OSCILLATION THEOREMS FOR THE REAL, SELF-ADJOINT LINEAR SYSTEM OF THE SECOND ORDER*

BY

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Introduction

It is the object of this paper to determine the number of oscillations of a linear combination of the form (2) for the systems (3) and (4). From these results, an oscillation theorem for the solution $u_p(x)$, corresponding to the pth characteristic number of (4), is obtained.

Given the second order self-adjoint linear differential equation

$$(1) \qquad \frac{d}{dx}\left[K(x,\lambda)\frac{du}{dx}\right] - G(x.\lambda)u = 0$$

and two linear combinations of a solution which does not vanish identically and its first derivative,

$$(2) \quad L_i[u(x,\lambda)] = \alpha_i(x,\lambda)u(x,\lambda) - \beta_i(x,\lambda)K(x,\lambda)u_x(x,\lambda) \quad (i=1,2),$$

we shall impose the following conditions and shall assume that they are satisfied throughout this paper:

I. $K(x,\lambda)$, $G(x,\lambda)$, $\alpha_i(x,\lambda)$, $\beta_i(x,\lambda)$, $\alpha_{ix}(x,\lambda)$, $\beta_{ix}(x,\lambda)$† are continuous, real functions of x in the interval

$$(X) \qquad (a \leqq x \leqq b)$$

and for all real values of λ in the interval

$$(\Lambda) \qquad (\mathcal{L}_1 < \lambda < \mathcal{L}_2).$$

II. $K(x,\lambda)$ is positive everywhere in (X,Λ) and

$$|\alpha_i| + |\beta_i| > 0$$

in (X,Λ).

III. For each value of x in (X), K and G decrease (or do not increase)

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† $f_{ix}(x,\lambda) = \frac{\partial}{\partial x} f_i(x,\lambda)$.

as λ increases. In no sub-interval of (X) are K and G simultaneously independent of λ and in no sub-interval of (X) is G identically zero.

IV. Either $\beta_i \equiv 0$ for all values of x and λ in (X, Λ); or else $\beta_i \neq 0$ in $a < x < b$ for all λ's in (Λ) and one of the following is true,

either (*a*) $\beta_i(a) \equiv 0$ for all λ's in (Λ), $\beta_i(b) \neq 0$, $-\alpha_i(b)/\beta_i(b)$ decreases (or does not increase) as λ increases;

or (*b*) $\beta_i(b) \equiv 0$, $\beta_i(a) \neq 0$ for all λ's in (Λ), $\alpha_i(a)/\beta_i(a)$ decreases (or does not increase) as λ increases;

or (*c*) $\beta_i(a) \neq 0$, $\beta_i(b) \neq 0$, $\alpha_i(a)/\beta_i(a)$ and $-\alpha_i(b)/\beta_i(b)$ decrease (or do not increase) as λ increases.

V.*
$$\lim_{\lambda \to \mathfrak{L}_1} -\frac{\min G}{\min K} = -\infty,$$
$$\lim_{\lambda \to \mathfrak{L}_2} -\frac{\max G}{\max K} = +\infty.$$

I. The Sturmian system

Concerning the system

$$(3) \qquad \frac{d}{dx}\left[K(x,\lambda)\frac{du}{dx}\right] - G(x,\lambda)u = 0,$$
$$L_1[u(a,\lambda)] = 0, \qquad L_1[u(b,\lambda)] = 0,$$

Sturm's oscillation theorem† may be stated with Bôcher‡ substantially as follows:

The system (3) *satisfying conditions I–V has an infinite set of characteristic numbers such that*

$$\mathfrak{L}_1 < \lambda_0 < \lambda_1 < \lambda_2 < \cdots < \mathfrak{L}_2$$

and $U(x, \lambda_p)$, *the pth characteristic function, vanishes exactly p times on* $a < x < b$.

We seek to determine the number of oscillations of $L_1[U(x, \lambda_p)]$. We notice first that if $\beta_1 \equiv 0$ for all values of x and λ in (X, Λ), then all the zeros of $U(x, \lambda_p)$ and $L_1[U(x, \lambda_p)]$ coincide, since $\alpha_1(x, \lambda) \neq 0$. Hence we have for $\beta_1 \equiv 0$ precisely p zeros of $L_1[U(x, \lambda_p)]$ on $a < x < b$.

