ON THE ZEROS OF SOLUTIONS OF HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS*

BY

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1. Introduction

In this paper I shall first prove a general separation theorem for the zeros of solutions of the general homogeneous linear differential equation. I shall then generalize Birkhoff's theorems of oscillation and comparison for equations of the third order† by proving two series of theorems, one for equations of odd order and one for equations of even order. This latter series of theorems is then applied to the study of the zeros of the solutions of self-adjoint linear homogeneous differential equations of the fourth order. It would be sufficient for my purposes if I were to assume that the coefficients and solutions of the equations considered, together with a sufficient number of their derivatives, were defined and continuous for all values of the independent variable considered. In order to simplify the statements of my results I shall assume these coefficients and solutions analytic.

We may, without loss of generality, take our nth order equation in the form

$$y^{(n)} + \sum_{i=2}^{n} p_i\, y^{(n-i)} = 0 \qquad (a \leqq x \leqq b) \tag{1}$$

and assume that the wronskian of any fundamental system of those solutions of (1) whose zeros we are studying is identically equal to one; i.e.,

$$W(x) \equiv \begin{vmatrix} y_1, & y_2, & \cdots, & y_n \\ y_1', & y_2', & \cdots, & y_n' \\ \cdot & \cdot & \cdot & \cdot \\ y_1^{(n-1)}, & y_2^{(n-1)}, & \cdots, & y_n^{(n-1)} \end{vmatrix} \equiv 1.$$

The equation adjoint to (1) is

$$z^{(n)} + \sum_{i=2}^{n} (-1)^i (p_i z)^{(n-i)} = 0. \tag{2}$$

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† Birkhoff, *On the solutions of ordinary linear homogeneous differential equations of the third order*, Annals of Mathematics, ser. 2, vol. 12 (1911), pp. 103–127.

This equation is satisfied by $z_i(x)$, the cofactor of $y_i^{(n-1)}(x)$ in $W(x)$, $i = 1, 2, \cdots, n$. By the properties of $z_i(x)$ as a cofactor,

$$\sum_{i=1}^{n} y_i^{(k)}(x) z_i(x) \equiv 0 \qquad (k = 0, 1, \cdots, n-2), \tag{3}$$

$$\sum_{i=1}^{n} y_i^{(n-1)}(x) z_i(x) \equiv 1. \tag{4}$$

Throughout this paper I shall denote determinants of the form

$$\begin{vmatrix} y_1^{(k_1)}, & y_2^{(k_1)}, & \cdots, & y_n^{(k_1)} \\ y_1^{(k_2)}, & y_2^{(k_2)}, & \cdots, & y_n^{(k_2)} \\ \cdot & \cdot & \cdot & \cdot \\ y_1^{(k_n)}, & y_2^{(k_n)}, & \cdots, & y_n^{(k_n)} \end{vmatrix}$$

by the symbol:

$$(y_1^{(k_1)}, y_2^{(k_2)}, \cdots, y_n^{(k_n)}).$$

2. A general separation theorem

THEOREM 1. *If x_1 and x_2 are consecutive zeros of $y_1(x)$, and if $y_2(x)$ does not vanish for either of these values of x, then the number of zeros of $y_2(x)$ between x_1 and x_2 plus the number of zeros of (y_1, y_2') in the same interval is odd.*

If for definiteness we let

$$y_1(x) > 0 \qquad (x_1 < x < x_2),$$

then

$$y_1(x_1) = y_1'(x_1) = \cdots = y_1^{(j-1)}(x_1) = 0, \qquad y_1^{(j)}(x_1) > 0,$$

$$y_1(x_2) = y_1'(x_2) = \cdots = y_1^{(k-1)}(x_2) = 0, \qquad (-1)^{(k)} y_1^{(k)}(x_2) > 0,$$

where $j < n$ and $k < n$ since $W(x) \neq 0$. Differentiating the function $f(x) \equiv y_2(y_1, y_2')$ repeatedly, we find that

