INVARIANTS OF INFINITE GROUPS IN THE PLANE*

BY

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In a previous paper published in these Transactions† the writer discussed the general question of the invariants of differential configurations in the plane under groups—finite and infinite—of point or contact transformations. The present paper deals with the application of the general results therein obtained to the various types of infinite groups of point and contact transformations.

I. Point transformations

The problem of the determination of the different types of such groups was solved by Lie.‡ Adopting his results as a basis of classification we subdivide infinite groups of point transformations into the following types:

(A) The entire group of point transformations.

(B) Those reducible by change of coördinates to the group which multiplies areas by a constant.

(C) Those reducible to the area-preserving group.

(D) Those which leave invariant one—and only one—differential equation of the first order.

(E) Those which leave invariant two differential equations of the first order.

The types (A) and (C) have already been dealt with from the point of view of their invariants. It was found that for type (A) the smallest number of curves—denoted by λ_n—having contact of zero order necessary for the existence of an invariant of order n was $2n + 2$, while for (C) it was $n + 3$. Explicit expressions for such invariants were calculated as far as the third order.

We now proceed to consider the remaining types.

Type (B). This is the only class of infinite groups of point transformations defined exclusively by differential equations of the second order. Its simplest representative is the group

$$\xi_{xx} + \eta_{xy} = 0; \qquad \xi_{xy} + \eta_{yy} = 0,$$

which transforms areas in a constant ratio.

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† Vol. 19 (1918), pp. 223–250.

‡ *Ueber unendliche continuirliche Gruppen*, Videnskabs-Selskabet Christiania, 1883.

The equivalent complete system of linear differential equations does not differ from that of the group (A) until it is extended to the third order. The additional equations corresponding to the third and higher orders are identical with those of the area-preserving group. *Hence for the type* (B) *we have* $\lambda_1 = 4;\ \lambda_2 = 6;\ \lambda_n = n + 3\,(n > 2)$.

It is a simple matter to obtain the first three invariants. The first two are identical with those of the group (A).* If we compare the complete system with that of the area-preserving group we find that the only difference is that the equation

$$V \equiv \sum 2y_i' \frac{\partial f}{\partial y_i'} + \sum 3y_i'' \frac{\partial f}{\partial y_i''} + \sum 4y_i''' \frac{\partial f}{\partial y_i'''} = 0,$$

characteristic of the area-preserving group is replaced by the pair of equations.

$$U_1 \equiv \sum y_i' \frac{\partial f}{\partial y_i'} + \sum y_i'' \frac{\partial f}{\partial y_i''} + \sum y_i''' \frac{\partial f}{\partial y_i'''} = 0,$$

$$U_2 \equiv \sum y_i' \frac{\partial f}{\partial y_i'} + \sum 2y_i'' \frac{\partial f}{\partial y_i''} + \sum 3y_i''' \frac{\partial f}{\partial y_i'''} = 0.$$

Since $V = U_1 + U_2$, it is clear that the group (B) differs only from the area-preserving group by the addition of the equation $U_1 = 0$. We have thus proved the

THEOREM. *The invariants of the group* (B) *are those invariants of the area-preserving group which are homogeneous and of degree zero.*

Of the three invariants of the area-preserving group I_4, J_5, K_6—given in our previous paper—I_4 was homogeneous of degree zero; J_5 was homogeneous of degree $-\frac{1}{3}$; and K_6 was homogeneous of degree -1. It is clear, then, that

$$K_6\, J_5^{-3}$$

is a third-order invariant of the group (B).

It is to be remarked that in the expression

$$J_5^{-6}\,(A_6 K_6 + A_7 K_7 + A_8 K_8)$$

for the third-order invariant of the entire group (A)† each of the three terms

$$J_5^{-6} A_6 K_6, \qquad J_5^{-6} A_7 K_7, \qquad J_5^{-6} A_8 K_8$$

is a third-order invariant of the group (B).

Type (D). The canonical form of the equation-system defining this type is the single equation $\xi_y = 0$, and the finite equations of the group are

$$X = \phi(x), \qquad Y = \psi(x, y),$$

ϕ and ψ being arbitrary functions.

* Page 230.
† Page 244.

The most important subgroup not of type (E) is

$$X = \phi(x), \qquad Y = \frac{\psi_1(x) + \psi_2(x)y}{\psi_3(x) + \psi_4(x)y}.$$

This was shown by the writer to be the largest group having the following properties:

1. For the existence of an infinite system of absolute invariants only a finite number of curves are required.* (In this case $\lambda_n = 5$, which is the highest value of λ_n independent of n for any group.)

2. Though the base-points of the configuration may have only one coördinate in common, invariants exist under the group.†

For type (D) we shall content ourselves with merely stating the following results, which are readily verified.

THEOREM. *For the most general group leaving invariant a differential equation of the first order* $\lambda_1 = 3$. *For* $n > 1$, $\lambda_n = n + 3$.