With Bôcher§ we define

$$\{\alpha_1 \beta_1\} = \beta_1 \alpha_{1x} - \alpha_1 \beta_{1x} + \frac{\alpha_1^2}{K} - \beta_1^2 G.$$

* This condition may be replaced by other sets of conditions. See Bôcher, *Leçons sur les Méthodes de Sturm* (hereafter referred to as *Leçons*) (1917), chap. III, paragraphs 13–15.

† Journal de Mathématiques pures et appliquées, vol. 1 (1836), p. 106 ff.

‡ *Leçons*, p. 63 ff.

§ *Leçons*, p. 45.

Let $\beta_1 \neq 0$ in (X, Λ), and consider the zeros of $U(x, \lambda)$ and $L_1[U(x, \lambda)]$. For $\lambda = \lambda_0$, $L_1[U(a, \lambda_0)] = 0$ and $L_1[U(b, \lambda_0)] = 0$, but $U(x, \lambda_0)$ does not vanish on (X). Hence $\{\alpha_1 \beta_1\}_{\lambda=\lambda_0}$ must vanish for some value of x in (X), since if $\{\alpha_1 \beta_1\}_{\lambda=\lambda_0} < 0$ throughout (X), $L_1[U(x, \lambda_0)]$ could vanish but once in (X);* and if $\{\alpha_1 \beta_1\}_{\lambda=\lambda_0} > 0$ for every x in (X), $U(x, \lambda_0)$ would vanish once in (X).† If $\{\alpha_1 \beta_1\} > 0$ for $\lambda \geqq \lambda_1$ for all values of x in (X), then the zeros of $L_1[U(x, \lambda)]$ and $U(x, \lambda)$ separate.† But $U(x, \lambda_p)$ vanishes exactly p times on $a < x < b$. Hence $L_1[U(x, \lambda_p)]$ vanishes $p - 1$ times on $a < x < b$ for $p \geqq 1$.

If $\beta_1(a, \lambda) \equiv 0$ in (Λ), $\beta_1(x, \lambda) \neq 0$ for $a < x \leqq b$, then $\alpha(a) \neq 0$ in (Λ). Then the zeros of $L_1[U(x, \lambda_p)]$ and $U(x, \lambda_p)$ separate† each other on $a < x \leqq b$, provided $\{\alpha_1 \beta_1\}_{\lambda=\lambda_p} > 0$. Let x_1 be the first zero of $U(x, \lambda_p)$ on $a < x \leqq b$. Then

$$L_1[U(x_1, \lambda_p)] = -\beta_1(x_1, \lambda_p) K(x_1, \lambda_p) U_x(x_1, \lambda_p).$$

Now direct computation shows that

$$L_{1x}[U(a)] = \{\alpha_1 \beta_1\}_{x=a} \frac{K(a) U_x(a)}{\alpha_1(a)}.$$

But $K(a) U_x(a)$ and $K(x_1, \lambda_p) U_x(x_1, \lambda_p)$ have opposite signs, and $\operatorname{sgn} \beta_1(x_1, \lambda_p) = \operatorname{sgn} \beta_1(b)$.‡ Hence

$$\operatorname{sgn} L_1[U(x_1, \lambda_p)] = \operatorname{sgn} L_{1x}[U(a)] \cdot \operatorname{sgn} \beta_1(b) \operatorname{sgn} \alpha_1(a).$$

Thus $L_1[U(x, \lambda_p)]$ vanishes or does not vanish in $a < x < x_1$ according as $\alpha_1(a) \beta_1(b)$ is negative or is positive respectively. Hence if $\alpha_1(a) \beta_1(b) > 0$, $L_1[U(x, \lambda_p)]$ has $p - 1$ zeros on $a < x < b$, and if $\alpha_1(a) \beta_1(b) < 0$, $L_1[U(x, \lambda_p)]$ has p zeros on $a < x < b$, for $p \geqq 1$.