$$f^{(m)}(x_1) = 0, \qquad m < j-1,$$

$$f^{(m)}(x_2) = 0, \qquad m < k-1,$$

$$f^{(j-1)}(x_1) = -y_2^2(x_1) y_1^{(j)}(x_1) < 0,$$

$$(-1)^k f^{(k-1)}(x_2) = -(-1)^k y_2^2(x_2) y_1^{(k)}(x_2) < 0,$$

or

$$(-1)^{k-1} f^{(k-1)}(x_2) > 0.$$

Hence the number of zeros of $f(x)$ between x_1 and x_2 is odd. Therefore the number of zeros of $y_1(x)$ plus the number of zeros of (y_1, y_2') between x_1 and x_2 is odd.

Theorem 1 may be geometrically interpreted by supposing the numbers $y_1(x), y_2(x), \cdots, y_n(x)$ to be the homogeneous coördinates of a point on an analytic curve in space of $n-1$ dimensions. This curve is the integral curve of equation (1). The condition $W(x) \neq 0$ then means that the osculating $(n-2)$-way plane spread is never stationary. The determinants (y_i, y_j') $(i, j = 1, 2, \cdots, n)$, $n(n-1)/2$ inn umber, are the homogeneous line coördinates of the tangent to the curve at the point $(y_1, y_2, \cdots, y_n)$. The vanishing of (y_1, y_2') at a point on the curve means that the tangent at the point meets the $(n-3)$-way plane intersection of the $(n-2)$-way plane spreads, whose equations are $y_1 = 0$ and $y_2 = 0$.

Hence Theorem 1 may be read as follows:

If the integral curve of equation (1) does not meet the $(n-3)$-way plane spread whose equations are $y_1 = 0$, $y_2 = 0$, then between two intersections of the curve with the $(n-2)$-way plane spread whose equation is $y_1 = 0$, there are an odd number of intersections with the $(n-2)$-way plane spread whose equation is $y_2 = 0$, and points of tangency with elements of the pencil of $(n-2)$-way plane spreads whose equation is $\alpha y_1 + \beta y_2 = 0$, where α and β are parameters.

For equations of the second order this theorem is equivalent to Sturm's theorem* that the zeros of two linearly independent solutions alternate, since $(y_1, y_2') \neq 0$ in this case. For equations of the third order it is equivalent to Birkhoff's general separation theorem.†

3. Regular intervals

Let

$$\phi(x, \xi) \equiv \sum_{i=1}^{n} z_i(\xi) y_i(x), \tag{5}$$

then from (3) and (4) we have

$$\frac{\partial^k}{\partial x^k}\phi(x, \xi) = 0, \qquad x = \xi \qquad (k = 0, 1, 2, \cdots, n-2), \tag{6}$$

and

$$\frac{\partial^{n-1}}{\partial x^{n-1}}\phi(x, \xi) = 1, \qquad x = \xi. \tag{7}$$

From (6) and (7) we can prove by substitution that if $\eta(x)$ is a function satisfying

$$\eta(x) \equiv y(x) - \int_a^x \phi(x, \xi) R(\xi) \eta(\xi) d\xi, \tag{8}$$

where $y(x)$ is an arbitrary solution of (1) and $R(x)$ is an arbitrary analytic function of x, then $\eta(x)$ is a solution of the linear differential system

* Cf. C. Sturm, Journal de mathématiques pures et appliquées, vol. 1 (1836), p. 131.

