PARALLEL MAPS OF SURFACES*

BY

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1. **Introduction.** Parallel maps, constituted by two surfaces in one-to-one point correspondence with the normals at corresponding points parallel, have been frequently studied, especially in particular cases. The problem of the determination of maps of this type which are conformal, first proposed and solved by Christoffel,† has received, perhaps, the greatest attention. Certain aspects of equiareal maps have been considered by Guichard‡ and Razzaboni.§ Of importance in connection with infinitesimal deformations of a surface are the parallel maps such that the asymptotic lines on each surface correspond to a conjugate system on the other.|| The surfaces of a map of this kind, called by Bianchi *associate surfaces*, have been carefully investigated by Eisenhart.¶

The congruence of lines joining corresponding points of the surfaces of a parallel map have been found to have interesting properties. In particular, its two families of developables, according as they are distinct or coincident, intersect the surfaces in basic, i. e., corresponding, conjugate systems or in corresponding families of asymptotic lines.**

Despite these diverse investigations there is extant no general theory of parallel maps which is complete or thorough. To give such a theory is the primary purpose of this paper. Parallel maps are first classified as directly or inversely parallel and then as hyperbolic, elliptic, or parabolic, after the manner of classifying one-dimensional projective correspondences, and each map is characterized by an invariant I, analogous to the invariant of such a correspondence.

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† *Über einige allgemeine Eigenschaften der Minimumsflächen*, Journal für Mathematik, vol. 67 (1867), pp. 218–228; cf. Darboux, *Leçons*, 1st edition, vol. 2, p. 239; ibid., vol. 1, p. 326.

‡ *Sur les surfaces qui se correspondent avec parallèlisme des plans tangents et conservation des aires*, Comptes Rendus, vol. 136 (1903), pp. 151–153.

§ *Sulla rappresentazione equivalente di una superficie su di un' altra per parallelismo delle normali*, Rendiconti della Reale Accademia delle Scienze dell' Istituto di Bologna, ser. 6, vol. 9 (1912), pp. 45–59.

|| Bianchi, *Lezioni*, vol. 2, p. 10; German translation, 2d edition, p. 299.

¶ *Associate surfaces*, Mathematische Annalen, vol. 62 (1906), pp. 504–538.

** Cf. Darboux, *Leçons*, 1st edition, vol. 2, p. 234.

In the case of parabolic maps, for which $I = 1$, a second invariant, J, of purely metrical significance, is introduced.

The essential element in this classification is the invariant I. It has its most striking geometrical interpretation as the cross ratio in which the focal points of a line of the associated congruence divide the corresponding points of the surfaces. With its introduction the order in which the surfaces are taken becomes relevant, so that the map takes on the aspect of a transformation rather than that of a correspondence. If I is the invariant of the map of the one surface on the other, the reciprocal of I is the invariant of the map of the second surface on the first. The involutorial maps, for which $I = -1$, are those of associate surfaces of Bianchi.

As fundamental must be considered questions of existence of maps under given conditions. It is found that it is impossible to prescribe one surface of a non-parabolic map, the basic conjugate system on it, and the invariant I; nor is it possible to specify the spherical representation of the map, the curves on the sphere which are to represent the basic conjugate systems, and the invariant I. It is, however, possible to prescribe one of the surfaces and either the basic conjugate system on it or the invariant I, or to prescribe the spherical representation of the map, specifying the curves which are to represent the basic conjugate systems. In the case of parabolic maps, for which the corresponding families of asymptotic lines and the invariant J play the fundamental roles, similar results are found.

The classification, the discussion of the invariants, and the existence theorems are given in Parts I and IV. In Part II the theory is applied to maps of special type. Particular attention is accorded equiareal non-parabolic maps and parabolic maps of ruled surfaces. The consideration of the former gives rise to a generalization to arbitrary translation surfaces of the concept of associate minimal surfaces, with a well-rounded theory of parallel maps of translation surfaces as a result.

Part III is given to the investigation of the basic conjugate systems, the corresponding orthogonal systems (of a non-conformal map), and the isometrically mapped systems (of a non-isometric map). It is found that these systems, and certain angles associated with them, have interesting interrelationships, especially in the case of an equiareal map.

I. General theory

2. **Classification. The invariant** I. In studying parallel maps we shall restrict ourselves to the case in which the surfaces, S and S', and the correspondence between them are real and analytic. It is assumed further that S and S' are not developables and that the trivial case of two homothetic surfaces is excluded.

The normals at corresponding points shall be directed in the same sense. According, then, as corresponding directions of rotation about corresponding points are the same or opposite, the map shall be called *directly* parallel or *inversely* parallel.

Since in the neighborhoods of corresponding points the map is projective, there exists a unique system of curves on each surface such that the curves of the two systems correspond and have in corresponding points parallel tangents. According as the families of curves, C_1 and C_2, constituting the system on S, or the corresponding families, C_1' and C_2', of the system on S', are real and distinct, real and coincident, or conjugate-imaginary, the map shall be termed *hyperbolic*, *parabolic*, or *elliptic*. Evidently an inversely parallel map is always hyperbolic, whereas a directly parallel map may be of any one of the three types.

The invariant, I, of the projective correspondence established by a non-parabolic map between the directions of departure from corresponding points, P and P', shall be called the *invariant* of the map. More specifically, if the corresponding parallel directions at P (or P') are denoted, in a fixed order, by d_1 and d_2, and if d, at P, and d', at P', are arbitrary corresponding directions, I shall be defined as the cross ratio, constant for a given pair of points P, P', in which d and d' divide d_1 and d_2:

$$I = (d_1, d_2; d, d'). \tag{1}$$

The invariant I is a point function on each surface. For an inversely parallel map, it is negative; for a directly parallel map, it is positive, $\not\equiv 1$, or it is of the form $e^{\alpha i}$, $\alpha \not\equiv 2n\pi$, according as the map is hyperbolic or elliptic. Finally, it is agreed that, for a parabolic map, $I \equiv 1$.

The type of the function I determines the type of the map, unless $I \equiv -1$, when the map is either inversely parallel and hyperbolic or directly parallel and elliptic.

If I, as defined, is taken as the invariant of the map of S on S', the reciprocal of I, the cross ratio $(d_1, d_2; d', d)$ is the invariant of the (inverse) map of S' on S.

Inasmuch as the corresponding parallel families of curves, C_1, C_2 and C_1', C_2', constitute basic, i. e., corresponding, conjugate systems in the non-parabolic case and, in the parabolic case, coincide on each surface with one family of asymptotic lines (§1), whether these families are real or imaginary depends, in part, on whether the asymptotic lines on the two surfaces are real or imaginary. Accordingly, we consider the total curvatures of the surfaces.

For this purpose we define a directed element of area on a surface, $x = x(u, v)$:

$$x_1 = x_1(u, v), \qquad x_2 = x_2(u, v), \qquad x_3 = x_3(u, v).$$

According as the tangents to the parametric curves in the directions of increasing u and increasing v form with the directed normal a trihedral of the same disposition as the coördinate axes or of opposite disposition the element of area shall be positive or negative. In order that the direction cosines of the directed normal may always have their usual values, $1/H$ times the vector product of x_u and x_v, the determination H of the square root of $EG - F^2$ must be taken as positive in the first case and negative in the second. Then the formula,

$$dA = Hdudv,$$

always gives the directed element of area, dA. It follows that the corresponding directed element of area, $d\mathfrak{A}$, of the spherical representation is

$$d\mathfrak{A} = KHdudv = KdA.$$

Assume, now, that the map is established by assigning the same curvilinear coördinates to corresponding points, x and x', of the two surfaces:

(2) $$x = x(u, v), \qquad x' = x'(u, v).$$

Since the surfaces have the same spherical representation, $d\mathfrak{A} = d\mathfrak{A}'$, or

(3) $$KdA = K'dA'.$$

But the map is directly or inversely parallel according as $dAdA'$ is positive or negative. Hence KK' is positive in the first case, negative in the second.

The total curvatures of the two surfaces have the same or opposite signs according as the map is directly or inversely parallel.

If, in the case of a directly parallel map, the curvatures are both positive, the asymptotic lines on both surfaces are imaginary; therefore the families C_1, C_2, and C_1', C_2', are real and distinct and the map is hyperbolic. If, however, the curvatures are both negative, the map is hyperbolic or elliptic, according as the pairs of (real) asymptotic directions at corresponding points do not or do separate one another, and is parabolic if a direction of one pair is parallel to one of the other pair.

We can now give a complete classification of parallel maps:

	Curvatures of S, S'	*Type of map*	*Invariant*
Inversely parallel	Opposite in sign	Hyperbolic	$I < 0,$
Directly parallel	Both positive	Hyperbolic	$I > 0, \not\equiv 1,$
	Both negative	Hyperbolic	$I > 0, \not\equiv 1,$
		Parabolic	$I \equiv 1,$
		Elliptic	$I = e^{\alpha i}, \not\equiv 1.$

It is to be noted that, though the value of I in general determines the type of map, it does not always fix the signs of the curvatures of the surfaces.

3. **Geometric interpretation of the invariant** *I*. *Non-parabolic maps.* On the surfaces (2) let the corresponding parallel curves be parametric, C_1, C_1' the u-curves and C_2, C_2' the v-curves. Then scalar functions $\lambda(u, v)$, $\mu(u, v)$ exist such that

$$x_u' = \lambda x_u, \qquad x_v' = \mu x_v. \tag{4}$$

It is readily found that

$$I = \frac{\lambda}{\mu}. \tag{5}$$

In the hyperbolic case the parameters u and v and the functions λ and μ are real; the map is directly or inversely parallel according as $\lambda\mu > 0$ or $\lambda\mu < 0$. In the elliptic case u and v are conjugate-imaginary, and λ and μ are conjugate-imaginary functions of u and v.

Corresponding u-curves, C_1, C_1', are in Combescurian correspondence: the directions of their trihedrals at corresponding points are respectively parallel. Moreover, if the two curves are directed in the same sense,* corresponding directions of the two trihedrals can be similarly oriented. Then the ratio of the element of arc of C_1' to the element of arc of C_1 at corresponding points is λ. This is also the ratio of the radii of curvature of C_1' and C_1, and of their radii of torsion, of geodesic curvature and geodesic torsion.† We shall call λ the *Combescurian ratio* of the curves C_1' and C_1. Similarly μ is defined as the Combescurian ratio of two corresponding curves C_2', C_2.

THEOREM 1. *The invariant I of a non-parabolic map is the quotient of the Combescurian ratios of corresponding parallel curves C_1', C_1 and of corresponding parallel curves C_2', C_2.*

It follows that I is equal to the ratio $R_{C_1'}/R_{C_2'}$, of the radii of normal curvature of S' in the basic conjugate directions divided by the corresponding ratio, R_{C_1}/R_{C_2}, formed for S. In particular, if the basic conjugate systems are the lines of curvature, I is equal to the ratio r_1'/r_2' of the radii of principal curvature of S' divided by the corresponding ratio r_1/r_2, formed for S. Consequently, *the invariant of the parallel map of a surface on its spherical representation is the ratio, r_2/r_1, of the principal curvatures.*

A quite different interpretation of I comes to light in considering the congruence formed by the lines joining corresponding points, P and P', of S and S'. That the developables of this congruence intersect S and S' in the basic conjugate systems (§1) follows from the equations

* In the hyperbolic case, for example, let C_1 be oriented in the direction of increasing u on the surface S; then C_1 is oriented in the same sense, which is that of increasing or decreasing u on the surface S', according as $\lambda > 0$ or $\lambda < 0$.

