ASSOCIATED SETS OF POINTS*

BY

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Introduction

Two sets of n points ordered with respect to each other, the one, P_n^k, in a linear space S_k, determined by the equations

$$(up_1) = 0, \qquad (up_2) = 0, \qquad \cdots, \qquad (up_n) = 0,$$

and the other Q_n^{n-k-2}, in a linear space S_{n-k-2}, determined by the equations

$$(vq_1) = 0, \qquad (vq_2) = 0, \qquad \cdots, \qquad (vq_n) = 0,$$

are called *associated sets* if the factors of proportionality in the coördinates of the points can be so chosen that an identity in u, v exists of the following form:

$$(up_1)(vq_1) + (up_2)(vq_2) + \cdots + (up_n)(vq_n) \equiv 0. \tag{1}$$

This relation, obviously mutual, between the two sets is such that either set uniquely defines the other to within projective modifications. Some general properties of such sets have been given by the writer.‡

A characteristic algebraic property of two associated sets is that complementary determinants formed from the matrices of the coördinates of the two sets of points when taken so that (1) is satisfied are proportional. A characteristic geometric property is the following: On $k + 3$ of the points of P_n^k there is a unique rational norm curve N^k upon which the $k + 3$ points determine a set of $k + 3$ parameters; on the complementary set of $n - k - 3$ points of Q_n^{n-k-2} there is a pencil of linear spaces S_{n-k-3} whose members on the remaining $k + 3$ points determine a set of $k + 3$ parameters; these two sets of $k + 3$ parameters are projective.

Unless $k = n - k - 2$ the associated sets are in spaces of different dimension. Conventional methods of passing from one space to another are the process of *mapping* the space of lower dimension upon that of higher dimension, and the process of *projecting* from the space of higher dimension upon the

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‡ A. B. Coble, *Point sets and allied Cremona groups* (I), these Transactions, vol. 16 (1915), p. 155, in particular §§ 1, 2 and theorems (25), (26); also (II), vol. 17 (1916), p. 345, § 4 (16). These are cited as P. S. I or II.

one of lower dimension. Thus in the simple case of P_n^1, n points $x_0^{(i)}$, $x_1^{(i)}$ $(i = 1, \cdots, n)$ on a line, the line is mapped by means of the totality of binary $(n-3)$-ics in x, i.e., by $y_0 = (\alpha_0 x)^{n-3}, \cdots, y_{n-3} = (\alpha_{n-3}x)^{n-3}$, upon the points y of a rational norm curve N^{n-3} in S_{n-3} in such a way that P_n^1 is mapped upon its associated Q_n^{n-3}. On the other hand Q_n^{n-3} is projected from any S_{n-5} which is $(n-4)$-secant to N^{n-3} upon its associated P_n^1.

Two problems considered in this paper are: When $n - k - 2 \geqq k$ can the space S_k be mapped upon the space S_{n-k-2} so that the set P_n^k is mapped upon the set Q_n^{n-k-2}?; when $n - k - 2 > k$ can the set Q_n^{n-k-2} be projected upon the set P_n^k? For $k = 2$ the first problem is solved in § 1, the second in § 2. For $k = 3$ the first problem is solved in § 3. For the general set P_9^3 there appears to be no solution to the second problem and this probably would be true of further sets also.

In § 4 *particular* sets, i.e., those for which n, k have particular values, are considered. Each of these presents its own peculiarities. Also *special* sets, i.e., those which for given n, k satisfy in addition some projective conditions, receive some attention. Those conditions which are invariant under regular Cremona transformation of the set (cf. P. S. II, § 4) are especially emphasized. Their form in the two sets is often very diverse. Thus if P_9^3 is on a quadric with a node, then the associated Q_9^4 is on a rational quintic curve and conversely. In this section the discussion is carried through the values $n \leqq 10$.

The results obtained for the sets of nine and ten nodes of the rational sextic and of the symmetroid are useful in connection with the author's investigations of the modular functions of genus four attached to these figures.*

1. Mapping of P_n^2 upon its associated Q_n^{n-4}

The space S_2 is mapped upon the space S_{n-4} by means of a linear system Σ of ∞^{n-4} plane curves. The points of the plane are mapped upon a 2-way in S_{n-4} of order λ where λ is the number of variable intersections of two curves of Σ. The intersections of this 2-way by the linear S_{n-5}'s contained in S_{n-4} correspond in S_2 to the curves of Σ. We have therefore to find a system Σ so related to the set of points P_n^2 that the additional condition that three points of P_n^2 are on a line has as a consequence that there must exist a curve of Σ on the remaining $n-3$ points of P_n^2 and therefore also that the corresponding $n-3$ points of Q_n^{n-4} lie upon an S_{n-5} in S_{n-4}. This ensures the proportionality of complementary determinants in the matrices of the two point sets. Of course this requirement may not define the system Σ and we seek merely a simple system Σ with the required property.

