$ is limited with respect to $\{r_{ij}^{(\alpha)} \}$ for the interval (a, b). It is obvious that the matrix F may be assumed to be limited. The matrix T

$$T = V'V + F'F + R'R$$

is positive definite, symmetric, and limited, and the matrix AT is symmetric.

Theorem VII. *If A is a limited matrix, not identically zero, and if there exists a positive definite, symmetric, limited matrix T such that AT is symmetric, then there exist real characteristic linear forms and linear differential forms of the equations* (5) *and* (6), *which form a complete system with limited linear forms $F_i(x)$ such that $AF_i(x) = 0$.*

Corresponding to the matrix T there exists by Theorem I a biorthogonal system of limited linear forms $\{\bar{U}_i(x)\}$ and $\{ \bar{V}_i(x)$, with the matrix $\bar{V}$ limited, $\{ \bar{U}_i(x) \}$ complete, and $\bar{V}_i(x) = T\bar{U}_i(x)$. The matrix

$$\bar{B} = \bar{U}A\bar{V}'$$

exists and is symmetric. To show that it is limited we introduce the biorthogonal system $(U_i(x),\, V_i(x),\, dQ^{(\alpha)}(\lambda\,;\, x),\, dR^{(\alpha)}(\lambda\,;\, x))$ defined by (8). Let $\bar{l}_{ik} = \sum_j \bar{u}_{kj}\, v_{ij}$ and $\bar{p}_i^{(\alpha)}(\lambda) = \sum_j \bar{u}_{ij}\, r_j^{(\alpha)}(\lambda)$, then by E and H § 2 and Theorem V the matrix $\bar{L}$ is limited and $\{ d\bar{P}^{(\alpha)} \}$ is limited with respect to $\{ r_{ij}^{(\alpha)} \}$ on (a, b). Applying relation (11) we obtain

$$\bar{U} = U'\bar{L} + Q'\bar{P}, \qquad \bar{V} = V'\bar{L} + R'\bar{P}.$$

Substitute these expressions in the elements of $\bar{B}$, and, since the summation signs may be interchanged on account of the form of $\bar{U}$ and $\bar{V}$, we obtain

$$\bar{B} = \bar{U}A\bar{V}' = \bar{L}'(UAV'\bar{L}) + \bar{L}(UAR'\bar{P}) + \bar{P}'QAV'\bar{L} + \bar{P}'QAR'\bar{P}.$$

It is easy to see that B is limited if zero is not a limiting point of the point spectrum and does not belong to the continuous spectrum, for then the matrix U is limited, and the matrix $\{ dQ^{(\alpha)} \}$ is limited with respect to $\{ r_{ij}^{(\alpha)} \}$.

The first term of $\bar{B}$ is a limited matrix if the matrix $B = UAV'$ is limited. The following argument shows that B is limited in any case. The maximum value of $|\sum_{i,k=1}^{i,k=n} b_{ik}\, x_i\, x_k|$ for $\sum_{i=1}^{i=n} x_i^2 \leqq 1$, is equal to the absolute value of that characteristic number which is largest in absolute value and satisfies

$$\lambda^{(n)} z_i = \sum_{k=1}^{n} \left(\frac{1}{\sqrt{\lambda_i}} \sum_{j,l} l_{ij}\, a_{jl}\, l_{kl} \sqrt{\lambda_k} \right) z_k \qquad (i = 1, 2, \cdots, n).$$

We may assume that $\sum_{i=1}^{i=n} \lambda_i z_i^2 = 1$, and hence

$$\lambda^{(n)} = \sum_{i,k=1}^{n} \left(\sum_{j,l} l_{ij}\, a_{jl}\, l_{kl} \right) \sqrt{\lambda_i}\, z_i \sqrt{\lambda_k}\, z_k .$$

Since the matrix LAL' is limited (D and E § 2), $|\lambda^{(n)}| \leqq M$ for all values of n, and the matrix B is limited.

If C denotes the second term, C' denotes the third term. If zero is not a limiting point of the point spectrum, C is obviously limited, and hence also C'. If zero does not belong to the continuous spectrum, C' is limited, and hence C is also limited. To consider the remaining case, let

$$C^{(\epsilon)} = \bar{L}'(UAR'\bar{P}_\epsilon^b),$$

where $b > \epsilon > 0$, and construct $C^{(\epsilon)}\, C^{(\epsilon)\prime}$, which may be expressed as follows

$$C^{(\epsilon)}\, C^{(\epsilon)\prime} = \bar{L}'UAR'\bar{P}_\epsilon^b\, \bar{P}'Q_\epsilon^b\, AV'\bar{L}.$$

From the orthogonal property (4) of $\{ d\bar{P}^{(\alpha)}(\lambda\,;\,x) \}$, and results obtained in proving the first term limited, it follows that

$$|C^{(\epsilon)}\, C^{(\epsilon)\prime}(x, x)| \leqq M^2,$$

and since $\lim_{\epsilon=0} C_{ik}^{(\epsilon)} = C_{ik}$, the matrices C and C' are limited (F § 2) in all cases.