† Cf. Birkhoff, loc. cit., p. 109.

$$\eta^{(n)} + \sum_{i=2}^{n} p_i \eta^{(n-i)} + R\eta = 0, \tag{9}$$

$$\eta^{(k)}(\alpha) = y^{(k)}(\alpha) \qquad (k = 0, 1, \cdots, n-1). \tag{10}$$

Interchanging the roles of (1) and (9) we may define

$$\psi(x, \xi) \equiv \sum_{i=2}^{n} \zeta_i(\xi)\eta_i(x), \tag{5'}$$

where $\eta_i(x)$, $(i = 1, 2, \cdots, n)$, is a fundamental system of solutions of (9) and the ζ's are derived from the η's as the z's were derived from the y's. Then

$$\frac{\partial^k}{\partial x^k}\psi(x, \xi) = 0, \qquad x = \xi \quad (k = 0, 1, 2, \cdots, n-2), \tag{6'}$$

$$\frac{\partial^{n-1}}{\partial x^{n-1}}\psi(x, \xi) = 1, \qquad x = \xi, \tag{7'}$$

and

$$y(x) = \eta(x) + \int_\alpha^x \psi(x, \xi)R(\xi)y(\xi)d\xi. \tag{8'}$$

Now, if we set $y(x) = \phi(x, \alpha)$ in (8), we have

$$\psi(x, \alpha) = \phi(x, \alpha) - \int_\xi^x \phi(x, \xi)R(\xi)\psi(\xi, \alpha)d\xi, \tag{11}$$

since $\psi^{(k)}(x, \alpha) = \phi^{(k)}(x, \alpha)$, $x = \xi$, $(k = 0, 1, 2, \cdots, n-1)$, by (6), (7), (6′), and (7′). It will now be possible to compare solutions of (1) and (9) in intervals which satisfy the following

Definition. An interval $(a \leqq x \leqq b)$ shall be said to be regular with respect to (1) and of the first [or second] kind whenever the following conditions are fulfilled, as x and ξ vary throughout the interval:

Kind of Regular Interval	n Odd	n Even
First	$\phi(x, \xi) \geqq 0,\ x > \xi$	$\phi(x, \xi) > 0,\ x > \xi$
Second	$\phi(x, \xi) \geqq 0,\ x < \xi$	$\phi(x, \xi) < 0,\ x < \xi$.

The proof of the following theorem does not differ essentially from the proof of the special case for the equation of the third order, which Birkhoff has given.* For this reason I do not give the proof here.

THEOREM 2. *If an interval is a regular interval of the first* [*second*] *kind for an equation* (1) *of odd order, then it is also a regular interval of the same kind for the equation* (9), *provided that the inequality* $R(x) \leqq 0$ [$R(x) \geqq 0$] *obtains throughout the interval.*

THEOREM 3. *Any regular interval of either kind for an equation* (1) *of even*

* Loc. cit., p. 117.

order throughout which $R(x) \leqq 0$ *is also a regular interval of the same kind for equation* (9).

If α is any number in our interval, then, by (6′) and (7′), $\psi(x, \alpha)$ is positive if x is in the immediate right-hand neighborhood of α. If $\psi(x, \alpha) \leqq 0$ for any value of x greater than α, then, since $\psi(x, \alpha)$ is continuous, there exists a number, x_0, greater than α, such that $\psi(x_0, \alpha) = 0, \psi(x, \alpha) > 0$, $(\alpha < x < x_0)$.

If our interval is regular and of the first kind for (1), then $\phi(x_0, \xi) > 0$, $(\alpha \leqq \xi < x_0)$. If we now let $R(x) \leqq 0$ throughout our interval and substitute x_0 for x in (11), we have

$$0 = \phi(x_0, \alpha) - \int_\alpha^{x_0} \phi(x_0, \xi) R(\xi) \psi(\xi, \alpha) d\xi,$$

or zero equal to the sum of a positive quantity and an integral which cannot be negative. Therefore the assumption that $\psi(x, \alpha) \leqq 0$ for any value of x greater than α leads to an absurdity. Hence $\psi(x, \alpha) > 0$ for $x > \alpha$, where α is any value of ξ in our interval, and our interval is regular and of the first kind for the equation (9).

Similarly our theorem may be proven for regular intervals of the second kind.