† It is assumed that C_1 and C_1' are not minimal. For application of the facts here given, cf., e. g., Karl Peterson, *Über Curven und Flächen*, pp. 48–50.

$$(6)\quad \frac{\partial}{\partial u}\,\frac{x' - \lambda x}{1-\lambda} = \frac{\lambda_u}{(1-\lambda)^2}(x'-x), \qquad \frac{\partial}{\partial v}\,\frac{x' - \mu x}{1-\mu} = \frac{\mu_v}{(1-\mu)^2}(x'-x),$$

which are readily deduced from (4). From (6) it is clear, further, that the focal points, F_1 and F_2, associated respectively with the developables $D_1 : v =$ const. and the developables $D_2 : u =$ const., divide the line-segment $P'\,P$ in the ratios $-\lambda$ and $-\mu$. Consequently, the cross ratio of P', P and F_1, F_2 is equal to I.

THEOREM 2. *The invariant of a non-parabolic map equals the cross ratio in which corresponding points P' and P of the surfaces S' and S are divided by those focal points, F_1 and F_2, of the associated congruence which lie on $P'\,P$.**

Evidently λ and μ, as well as I, are invariant under a change of curvilinear coördinates and under a homothetic transformation of both surfaces. It is important to note, however, that, unlike I, they are not invariant under *different* homothetic transformations applied, one to S, the other to S'.

Parabolic maps. Let the surfaces be represented by (2), with the corresponding parallel families, A and A', of asymptotic lines as the u-curves and any other corresponding families of real curves as the v-curves. Real scalar functions $\lambda(u, v)$, $\nu(u, v)$ exist such that

$$(7)\qquad x'_u = \lambda x_u, \quad x'_v = \nu x_u + \lambda x_v.$$

The Combescurian ratio, λ, of curves A' and A is an invariant similar in type to the λ and μ of the non-parabolic case. The function ν depends on the choice of parameters.

The two families of developables of the related congruence coincide and intersect the surfaces in the asymptotic lines A and A'. The single focal point

* In particular, if S and S' are associate surfaces ($I = -1$), F_1, F_2 divide P, P' harmonically. This fact has been noted by Bianchi, loc. cit., German translation, p. 301.

If $\lambda \equiv 1$ (or $\mu \equiv 1$), corresponding parallel curves C_1, C_1' (or C_2, C_2') are congruent by a translation; the developables D_1 (or D_2) are cylinders, the segments $P'P$ on the rulings of a cylinder are all equal, and the focal points F_1 (or F_2) are at infinity. If $\lambda_u \equiv 0$ but $\lambda \not\equiv 1$ (or $\mu_v \equiv 0$ but $\mu \not\equiv 1$), corresponding curves C_1, C_1' (or C_2, C_2') are homothetic but not congruent; the developables D_1 (or D_2) are cones, the segments $P'P$ on the rulings of a cone are all divided by the vertex F_1 (or F_2) in the same ratio, and the locus of the points F_1 (or F_2) is a curve. If the map is elliptic, the first of the two cases considered is impossible and the second occurs only when both $\lambda_u \equiv 0$ and $\mu_v \equiv 0$.

In general, if $\lambda_u \equiv \mu_v \equiv 0$, but neither λ nor μ is constant, equations (4) can be written as $x'_u = vx_u$, $x'_v = ux_v$, where the original parameters have been replaced by $u_1 = \mu(u)$, $v_1 = \lambda(v)$, and the subscripts subsequently dropped. Hence the surfaces S and S' are represented by

$$x = \frac{\varphi(v) - f(u)}{u - v}, \qquad x' = \frac{u\varphi(v) - vf(u)}{u - v},$$

where $f(u)$ and $\varphi(v)$ are arbitrary real triples in the hyperbolic case and arbitrary conjugate-imaginary triples in the elliptic case.

If λ and μ are both constant, S and S' are translation surfaces; cf. §9.

on a line of the congruence divides the line-segment $P'P$ in the ratio $-\lambda$. Theorem 2 can be considered to hold in a limiting form.*

4. **Fundamental theorem for a non-parabolic map.** From equations (4) follows

$$(\lambda - \mu)x_{uv} + \lambda_v x_u - \mu_u x_v = 0. \tag{8}$$

But
$$x_{uv} = \begin{Bmatrix} 12 \\ 1 \end{Bmatrix} x_u + \begin{Bmatrix} 12 \\ 2 \end{Bmatrix} x_v.$$

Consequently,

$$(\lambda - \mu)\begin{Bmatrix} 12 \\ 1 \end{Bmatrix} + \lambda_v = 0, \qquad (\lambda - \mu)\begin{Bmatrix} 12 \\ 2 \end{Bmatrix} - \mu_u = 0, \tag{9}$$

or

$$\left(1 - \frac{1}{I}\right)\begin{Bmatrix} 12 \\ 1 \end{Bmatrix} + \frac{\partial \log \lambda}{\partial v} = 0, \qquad (I - 1)\begin{Bmatrix} 12 \\ 2 \end{Bmatrix} - \frac{\partial \log \mu}{\partial u} = 0. \tag{10}$$

Differentiating the first of these equations with respect to u, the second with respect to v, and adding, we obtain the condition

$$\frac{\partial^2 \log I}{\partial u \partial v} + \frac{\partial}{\partial u}\left[\left(1 - \frac{1}{I}\right)\begin{Bmatrix} 12 \\ 1 \end{Bmatrix}\right] + \frac{\partial}{\partial v}\left[(I - 1)\begin{Bmatrix} 12 \\ 2 \end{Bmatrix}\right] = 0 \tag{11}$$

on the invariant I and the differential coefficients of the first order of the surface S.

Given, conversely, a surface S referred to a conjugate system, and a function $I(u, v)$, such that (11) is satisfied. Then the equation $\lambda = I\mu$ and equations (10), in λ and μ, are compatible; from them, by means of a quadrature, λ and μ are determined uniquely except for the same multiplicative constant k. Since these values of λ and μ satisfy (9) and hence (8), equations (4) are integrable and determine x' uniquely except for the multiplicative constant k and an additive triple.†

* The case, $\lambda_u \equiv 0$, in which it turns out that S and S' are ruled surfaces, is given special attention in §13.

† The values found for λ and μ are

$$\lambda = ke^M, \qquad M = \int\left[(I-1)\begin{Bmatrix} 12 \\ 2 \end{Bmatrix} + \frac{\partial \log I}{\partial u}\right] du + \left(\frac{1}{I} - 1\right)\begin{Bmatrix} 12 \\ 1 \end{Bmatrix} dv,$$

$$\mu = ke^N, \qquad N = \int (I-1)\begin{Bmatrix} 12 \\ 2 \end{Bmatrix} du + \left[\left(\frac{1}{I} - 1\right)\begin{Bmatrix} 12 \\ 1 \end{Bmatrix} - \frac{\partial \log I}{\partial v}\right] dv.$$

Hence
$$y = k\int e^M x_u du + e^N x_v dv.$$

THEOREM 3. *If a conjugate system on a given surface S and a point function $I(u, v)$ on S are chosen, a necessary and sufficient condition that there exist a surface S' on which S is mapped by parallel normals with the given conjugate system basic and the given function I as the invariant, is that, when S is referred to the conjugate system, E, F, G, and I satisfy* (11). *The surface S' is then determined to within its homothetics and its point coördinates can be found by quadratures.*

It follows that, if a surface S and a conjugate system on it are given, there exist infinitely many surfaces S' on which S is mapped by parallel normals with the given conjugate system basic. The determination of the surfaces S' depends, when S is referred to the conjugate system, on the solution of (11) for I and on subsequent quadratures.

Equation (11) is invariant under a change of parameters of the form $u = u(u')$, $v = v(v')$ or of the form $u = u(v')$, $v = v(u')$, provided that in the latter case I is replaced by $1/I$. Accordingly, $\begin{Bmatrix} 12 \\ 1 \end{Bmatrix}$ and $\begin{Bmatrix} 12 \\ 2 \end{Bmatrix}$ can be thought of as the invariants of (11). If either is zero, the equation can be integrated by quadratures.

By means of the formulas*

$$(12) \qquad \begin{Bmatrix} 12 \\ 1 \end{Bmatrix} = I\begin{Bmatrix} 12 \\ 1 \end{Bmatrix}', \qquad \begin{Bmatrix} 12 \\ 2 \end{Bmatrix} = \frac{1}{I}\begin{Bmatrix} 12 \\ 2 \end{Bmatrix}',$$

where the symbols with primes refer to the surface S', we deduce from (11) the equation

$$(13) \qquad \frac{\partial^2 \log I}{\partial u \partial v} + \frac{\partial}{\partial u}\left[(I-1)\begin{Bmatrix} 12 \\ 1 \end{Bmatrix}'\right] + \frac{\partial}{\partial v}\left[\left(1-\frac{1}{I}\right)\begin{Bmatrix} 12 \\ 2 \end{Bmatrix}'\right] = 0,$$

which is the basis for a theorem analogous to Theorem 3 when the *transformed* surface S', and a conjugate system and a point function $I(u, v)$ on it, are given.

By the use of (12) either (11) or (13) can be put into the symmetric form

$$(14) \qquad \frac{\partial^2 \log I}{\partial u \partial v} + \left[\frac{\partial}{\partial u}\begin{Bmatrix} 12 \\ 1 \end{Bmatrix} - \frac{\partial}{\partial v}\begin{Bmatrix} 12 \\ 2 \end{Bmatrix}\right] - \left[\frac{\partial}{\partial u}\begin{Bmatrix} 12 \\ 1 \end{Bmatrix}' - \frac{\partial}{\partial v}\begin{Bmatrix} 12 \\ 2 \end{Bmatrix}'\right] = 0,$$

bearing on both S and S'. The quantities in the square brackets are the differences of the point invariants of the basic conjugate systems on S and S'. Since

* These formulas are most simply deduced by noting that

$$\begin{Bmatrix} 12 \\ 1 \end{Bmatrix} = \frac{(x_{uv}x_v|x_u x_v)}{H^2}, \qquad \begin{Bmatrix} 12 \\ 2 \end{Bmatrix} = \frac{(x_u x_{uv}|x_u x_v)}{H^2},$$

where $(x_{uv}x_v|x_u x_v)$, for example, is the scalar product of the vector products of x_{uv}, x_v and x_u, x_v.

these invariants are unchanged by a projective transformation of S and S',* (14) is invariant under such a transformation.

5. **The invariant J and the fundamental theorem for a parabolic map.** A parabolic projective transformation of a pencil of lines into itself, when the vertex of the pencil is chosen as the origin of a system of rectangular coördinates in the plane of the pencil and the double line is taken as the axis of x, is represented by an equation of the form

$$\frac{1}{\lambda'} - \frac{1}{\lambda} = J,$$

where λ and λ' are, respectively, the slopes of the given and transformed lines. The constant J, equal to the difference of the cotangents of the angles which the transformed and given lines make with the directed double line, is a metrical invariant of the transformation, and, taken with the double line, determines the transformation uniquely. The invariant of the inverse transformation is $-J$ and that of the identity, zero.