* A part of this work appears in abstract in the Proceedings of the National Academy of Sciences, vol. 7 (1921), (I) p. 245; (II) p. 334. These are cited as Proc. I or II.

The cases where n is even and n is odd are slightly different and we begin with the mapping of a P^2_{2j+3} upon its associated Q^{2j-1}_{2j+3}. In S_2 pass through P^2_{2j+3} a proper curve C of order j with a $(j-2)$-fold point at a point r. Also pass through P^2_{2j+3} a proper curve D of order $(j+1)$ with a $(j-1)$-fold point at r which meets C in $(2j-5)$ points $s_1, s_2, \cdots, s_{2j-5}$. If L_1, L_2 are distinct lines on r, then D, CL, CL_2 cut out the same set S upon C, so that the set S lies in an I^{2j-5}_{j-3} on C. The choice of the points r, s thus depends upon $2j-5$ constants when P^2_{2j+3} is given. Let A, B respectively be arbitrary sets of $(j-2)$ and $(j-3)$ lines on r. Then in $AC + BD = 0$ we have a system of curves of order $(2j-2)$ with a $(2j-4)$-fold point at r, on P^2_{2j+3}, and on the points S. The parameters in A and B are essential. For if $AC + BD \equiv A'C + B'D$ then $(A - A')C \equiv (B' - B)D$, whence $A \equiv A'$ and $B \equiv B'$, since C and D are proper curves. Thus the system $AC + BD = 0$ contains ∞^{2j-4} curves. If three points of P^2_{2j+3} are on a line, a curve of the system can be passed through $(2j-4)$ further points of the line, which therefore will contain the line as a factor. The complementary factor will be a curve of the required system Σ of order $(2j-3)$ with a $(2j-4)$-fold point at r and on the set S, and this curve will pass through the complementary set of $2j$ points of P^2_{2j+3}. Hence the system Σ will map the set P^2_{2j+3} upon its associated set.

For the case n even, or a P^2_{2j+2}, we pass through P^2_{2j+2} two proper curves C, D of order j with a common $(j-2)$-fold point r which meet again in $(2j-6)$ points S. Here the choice of r, s depends upon $2j-6$ constants. Let A, B be arbitrary sets of $(j-3)$ lines on r. Then in $AC + BD = 0$ we have a linear system of dimension $(2j-5)$ of curves of order $(2j-3)$ with a $(2j-5)$-fold point at r, on P^2_{2j+2}, and on the set S. If three of the points of P^2_{2j+2} are on a line, one curve of the system contains this line as a factor, whence one curve of the required system Σ of order $(2j-4)$ with a $(2j-5)$-fold point at r and on S will pass through the complementary set of $(2j-1)$ points of P^2_{2j+2}. This system Σ therefore effects the required mapping. Hence

THEOREM 1. *The plane set of points P^2_n is mapped upon its associated Q^{n-4}_n by a linear system of curves of order $(n-6)$ with an $(n-7)$-fold point at r and on a set of $(n-8)$ points S in such a way that the plane is mapped upon the normal 2-way, M^{n-5}_2, of order $(n-5)$ in S_{n-4}. If n is even the points S are the further intersections of two proper curves of order $(n-2)/2$ with a common $(n-6)/2$-fold point at the arbitrarily chosen point r and on the given set P^2_n. If n is odd the points S are the further intersections of two proper curves of order $(n-3)/2$ and $(n-1)/2$ with respectively $(n-7)/2$- and $(n-5)/2$-fold points at r and on P^2_n. For given P^2_n the choice of the points r, S depends upon $(n-8)$ constants.*

The mapping described above becomes evanescent for $n = 6$ and $n = 7$. In the case of P_6^2 let a pencil of cubics on P_6^2 meet again in s_1, s_2, s_3. Then conics on S map P_6^2 upon its associated Q_6^2. For if three of the points of P_6^2 are on a line, the complementary three are on a conic with s_1, s_2, s_3 and therefore map into three points of Q_6^2 on a line. Hence

THEOREM 2. *Six corresponding point pairs of a quadratic transformation are associated P_6^2, Q_6^2 if P_6^2 and the singular triangle of the transformation are the base points of a pencil of cubics.*

In the case of P_7^2 we pass a pencil of cubics through P_7^2 to meet again in s_1, s_2. Then conics on s_1, s_2 map P_7^2 upon its associated Q_7^3 in S_3. In this mapping the plane becomes a quadric on Q_7^3 and the points on the line $\overline{s_1 s_2}$ become the directions on this quadric about the eighth base point of the net of quadrics on Q_7^3. Thus to the ∞^2 possible choices of the pair s_1, s_2 there correspond the set Q_7^3 and the ∞^2 quadrics on it.