If $\beta_1(b, \lambda) \equiv 0$ in (Λ), $\beta_1(x, \lambda) \neq 0$ for $a \leqq x < b$, then a similar argument can be made with the following result: if $\alpha_1(b) \beta_1(a) > 0$, $L_1[U(x, \lambda_p)]$ has $p - 1$ zeros on $a < x < b$, and if $\alpha_1(b) \beta_1(a) < 0$, $L_1[U(x, \lambda_p)]$ has p zeros on $a < x < b$, for $p \geqq 1$. Therefore

OSCILLATION THEOREM I. *If $U(x, \lambda_p)$ is the pth characteristic function of the system* (3) *satisfying conditions I–V and if $\{\alpha_1 \beta_1\} > 0$ for $\lambda \geqq \lambda_1$ for every x in (X), then $L_1[U(x, \lambda_p)]$ will vanish on $a < x < b$ for $p \geqq 1$,*
p times if
either $\beta_1 \equiv 0$ in (X, Λ);
or $\beta_1(a) \equiv 0$ in (Λ), $\beta_1(x, \lambda) \neq 0$ in $a < x \leqq b$ and $\alpha_1(a) \beta_1(b) < 0$;
or $\beta_1(b) \equiv 0$ in (Λ), $\beta_1(x, \lambda) \neq 0$ in $a \leqq x < b$ and $\alpha_1(b) \beta_1(a) < 0$;
$p - 1$ times if

* Bôcher, *Leçons*, p. 51.
† Ibid., p. 50.
‡ sgn f = sign of f.

either $\beta_1 \neq 0$ *in* (X, Λ);

or $\beta_1(a) \equiv 0$ *in* (Λ), $\beta_1(x, \lambda) \neq 0$ *in* $a < x \leqq b$ *and* $\alpha_1(a)\beta_1(b) > 0$;

or $\beta_1(b) \equiv 0$ *in* (Λ), $\beta_1(x, \lambda) \neq 0$ *in* $a \leqq x < b$ *and* $\alpha_1(b)\beta_1(a) > 0$.

Let $L[u(x, \lambda)] = \alpha(x, \lambda)u(x, \lambda) - \beta(x, \lambda)K(x, \lambda)u_x(x, \lambda)$, where $\alpha, \beta, \alpha_x, \beta_x$ are continuous in (X, Λ). Then we may state

OSCILLATION THEOREM II. *If* $U(x, \lambda_p)$ *is the pth characteristic function of* (3) *satisfying conditions I–V, where* $\{\alpha_1 \beta_1\} > 0$, $\{\alpha\beta\} > 0$ *for* $\lambda \geqq \lambda_1$ *and* $(\alpha_1\beta - \alpha\beta_1) \neq 0$ *in* (X, Λ), *then* $L[U(x, \lambda_p)]$ *will vanish on* $a < x < b$, *for* $p \geqq 1$,

p times if

either $\beta_1 \neq 0$ *in* (X, Λ);

or $\beta_1(a) \equiv 0$ *in* (Λ), $\beta_1(x, \lambda) \neq 0$ *in* $a < x \leqq b$ *and* $\alpha_1(a)\beta_1(b) > 0$;

or $\beta_1(b) \equiv 0$ *in* (Λ), $\beta_1(x, \lambda) \neq 0$ *in* $a \leqq x < b$ *and* $\alpha_1(b)\beta_1(a) > 0$;

$p + 1$ *times if*

either $\beta_1 \equiv 0$ *in* (X, Λ);

or $\beta_1(a) \equiv 0$ *in* (Λ), $\beta_1(x, \lambda) \neq 0$ *in* $a < x \leqq b$ *and* $\alpha_1(a)\beta_1(b) < 0$;

or $\beta_1(b) \equiv 0$ *in* (Λ), $\beta_1(x, \lambda) \neq 0$ *in* $a \leqq x < b$ *and* $\alpha_1(b)\beta_1(a) < 0$.

Proof: The zeros of $L_1[U(x, \lambda_p)]$ and $L[U(x, \lambda_p)]$ separate one another on (X).* But $L_1[U(a, \lambda_p)] = 0$ and $L_1[U(b, \lambda_p)] = 0$. Hence $L[U(x, \lambda_p)]$ has one more zero on $a < x < b$, for $p \geqq 1$, than $L_1[U(x, \lambda_p)]$. This proves Theorem II.