As the invariant $J(u, v)$ of a parabolic parallel map we define the difference of the cotangents of the angles which two corresponding curves, C and C', make at corresponding points with the parallel, similarly directed,† asymptotic lines, A and A', through these points:

$$J(u, v) = \cot(C', A') - \cot(C, A).$$

It is clear that J is invariant under different homothetic transformations applied, one to S, the other to S'.

THEOREM 4. *The invariant J of a parabolic map equals twice the difference of the cotangents of the angles which the non-corresponding asymptotic lines, $\overline{A}'$ and $\overline{A}$, make with the basic, similarly directed, asymptotic lines, A' and A*:

$$J(u, v) = 2[\cot(\overline{A}', A') - \cot(\overline{A}, A)].$$

It follows that two surfaces cannot correspond by a parabolic map so that the angles between the asymptotic lines at corresponding points are equal. In particular, *two minimal surfaces can never correspond by a parabolic map.*

From equations (7), taking as C and C' corresponding v-curves, we find that

$$J = \frac{\nu}{\lambda}\frac{E}{H}. \tag{15}$$

* Voss, *Zur Theorie der Krümmung der Flächen*, Mathematische Annalen, vol. 39 (1891) (pp. 179–256), p. 196.

† Orient A in the direction of increasing u on the surface S and give to A' the same orientation.

In developing the conditions for the integrability of (7) we have, first, that

$$(16) \qquad \nu x_{uu} + (\nu_u - \lambda_v)x_u + \lambda_u x_v = 0.$$

Hence

$$(17) \qquad \nu\begin{Bmatrix}11\\1\end{Bmatrix} + \nu_u - \lambda_v = 0, \qquad \nu\begin{Bmatrix}11\\2\end{Bmatrix} + \lambda_u = 0.$$

These equations are equivalent to the equations

$$(18) \quad \frac{\nu}{\lambda}\left(\begin{Bmatrix}11\\1\end{Bmatrix} - \frac{\nu}{\lambda}\begin{Bmatrix}11\\2\end{Bmatrix}\right) + \frac{\partial}{\partial u}\left(\frac{\nu}{\lambda}\right) - \frac{\partial \log \lambda}{\partial v} = 0, \quad \frac{\nu}{\lambda}\begin{Bmatrix}11\\2\end{Bmatrix} + \frac{\partial \log \lambda}{\partial u} = 0,$$

from which follows, by differentiation and the introduction of J, the condition

$$(19) \quad \frac{\partial^2}{\partial u^2}\left(J\frac{H}{E}\right) + \frac{\partial}{\partial u}\left[J\frac{H}{E}\left(\begin{Bmatrix}11\\1\end{Bmatrix} - J\frac{H}{E}\begin{Bmatrix}11\\2\end{Bmatrix}\right)\right] + \frac{\partial}{\partial v}\left(J\frac{H}{E}\begin{Bmatrix}11\\2\end{Bmatrix}\right) = 0.$$

Given, conversely, a surface S, referred to one family of asymptotic lines as the u-curves, and a function $J(u, v)$, such that (19) is satisfied. Then equations (18) in λ, where ν/λ has been replaced by JH/E, are integrable and determine λ except for a multiplicative constant k. Since this value of λ and the resulting value $\nu = \lambda JH/E$ for ν satisfy (16), x' can be determined from (7) except for the constant multiplier k and an additive triple.

THEOREM 5. *If one family of asymptotic lines on a given surface S, of negative curvature, and a point function $J(u, v)$ on S are chosen, a necessary and sufficient condition that there exist a surface S' on which S is mapped by parallel normals with the given asymptotic lines basic and the given function $J(u, v)$ as the invariant is that, when S is referred to the asymptotic lines as the u-curves, E, F, G, and J satisfy* (19). *The surface S' is then determined to within its homothetics and its point coördinates can be found by quadratures.*

It is evident that there exist infinitely many surfaces S' on which a given surface S, of negative curvature, is mapped by parallel normals so that a chosen family of asymptotic lines on S is basic. When S is ruled and the given asymptotic lines are the rulings, i. e., when $\begin{Bmatrix}11\\2\end{Bmatrix} = 0$, the surfaces S' can be determined by quadratures; cf. §13.

By means of the formulas,

$$(20) \quad \frac{H}{E} = \frac{H'}{E'}, \quad \begin{Bmatrix}11\\1\end{Bmatrix} - J\frac{H}{E}\begin{Bmatrix}11\\2\end{Bmatrix} = \begin{Bmatrix}11\\1\end{Bmatrix}' + J\frac{H'}{E'}\begin{Bmatrix}11\\2\end{Bmatrix}', \quad \begin{Bmatrix}11\\2\end{Bmatrix} = \begin{Bmatrix}11\\2\end{Bmatrix}',$$

we could deduce from (19) an equation similar in form to (19), but involving only

E', F', G', and J, and thus obtain a theorem, analogous to Theorem 5, when the *transformed* surface S' is given.

II. Maps of special type

6. **Differential coefficients of the surfaces.** Throughout Parts II and III we assume that the map is established by equations (4) or (7) of §2, according as it is non-parabolic or parabolic.

From equations (4) for a *non-parabolic* map we find that*

$$(21)\qquad \begin{array}{llll} E' = \lambda^2 E, & F' = \lambda\mu F, & G' = \mu^2 G, & H' = \lambda\mu H; \\ e' = \lambda e, & f' = f = 0, & g' = \mu g, & h'^2 = \lambda\mu h^2; \end{array} \qquad K' = \frac{1}{\lambda\mu}K.$$

For a *parabolic* map we have, from (7):

$$(22)\qquad \begin{array}{lll} E' = \lambda^2 E, & F' = \lambda\nu E + \lambda^2 F, & G' = \nu^2 E + 2\lambda\nu F + \lambda^2 G; \\ e' = e = 0, & f' = \lambda f, & g' = \mu f + \lambda g; \\ H' = \lambda^2 H, & h'^2 = \lambda^2 h^2, & K' = \dfrac{1}{\lambda^2}K. \end{array}$$

7. $I = -1$. **Associate surfaces of Bianchi.** In taking up special cases we consider, first, the involutory maps, for which $I = -1$.

THEOREM 6. *The pairs of asymptotic directions at two corresponding points of the two surfaces of a parallel map separate one another harmonically if and only if* $I = -1$.

The relationship is clearly impossible in the parabolic case. In the non-parabolic case, since the asymptotic directions on S' are defined, according to (21), by $Iedu^2 + gdv^2 = 0$, the directions on S parallel to them are given by $edu^2 + Igdv^2 = 0$;† but these directions separate harmonically the asymptotic directions on S if and only if $I = -1$.

A necessary and sufficient condition that the surfaces of a non-parabolic map be associate surfaces of Bianchi (cf. §1) is that $eg' + e'g = 0$, i. e., by (21), that $(I + 1)eg = 0$, or, by virtue of initial assumptions (§2), that $I = -1$. But a parabolic map can never have the property in question. Consequently:

THEOREM 7. *Two surfaces mapped by parallel normals are associate surfaces of Bianchi if and only if the invariant I of the map is* -1.

For the purpose of stating a more general characteristic property of involu-

* The quantities e, f, g are the differential coefficients of the second order, and $h^2 = eg - f^2$.

† The direction at a point of S parallel to the direction du/dv at the corresponding point of S' is Idu/dv.

tory maps we agree to call two directions on S *cross-parallel* to the corresponding directions on S' if each is parallel to the one to which it does not correspond.

THEOREM 8. *If $I = -1$, each two directions on S which are in involution with the basic conjugate directions are cross-parallel to the corresponding directions on S'. If $I \neq -1$, a pair of directions is never cross-parallel to the corresponding pair.**

When $I = -1$, equations (11) and (13) become

$$(23) \qquad \frac{\partial}{\partial u}\begin{Bmatrix}12\\1\end{Bmatrix} - \frac{\partial}{\partial v}\begin{Bmatrix}12\\2\end{Bmatrix} = 0, \qquad \frac{\partial}{\partial u}\begin{Bmatrix}12\\1\end{Bmatrix}' - \frac{\partial}{\partial v}\begin{Bmatrix}12\\2\end{Bmatrix}' = 0.$$

THEOREM 9. *The basic conjugate systems on two associate surfaces of Bianchi have equal point invariants.*

From Theorem 3 we obtain, when $I = -1$:

THEOREM 10. *If a conjugate system with equal point invariants on a given surface S is chosen, there exists a surface S' unique to within its homothetics, which is associate to S in the sense of Bianchi, so that the given conjugate system on S is basic.*

When S is referred to the conjugate system, the coördinates of S' are found to be

$$x' = k\int e^{-2\varphi}(x_u\,du - x_v\,dv),$$

where

$$(24) \qquad \varphi = \int \begin{Bmatrix}12\\2\end{Bmatrix}du + \begin{Bmatrix}12\\1\end{Bmatrix}dv.$$

The converse of Theorem 9 is not true, as (14) shows. We can, however, prove the theorem:

THEOREM 11. *If the basic conjugate systems of a non-parabolic map have equal point invariants and I is constant, the surfaces are, in general, associate surfaces of Bianchi.*

For, in this case (11) reduces to

$$\frac{\partial}{\partial u}\begin{Bmatrix}12\\1\end{Bmatrix} + I\frac{\partial}{\partial v}\begin{Bmatrix}12\\2\end{Bmatrix} = 0,$$

which in conjunction with the first of equations (23) gives, in general, $I = -1$.

* In particular, the asymptotic directions on each surface of an involutory map are cross-parallel to the conjugate directions corresponding to them on the other surface. This fact and Theorems 9 and 10 are well known; cf., e. g., Eisenhart, *Differential Geometry*, pp. 380, 381.

An exception occurs when

$$\frac{\partial}{\partial u}\left\{\begin{matrix}12\\1\end{matrix}\right\} = 0, \qquad \frac{\partial}{\partial v}\left\{\begin{matrix}12\\2\end{matrix}\right\} = 0.$$

Then, according to (12), the corresponding equations hold for S' and it is found, after a suitable change of parameters, that the point coördinates of S and S' are, respectively, solutions of the equations:

$$\theta_{uv} = \theta_u + \theta_v, \qquad \theta_{uv} = \frac{1}{I}\theta_u + I\,\theta_v.$$

8. **Conjugate systems with equal point invariants.** From (14) we conclude the following:

THEOREM 12. *If the basic conjugate system on one surface of a non-parabolic map has equal point invariants, that on the other surface has also, if and only if I is of the form $U(u)/V(v)$.*

If, in Theorem 3, the conjugate system chosen on the given surface S has equal point invariants, condition (11) becomes

$$\frac{\partial^2 \log I}{\partial u \partial v} - \frac{\partial}{\partial u}\left[\frac{1}{I}\left\{\begin{matrix}12\\1\end{matrix}\right\}\right] + \frac{\partial}{\partial v}\left[I\left\{\begin{matrix}12\\2\end{matrix}\right\}\right] = 0. \tag{25}$$

If we demand, further, that the basic conjugate system on the surface S' to be determined also have equal point invariants, (25) is replaced by the requirement that I be of the form $U(u)/V(v)$ and satisfy the equation

$$\frac{\partial}{\partial u}\left(\frac{1}{I}\varphi_v\right) = \frac{\partial}{\partial v}\left(I\varphi_u\right),$$

where φ is defined by (24).