The fourth term of $\bar{b}_{ik}$ is limited if zero does not belong to the continuous spectrum. In case it does, let ϵ and $\bar{\epsilon}$ be positive and less than b; divide (ϵ, b) into n subintervals Δ_i and $(\bar{\epsilon}, b)$ into n subintervals $\bar{\Delta}_i$. Let

$$d_{ik}^{(n,\bar{n})} = \sum_{\alpha,j} \sum_{m=1}^{n} \frac{\Delta_m\, \bar{p}_i^{(\alpha)}\, \Delta_m\, q_j^{(\alpha)}}{\Delta_m\, r_0^{(\alpha)}} \sum_{l,\beta} a_{jl} \sum_{\bar{m}=1}^{\bar{n}} \frac{\Delta_{\bar{m}}\, r_l^{(\beta)}\, \Delta_m\, \bar{p}_k^{(\beta)}}{\Delta_{\bar{m}}\, r_0^{(\beta)}}$$

$$= \sum_{\alpha,j} \sum_{m=1}^{n} \frac{\Delta_m\, \bar{p}_i^{(\alpha)}\, \Delta_m\, r_j^{(\alpha)}}{\Delta_m\, r_0^{(\alpha)}} \sum_{l,\beta} a_{jl} \sum_{\bar{m}=1}^{\bar{n}} \frac{\Delta_{\bar{m}}\, q_l^{(\beta)}\, \Delta_{\bar{m}}\, \bar{p}_k^{(\beta)}}{\Delta_{\bar{m}}\, r_0^{(\beta)}},$$

then $D^{(n,\bar{n})}(x, x)$ is completely continuous (L § 2) for each value of n and $\bar{n}$, since it is composed of a limited form $A(x, x)$ and completely continuous forms (L § 2); but $D^{(n,\bar{n})}$ is not symmetric for n and $\bar{n}$ are not equal. Form

$$\sum_j d_{ij}^{(n,\bar{n})}\, d_{kj}^{(n,\bar{n})} = \sum_{\alpha,j} \sum_{m=1}^{n} \frac{\Delta_m\, \bar{p}_i^{(\alpha)}\, \Delta_m\, q_j^{(\alpha)}}{\Delta_m\, r_0^{(\alpha)}} \sum_{l,\beta} \bar{a}_{jl} \sum_{\bar{m}=1}^{n} \frac{\Delta_{\bar{m}}\, r_l^{(\beta)}\, \Delta_{\bar{m}}\, \bar{p}_k^{(\beta)}}{\Delta_{\bar{m}}\, r_0^{(\beta)}},$$

where $|\bar{A}(x, x)| \leqq M^2 \sum_i x_i^2$ and M is independent of $\bar{n}$. The maximum value of $|D^{(n, \bar{n})} D^{(n, \bar{n})'}(x, x)|$ for $\sum_i x_i^2 = 1$, since it is a quadratic form, is equal to the absolute value of that characteristic number μ, which is largest in absolute value, for the equations

$$\mu z_i = \sum_{\alpha, j} \sum_{m=1}^{n} \frac{\Delta_m \bar{p}_i^{(\alpha)}}{\sqrt{\Delta_m r_0^{(\alpha)}}} \cdot \frac{1}{\sqrt{\lambda_m^{(\alpha)}}} \frac{\Delta_m q_j^{(\alpha)}}{\sqrt{\Delta_m p_0^{(\alpha)}}} \sum_{l, \beta} \bar{a}_{jl} \sum_{\bar{m}=1}^{n} \frac{\Delta_{\bar{m}} r_l^{(\beta)}}{\lambda_{\bar{m}}^{(\beta)} \sqrt{\Delta_{\bar{m}} p_0^{(\beta)}}} \cdot \sqrt{\lambda_{\bar{m}}^{\beta}} \frac{\Delta_{\bar{m}} \bar{p}_k^{(\beta)}}{\sqrt{\Delta_{\bar{m}} r_0^{(\beta)}}} z_k,$$

where $\{ z_i \}$ is of finite norm and $\lambda_m^{(\alpha)} \Delta_m p_0^{(\alpha)} = \Delta_m r_0^{(\alpha)}$. Multiply both sides by $\sqrt{\lambda_m^{(\alpha)}} \Delta_m p_i^{(\alpha)} / \sqrt{\Delta_m r_0^{(\alpha)}}$ and sum with respect to i. Then multiply both sides by $\sum_i \sqrt{\lambda_m^{(\alpha)}} \Delta_m p_i^{(\alpha)} z_i / \sqrt{\Delta_m r_0^{(\alpha)}}$ and sum with respect to m and α. We may assume that