We observe also that the cases $n = 8$, $n = 9$ are exceptional in that for P_8^2 r is the ninth base point of the pencil of cubics on P_8^2 and that for P_9^2 r is a point on the cubic determined by P_9^2. For further cases r may be taken in general position.

2. THE PROJECTION OF Q_{k+4}^k UPON ITS ASSOCIATED P_{k+4}^2

We now consider the set Q_{k+4}^k as given in S_k and ask for spaces L of dimension $k - 3$ such that under projection from L, the set Q will become its associated set in the plane. Two lemmas are needed.

LEMMA 1. *The S_{k-2} π determined by L and q_1 is a $(k - 1)$-secant space of the norm curve N_1^k on $q_2, \cdots, q_{k+4}$.*

For if τ is the parameter of the pencil of S_{k-1}'s on π and t the parameter on N_1^k the incidence condition of S_{k-1} τ and point t is a $(1, k)$ relation on τ, t which in general would have only $k + 1$ pairs τ, t in common with any $(1, 1)$ relation on τ, t. If this $(1, 1)$ relation is the projectivity mentioned in the introduction between the parameter τ of the line pencil on p_1 in S_2 and the parameter t of N_1^k, then it is satisfied by the $k + 3$ pairs t, τ determined by $q_2, \cdots, q_{k+4}$. Therefore the projectivity determines a $(1, 1)$ relation which is a factor of the $(1, k)$ relation. The complementary factor of degree $k - 1$ in t determines the points of N_1^k on π. Thus the $k + 4$ norm curves on the sets of $k + 3$ points q selected from Q_{k+4}^k are projected from L into $k + 4$ rational k-ics in the plane on the points of P_{k+4}^2 and with respectively a $(k - 1)$-fold point at each point of P_{k+4}^2. This remark is utilized in Theorem 5.

LEMMA 2. *Quadrics on $q_2, \cdots, q_{k+4}$ cut π in quadrics apolar to a unique quadric Q_π in π and L in π is the polar S_{k-3} of q_1 as to Q_π.*

For the $\binom{k}{2}$ linearly independent quadrics on N_1^k cut π in $\binom{k}{2}$ sections on the $k-1$ points common to π and N_1^k, whence of these only $\binom{k}{2} - (k-1)$ are linearly independent in π. Therefore $k-1$ quadrics on N_1^k contain π and the $\binom{k+2}{2} - (k+3)$ quadrics in S_k on $q_2, \cdots, q_{k+4}$ cut π in at most $\binom{k-2}{2} - (k+3) - (k-1) = \binom{k}{2} - 1$ linearly independent quadrics all of which are apolar to at least one quadric Q_π in π. Moreover the S_2 on three points of $q_2, \cdots, q_{k+4}$ and the S_{k-1} on the remaining k points meet π respectively in a point and S_{k-3} which are pole and polar as to Q_π and thereby Q_π is uniquely determined. For any S_{k-1} on S_2 together with the given S_{k-1} constitute a quadric on $q_2, \cdots, q_{k+4}$ and meet π in a pair of S_{k-3}'s apolar to Q_π. Finally, if three points of $q_2, \cdots, q_{k+4}$ are in an S_{k-1} with L and therefore project from L into three points of a line in S_2, then the remaining k points and q_1 must be in an S_{k-1} which meets π in an S_{k-3} on q_1. Hence the point, S_{k-3} of π mentioned above are such that when the point is on L then the S_{k-3} must be on q_1, which requires that q_1, L be pole and polar as to Q_π.

In order to put all the points of the set Q on the same footing we now prove

THEOREM 3. *Given Q_{k+4}^k in S_k there exist ∞^{k-3} spaces L of dimension $k-3$ such that all the quadrics on L and any $k+3$ of the points Q meet again at the remaining point of Q, or also such that all the quadrics on the points Q and $\binom{k-1}{2} - 1$ points of L contain L. From any one of these spaces L the set Q_{k+4} is projected into its associated P_{k+4}^2.*