THEOREM 4. *If* $(a \leqq x \leqq b)$ *is a regular interval of the first* [*second*] *kind, for an equation* (9) *of odd order, throughout which* $R(x) < 0$ $[R(x) > 0]$, *except for at most a finite number of zeros, and if* $y(x)$ *and* $\eta(x)$ *are non-identically vanishing solutions of* (1) *and* (9), *respectively, such that*

$$y^{(k)}(\alpha) = \eta^{(k)}(\alpha), \qquad a \leqq \alpha \leqq b \quad (k = 0, 1, \cdots, n-1),$$

then between α *and the least* [*greatest*] *zero of* $\eta(x)$ $[y(x)]$ *which is greater* [*less*] *than* α, *there exists at least one zero of* $y(x)$ $[\eta(x)]$ *at which* $y(x)$ $[\eta(x)]$ *changes sign.*

For definiteness, let the first of the numbers $\eta^{(k)}(\alpha)$ which does not vanish be positive. Then, in the immediate right-hand neighborhood of $x = \alpha$, both $\eta(x)$ and $y(x)$ are positive. Now, let $\bar{x}$ be such that $\eta(\bar{x}) = 0$, $\eta(x) > 0$ $(\alpha \leqq x < \bar{x})$. Then, substituting $x = \bar{x}$ in (8′), we have

$$y(\bar{x}) = 0 + \int_\alpha^{\bar{x}} \psi(\bar{x}, \xi) R(\xi) y(\xi) d\xi.$$

Now, if our interval is regular and of the first kind, $\psi(\bar{x}, \xi) \geqq 0 (\bar{x} > \xi)$ and $R(\xi) < 0$ $(\alpha \leqq \xi \leqq \bar{x})$, except for at most a finite number of zeros. Hence, if we suppose that $y(x)$ does not change sign between α and $\bar{x}$, we have $y(\xi) \geqq 0 (\alpha \leqq \xi \leqq \bar{x})$ and a non-negative quantity $y(\bar{x})$ equal to an integral that must be negative, which is absurd. Therefore, $y(x)$ changes sign at least once between α and $\bar{x}$.

If our interval is regular and of the second kind, a similar proof will hold.

If we change the sign of $R(x)$, we interchange $y(x)$ and $\eta(x)$ in the conclusion.

THEOREM 5. *If* $(a \leqq x \leqq b)$ *is a regular interval of the first* [*second*] *kind, for an equation* (9) *of even order, throughout which* $R(x) < 0$, *except for at most a finite number of zeros, and if* $y(x)$ *and* $\eta(x)$ *are non-identically vanishing solutions of* (1) *and* (9), *respectively, such that*

$$y^{(k)}(\alpha) = \eta^{(k)}(\alpha), \qquad (a \leqq \alpha \leqq b) \quad (k = 0, 1, \cdots, n-1),$$

then between α *and the least* [*greatest*] *zero of* $\eta(x)$ [$y(x)$], *which is greater* [*less*] *than* α, *there exists at least one zero of* $y(x)$ [$\eta(x)$] *at which* $y(x)$ [$\eta(x)$] *changes sign.*

A proof similar to that of Theorem 4 proves this theorem, and, as in Theorem 4, a change in sign of $R(x)$ interchanges $y(x)$ and $\eta(x)$ in our conclusion.

4. SELF-ADJOINT EQUATIONS OF THE FOURTH ORDER

The general self-adjoint linear homogeneous differential equation of the fourth order, with the coefficient of the first term identically equal to 1, can be written in the form

$$(12) \qquad \eta^{\mathrm{IV}} + 10p_2\,\eta'' + 10p_2'\,\eta' + (3p_2'' + 9p_2^2 + R)\,\eta = 0,$$

where $p_2(x)$ is not the function used in (1). Here the form of the coefficient of η is not dictated by the condition of self-adjointness, but is chosen for the purpose of relating equation (12) to the equation