To determine all the surfaces S, to each of which there corresponds a surface S' by a parallel map with a prescribed invariant $I = U(u)/V(v)$ so that the basic conjugate systems on both surfaces have equal point invariants, it is necessary, first, to integrate the equation in φ:

$$\frac{\partial}{\partial u}\left(\frac{V}{U}\varphi_v\right) = \frac{\partial}{\partial v}\left(\frac{U}{V}\varphi_u\right),$$

and then to integrate the equation in x : $x_{uv} = \varphi_v x_u + \varphi_u x_v$.

9. **Translation surfaces.** If one of the two surfaces of a non-parabolic map is a translation surface whose generators form the basic conjugate system, then, by (12), the other is also a translation surface with its generators basic. According to (9) this case occurs if and only if $\lambda_v = 0$ and $\mu_u = 0$, i. e., if and only if

the Combescurian ratio of the basic curves C_1' and C_1 is constant along the basic curves C_2' and C_2, and vice versa. Evidently $I = \lambda(u)/\mu(v)$, or

$$\frac{\partial^2 \log I}{\partial u \partial v} = 0. \tag{26}$$

THEOREM 13. *Two translation surfaces can be mapped by parallel normals so that the generators correspond if and only if two non-congruent generators of one can be put simultaneously into Combescurian correspondence with two non-congruent generators of the other. If both correspondences are homothetic, the invariant of the map is constant, and conversely.**

To prove the converse last stated, we note that, if $I = \lambda(u)/\mu(v)$ is constant, λ and μ are constant.

THEOREM 14. *The only surfaces on which a given translation surface*

$$x = U(u) + V(v) \tag{27}$$

is mapped by parallel normals so that its generators are basic are the translation surfaces

$$x' = \int \lambda(u)U'(u)du + \int \mu(v)V'(v)dv, \tag{28}$$

where $\lambda(u)$ and $\mu(v)$ are arbitrary scalar functions, not zero.

For, in this case (11) reduces to (26) and (9) to $\lambda_v = 0$, $\mu_u = 0$.

10. **Conformal and isometric maps.** The following three theorems, whose content except in so far as it affects the value of I is well known,† are stated without proof for later reference.

THEOREM 15. *A directly parallel map is conformal if and only if the two surfaces are minimal surfaces. Conversely, any two minimal surfaces can be brought into a correspondence by parallel normals; the map is always directly conformal and elliptic and, when the minimal lines are parametric, I is of the form $U(u)/V(v)$.*

THEOREM 16. *A directly parallel map is isometric if and only if the surfaces are associate minimal surfaces. The map is always elliptic; its invariant is constant and has the value -1 when and only when the surfaces are adjoint.*

THEOREM 17. *An inversely parallel map is conformal if and only if the surfaces are associate surfaces of Bianchi whose lines of curvature correspond. The surfaces are isometric surfaces; that is, the lines of curvature on each form an isometric system.*

* Suitable changes must be made in the conditions imposed if the surfaces are doubly covered by single families of generators.

† Cf. the reference to Christoffel, §1. The proofs can be readily constructed by use of equations (21), (22) and §§ 7, 9.

According to Theorem 10, to a given isometric surface S there corresponds, by an inversely parallel conformal map, an isometric surface S' determined to within its homothetics:*

$$x' = k\int e^{-\varphi}(x_u du - x_v dv), \qquad \text{where} \qquad \varphi = \int \frac{\partial \log G}{\partial u} du + \frac{\partial \log E}{\partial v} dv.$$

If two isometric surfaces are mapped by parallel normals so that the lines of curvature correspond, I is of the form $U(u)/V(v)$; the map is not necessarily conformal (or inversely parallel). However, by Theorem 11, if we stipulate that I be constant, the map is, in general, conformal. The exceptional case occurs when E, G, E', G' are all functions of the form $U(u)/V(v)$; the linear element of the surface S, referred to its lines of curvature, can then be written

$$ds^2 = U(u)V(v)(du^2 + dv^2),$$

whereupon that of S' becomes

$$ds'^2 = k^2 U^I V^{\frac{1}{I}} (I^2 du^2 + dv^2).$$

It is to be noted that surfaces of revolution are of the type in question.†

A necessary and sufficient condition that an inversely parallel map be isometric is, by (21), that $\lambda = \pm 1$, $\mu = \mp 1$, and $F = 0$. The surfaces are, then, translation surfaces with orthogonal generators and can be represented by equations of the form:

$$(29) \qquad x = U(u) + V(v), \quad x' = \pm U(u) \mp V(v).$$

It can be shown that they are congruent cylinders and that the correspondence is equivalent in the one case to a reflection in a plane and a translation, and in the other to a reflection in a line.

11. **Equiareal non-parabolic maps.** A non-parabolic map is equiareal, by (21), if and only if the product, $\lambda\mu$, of the Combescurian ratios of corresponding parallel curves is $+1$ or -1, according as the map is directly or inversely parallel. From (21), or (3), we have also:

THEOREM 18. *A non-parabolic map is equiareal when and only when the total curvatures of the surfaces at corresponding points are equal or opposite, according as the map is directly or inversely parallel.*‡

* When the parameters on S are isometric, E and G are equal and the representation of S' reduces to that usually given; cf. Eisenhart, *Differential Geometry*, p. 388.

† Since a surface whose lines of curvature have equal point invariants is isometric, the results of §8 also are applicable here.

‡ This theorem is, of course, well known; cf. the references to Guichard and Razzaboni, §1. The remaining results of the section are believed to be new.

Let the surface S:

$$x = U(u) + V(v)$$

be a translation surface referred to its generators. If these are real, there are two distinct one-parameter families of translation surfaces which we shall call *associate* to S, namely the families

$$(A) \qquad x' = \lambda U(u) - \frac{1}{\lambda} V(v),$$

$$(B) \qquad x' = \lambda U(u) + \frac{1}{\lambda} V(v),$$

where in each case λ is an arbitrary real constant, $\neq 0$. The family (A) does not contain the given surface, whereas the family (B) does.

If the generators of S are conjugate-imaginary, the surfaces associate to it shall be defined by the representation

$$(C) \qquad x' = e^{\alpha i} U(u) + e^{-\alpha i} V(v),$$

where α is an arbitrary real constant.

A special case of the family (C) is that of surfaces associate to a minimal surface. As in that case, so here also, the path curve of a chosen point (u, v) of a variable surface of the family is an ellipse whose center is at the origin of coördinates, O. A path curve for the family (B) is a hyperbola whose center is at O and whose asymptotes are the lines through O and the points u and v of the curves $y = U(u)$, $z = V(v)$; the conjugate hyperbola is the corresponding path curve for the family (A).

If S is minimal, the surfaces (C) for which $\alpha = \pm \pi/2$ are known as its adjoints. Accordingly, we define, as the adjoints of an arbitrary translation surface S, the surfaces

$$x' = \pm U(u) \mp V(v), \quad \text{or} \quad x' = \pm iU(u) \mp iV(v),$$

according as the generators of S are real or imaginary.

The translation surfaces (27) and (28) are associate if and only if λ and μ of (28) are constants such that $\lambda\mu = \pm 1$; they are adjoint if and only if $\lambda = \pm 1$, $\mu = \mp 1$, or $\lambda = \pm i$, $\mu = \mp i$. Thus:

THEOREM 19. *A non-parabolic map is equiareal with constant invariant when and only when the surfaces are associate translation surfaces. It is equiareal with invariant* -1 *if and only if the surfaces are adjoint translation surfaces.*

If the map is hyperbolic, the surfaces have real generators and belong both to a family of type (B), or one to a family of type (B) and the other to the corresponding family of type (A), according as the map is directly or inversely parallel. In the elliptic case the surfaces have imaginary generators and both belong to a family of type (C).

From this discussion and Theorem 18 we conclude the following:

THEOREM 20. *The total curvatures of two associate translation surfaces at corresponding points are equal, if the generators are imaginary; if the generators are real, they are equal or opposite, according as the surfaces belong to the same or different families.*

Associate translation surfaces constitute the only equiareal maps of translation surfaces such that the generators correspond. For, if $\lambda(u)$ and $\mu(v)$ of (28) satisfy $\lambda\mu = \pm 1$, λ and μ are constant.

12. **Summary of special maps for which** $I = -1$. It is instructive to collect from §§10, 11 the results which concern associate surfaces of Bianchi.

These surfaces, in the case of an inversely parallel map, are: (a) if the map is equiareal, adjoint translation surfaces with real generators; (b) if the map is isometric, adjoint translation surfaces with orthogonal generators, i. e., cylinders; (c) if the map is conformal, isometric surfaces.

If the map is directly parallel, the surfaces, in the three cases considered, are: (a) adjoint translation surfaces with conjugate-imaginary generators; (b) adjoint translation surfaces with minimal generators, i. e., adjoint minimal surfaces; (c) the same as in (b),—a directly conformal parallel map for which $I = -1$ is necessarily isometric.

13. **Parabolic maps of ruled surfaces.** The surfaces of a parabolic map are, by (16), ruled surfaces whose rulings correspond if and only if $\lambda_u = 0$, i. e., if and only if along each pair of basic asymptotic lines the Combescurian ratio is constant. From §3 we find also:

THEOREM 21. *The focal surface of the congruence associated with a parabolic map degenerates into a curve if and only if the surfaces of the map are ruled surfaces whose generators correspond.*

Then every developable of the congruence is a plane determined by two corresponding generators, and the line-segments joining corresponding points of these generators are all divided by the focal point in the plane in the same ratio, $-\lambda(v)$.

In deducing the equations of the general pair of ruled surfaces mapped by parallel normals so that the generators correspond, we distinguish two cases.

Case 1: $\lambda(v) \neq$ *const.* Introducing instead of u in the equations of a parabolic map a new parameter $\bar{u}$ such that $\partial\bar{u}/\partial u = \lambda'(v)u/v$, and subsequently dropping the bars, we find that (7) becomes*

$$x'_u = \lambda(v)x_u, \qquad x'_v = \lambda'(v)ux_u + \lambda(v)x_v,$$

* Throughout the paragraph we denote the derivatives of functions of a single variable, e. g., $\lambda(v)$, by using primes.

and that (16) reduces to $x_{uu} = 0$. It follows, then, that

$$(30) \qquad x = \psi(v)u + \varphi(v), \qquad x' = \lambda(v)\psi(v)u + f(v),$$

where
$$f'(v) = \lambda(v)\varphi'(v).$$

Equations (30), *where* $\lambda(v)$ *is an arbitrary scalar function, not constant, and* $f(v)$, $\varphi(v)$, $\psi(v)$ *are arbitrary triples of functions such that* $f' \equiv \lambda\varphi'$, *define the general parabolic map of two ruled surfaces whose generators correspond, in the case that the Combescurian ratio of corresponding rulings is not constant.*

The map of S on S' is equivalent to what might be called a pseudo-homothetic transformation,* consisting of a Combescurian transformation of ratio $\lambda(v)$ of the directrix curve, $X = \varphi(v)$, of S into the directrix curve, $X' = f(v)$, of S', and a homothetic transformation of ratio $\lambda(v)$ carrying the ruling through X into the ruling through X' so that the point X goes into the point X'.