II. THE GENERAL SELF-ADJOINT SYSTEM

Consider the system

$$\frac{d}{dx}\left[K(x, \lambda)\frac{du}{dx}\right] - G(x, \lambda)u = 0, \tag{4}$$

$$L_1[u(a, \lambda)] = L_1[u(b, \lambda)], \qquad L_2[u(a, \lambda)] = L_2[u(b, \lambda)],$$

where L_1 and L_2 are defined as in (2) and $\beta_2 \neq 0$† in (X, Λ). We impose further conditions:

VI. $$\{\alpha_i \beta_i\} \neq 0 \text{ for } \lambda \geqq \lambda_1,$$

VII. $$\alpha_1\beta_2 - \alpha_2\beta_1 \equiv -1 \text{ in } (X, \Lambda),$$

VIII.‡ $$\begin{vmatrix} \alpha_1(a) & \beta_1(a) & \alpha_1(b) & \beta_1(b) \\ \alpha_2(a) & \beta_2(a) & \alpha_2(b) & \beta_2(b) \\ \Delta\alpha_1(a) & \Delta\beta_1(a) & \Delta\alpha_1(b) & \Delta\beta_1(b) \\ \Delta\alpha_2(a) & \Delta\beta_2(a) & \Delta\alpha_2(b) & \Delta\beta_2(b) \end{vmatrix} \geqq 0.$$

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* Bôcher, *Leçons*, p. 50.

† This restriction does not involve a loss of generality, since if $\beta_1 \neq 0$, $\beta_2 \equiv 0$ then we may interchange L_1 and L_2.

‡ $\Delta f = f(\lambda + \Delta\lambda) - f(\lambda)$ for $\Delta\lambda > 0$.

In a recent paper* the writer proved the following theorem concerning the system (4), satisfying conditions I–VIII:

There exists one and only one characteristic number of (4) *between every pair of characteristic numbers of the Sturmian system* (3). *If* λ_p *represents the ordered characteristic numbers of* (3) *and* l_p *those of* (4) (*account being taken of their multiplicity*) *then*

Case I. l_p *is in the interval* $(\lambda_p, \lambda_{p+1})$ *if* $L_2[U(b, \lambda_0)] \cdot \phi(\mathcal{L}_1 + \epsilon) > 0$.†

Case II. l_p *is in the interval* $(\lambda_{p-1}, \lambda_p)$, $p \geqq 1$, *if*

$$L_2[U(b, \lambda_0)] \cdot \phi(\mathcal{L}_1 + \epsilon) < 0.$$

Let $u_p(x) = u(x, l_p)$ be the pth characteristic function of (4). We proceed to consider the number of oscillations of $L_1[u_p(x)]$. We notice first that if $\lambda = l_p$ is a double characteristic number, then l_p coincides with λ_{p-1}, λ_p, or λ_{p+1}, and $L_1[u_p(x)]$ will have the number of oscillations designated by Theorem I.

If $\lambda = l_p$ is a simple value, we may discriminate between Case I and Case II, following a method due to Birkhoff.‡ If $\beta_1 \equiv 0$ in (X, Λ), the sign of $L_2[U(b, \lambda_0)]$ is the same as $-\alpha_1(a)\alpha_1(b)$, which is negative, since $\alpha_1 \neq 0$ in (X, Λ). Hence $L_2[U(b, \lambda_0)]$ is negative. Also $\phi(\mathcal{L}_1 + \epsilon)$ has the sign of $-\beta_2(a)\beta_2(b)$, but $\beta_2 \neq 0$ in (X, Λ). Hence $\phi(\mathcal{L}_1 + \epsilon)$ is negative, and we have Case I where l_p is on the interval $\lambda_p < \lambda < \lambda_{p+1}$. By Theorem I, $L_1[U(x, \lambda_p)]$ vanishes exactly p times on $a < x < b$ for $p \geqq 1$, and $L_1[U(x, \lambda_{p+1})]$ vanishes $p + 1$ times on $a < x < b$ for $p \geqq 1$. But $L_1[U(b, \lambda)] \neq 0$ for $\lambda_p < \lambda < \lambda_{p+1}$. Hence $L_1[U(x, \lambda)]$ vanishes $p + 1$ times on $a < x < b$ for $\lambda_p < \lambda < \lambda_{p+1}$. But the roots of $L_1[U(x, \lambda)]$ and $L_1[u(x, \lambda)]$§ separate one another, and $L_1[U(a, \lambda)] = 0$. Hence $L_1[u(x, \lambda)]$ vanishes $p + 1$ or $p + 2$ times on $a < x < b$ for $\lambda_p < \lambda < \lambda_{p+1}$, $p \geqq 1$. Hence $L_1[u_p(x)]$ vanishes either $p + 1$ or $p + 2$ times on $a < x < b$, $p \geqq 1$. But from (4) the number of zeros of $L_1[u_p(x)]$ is always even. Therefore we have $p + 2$ roots if p is even and $p + 1$ roots for p odd, $p \geqq 1$.