$$(13) \qquad y^{\mathrm{IV}} + 10p_2\,y'' + 10p_2'\,y' + (3p_2'' + 9p_2^2)\,y = 0,$$

which is satisfied by the cube of any solution of*

$$(14) \qquad u'' + p_2\,u = 0.$$

Furthermore we know that if $y(x)$ is any solution of (13) then $\eta(x)$ as defined by (8) satisfies (12). If $\phi_2(x, \xi)$ is the solution of (14) defined by (5), then†

$$\phi_2(x, \xi) = 0, \qquad \frac{\partial}{\partial x}\phi_2(x, \xi) = 1, \qquad x = \xi,$$

and

$$\phi_2^3(x, \xi) = \frac{\partial}{\partial x}\phi_2^3(x, \xi) = \frac{\partial^2}{\partial x^2}\phi_2^3(x, \xi) = 0, \qquad \frac{\partial^3}{\partial x^3}\phi_2^3(x, \xi) = 6, \qquad x = \xi.$$

* Cf. Brioschi, Acta Mathematica, vol. 14 (1890), p. 236.

† If $u_1(x)$ and $u_2(x)$ are two solutions of (14) for which $(u_1, u') \equiv 1$, then

$$\phi_2(x, \xi) = -u_2(\xi)\,u_1(x) + u_1(\xi)\,u_2(x).$$

Therefore if $\phi_4(x, \xi)$ is the solution of (13) defined by (5), then

$$\phi_4(x, \xi) = \tfrac{1}{6}\phi_2^3(x, \xi).$$

Hence by the definition of regular intervals any interval which is regular for (14) will be regular for (13) and conversely. Therefore we can substitute (14) for (1) and (12) for (9) in Theorem 3 and obtain

THEOREM 6. *Any regular interval for* (14) *is a regular interval of both kinds for* (12) *provided that* $R(x) \leqq 0$ *throughout our interval.*

Similarly we may substitute (13) for (1), (14) for the first (9) and (12) for the second (9) in Theorem 5 and obtain

THEOREM 7. *If* $(\xi_1 \leqq x \leqq \xi_2)$ *is a regular interval for* (14) *throughout which* $R(x) < 0$ *except for at most a finite number of zeros, and if* $y(x)$ *and* $\eta(x)$ *are non-identically vanishing solutions of* (13) *and* (12) *respectively such that* $y^{(k)}(\alpha) = \eta^{(k)}(\alpha)$, $\xi_1 \leqq \alpha \leqq \xi_2$, $(k = 0, 1, 2, 3)$, *then between* α *and the least* [*greatest*] *zero of* $\eta(x)$ [$y(x)$] *which is greater* [*less*] *than* α *there exists at least one zero of* $y(x)$ [$\eta(x)$] *at which* $y(x)$ [$\eta(x)$] *changes sign.*

In applying these two theorems it is to be noted that sometimes the regular intervals for equation (14) are bounded by two consecutive zeros of a solution of (14), but that if (14) has non-oscillatory solutions then they are not so bounded.

Definitions. The forward interval of oscillation at $x = \alpha$ *for a given equation is the least interval* (α, β) *such that all solutions vanishing for* $x = \alpha$ *will vanish again in* (α, β).

The backward interval of oscillation at $x = \alpha$ *is the least interval* (β, α) *such that all solutions vanishing for* $x = \alpha$ *will vanish again in* (β, α).

THEOREM 8. *If* $R(x) < 0$ *except for at most a finite number of zeros and if equation* (14) *possesses a backward interval of oscillation at* $x = \alpha$, *then equation* (12) *possesses a backward interval of oscillation at* $x = \alpha$ *which is not greater than the backward interval of oscillation for equation* (14).