Along the directrix curves the map has a singular behavior, in that in the neighborhoods of corresponding points of these curves it is equivalent, to a first approximation, to a homothetic transformation of ratio $\lambda(v)$; in particular, corresponding directions at these points are, without exception, parallel.

From the equations

$$z = \frac{x' - \lambda x}{1 - \lambda} = \frac{X' - \lambda X}{1 - \lambda}, \qquad \frac{dz}{dv} = \frac{\lambda'(X' - X)}{(1 - \lambda)^2},$$

we conclude that *the focal curve of the associated congruence is the envelope of the lines joining corresponding points* X', X *of the directrix curves of the surfaces.* It is also the locus of a point dividing the segment $X'X$ in the ratio $-\lambda(v)$

Case 2: $\lambda = const.$ Here we replace u by a new parameter $\bar{u}$ such that $\partial\bar{u}/\partial u = 1/\nu$. Equations (7) become

$$x'_u = \lambda x_u, \qquad x'_v = x_u + \lambda x_v,$$

and (16) reduces again to $x_{uu} = 0$. Hence we can write

$$(31) \qquad x = \zeta'(v)u + \xi(v), \qquad x' = \lambda x + \zeta(v).$$

Equations (31), *where* λ *is an arbitrary constant,* $\neq 0$, *and* $\xi(v)$, $\zeta(v)$ *are arbitrary triples of functions, define the general parabolic map of two ruled surfaces whose generators correspond, in the case that the Combescurian ratio of the generators is constant.*

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* If λ were constant the surfaces would actually be homothetic.

The map is equivalent to a pseudo-homothetic transformation whose coefficient of stretching, λ, is constant, but whose parameters of translation vary from ruling to ruling. These parameters can be thought of as the point coordinates of the curve $\zeta = \zeta(v)$, whose tangents furnish the directions of the corresponding rulings of the two surfaces. It is to be noted that in this case there is no pair of curves along which the map has a singular behavior.

From the equations,

$$z = \frac{x' - \lambda x}{1 - \lambda} = \frac{\zeta}{1 - \lambda}, \qquad \frac{dz}{dv} = \frac{\zeta'}{1 - \lambda},$$

it follows that *the focal curve of the associated congruence is the envelope of lines which are parallel to corresponding generators and divide their common perpendiculars in the ratio* $-\lambda$. It is homothetic to the curve $\zeta = \zeta(v)$.

If $\lambda = -1$, the line-segments joining corresponding points of the two surfaces are all bisected by the focal curve. If $\lambda = 1$, the lines joining corresponding points of parallel rulings are all parallel; there is no focal curve. In these two cases and in these only the map is equiareal.

THEOREM 22. *A parabolic map is equiareal if and only if the surfaces are ruled surfaces whose rulings correspond and have the Combescurian ratio* ± 1.

Equations (30) and (31), by proper manipulation, can both be written:

$$(32) \quad x = [F'(v) - \lambda(v)\,\Phi'(v)]u + \Phi(v), \quad x' = \lambda(v)[F'(v) - \lambda(v)\Phi'(v)]u + F(v),$$

where $\lambda(v)$ is an arbitrary scalar function or constant, $\neq 0$, and $F(v)$, $\Phi(v)$ are arbitrary triples of functions for which $F' \not\equiv \lambda\Phi'$.

III. FUNDAMENTAL SYSTEMS OF CURVES FOR A NON-CONFORMAL MAP

14. **Corresponding orthogonal systems. Isometrically mapped systems.** Let the equations $x = x(u, v)$, $x' = x'(u, v)$, represent an arbitrary real map, not necessarily by parallel normals, of two arbitrary real surfaces. If the map is not conformal, there exist on the surfaces unique orthogonal systems of curves which correspond. If the map is not isometric, there exists on each surface a unique system such that the curves of the two systems correspond and are mapped isometrically. The two families of curves of each system are not necessarily real or distinct; for a conformal map, for example, they consist of the minimal curves.

THEOREM 23. *In a non-conformal map the corresponding orthogonal systems bisect the angles between the isometrically mapped systems, or, if the families constituting the latter are coincident, the corresponding orthogonal systems consist of these families and their orthogonal trajectories.*

For, if the families of isometrically mapped curves are distinct and parametric,

$$E' = E \neq 0, \quad F' \neq F, \quad G' = G \neq 0.$$

The differential equation of the corresponding orthogonal systems is, then, $Edu^2 - Gdv^2 = 0$.

If the families of isometrically mapped curves coincide, in the u-curves, and if on S their orthogonal trajectories are the v-curves, $E' = E$ and $F = 0$. But it follows, then, that $F' = 0$.

In the second case the map is never equiareal, since, by hypothesis, $G' \neq G$. In the first case the map is equiareal if and only if $F' = -F$, or $F'/\sqrt{E'G'} = -F/\sqrt{EG}$.

THEOREM 24. *In a non-isometric map the angles between the isometrically mapped curves at corresponding points are supplementary when and only when the map is equiareal.*

Since the map is not isometric, F and F' are not both zero. Hence the isometrically mapped curves never form orthogonal systems on both surfaces. Theorem 24 gains in significance in light of the following fact.

THEOREM 25. *If a non-isometric map is equiareal the families of isometrically mapped curves are real and distinct.*

For, if the corresponding orthogonal systems of curves are parametric, $F' = F = 0$, $E'G' = EG$, and E, E', G, G' are all real and positive. Hence, for the differential equation of the isometrically mapped systems,

$$(E' - E)du^2 + (G' - G)dv^2 = 0,$$

it follows that $(E' - E)(G' - G) < 0$.

Parallel maps. If a parallel map is non-parabolic; the differential equations of the corresponding orthogonal systems and the isometrically mapped systems are, respectively,

$$\lambda EFdu^2 + (\lambda + \mu)EGdudv + \mu FGdv^2 = 0, \tag{33}$$

$$(\lambda^2 - 1)Edu^2 + 2(\lambda\mu - 1)Fdudv + (\mu^2 - 1)Gdv^2 = 0. \tag{34}$$

In discussing these systems it will be tacitly assumed that the map is non-conformal or non-isometric, according to the case in point.

THEOREM 26. *In a parallel map the corresponding orthogonal systems bisect the angles of the basic conjugate systems if and only if* $I = -1$.

This follows from (33), or from Theorem 8, in the non-parabolic case. In the parabolic case, the relationship, in the limiting form it would then have, is clearly impossible.

Since, when $I = -1$, the corresponding orthogonal systems bisect the angles of both the basic conjugate systems and the isometrically mapped systems, it is pertinent to inquire when the latter systems coincide.

THEOREM 27. *In a parallel map the isometrically mapped systems coincide with the corresponding conjugate systems when and only when the two surfaces are adjoint translation surfaces with real generators.**

For, equation (34) reduces to $dudv = 0$, if and only if $\lambda = \pm 1$, $\mu = \mp 1$. Moreover, if the isometrically mapped curves of a parabolic map coincided with the single families of basic parallel curves, the corresponding orthogonal systems would consist, by Theorem 23, of these families and their orthogonal trajectories; there would then be two families of corresponding parallel curves,—a contradiction.

The corresponding orthogonal systems coincide with the basic conjugate systems, if and only if the latter are the lines of curvature. In this case, if the lines of curvature are to be the only corresponding orthogonal systems, the invariant cannot be -1; cf. Theorem 17.

From Theorem 23 it is clear that the corresponding orthogonal systems never coincide with the isometrically mapped systems.

15. **Angles associated with the fundamental systems of curves.** Let θ be the angle between corresponding tangents to corresponding curves of the corresponding orthogonal systems of a non-conformal, non-parabolic map.† We find that

$$\cos^2\theta = \frac{(I+1)^2H^2}{(I-1)^2EG + 4IH^2}, \qquad \tan^2\theta = \left(\frac{I-1}{I+1}\right)^2 \frac{F^2}{H^2}. \tag{35}$$

Consequently, if the angle between the curves of the basic conjugate systems is denoted by φ,‡

$$\tan^2\theta\tan^2\varphi = \left(\frac{I-1}{I+1}\right)^2. \tag{36}$$

If the corresponding orthogonal systems are the basic conjugate systems, $\theta = 0$ (or π) and $\varphi = \pi/2$, and conversely. If the curves of the corresponding orthogonal systems are cross-parallel, $\theta = \pi/2$ and $I = -1$, and conversely;

* Cf. Theorem 19. In the case of two adjoint translation surfaces with conjugate-imaginary generators, the generators are mapped pseudo-isometrically: the ratio of the squares of corresponding elements of arc is -1.

† If the map is inversely parallel there are two angles θ, supplementary to each other; which is chosen is immaterial for our purposes.

‡ In the elliptic case, F is positive and H^2, $\tan^2\varphi$ and $\left(\frac{I-1}{I+1}\right)^2$ are negative.

cf. Theorem 8. In all other cases the three expressions in (36) have finite values, not zero.

THEOREM 28. *If corresponding curves, C and C′, of the corresponding orthogonal systems of a non-conformal, non-parabolic map are neither parallel nor perpendicular at corresponding points, the angle θ between C and C′, the angle φ between the curves of the basic conjugate systems, and the invariant I are related by* (36).

Under the restrictions, $\theta \neq 0, \pi/2$, two of the quantities θ, φ, I determine the third; if two are constant, so is the third.

16. **The angles for an equiareal map.** *Non-parabolic case.* Assume, now, that the map is equiareal and let ψ be the angle between the isometrically mapped curves on either surface.

For an *inversely parallel* map,

$$\sin^2\psi = -\frac{4IH^2}{(I-1)^2EG}, \qquad \tan^2\psi = -\frac{4IH^2}{(I-1)^2EG+4IH^2}.$$

From these equations and (35) we find that

$$\frac{\tan^2\psi}{\cos^2\theta} = -\frac{4I}{(I+1)^2}, \qquad \frac{\sin^2\psi}{\sin^2\varphi} = -\frac{4I}{(I-1)^2}. \tag{37}$$

Elimination of ψ yields (36). Elimination of I gives the equation

$$\sin^2\theta = \frac{\cos^2\varphi}{\cos^2\psi}. \tag{38}$$

Of the various conclusions which can be drawn from (37) and (38), the following appear to be the most important. The map is assumed throughout to be inversely parallel, equiareal, and non-conformal.