If $\beta_1(a) \equiv 0$ in (Λ) and $\beta_1(b) \neq 0$, the sign of $L_2[U(b, \lambda_0)]$ is that of $\alpha_1(a)/\beta_1(b)$, and $\phi(\mathcal{L}_1 + \epsilon)$ has the sign of $\beta_1(b)/\alpha_1(a)$. Hence we have Case I where l_p is on the interval $\lambda_p < \lambda < \lambda_{p+1}$. If $\alpha_1(a)\beta_1(b) < 0$, by Theorem I, $L_1[U(x, \lambda_p)]$ vanishes p times on $a < x < b$ and $L_1[U(x, \lambda_{p+1})]$ vanishes $p + 1$ times on $a < x < b$. Reasoning exactly as above, we find that $L_1[u_p(x)]$ vanishes $p + 2$ times on $a < x < b$ if p is even and $p + 1$

* *Existence Theorems for the General, Real, Self-Adjoint Linear System of the Second Order*, these Transactions, vol. 19 (1918), p. 94.

† $\phi(\lambda) = 0$ is the characteristic equation of (4) whose roots are the characteristic numbers, l_p. $L_1 + \epsilon$ is a value of λ in (Λ) near L_1.

‡ *Existence and Oscillation Theorem for a Certain Boundary Value Problem*, these Transactions, vol. 10 (1909), pp. 259–270.

§ Bôcher, *Leçons*, p. 48.

times if p is odd, $p \geqq 1$. If $\alpha_1(a)\beta_1(b) > 0$, by Theorem I, $L_1[U(x,\lambda_p)]$ vanishes $p-1$ times on $a < x < b$, and $L_1[U(x,\lambda_{p+1})]$ vanishes p times on $a < x < b$. But $L_1[U(b,\lambda)] \neq 0$ for $\lambda_p < \lambda < \lambda_{p+1}$, and $L_1[U(a,\lambda)] = 0$. The roots of $L_1[U(x,\lambda)]$ and $L_1[u(x,\lambda)]$ separate one another. Hence $L_1[u_p(x)]$ vanishes either p or $p+1$ times on $a < x < b$, $p \geqq 1$. But from (4) the number of zeros of $L_1[u_p(x)]$ is always even. Therefore we have p roots if p is even and $p+1$ roots if p is odd.

If $\beta_1(b) \equiv 0$ in (Λ) and $\beta_1(a) \neq 0$, the sign of $L_2[U(b,\lambda_0)]$ is that of $-\beta_1(a)/\alpha_1(b)$, and $\phi(\mathcal{L}_1+\epsilon)$ has the sign of $-\beta_1(a)/\alpha_1(b)$. Hence we have Case I again. If $\alpha_1(b)\beta_1(a) < 0$ we proceed as before and obtain $p+2$ zeros of $L_1[u_p(x)]$ on $a < x < b$ if p is even and $p+1$ zeros if p is odd, $p \geqq 1$. If $\alpha_1(b)\beta_1(a) > 0$, we obtain p zeros of $L_1[u_p(x)]$ on $a < x < b$ if p is even and $p+1$ zeros if p is odd, $p \geqq 1$.