$$\sum_{\alpha, m} \left(\sum_i \frac{\sqrt{\lambda_m^{(\alpha)}} \Delta_m p_i^{(\alpha)}}{\sqrt{\Delta_m r_0^{(\alpha)}}} z_i \right)^2 = 1,$$

and since

$$\left| \sum_{\alpha, j, m} x_m^{(\alpha)} \frac{\Delta_m q_j^{(\alpha)}}{\sqrt{\Delta_m p_0^{(\alpha)}}} \sum_{l, \beta} \bar{a}_{jl} \sum_{\bar{m}} \frac{\Delta_{\bar{m}} r_l^{(\beta)}}{\lambda_{\bar{m}}^{(\beta)} \sqrt{\Delta_{\bar{m}} p_0^{(\beta)}}} x_{\bar{m}}^{(\beta)} \right| \leqq M^2 \sum_{\alpha} \sum_{m} (x_m^{(\alpha)})^2,$$

it follows that $|\mu| \leqq M^2$ for all values of n and $\bar{n}$, and hence

$$|D^{(n, \bar{n})}(x, x)| \leqq M \sum_i x_i^2,$$

and also

$$|\bar{P}' Q_\epsilon^b A R' \bar{P}_\epsilon^b (x, x)| \leqq M \sum_i x_i^2,$$

and in consequence of (F § 2) the last term of $\bar{b}_{ik}$ is limited.

Since $\bar{B}$ is limited and symmetric, there exist (I § 2) characteristic linear forms $\{ \bar{\bar{L}}_i(x) \}$ corresponding to real characteristic numbers $\bar{\bar{\lambda}}_i$ of the equations

$$\bar{\bar{\lambda}}_i \bar{\bar{l}}_{ik} = \sum_j \bar{b}_{kj} \bar{\bar{l}}_{ij}, \tag{12}$$

and characteristic linear differential forms $\{ d\bar{\bar{P}}^{(\beta)}(\lambda; x) \}$ limited with respect to $\{ \bar{\bar{p}}_0^{(\beta)} \}$ for an interval $(\bar{\bar{a}}, \bar{\bar{b}})$, of the equations

$$\int_{\Delta} \lambda \, d\bar{\bar{p}}_k^{(\beta)} = \sum_j \bar{b}_{kj} \Delta \bar{\bar{p}}_j^{(\beta)}, \tag{13}$$

where Δ is any subinterval Δ of $(\bar{\bar{a}}, \bar{\bar{b}})$. Let

$$\bar{\bar{v}}_{ik} = \sum_j \bar{\bar{l}}_{ij} \bar{v}_{jk}, \qquad \bar{\bar{r}}_k^{(\beta)}(\lambda) = \sum_j \bar{\bar{p}}_j^{(\beta)}(\lambda) \bar{v}_{jk}.$$

Then $\{ \bar{\bar{V}}_i(x) \}$ are characteristic linear forms, corresponding to $\{ \bar{\bar{\lambda}}_i \}$, of the equations

$$\bar{\bar{\lambda}}_i \bar{\bar{v}}_{ik} = \sum_j a_{kj} \bar{\bar{v}}_{ij}, \tag{14}$$

and $\{ d\overline{\overline{R}}^{(\beta)}(\lambda; x) \}$ are characteristic linear differential forms, for the continuous spectrum $(\bar{\bar{a}}, \bar{\bar{b}})$, of the equations

$$(15) \qquad \int_\Delta \lambda \, d\bar{\bar{r}}_k^{(\beta)}(\lambda) = \sum_j a_{kj} \Delta \bar{\bar{r}}_j^{(\beta)}$$

where Δ is any subinterval of $(\bar{\bar{a}}, \bar{\bar{b}})$.

Apply (7) and (11) to $B(x^*, z^*)$, where $x_i^* = \overline{V}_i(x)$, $z_i^* = \overline{V}_i(z)$ and we obtain

$$(16) \quad A(x, y) = \sum_j \bar{\bar{\lambda}}_j \overline{\overline{V}}_j(x) \overline{\overline{V}}_j(z) + \sum_\beta \int_{\bar{\bar{a}}}^{\bar{\bar{b}}} \lambda \cdot \frac{d\overline{\overline{R}}^{(\beta)}(\lambda; x) \, d\overline{\overline{R}}^{(\beta)}(\lambda; z)}{d\bar{\bar{p}}_0^{(\beta)}(\lambda)},$$

for all sequences $\{ x_i \}$ of finite norm, and all sequences $\{ y_i \}$ such that $y_i = \sum_k t_{ik} z_k$ where $\{ z_i \}$ is of finite norm. From this development (16) it follows that any limited linear form $F(x)$ which is orthogonal to $\{ \overline{\overline{V}}_i(x) \}$ and $\{ d\overline{\overline{R}}^{(\beta)}(\lambda; x) \}$ is such that $AF(x) = 0$, and the theorem is proved.