THEOREM 29. *The square of the sine of the angle θ between corresponding curves of corresponding orthogonal systems equals the ratio of the squares of the cosines of the angles, φ and ψ, between the basic conjugate curves and the isometrically mapped curves, respectively.*

THEOREM 30. *The basic conjugate systems are the lines of curvature if and only if*

$$\tan^2\psi = -\frac{4I}{(I+1)^2}. \tag{39}$$

THEOREM 31. *If corresponding curves of the corresponding orthogonal systems are neither parallel nor perpendicular at corresponding points, any two of the four* *quantities, θ, φ, ψ, I, are determined when the other two are known and if any two are constant, so are the other two.*

In the case of a *directly parallel* map,

$$\sin^2\psi = \frac{4IH^2}{(I-1)^2EG+4IH^2}, \qquad \tan^2\psi = \frac{4IH^2}{(I-1)^2EG}.$$

Hence

$$\frac{\sin^2\psi}{\cos^2\theta} = \frac{4I}{(I+1)^2}, \qquad \frac{\tan^2\psi}{\sin^2\varphi} = \frac{4I}{(I-1)^2}. \tag{40}$$

As before, elimination of ψ gives (36). Elimination of I leads to

$$\sin^2\theta = \cos^2\varphi\cos^2\psi. \tag{41}$$

Thus, for a directly parallel non-parabolic map which is equiareal but not conformal, Theorem 31 is duplicated and Theorem 29 is valid if "ratio" is replaced by "product." Moreover, if the map is hyperbolic, Theorem 30 holds when the minus sign in (39) is replaced by a plus sign. If the map is elliptic, we find the following:

THEOREM 32. *Corresponding curves of corresponding orthogonal systems are cross-parallel, i. e., the surfaces are adjoint translation surfaces with imaginary generators,* if and only if*

$$\cos^2\varphi = \sec^2\psi. \tag{42}$$

Of equations (40), (41), the only one which involves angles which are real for both types of map is the first equation of (41); from it we easily show that the map is hyperbolic or elliptic, according as

$$\frac{\sin^2\psi}{\cos^2\theta} < 1 \quad \text{or} \quad \frac{\sin^2\psi}{\cos^2\theta} > 1. \tag{43}$$

Parabolic case. We are dealing here with the type of map considered in Theorem 22. One family of isometrically mapped curves on each surface coincides with the basic asymptotic lines,—the generators of the ruled surfaces. Consequently, since the corresponding orthogonal systems bisect the angles of the isometrically mapped systems, it follows that $\theta \pm \psi = \pi/2$.

THEOREM 33. *In a parabolic equiareal map the angle θ between corresponding curves of the corresponding orthogonal systems is the complement of, or differs by $\pi/2$ from, the angle ψ between the isometrically mapped curves.*

* Cf. Theorem 19 and the footnote to Theorem 27. The relation (42) takes the place, in this case, of the relation, $\cos^2\varphi = \cos^2\psi$, which holds in the case of adjoint translation surfaces with real generators.

Since, then, $\sin^2\psi = \cos^2\theta$, we obtain, in light of (43), the following:

THEOREM 34. *An equiareal, directly parallel map which is not conformal is hyperbolic, parabolic, or elliptic, according as* $\sin^2\psi/\cos^2\theta$ *is less than, equal to, or greater than unity.*

IV. CONTINUATION OF GENERAL THEORY

17. **Classification when parametric curves are arbitrary. Conditions of integrability.** If the surface $x = x(u, v)$, referred to two arbitrary real or conjugate-imaginary families of curves, is mapped by parallel normals on the surface $x' = x'(u, v)$, then

$$x'_u = \alpha\, x_u + \beta\, x_v, \qquad x'_v = \gamma\, x_u + \delta\, x_v, \tag{44}$$

where the pairs of functions $\alpha(u, v)$, $\delta(u, v)$ and $\beta(u, v)$, $\gamma(u, v)$ are real or conjugate-imaginary, according as u and v are real or conjugate-imaginary, and where

$$D = \alpha\delta - \beta\gamma \neq 0.$$

The basic conjugate systems are defined by the equation

$$\beta du^2 + (\delta - \alpha)dudv - \gamma dv^2 = 0, \tag{45}$$

whose discriminant is

$$\Delta = (\delta - \alpha)^2 + 4\beta\gamma = (\delta + \alpha)^2 - 4D.$$

Factoring (45) we obtain:*

$$2du + \left(\frac{\delta-\alpha}{\beta} + \sqrt{\frac{\Delta}{\beta^2}}\right) dv = 0, \quad 2du + \left(\frac{\delta-\alpha}{\beta} - \sqrt{\frac{\Delta}{\beta^2}}\right)dv = 0.$$

The cross ratio in which the directions defined by these equations, in the order given, are divided by the corresponding directions dx, dx', is

$$I = \frac{\dfrac{\alpha+\delta}{\beta} - \sqrt{\dfrac{\Delta}{\beta^2}}}{\dfrac{\alpha+\delta}{\beta} + \sqrt{\dfrac{\Delta}{\beta^2}}}. \tag{46}$$

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* The case $\beta = \gamma = 0$ is that already treated. We assume, then, that β and γ are not both zero and, in particular, that $\beta \neq 0$. The equations are written with each term of zero degree in α, β, γ, δ so that, when the surface S' is replaced by the surface $z = kx'$, k a constant, the invariant (46) will preserve its value, regardless of whether k is positive or negative.

We can now give, under our present hypothesis of arbitrary parametric curves, a complete classification of parallel maps. The map is directly or inversely parallel according as $D > 0$, or $D < 0$. It is hyperbolic, parabolic, or elliptic, according as $\Delta > 0$, $\Delta = 0$, or $\Delta < 0$.

In developing the conditions of integrability of (44), we choose as the parametric curves on S the asymptotic lines. Then, since (45) represents a conjugate system, $\alpha = \delta$, and our equations become

$$x'_u = \alpha x_u + \beta x_v, \qquad x'_v = \gamma x_u + \alpha x_v, \tag{47}$$

$$\beta du^2 - \gamma dv^2 = 0, \tag{48}$$

$$I = \frac{\dfrac{\alpha}{\beta} - \sqrt{\dfrac{\gamma}{\beta}}}{\dfrac{\alpha}{\beta} + \sqrt{\dfrac{\gamma}{\beta}}} = \frac{\dfrac{\alpha}{\gamma} - \sqrt{\dfrac{\beta}{\gamma}}}{\dfrac{\alpha}{\gamma} + \sqrt{\dfrac{\beta}{\gamma}}}, \tag{49}$$

where the different square roots in (49) are reciprocals of one another.*

Since
$$x'_{uv} = x'_{vu},$$

$$\gamma x_{uu} - \beta x_{vv} + (\gamma_u - \alpha_v)x_u + (\alpha_u - \beta_v)x_v = 0. \tag{50}$$

On substituting for x_{uu} and x_{vv} their values in terms of x_u and x_v, we obtain, as necessary and sufficient conditions for the integrability of (47), the equations:

$$\begin{aligned} \gamma_u - \alpha_v + \begin{Bmatrix} 11 \\ 1 \end{Bmatrix} \gamma - \begin{Bmatrix} 22 \\ 1 \end{Bmatrix} \beta &= 0, \\ \beta_v - \alpha_u - \begin{Bmatrix} 11 \\ 2 \end{Bmatrix} \gamma + \begin{Bmatrix} 22 \\ 2 \end{Bmatrix} \beta &= 0. \end{aligned} \tag{51}$$

18. **Existence of a map when one surface and I are given.** Given a surface S, referred to its asymptotic lines, and a point function $I(u, v)$, $\neq 1$, on the surface. Will there exist a surface S' on which S is mapped by parallel normals so that the invariant of the map is the given function I?

The problem is essentially that of solving the system of equations (49) and (51) for α, β, γ, when E, F, G, and I are given. Since we need, in particular, to determine the basic conjugate system (48), we seek an eliminant of (49) and (51) which involves, of the unknowns, only the ratio β/γ.

* The assumption, $\beta\gamma \neq 0$, of (49) rules out parabolic maps; but these have been considered in detail, in §5, in the typical case $\beta = 0$.

Case 1: $I = -1$. Here $\alpha = 0$ and equations (51) can be written as

$$(52)\qquad -\frac{\partial \log \gamma}{\partial u} = \begin{Bmatrix} 11 \\ 1 \end{Bmatrix} - \begin{Bmatrix} 22 \\ 1 \end{Bmatrix}\frac{\beta}{\gamma}, \quad \frac{\partial \log \beta}{\partial v} = \begin{Bmatrix} 11 \\ 2 \end{Bmatrix}\frac{\gamma}{\beta} - \begin{Bmatrix} 22 \\ 2 \end{Bmatrix}.$$

By differentiation, and application of the identity*

$$\frac{\partial}{\partial u}\begin{Bmatrix} 22 \\ 2 \end{Bmatrix} = \frac{\partial}{\partial v}\begin{Bmatrix} 11 \\ 1 \end{Bmatrix},$$

we obtain

$$(53)\qquad \frac{\partial^2 \log \beta/\gamma}{\partial u \partial v} = \frac{\partial}{\partial v}\left[\frac{\gamma}{\beta}\begin{Bmatrix} 11 \\ 2 \end{Bmatrix}\right] - \frac{\partial}{\partial v}\left[\frac{\beta}{\gamma}\begin{Bmatrix} 22 \\ 1 \end{Bmatrix}\right].$$

Each solution $\overline{\beta/\gamma}$ of this equation leads, by quadratures, to a surface S' associate to S in the sense of Bianchi and unique to within its homothetics. For, β and γ can be found from (52) and hence x' from $x'_u = \beta x_v$, $x'_v = \gamma x_u$.

Case 2: $I \neq -1$. In this case we rewrite (49) as

$$(54)\qquad \sqrt{\frac{\beta}{\gamma}} = \psi\frac{\alpha}{\gamma} \quad \text{or} \quad \sqrt{\frac{\gamma}{\beta}} = \psi\frac{\alpha}{\beta},$$

where

$$\psi = \frac{1-I}{1+I},$$

and (51) as

$$\frac{\beta_v}{\beta} + \begin{Bmatrix} 22 \\ 2 \end{Bmatrix} - \begin{Bmatrix} 11 \\ 2 \end{Bmatrix}\frac{\gamma}{\beta} = \frac{\alpha_u}{\beta}, \quad \frac{\gamma_u}{\gamma} + \begin{Bmatrix} 11 \\ 1 \end{Bmatrix} - \begin{Bmatrix} 22 \\ 1 \end{Bmatrix}\frac{\beta}{\gamma} = \frac{\alpha_v}{\gamma}.$$

From (54),

$$\psi\sqrt{\frac{\beta}{\gamma}}\frac{\alpha_u}{\beta} = \frac{\gamma_u}{\gamma} - \frac{1}{2}\frac{\partial}{\partial u}\log\psi^2\frac{\gamma}{\beta}, \quad \psi\sqrt{\frac{\gamma}{\beta}}\frac{\alpha_v}{\gamma} = \frac{\beta_v}{\beta} - \frac{1}{2}\frac{\partial}{\partial v}\log\psi^2\frac{\beta}{\gamma}.$$

Hence

$$-\psi\sqrt{\frac{\beta}{\gamma}}\frac{\beta_v}{\beta} + \frac{\gamma_u}{\gamma} = \frac{1}{2}\frac{\partial}{\partial u}\log\psi^2\frac{\gamma}{\beta} + \psi\sqrt{\frac{\beta}{\gamma}}\left[\begin{Bmatrix} 22 \\ 2 \end{Bmatrix} - \begin{Bmatrix} 11 \\ 2 \end{Bmatrix}\frac{\gamma}{\beta}\right],$$

$$\frac{\beta_v}{\beta} - \psi\sqrt{\frac{\gamma}{\beta}}\frac{\gamma_u}{\gamma} = \frac{1}{2}\frac{\partial}{\partial v}\log\psi^2\frac{\beta}{\gamma} + \psi\sqrt{\frac{\gamma}{\beta}}\left[\begin{Bmatrix} 11 \\ 1 \end{Bmatrix} - \begin{Bmatrix} 22 \\ 1 \end{Bmatrix}\frac{\beta}{\gamma}\right].$$

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* Cf., e. g., Eisenhart, *Differential Geometry*, p. 189.