If $\beta_1 \neq 0$ in (X,Λ), the sign of $L_2[U(b,\lambda_0)]$ is that of $\beta_1(a)/\beta_1(b)$, which is positive. The sign of $\phi(\mathcal{L}_1+\epsilon)$ is that of $\beta_1(a)\beta_2(b) - \beta_2(a)\beta_1(b)$ if $\beta_1(a)\beta_2(b) - \beta_2(a)\beta_1(b)$ does not vanish. If

$$\beta_1(a)\beta_2(b) - \beta_2(a)\beta_1(b) = 0,$$

the sign of $\phi(\mathcal{L}_1+\epsilon)$ is that of $\beta_1(a)\beta_1(b)$, which is positive. Accordingly, if $\beta_1(a)\beta_2(b) - \beta_2(a)\beta_1(b) \geqq 0$, we have Case I, and l_p is on $\lambda_p < \lambda < \lambda_{p+1}$. By Theorem I, $L_1[U(x,\lambda_p)]$ vanishes $p-1$ times, and, proceeding as above, we find that $L_1[u_p(x)]$ has p zeros if p is even and $p+1$ zeros if p is odd, $p \geqq 1$. Likewise, if $\beta_1(a)\beta_2(b) - \beta_2(a)\beta_1(b) < 0$, l_p is on $\lambda_{p-1} < \lambda < \lambda_p$, and $L[u_p(x)]$ will vanish p times if p is even and $p-1$ times if p is odd, $p \geqq 1$.

Summarizing the various kinds of coefficients of the boundary conditions of (4) as follows:

A: $\beta_1 \equiv 0$ in (X,Λ);

B^-: Either $\beta_1(a) \equiv 0$ in (Λ), $\beta_1(x,\lambda) \neq 0$ in $a < x \leqq b$, and $\alpha_1(a)\beta_1(b) < 0$;

or $\beta_1(b) \equiv 0$ in (Λ), $\beta_1(x,\lambda) \neq 0$ in $a \leqq x < b$, and $\alpha_1(b)\beta_1(a) < 0$;

B^+: Either $\beta_1(a) \equiv 0$ in (Λ), $\beta_1(x,\lambda) \neq 0$ in $a < x \leqq b$, and $\alpha_1(a)\beta_1(b) > 0$;

or $\beta_1(b) \equiv 0$ in (Λ), $\beta_1(x,\lambda) \neq 0$ in $a \leqq x < b$, and $\alpha_1(b)\beta_1(a) > 0$;

C^+: $\beta_1 \neq 0$ in (X,Λ) and $\beta_1(a)\beta_2(b) - \beta_2(a)\beta_1(b) \geqq 0$ in (Λ);

C^-: $\beta_1 \neq 0$ in (X,Λ) and $\beta_1(a)\beta_2(b) - \beta_2(a)\beta_1(b) < 0$ in (Λ);

we state

Oscillation theorem III. *If $u_p(x)$ is the pth characteristic function corresponding to a simple value of the system (4) satisfying conditions I–VIII,* *then the number of zeros of* $L_1[u_p(x)]$ *on* $a < x < b$ *for* $p \geqq 1$ *is given by the following table:*

Case	*Number of zeros* $p = 2m$	$p = 2m + 1$
A or B^-	$p + 2$	$p + 1$
B^+ or C^-	p	$p + 1$
C^+	p	$p - 1$.

Furthermore, we notice that, since $\alpha_1 \beta_2 - \alpha_2 \beta_1 \neq 0$ in (X, Λ), the zeros of $L_1[u_p(x)]$ and $L_2[u_p(x)]$ separate one another on X,* and, if $\lambda = l_p$ is a double value, the number of zeros of $L_2[u_p(x)]$ is given by Theorem II. If $\lambda = l_p$ is a simple value, both $L_1[u_p(x)]$ and $L_2[u_p(x)]$ by (4) have an even number of zeros on (X). Hence they oscillate the same number of times. Therefore

COROLLARY. *The number of zeros of* $L_2[u_p(x)]$ *on* $a < x < b$, *for* $p \geqq 1$, *is precisely the number given in the table of Theorem III.*

From the foregoing results we may deduce the number of oscillations of $u_p(x)$. If $\lambda = l_p$ is a double value, then $u_p(x)$ will differ from $U(x, \lambda_{p-1})$, $U(x, \lambda_p)$, or $U(x, \lambda_{p+1})$ by at most a non-vanishing factor, and the number of zeros of $u_p(x)$ will be given by the Sturmian Oscillation Theorem.