Any solution of (13), being a homogeneous binary form of the third degree with real constant coefficients in any pair of linearly independent solutions of (14), has at least one real linear factor. This factor vanishes once and only once in the interval $(x_1 \leqq x < \alpha)$ where x_1 is the zero of $\phi_2(x, \alpha)$ which immediately precedes α. Therefore, equation (13) possesses a backward interval of oscillation at $x = \alpha$. Since $\phi_2^3(x, \alpha)$ satisfies (13), this interval is equal to the backward interval of oscillation for (14).

By the preceding theorem any solution, $\eta(x)$, of (12) which vanishes for $x = \alpha$, vanishes and changes sign at least once between α and the greatest zero less than α of that solution, $y(x)$, of (13) which satisfies the conditions (10). Therefore, $\eta(x)$ vanishes and changes sign at least once in the interval $(x_1 \leqq x < \alpha)$. Of all the zeros, infinite in number, immediately preceding α, of the various solutions of (12) which vanish at $x = \alpha$ there must be a lower limiting value, $\bar{x}$, which is not less than x_1, such that all solutions of (12) which vanish for $x = \alpha$ vanish again in the interval $(\bar{x} \leqq x < \alpha)$. This interval is the required interval of oscillation.

5. A SEPARATION THEOREM FOR SELF-ADJOINT EQUATIONS OF THE FOURTH ORDER

If $\eta_i(x)$ and $\eta_j(x)$ satisfy equation (12) then we can prove by differentiating or by Lagrange's identity* that

$$(15)\qquad P(\eta_i, \eta_j) \equiv (\eta_i, \eta_j''') - (\eta_i', \eta_j'') + 10p_2(\eta_i, \eta_j')$$

is a constant.

Again it is well known that the six functions

$$(\eta_i, \eta_j') \qquad (i, j = 1, 2, 3, 4, i < j)$$

satisfy a linear differential equation of the fifth order.† Therefore, they must be linearly dependent. If we expand the following linear combination of identically vanishing determinants,

$$(16)\quad (\eta_1, \eta_2''', \eta_3, \eta_4') - (\eta_1', \eta_2'', \eta_3, \eta_4') + 10p_2(\eta_1, \eta_2', \eta_3, \eta_4') \equiv 0,$$

in terms of the two-rowed determinants of the first two rows we have the linear relation

$$(17)\quad \begin{aligned} &P(\eta_1, \eta_2)(\eta_3, \eta_4') - P(\eta_1, \eta_3)(\eta_2, \eta_4') + P(\eta_1, \eta_4)(\eta_2, \eta_3') \\ &\quad + P(\eta_2, \eta_3)(\eta_1, \eta_4') - P(\eta_2, \eta_4)(\eta_1, \eta_3') + P(\eta_3, \eta_4)(\eta_1, \eta_2') \equiv 0. \end{aligned}$$

Furthermore, it can be shown by actual expansion of terms that

$$(18)\qquad \zeta_1 \equiv P(\eta_2, \eta_3)\eta_4 + P(\eta_3, \eta_4)\eta_2 + P(\eta_4, \eta_2)\eta_3,$$

the other ζ's being obtained by cyclic permutations of the subscripts. Substituting ζ_1 in (17) we have

$$(19)\qquad (\eta_1, \zeta_1') + \begin{vmatrix} \eta_2, & \eta_3, & \eta_4 \\ \eta_2', & \eta_3', & \eta_4' \\ P(\eta_1, \eta_2), & P(\eta_1, \eta_3), & P(\eta_1, \eta_4) \end{vmatrix} \equiv 0,$$

where our three-rowed determinant does not involve $\eta_1(x)$ explicitly. Solving for (η_1, ζ_1') and substituting in Theorem 1, we have a general separation theorem for self-adjoint equations of the fourth order.

THEOREM 9. *If $\eta_i(x)$ $(i = 1, 2, 3, 4)$ are linearly independent solutions of* (12) *then between two consecutive zeros of $\eta_1(x)$ at which*

* Cf. Bôcher, *Leçons sur les Méthodes de Sturm*, Paris, 1917, p. 23.