Solving for β_v/β and γ_u/γ, we obtain

$$(55) \qquad (1-\psi^2)\frac{\partial \log \beta}{\partial v} = A, \qquad (1-\psi^2)\frac{\partial \log \gamma}{\partial u} = B,$$

where

$$A = \frac{1}{2}\frac{\partial \log \psi^2 \frac{\beta}{\gamma}}{\partial v} + \frac{\partial}{\partial u}\left(\psi\sqrt{\frac{\gamma}{\beta}}\right) + \psi\sqrt{\frac{\gamma}{\beta}}\left[\left\{\begin{matrix}11\\1\end{matrix}\right\} - \left\{\begin{matrix}22\\1\end{matrix}\right\}\frac{\beta}{\gamma}\right] + \psi^2\left[\left\{\begin{matrix}22\\2\end{matrix}\right\} - \left\{\begin{matrix}11\\2\end{matrix}\right\}\frac{\gamma}{\beta}\right],$$

$$B = \frac{1}{2}\frac{\partial \log \psi^2 \frac{\gamma}{\beta}}{\partial u} + \frac{\partial}{\partial v}\left(\psi\sqrt{\frac{\beta}{\gamma}}\right) + \psi^2\left[\left\{\begin{matrix}11\\1\end{matrix}\right\} - \left\{\begin{matrix}22\\1\end{matrix}\right\}\frac{\beta}{\gamma}\right] + \psi\sqrt{\frac{\beta}{\gamma}}\left[\left\{\begin{matrix}22\\2\end{matrix}\right\} - \left\{\begin{matrix}11\\2\end{matrix}\right\}\frac{\gamma}{\beta}\right].$$

From (55) we have, finally, the equation

$$(56) \qquad \frac{\partial^2 \log \beta/\gamma}{\partial u \partial v} = \frac{\partial}{\partial u}\frac{A}{1-\psi^2} - \frac{\partial}{\partial v}\frac{B}{1-\psi^2},$$

which involves β and γ only in their ratio.

In discussing these results, we note first the effect of changing the determination of the square root, when an actual map is given. The order of the basic conjugate directions is reversed, I is replaced by $1/I$ and ψ changes sign. Thus (55) and (56) are unchanged.

If, for a fixed determination of the square root, β/γ is a particular solution of (56), equations (55) are integrable and determine β and γ except for the same multiplicative constant k; then α, unique except for the factor k, is obtained from (54). For these values of α, β, γ, equations (47) are integrable and determine x' except for the multiplier k and an additive triple.

Thus, in this case also, the determination of all the surfaces S' in question requires the solution of a partial differential equation of the second order and quadratures.

THEOREM 35. *Given a surface S and a point function I, $\neq 1$, on it, there exist infinitely many surfaces S' which correspond to S by a parallel map whose invariant is I. If S is referred to its asymptotic lines and a particular solution, β/γ, of* (56)*—or of* (53), *if $I = -1$—is chosen, S' is determined to within its homothetics and its point coördinates can be found by quadratures.*

A solution, β/γ, of (53) or (56) determines immediately, not only the basic conjugate systems (48), but also the asymptotic lines, $\beta du^2 + \gamma dv^2 = 0$, of the surface S'. It is possible to replace β/γ by a quantity of even more striking geometrical significance, namely by the ratio of the radii R_{c_1}, R_{c_2} of normal curvatures of the surface S in the conjugate directions,

$$\sqrt{\frac{\beta}{\gamma}}\,du + dv = 0, \quad \sqrt{\frac{\beta}{\gamma}}\,du - dv = 0,$$

which serve, in the order given, as the basic directions. We find that

$$R = \frac{R_{c_1}}{R_{c_2}} = -\frac{E - 2F\sqrt{\frac{\beta}{\gamma}} + G\frac{\beta}{\gamma}}{E + 2F\sqrt{\frac{\beta}{\gamma}} + G\frac{\beta}{\gamma}}, \tag{57}$$

whence

$$\sqrt{\frac{\beta}{\gamma}} = \frac{(1-R)F \pm \sqrt{(1-R)^2F^2 - (1+R)^2EG}}{(1+R)G}. \tag{58}$$

For a given non-parabolic map the ratio R in question is determined by (57); hence the value of $\sqrt{\beta/\gamma}$ is given by (58) for a proper choice of sign. Substituting this value in (53) or (56), according as I is or is not -1, we get a valid equation in E, F, G, I (or ψ), and R. Conversely, if this equation is satisfied for one choice of sign, then $\sqrt{\beta/\gamma}$, given by (58) for this choice of sign, satisfies (53) or (56).

THEOREM 36. *A necessary and sufficient condition that there exist a surface S' on which a given non-minimal* surface S is mapped by parallel normals, so that the invariant is a prescribed function I, $\neq 1$, and the ratio of the radii of normal curvatures in the ordered basic conjugate directions is a prescribed function R, is that, when S is referred to its asymptotic lines, E, F, G, I, and R satisfy* (56)—*or, when $I = -1$,* (53)—*for one choice of sign of the radical in* (58).

There are certain preliminary restrictions which can be placed on $I(u, v)$ and $R(u, v)$. If S is of positive curvature, I must be real, whereas, if S is of negative curvature, I may be real or of the form $e^{\varphi(u,v)i}$. If $I \neq -1$, R is of the same type as I, whereas, if $I = -1$, R may be of either type. In case R is real, it must satisfy the conditions†

$$\frac{r_2}{r_1} \leqq R \leqq \frac{r_1}{r_2}, \qquad R \neq -1,$$

where r_1 and r_2 are the principal radii of curvature of S.

* If S were minimal, R would be always -1 and hence not dependent on β/γ.

† The restriction, $R \neq -1$, serves to prevent the basic conjugate directions from coinciding in an asymptotic direction.

19. **Existence of a non-parabolic map when the spherical representation is given.** It is well known that a system of curves $\mathfrak{C}$ on the Gauss sphere represents a conjugate system of curves C on each of infinitely many surfaces S. It can be shown further* that, if the curves $\mathfrak{C}$ are parametric,† each solution $\theta = M$ of the equation

$$(59) \qquad \frac{\partial^2 \log \theta}{\partial u \partial v} = \frac{\partial}{\partial u}\left[\left\{ {12 \atop 1} \right\}_1 - \left\{ {11 \atop 2} \right\}_1 \frac{1}{\theta}\right] - \frac{\partial}{\partial v}\left[\left\{ {12 \atop 2} \right\}_1 - \left\{ {22 \atop 1} \right\}_1 \theta\right],$$

where the Christoffel symbols pertain to the sphere, determines one of the required surfaces S to within its homothetics, in that it leads to values for e and g, which satisfy the equation

$$(60) \qquad \frac{e}{g} = M,$$

and are unique except for the same multiplicative constant. The ratio R_{c_1}/R_c of the radii of normal curvature of the surface in the directions of the conjugate system C is‡

$$(61) \qquad \frac{R_{c_1}}{R_{c_2}} = \frac{\mathfrak{G}}{\mathfrak{E}} M,$$

the asymptotic lines on the surfaces are defined by $Mdu^2 + dv^2 = 0$, and its point coördinates can be found by quadratures when those of the sphere are known.

From these facts we obtain immediately the following result.

THEOREM 37. *Two surfaces determined, to within their homothetics, by two distinct solutions M and M' of* (59) *correspond by a non-parabolic parallel map whose basic conjugate systems are represented by the given curves on the sphere and whose invariant I has the value M'/M.*

The spherical representation of the basic conjugate systems of a non-parabolic map can, then, be prescribed at pleasure. However, it is impossible, in general, to prescribe at the same time the value of I; the conditions under which there will exist a map in this case are evident from Theorem 37.

* Author, *Spherical representation of conjugate systems and asymptotic lines*, to appear in the Annals of Mathematics.

† Explicit mention of this condition, which we shall always assume fulfilled, will henceforth be suppressed.

‡ $\mathfrak{E}$, $\mathfrak{F}$, $\mathfrak{G}$ are the differential coefficients of the first order of the sphere and $\mathfrak{H}^2 = \mathfrak{E}\mathfrak{G} - \mathfrak{F}^2$.

THEOREM 38. *A necessary and sufficient condition that a system of curves $\mathfrak{C}$ on the sphere represent the basic conjugate systems of two surfaces corresponding by a parallel map with prescribed invariant I is that there exist a function M such that M and IM both satisfy* (59). *The two surfaces are then determined to within their homothetics and their point coördinates can be found by quadratures, when those of the sphere are known.*

If for θ in (59) e/g and Ie/g are substituted in turn, the difference of the two equations obtained reduces, by virtue of the identities*

$$(62) \qquad -\frac{g}{e}\begin{Bmatrix}11\\2\end{Bmatrix}_1 = \begin{Bmatrix}12\\1\end{Bmatrix}, \qquad -\frac{e}{g}\begin{Bmatrix}22\\1\end{Bmatrix}_1 = \begin{Bmatrix}12\\2\end{Bmatrix},$$

to the fundamental equation (11) for a non-parabolic map.

If h and k are the point invariants of the conjugate system, then, by (62),

$$(63) \qquad h-k = -\frac{\partial}{\partial u}\left[\frac{g}{e}\begin{Bmatrix}11\\2\end{Bmatrix}_1\right] + \frac{\partial}{\partial v}\left[\frac{e}{g}\begin{Bmatrix}22\\1\end{Bmatrix}_1\right].$$

COROLLARY 1. *The given system $\mathfrak{C}$ gives rise to a map of invariant I for which the basic conjugate systems both have equal point invariants if and only if M exists such that M and IM both satisfy the equations*

$$(64) \qquad \frac{\partial^2 \log\theta}{\partial u \partial v} = \frac{\partial}{\partial v}\begin{Bmatrix}12\\1\end{Bmatrix}_1 - \frac{\partial}{\partial u}\begin{Bmatrix}12\\2\end{Bmatrix}_1, \qquad \frac{\partial}{\partial u}\left[\frac{1}{\theta}\begin{Bmatrix}11\\2\end{Bmatrix}_1\right] = \frac{\partial}{\partial v}\left[\theta\begin{Bmatrix}22\\1\end{Bmatrix}_1\right].$$

If, in particular, the surfaces are to be translation surfaces, the second of these equations must be replaced by the restrictions, $\begin{Bmatrix}11\\2\end{Bmatrix}_1 = 0$, $\begin{Bmatrix}22\\1\end{Bmatrix}_1 = 0$, on the curves $\mathfrak{C}$. Then each solution $\theta = M$ of the first equation determines a map, provided merely that the given function I is of the form $U(u)/V(v)$; cf. §9.

If the surfaces are to be associate surfaces of Bianchi, it suffices that (64) be compatible; for if $\theta = M$ is a common solution, so also is $\theta = IM = -M$.