If the invariant differential equation is

$$y' = f(x, y)$$

the invariant of the first order is obviously the cross-ratio $[y_1' \, y_2' \, y_3' f]$.

We have already found that in case all the arbitrary functions in the defining-equations of a group are functions of one variable, the value of λ_n for the group could at once be written down. There are then only three cases left under type (D), corresponding respectively to the values

$$x, \qquad ax + b, \qquad \frac{ax + b}{cx + d}$$

of $\phi(x)$, the second function $\psi(x, y)$ being completely arbitrary.

For all of these cases λ_n *takes the value* $n + 2$ *after a finite value of* n. This is an illustration of the result which is true in general that the presence of a finite number of arbitrary constants in the defining-equations of the group can only affect λ_n for a finite number of values of n. The subsequent values of λ_n depend only on the number of arbitrary *functions*.

Type (E). This last type includes some important groups, the most noteworthy being the conformal. The equilong group may be regarded as a degenerate case. Certain results have been obtained for the conformal and equilong groups by Kasner,‡ and for the equilong by the writer.§

* Page 235.

† Page 236.

‡ *Conformal Geometry, Proceedings of the Fifth International Congress*, Cambridge (1912) vol. 2, pp. 81–87; *Conformal classification of analytic arcs or elements: Poincaré's local problem of conformal geometry*, these Transactions, vol. 16 (1915), pp. 333–349; *Equilong invariants and convergence proofs*, Bulletin of the American Mathematica Society, vol. 23 (1917), pp. 341–347.

§ Loc. cit., pp. 244–246.

The general type of these groups is

$$(1) \qquad X = \phi(x), \qquad Y = \psi(y),$$

ϕ and ψ being arbitrary functions. The conformal group can be reduced to the form (I) by the transformation

$$x = u + iv, \qquad y = u - iv.$$

It is found that for the largest groups of type (E)*—that is, those involving two arbitrary functions—*$\lambda_1 = 2, \lambda_2 = 4, \lambda_n = 3$ *for* $n > 2$. This is true of both the conformal and equilong groups. In the latter case the three curves of the configuration need not pass through the same point, though for the most general type it is necessary that they do so.* We shall obtain invariants of the group (I) as far as the fourth order.†

The equations (1) may be taken in the form

$$X = \int e^{-\phi(x)} dx, \qquad Y = \int e^{\psi(y)} dy.$$

We obtain in succession

$$Y' = e^{\psi+\phi} y',$$

$$Y'' = e^{\psi+2\phi}(y'' + \psi' y'^2 + \phi' y'),$$

$$Y' Y''' - \tfrac{3}{2} Y''^2 = e^{2\psi+4\phi}[(y' y''' - \tfrac{3}{2} y''^2) + (\psi'' - \tfrac{1}{2}\psi'^2) y'^4 + (\phi'' + \tfrac{1}{2}\phi'^2) y'^2],$$

$$Y'^2 Y^{(\mathrm{iv})} - 6Y' Y'' Y''' + 6Y''^3 = e^{3\psi+5\phi}[(y'^2 y^{(\mathrm{iv})} - 6y' y'' y''' + 6y''^3) - 2\psi' y'^2 (y' y''' - \tfrac{3}{2} y''^2) - 2y'^2 (y'' + \psi' y'^2)(\phi'' + \tfrac{1}{2}\phi'^2) + (\psi''' + \psi'^3 + 3\psi' \psi'') y'^6 + (\phi''' + \phi' \phi'') y'^3].$$

The presence of the exponential factors on the right means that all the invariants are homogeneous and isobaric. This is also evident from the consideration that the groups $X = x, Y = ay$; $X = ax, Y = y$ are subgroups of (I). The first has only homogeneous invariants; the second only isobaric.

By eliminating the arbitrary functions from the above equations we find a relative invariant of each order for three curves passing through a common

* Loc. cit., p. 235.

† The fact that four curves through a common point, with distinct tangents, have a conformal invariant of second order was first stated in an article by Kasner, *The geometry of differential elements of the second order, etc.*, American Journal of Mathematics, vol. 28 (1906), p. 213. The invariant is there shown to be expressible as the anharmonic ratio of the lines connecting C_1 with C_2, C_3, C_4, O, where O is the common point, and C_1, C_2, C_3, C_4 are the centers of curvature of the four curves.

point. These are

$$I_1 = y_i', \qquad I_2 = |y_i'' y_i'^2 y_i'|,$$

$$I_3 = |y_i' y_i''' - \tfrac{3}{2} y_i''^2 y_i'^4 y_i'^2|,$$

$$I_4 = |y_i'^2 y_i^{(\text{iv})} - 6 y_i' y_i'' y_i''' + 6 y_i''^3 - 2K_1 y_i'^2 (y_i' y_i''' - \tfrac{3}{2} y_i''^2) - 2K_2 (y_i'' + K_1 y_i'^2) y_i'^6 y_i'^3|.$$