For a simple value we consider the following cases:

If $\beta_1 \equiv 0$ in (X, Λ) the zeros of $u_p(x)$ and $L_1[u_p(x)]$ coincide. Hence $u_p(x)$ has $p + 2$ zeros if $p = 2m$ and $p + 1$ zeros if $p = 2m + 1$.

If $\beta_1 \neq 0$ in (X, Λ) or $\beta_1(a) \equiv 0$, $\beta_1(b) \neq 0$ or $\beta_1(b) \equiv 0$, $\beta_1(a) \neq 0$, we write the first boundary condition of (4)

$$u_p(a) \cdot P(a) = u_p(b) \cdot P(b),$$

where

$$P_1(x) = \alpha_1(x) - \beta_1(x) K(x) u_p'(x)/u_p(x).$$

If $P_1(a) P_1(b) > 0$, $u_p(a) u_p(b)$ will be positive and $u_p(x)$ has an even number of roots. But the zeros of $u_p(x)$ and $L_1[u_p(x)]$ separate. Hence $u_p(x)$ and $L_1[u_p(x)]$ have the same number of zeros on $a < x < b$. If $P_1(a) P_1(b) < 0$, $u_p(a) u_p(b)$ will be negative and $u_p(x)$ has an odd number of zeros. Hence $u_p(x)$ will have one more or one less zero than $L_1[u_p(x)]$ on $a < x < b$.

It is also to be noticed that in Case I, since l_p is on $(\lambda_p, \lambda_{p+1})$, $u_p(x)$ can vanish not more than $p + 2$ times nor less than $p - 1$ times. In Case II, since l_p is on $(\lambda_{p-1}, \lambda_p)$, $u_p(x)$ can vanish not more than $p + 1$ times nor less than $p - 2$ times.†

Designating the condition $P_1(a) \cdot P_1(b) > 0$ by P^+ and the condition $P_1(a) \cdot P_1(b) < 0$ by P^-, we may state

* Bôcher, *Leçons*, p. 50.

† Cf. Cor., p. 96 of the paper of the author referred to above.

OSCILLATION THEOREM IV. *If $u_p(x)$ is the pth characteristic function corresponding to a simple value of the system* (4) *satisfying conditions I–VIII, then the number of zeros of $u_p(x)$ on $a < x < b$, for $p \geqq 1$, is given by the following table:*

Case	*Number of zeros* $p = 2m$	$p = 2m + 1$
A	$p + 2$	$p + 1$
$B^- P^+$	$p + 2$	$p + 1$
$B^- P^-$	$p + 1$	$p + 2$ *or* p
$B^+ P^+$	p	$p + 1$
$B^+ P^-$	$p + 1$ *or* $p - 1$	$p + 2$ *or* p
$C^+ P^+$	p	$p + 1$
$C^+ P^-$	$p + 1$ *or* $p - 1$	$p + 2$ *or* p
$C^- P^+$	p	$p - 1$
$C^- P^-$	$p + 1$ *or* $p - 1$	p *or* $p - 2$

Note: if in particular we choose

$$\bar{\alpha}_i(x, \lambda) = \frac{(b - x)\alpha_i(\lambda) + (x - a)\gamma_i(\lambda)}{b - a}$$

$$\bar{\beta}_i(x, \lambda) = \frac{(b - x)\beta_i(\lambda) + (a - x)\delta_i(\lambda)}{b - a}$$

the system (4) becomes identical with that of (4) of the paper by the writer to which reference has been made above. If $\beta_i \delta_i > 0$, it will be necessary to modify the boundary conditions of (4) by taking

$$L_i[u(a, \lambda)] = - L_i[u(b, \lambda)]$$

with condition VII replaced by

$$\alpha_1(a)\beta_2(a) - \alpha_2(a)\beta_1(a) = \alpha_2(b)\beta_1(b) - \alpha_1(b)\beta_2(b) = -1,$$

if only one of the boundary conditions is modified. If both boundary conditions are modified, condition VII remains unchanged. In either case conditions I–VI and VIII will be satisfied. The oscillation theorems will be true with the modification that $L_i[u_p(x)]$ will have an odd number of zeros.

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