* Cf., e. g., Eisenhart, *Differential Geometry*, p. 201.

Parallel maps whose basic systems are isothermal-conjugate* can be treated, in light of (60), by means of Theorem 38.

COROLLARY 2. *The given system $\mathfrak{C}$ gives rise to a map of invariant I for which the basic systems are isothermal-conjugate if, and only if, I is of the form $U(u)/V(v)$ and there exists a function M of this form such that I and IM both satisfy the equation*

$$\frac{\partial}{\partial u}\left[\begin{Bmatrix}12\\1\end{Bmatrix}_1 - \begin{Bmatrix}11\\2\end{Bmatrix}_1\frac{1}{\theta}\right] = \frac{\partial}{\partial v}\left[\begin{Bmatrix}12\\2\end{Bmatrix}_1 - \begin{Bmatrix}22\\1\end{Bmatrix}_1\theta\right].$$

Substituting e/g for θ in (59) and applying (63), we obtain the relation

$$\frac{\partial^2 \log e/g}{\partial u \partial v} = (h-k) + (h_1-k_1),$$

where h_1, k_1 are the plane invariants of the conjugate system. Hence:

THEOREM 39. *The basic conjugate systems of a non-parabolic map are both isothermal-conjugate if, and only if, the differences of their point invariants are equal to each other and to the difference of their common plane invariants taken in opposite order.*

Lines of curvature basic. If the given system of curves $\mathfrak{C}$ on the sphere is orthogonal, (59) can be replaced by†

$$2\frac{\partial^2 \log \theta}{\partial u \partial v} = \frac{\partial}{\partial u}\left[\left(\frac{1}{\theta} - 1\right)\frac{\partial \log \mathfrak{E}}{\partial v}\right] - \frac{\partial}{\partial v}\left[(\theta - 1)\frac{\partial \log \mathfrak{G}}{\partial u}\right]. \tag{65}$$

Each solution $\theta = R$ of this equation determines, to within its homothetics, a surface whose lines of curvature are represented by the curves $\mathfrak{C}$. Moreover, the ratio r_1/r_2 of the principal radii of curvature of the surface is precisely R:

$$\frac{r_1}{r_2} = R. \tag{66}$$

THEOREM 40. *Two surfaces determined to within their homothetics by two distinct solutions R and R′ of* (65) *are mapped by parallel normals so that the lines of*

* A conjugate system is termed isothermal-conjugate if, and only if, when it is parametric, e/g is of the form $U(u)/V(v)$.

† Author, loc. cit.

curvature correspond and are represented by the given orthogonal system on the sphere; the invariant I of the map has the value R'/R.

The analogue of Theorem 38, stated in brief, is as follows:

THEOREM 41. *A necessary and sufficient condition that there exist two surfaces corresponding by a parallel map with prescribed invariant I so that the lines of curvature are basic and are represented by a given orthogonal system $\mathfrak{E}$ on the sphere is that a function R exist such that R and IR both satisfy* (65).

From the identities

$$\frac{\partial \log E}{\partial v} = \frac{r_2}{r_1}\frac{\partial \log \mathfrak{E}}{\partial v}, \qquad \frac{\partial \log G}{\partial u} = \frac{r_1}{r_2}\frac{\partial \log \mathfrak{G}}{\partial u}, \tag{67}$$

follows the

COROLLARY. *The given orthogonal system $\mathfrak{E}$ gives rise to a map, of invariant I, of two isometric surfaces if, and only if, I is of the form $U(u)/V(v)$ and a function R exists such that $R^2\mathfrak{E}/\mathfrak{G}$ is of this form and R and IR satisfy the equation*

$$\frac{\partial}{\partial u}\left(\frac{1}{\theta}\frac{\partial \log \mathfrak{E}}{\partial v}\right) = \frac{\partial}{\partial v}\left(\theta\frac{\partial \log \mathfrak{G}}{\partial u}\right).$$

The equation,

$$\frac{\partial^2 \log r_1/r_2}{\partial u \partial v} = (h - k) - (h_1 - k_1),$$

which is obtained by substituting r_1/r_2 for θ in (65) and applying (67), gives the following interesting result:

THEOREM 42. *The ratios of the principal curvatures of two surfaces which are mapped by parallel normals so that the lines of curvature correspond (and are parametric) are both of the form $U(u)/V(v)$ if, and only if, the differences of the point invariants of the lines of curvature are equal to each other and to the difference of their common plane invariants.**

20. **Existence of a parabolic map when the spherical representation is given.** The u-curves of a parametric system of real curves on the sphere represent one family of asymptotic lines on each of infinitely many surfaces of negative curvature. In particular,† each solution $\theta = L$ of the equation

$$\frac{\partial^2\theta}{\partial u^2} - \frac{\partial}{\partial u}\left[\begin{Bmatrix}12\\1\end{Bmatrix}_1 - \left(\begin{Bmatrix}11\\1\end{Bmatrix}_1 - 2\begin{Bmatrix}12\\2\end{Bmatrix}_1\right)\theta - 2\begin{Bmatrix}11\\2\end{Bmatrix}_1\theta^2\right] + \frac{\partial}{\partial v}\left[\begin{Bmatrix}12\\2\end{Bmatrix}_1 - \begin{Bmatrix}11\\2\end{Bmatrix}_1\theta\right] = 0 \tag{68}$$

* The results of this section can be amplified by further applications of the theory set forth in the paper cited.

† Author, loc. cit.

determines one of the surfaces S to within its homothetics, in that it leads to values of f and g which satisfy the equation

$$\frac{g}{2f} = L \tag{69}$$

and are unique except for the same constant multiplier. The point coördinates of the surface can be found by quadratures when those of the sphere are known. Finally, the second family of asymptotic lines is defined by $du + Ldv = 0$, and the cotangent of the angle φ between the asymptotic lines is

$$\cot \varphi = \frac{\mathfrak{G}L - \mathfrak{F}}{\mathfrak{H}}. \tag{70}$$

THEOREM 43. *Two surfaces determined by two distinct solutions L and L' of* (68) *correspond by a parabolic map whose basic asymptotic lines are represented by the u-curves of the given system on the sphere and whose invariant J has the value*

$$J = 2(L' - L)\frac{\mathfrak{G}}{\mathfrak{H}}.$$

The value of J is found from (70) by applying Theorem 4. It is clear that we can choose arbitrarily the spherical representation of a parabolic map, specifying the family of curves which is to correspond to the basic asymptotic lines. We cannot at the same time choose J at pleasure, as the following theorem shows.

THEOREM 44. *A necessary and sufficient condition that the u-curves of a parametric system of real curves $\mathfrak{C}$ on the sphere represent the basic asymptotic lines on two surfaces corresponding by a parabolic map which has the given spherical representation and a prescribed invariant J is that there exist a function L such that L and $L + 2J\mathfrak{H}/\mathfrak{G}$ satisfy* (68).

To apply the theorem to ruled surfaces, we note that, for a surface for which $e = 0$,*

$$\begin{Bmatrix} 11 \\ 2 \end{Bmatrix} + \begin{Bmatrix} 11 \\ 2 \end{Bmatrix}_1 = 0, \tag{71}$$

and that the surface is ruled if, and only if, one and hence both of these symbols vanish.

* Cf., e. g., Eisenhart, *Differential Geometry*, p. 162.

COROLLARY. *The given system $\mathfrak{C}$ leads in the manner prescribed to a parabolic map, of invariant J, of two ruled surfaces whose rulings correspond if, and only if,* $\left\{ {11 \atop 2} \right\}_1 = 0$ *and L exists such that L and $L + 2J\mathfrak{H}/\mathfrak{C}$ both satisfy the equation*

$$\frac{\partial^2\theta}{\partial u^2} - \frac{\partial}{\partial u}\left[\left\{ {12 \atop 1} \right\}_1 - \left(\left\{ {11 \atop 1} \right\}_1 - 2\left\{ {12 \atop 2} \right\}_1\right)\theta\right] + \frac{\partial}{\partial v}\left\{ {12 \atop 2} \right\}_1 = 0.$$

It is to be noted that this equation, unlike (68), is linear in θ.

If for θ in (68) $g/2f$ and $g/2f + 2J\mathfrak{H}/\mathfrak{C}$ are substituted in turn, the difference of the two resulting equations can be reduced, by the use of (71) and analogous identities, to the fundamental equation (19) of a parabolic map.

21. **Miscellaneous theorems. Generalizations.** The product of two parallel maps, $S \longrightarrow S'$ and $S' \longrightarrow S''$, is a parallel map, $S \longrightarrow S''$. If the given maps are non-parabolic and the same conjugate system on the surface S' is basic for both, the product is non-parabolic; its basic conjugate systems, on the surfaces S and S'', are those of the given maps, and its invariant is the product of the invariants of the given maps. If the given maps are parabolic and the same family of asymptotic lines on S' is basic for both, the product is parabolic.

If a surface $x = x(u, v)$ is mapped by parallel normals on a surface $y = y(u, v)$ and also on a surface $z = z(u, v)$, it is mapped by parallel normals on every surface $my + nz$, where m and n are arbitrary constants, not both zero. If the given maps are non-parabolic and the same conjugate system on the surface x is basic for both, the map $x \longrightarrow my + nz$ is non-parabolic, and the conjugate system in question is basic for it; moreover, if the Combescurian ratios for the maps $x \longrightarrow y$ and $x \longrightarrow z$ are, respectively, λ, μ and λ', μ', those for the map $x \longrightarrow my + nz$ are $m\lambda + n\lambda'$, $m\mu + n\mu'$. Similar observations can be made in case the given maps are parabolic and the same family of asymptotic lines on the surface x is basic for both.

Of particular interest is the following theorem, which we state without proof

THEOREM 45. *If the maps $x \longrightarrow y$ and $x \longrightarrow z$ have the same invariant I, $\neq -1$, and just one map $x \longrightarrow my + nz$, $mn \neq 0$, has this invariant, the surfaces y and z are homothetic. However, if $I = -1$, the invariant of every map $x \longrightarrow my + nz$ is -1.**

The elements on which the general theory of parallel maps, given in §§2–4, 17, 18, is based, namely, conjugate systems, asymptotic lines, and the properties of one-dimensional linear correspondences, are invariant under projective

* This latter fact has been pointed out by Eisenhart, p. 508 of paper cited in §1.

transformations and polar reciprocations. Consequently, the theory can be generalized by transformations of these two types.

In the first case the transformed map consists of two surfaces whose tangent planes at corresponding points, P and P', intersect in a line l of a fixed plane. The basis of the classification is the projective correspondence of the points of l established by the pairs of corresponding tangents at P and P' to the surfaces. The associated congruence, as in the case of a parallel map, is made up of the lines PP'.

In the second case a line l joining corresponding points, P and P', of the transformed surfaces passes through a fixed point. The basis of the classification is the projective correspondence of the planes through l established by the pairs of corresponding tangents at P and P' to the surfaces. The associated congruence consists of the lines of intersection of the tangent planes at corresponding points P and P'.

In both cases the essential content of the existence theorems of §§4, 18 is valid for the transformed map. In particular, we recall (§4) that the fundamental equation (14) of a non-parabolic map is invariant under a projective transformation.

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