I_2, I_3, I_4 are three-rowed determinants, of which the ith row is given in each case $(i = 1, 2, 3)$. Moreover

$$K_1 = \begin{vmatrix} y_1' & y_1'' \\ y_2' & y_2'' \end{vmatrix} \div y' y' (y' - y'),$$

$$K_2 = \begin{vmatrix} y_1'^4 & y_1' y_1''' - \tfrac{3}{2} y_1''^2 \\ y_2'^4 & y_2' y_2''' - \tfrac{3}{2} y_2''^2 \end{vmatrix} \div y_1'^2 y_2'^2 (y_1'^2 - y_2'^2).$$

The magnifying factors of I_1, I_2, I_3, I_4 are respectively $e^{\psi+\phi}$, $e^{4\psi+5\phi}$, $e^{8\psi+10\phi}$, $e^{12\psi+14\phi}$. Hence we have the four absolute invariants

$$y_2' - y_1'; \qquad y_3' - y_1'; \qquad I_3 I_2^{-2}; \qquad I_4 I_3^{-1} I_1^{-4}.$$

II. Contact transformations

In our previous paper we dealt with the question of finite groups. There remain only those infinite groups that are not reducible to groups of point transformations. There are three types, whose characteristic functions are

$$(F) \quad ay + W(x, y'),$$

$$(G) \quad W(x, y'),$$

$$(H) \quad \Omega(x, y, y'),$$

where W and Ω are arbitrary functions of their arguments, and a is an arbitrary parameter.* (F) and (G) are imprimitive. (H) is the entire group of contact transformations in the plane.

We have shown that the group defined by (H) has invariants of the same form as those of the entire group of point transformations, except that the order of each derivative is increased by unity. $2n$ analytic curves having contact of the first order but subject only to that restriction have a single invariant of order $n + 1$ for every $n > 1$.†

The infinitesimal transformations of the type (G) are

$$(2) \qquad \xi \frac{\partial f}{\partial x} + \eta \frac{\partial f}{\partial y} + \pi \frac{\partial f}{\partial y'}$$

* Lie, Leipsiger Abhandlungen, vol. 21, p. 150.

† Loc. cit., Transa, p. 247.

where

$$\xi = W_{y'}, \qquad \eta = y' W_{y'} - W, \qquad \pi = - W_x.$$

It is to be noticed that $\xi_x + \pi_{y'} = 0$. If, then, we take x, y' as point coördinates in a plane, the group

$$\xi \frac{\partial f}{\partial x} + \pi \frac{\partial f}{\partial y'}$$

becomes the area-preserving group of point transformations. The extended complete system of (2) is identical with that of (3), since neither ξ nor π involve y. It follows, then, that the complete system equivalent to the group of contact transformations whose characteristic function is the same as that of the area-preserving group of point transformations except that the order of each derivative is increased by unity.

For the entire group of contact transformations it is necessary that all the curves of the configuration shall pass through a common point and have contact of at least the first order if the configuration is to have invariants. For the group $W(x, y')$ this is not necessary. Since all the arbitrary functions involve only x and y', the most general configuration having invariants consists of curves whose base-points have the same abscissa, and whose tangents are parallel. From our results for the area-preserving group, then, we have the

THEOREM. *If our configuration consists of curves whose base-points have the same abscissa and whose tangents are parallel, $n + 3$ curves have an invariant of order $n + 1$ under the group of contact transformations whose characteristic function is $W(x, y')$. The invariants are of the same form as those of the area-preserving group of point-transformations for configurations of $n + 3$ curves having contact of zero order. The order of each derivative, however, is increased by unity.*

The group $ay + W(x, y')$ has the infinitesimal transformations

$$\xi = W_{y'}; \qquad \eta = y' W_{y'} - ay - W; \qquad \pi = - W_x - ay'.$$

Hence we have

$$\xi_{xx} + \pi_{xy'} = 0,$$

$$\xi_{xy'} + \pi_{y'y'} = 0,$$

and it is clear that this group bears the same relation to the group (B) of point transformations as $W(x, y')$ bears to the area-preserving group. The results for this group, then, are as follows:

THEOREM. *The group of contact transformations whose characteristic function is $ay + W(x, y')$ has invariants for a configuration of $n + 3$ curves whose base-points have the same abscissa and whose tangents are parallel. There is one invariant of order $n + 1$, which is identical with the corresponding invariant* *of the group of point transformations which multiplies areas by a constant, except that the order of each derivative is increased by unity.*

And finally:

THEOREM. *The invariants of the group $ay + W(x, y')$ are the invariants of the group $W(x, y')$ for the same configuration which are homogeneous of degree zero.*

III. SUMMARY

We may resume the results for λ_n for the various types of infinite groups in the following table:

	Point Transformations					Contact		
Type	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*
λ_n	$2n+2$	$n+3$	$n+3$	$n+2$	3	$2n+2$	$n+3$	$n+3$

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