If the adjoint system of equations

$$(17) \qquad \bar{\bar{\lambda}}_i \bar{\bar{u}}_{ik} = \sum_j a_{jk} \bar{\bar{u}}_{ij}, \qquad \int_\Delta \lambda \, d\bar{\bar{q}}_k(\lambda) = \sum_j a_{jk} \Delta \bar{\bar{q}}_j$$

where Δ is any subinterval of $(\bar{\bar{a}}, \bar{\bar{b}})$, have characteristic forms $\{ \overline{\overline{U}}_i(x) \}$ and $\{ d\overline{\overline{Q}}^{(\beta)}(\lambda; x) \}$, then for the same values of λ the equations (12) and (13) have solutions, and $\{ T\overline{\overline{U}}_i(x) \}$ and $\{ Td\overline{\overline{Q}}^{(\beta)}(\lambda; x) \}$ satisfy (14) and (15).

If

$$\bar{\bar{u}}_{ik} = \sum_j \bar{\bar{l}}_{ij} \bar{u}_{jk}, \qquad \bar{\bar{q}}_k^{(\beta)}(\lambda) = \sum_j \bar{\bar{p}}_j^{(\beta)}(\lambda) \bar{u}_{jk}$$

exist and are the coefficients of limited systems, the forms $\{ \overline{\overline{U}}_i(x) \}$ and $\{ d\overline{\overline{Q}}^{(\beta)}(\lambda; x) \}$ are characteristic forms of (17) corresponding to $\{ \bar{\bar{\lambda}}_i \}$ and $(\bar{\bar{a}}, \bar{\bar{b}})$, form a biorthogonal system with $\{ \overline{\overline{V}}_i(x) \}$ and $\{ d\overline{\overline{R}}^{(\beta)}(\lambda; x) \}$, and the development (16) becomes

$$A(x, y) = \sum_j \bar{\bar{\lambda}}_j \overline{\overline{V}}_j(x) \overline{\overline{U}}_j(y) + \sum_\beta \int_{\bar{\bar{a}}}^{\bar{\bar{b}}} \lambda \cdot \frac{d\overline{\overline{R}}^{(\beta)}(\lambda; x) \, d\overline{\overline{Q}}^{(\beta)}(\lambda; y)}{d\bar{\bar{p}}_0^{(\beta)}},$$

where $\{x_i\}$ and $\{z_i\}$ are any sequences of finite norm, and $y_i = \sum_k t_{ik} z_k$.

Some sufficient conditions that (17) have characteristic forms are: 1), that the matrix U be limited, which is so in case the values of λ belonging to the spectrum are $> M > 0$; 2), that $A = TK$ where K is a symmetric limited matrix; 3), that $A(x, x)$ be a completely continuous function (K § 2).

The following example shows that for a limited matrix A such that AT is symmetric, where T is a positive definite, symmetric, and limited matrix, the equations (14) and (15) may have characteristic forms while the adjoint system (17) does not. Let

$$u_{ik} = \begin{cases} \lambda_i & i = k \\ -\lambda_i & i = k-1 \\ 0 & i \neq k, i \neq k-1, \end{cases}$$

where $\lambda_i = 2 - 1/2^{i-1}$. The only solutions of (17) correspond to λ_i, but they are not of finite norm. The equations (14) have solutions of finite norm for $\lambda = \lambda_i$ and they form a complete system. In fact, these equations (14) have solutions of finite norm for $|\lambda| < 2$. If $\lambda_i = 1$, the equations (17) have no solutions whatever, and the equations (14) have solutions of finite norm for $|1 - \lambda| < 1$. The solutions are $x_i = (1 - \lambda)^{i-1}$.

Theorems I and II enable us to give the following form to the Theorems VI and VII.

THEOREM VIII. *The necessary and sufficient condition that for a limited matrix A, there exist real characteristic forms and characteristic linear differential forms of the equations* (5) *and* (6), *such that they form a complete system with limited linear forms $F(x)$ such that $AF_i(x) \equiv 0$, is the existence of two matrices U and V such that U is the rear reciprocal of V', the matrix UAV' is symmetric, the rows of U are of finite norm, V is a limited matrix, and the systems $\{U_i(x)\}$ and $\{V_i(x)\}$ are complete.*

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