† Cf. Forsyth, Philosophical Transactions, vol. 179 (1888), p. 45?

$$\zeta_1(x)[\equiv -(\eta_2, \eta_3', \eta_4'')]$$

does not vanish, there exist an odd number of zeros of $\zeta_1(x)$ *and*

$$\begin{vmatrix} \eta_2, & \eta_3, & \eta_4 \\ \eta_2', & \eta_3', & \eta_4' \\ P(\eta_1, \eta_2), & P(\eta_1, \eta_3), & P(\eta_1, \eta_4) \end{vmatrix}$$

Interpreting this theorem geometrically by means of the curve whose parametric equations are $y_i = \eta_i(x)$ $(i = 1, 2, 3, 4)$ and the conical projection of this curve upon the plane $y_1 = 0$ from the vertex $(1, 0, 0, 0)$ we see that between two consecutive intersections of our curve with the plane $y_1 = 0$ at which the projected curve has no point of inflection there will exist on the projected curve an odd number of points of inflection and points at which the tangent to the curve passes through the point $[0, P(\eta_1, \eta_2), P(\eta_1, \eta_3), P(\eta_1, \eta_4)]$.

6. Applications

In this section I shall apply the theorems derived in Section 4 to two examples.

Example 1.

(*a*) $$\eta^{\text{IV}} + 10\eta'' + x\eta = 0, \qquad x \leqq 9.$$

We shall compare the solutions of this equation with the solutions of the equation

(*b*) $$y^{\text{IV}} + 10y'' + 9y = 0.$$

Here $R(x) = x - 9 \leqq 0$, and (*b*) is satisfied by $u^3(x)$ whenever

(*c*) $$u'' + u = 0.$$

Every interval of length π is an interval of oscillation for equations (*b*) and (*c*). Therefore, by Theorem 6, $(9 - \pi \leqq x \leqq 9)$ is a regular interval of both kinds for (*a*). Theorem 7 implies that if two solutions, $\eta(x)$ and $y(x)$, of (*a*) and (*b*) respectively, are such that

$$\eta^{(k)}(\alpha) = y^{(k)}(\alpha), \qquad 9 - \pi \leqq \alpha \leqq 9 \qquad (k = 0, 1, 2, 3),$$

then between α and the least [greatest] zero of $\eta(y)$ which is greater than α [less than α] there exists at least one zero of $y(\eta)$ at which $y(\eta)$ changes signs. By Theorem 8 we know that for $\alpha \leqq 9$, (*a*) possesses a backward interval of oscillation at $x = \alpha$ of length not greater than π.

Example 2.

(*a*) $$\eta^{\text{IV}} - x\eta = 0, \qquad x \geqq 0.$$

Here $R(x) = -x \leqq 0$. In this case our comparison equation (*b*) $$y^{IV} = 0$$

has no intervals of oscillation. If the particular solution $\eta_1(x)$ satisfies the boundary conditions

(*c*) $$\eta_1(\alpha) = 1, \quad \eta_1'(\alpha) = \eta_1''(\alpha) = \eta_1'''(\alpha) = 0, \quad \alpha \geqq 0,$$

then $y_1(x)$, the corresponding solution of (*b*), is identically equal to 1. Therefore, by Theorem 7, $\eta_1(x)$ cannot vanish for any value of x greater than α. Similarly none of the other principal solutions of (*a*) at $x = \alpha$ vanish for x greater than α.

If $\eta_2(x)$ satisfies the boundary conditions

(*c'*) $$\eta_2(1) = 0, \quad \eta_2'(1) = -1, \quad \eta_2''(1) = 0, \quad \eta_2'''(1) = 6,$$

then

$$y_2(x) = x(x-1)(x-2).$$

Therefore by Theorem 7, $\eta_2(x)$ has at least one zero in the interval $(0 \leqq x \leqq 1)$, no zero in the interval $(1 \leqq x \leqq 2)$, and may have zeros for x greater than 2.