For there are ∞^{k-1} S_{k-2}'s which are $(k-1)$-secant spaces of N_1^k each with ∞^{k-2} points, so that on q_1 there are ∞^{k-3} such spaces π. In any such space π choose L to be the polar S_{k-3} of q_1 as to the quadric Q_π determined as in Lemma 2. Then all the quadrics of S_k on $q_2, \cdots, q_{k+4}$ which contain L cut π in another S_{k-3} on q_1 and L has the first property described in the theorem. That all the S_{k-3}'s L of the theorem are found among the $(k-1)$-secant spaces π of N_1^k on q_1 is proved as follows. If, as given, quadrics on $q_2, \cdots, q_{k+4}$ and L meet again in q_1, then the $\binom{k+2}{2} - \binom{k-1}{2} - (k+3) = 2k-3$ linearly independent quadrics of this sort meet π $[L, q_1]$ in a linear system of S_{k-3}'s on p_1 of which only $k-2$ are linearly independent in π. Hence $k-1$ of the quadrics contain the S_{k-2} π and therefore meet in a N_1^k (necessarily on $q_2, \cdots, q_{k+4}$) which is $(k-1)$-secant to π. We observe that the configuration Q_{k+4}^k, L is the generalization of the set of eight base points of a net of quadrics as one of the points is enlarged in dimension. To prove the last statement in the theorem we note that if $q_2, \cdots, q_{k+2}$ are on an S_{k-1}, this S_{k-1} together with the S_{k-1} on L and q_{k+3}, q_{k+4} constitute a quadric which must contain q_1, whence in the projection p_1, p_{k+3}, p_{k+4} are on a line. Here the isolated position of p_1 is not material.

The above discussion suggests the following construction for the set in S_k when the set in the plane is given.

THEOREM 4. *Given the set P_{k+4}^2, let the parameter t of the line pencil on p_1 be* *introduced as a parameter on the linear system* Σ_1 *of* ∞^{k-3} *rational curves of order k with a* $(k-1)$*-fold point at* p_1. *Then* $t_2, \cdots, t_{k+4}$ *are the parameters of* $p_2, \cdots, p_{k+4}$ *on every curve of* Σ_1 *and the parameters of the multiple point* p_1 *determine a linear system of* ∞^{k-3} *binary* $(k-1)$*-ics all of which are apolar to a binary k-ic,* γ_1^k. *In* S_k *select a parameter system t on a norm curve* N_1^k. *Then the points of* N_1^k *with parameters* $t_2, \cdots, t_{k+4}$ *and the point of* S_k *determined by* γ_1^k *with reference to* N_1^k *constitute a set* $q_2, \cdots, q_{k+4}, q_1$ *associated with* P_{k+4}^2. .

This is indeed an immediate consequence of the fact that the curves of Σ_1 are the projections of N_1^k from the ∞^{k-3} spaces L. This same projection and the further fact that the choice of a single curve of the system Σ_1 is sufficient to determine the corresponding L lead to the following theorem, which is not readily apparent from the plane figure alone.

THEOREM 5. *The* $k+4$ *systems* Σ_i *of dimension* $k-3$ *of rational curves of order k with a* $(k-1)$*-fold point at* p_i *and simple points at the remaining points of* P_{k+4}^2 *are in one-to-one correspondence with each other.*

We shall see in § 4 that for Q_7^3 the $\infty^0 = 1$ space L is the point common to all of the ∞^2 elliptic quartics on Q_7^3; for Q_8^4 the ∞^1 spaces L are the common bisecants of all the ∞^1 elliptic quintics on Q_8^4; and for Q_9^5 the ∞^2 spaces L are the trisecant planes of the unique elliptic sextic on Q_9^5. For further sets no equally simple characterization of the spaces L has been obtained.

3. MAPPING OF P_n^3 UPON ITS ASSOCIATED Q_n^{n-5}

In order to map a set P_8^3 upon its associated Q_8^3 we need only to find a further set P_6^3 such that the set $P_{14}^3 = P_8^3 + P_6^3$ shall have the property that the linear system Σ of cubic surfaces on the 14 points shall have the dimension 6, i.e., that all the cubic surfaces on 13 of the points shall pass through the 14th. For then if 4 of the points of P_8^3 are in a plane π a cubic surface of the system Σ can be made to pass through 6 more points of π in general position and therefore to contain π as a factor. The remaining factor is a quadric on P_6^3 which contains the other four points of P_8^3. Hence the linear system of quadrics on P_6^3 will map P_8^3 upon its associated Q_8^3.

One symmetrical set of 14 points of such character may be obtained as follows. Given 6 points $r_1, \cdots, r_6$ of a plane, select a quartic curve with simple points at r and an octavic curve with triple points at r. These two curves meet elsewhere in 14 points. They are mapped from the plane by cubic curves on the points r into two space sextics of genus 3 with 14 common points. The two space sextics are on one cubic surface—the map of the plane —and only one since the two sextic curves could not lie at once on two cubic surfaces one of which is non-degenerate. Since each sextic curve is on 4 linearly independent cubic surfaces, there must be on their 14 common points $4+4-1=7$ linearly independent cubic surfaces and the set has the required property.

The trisecant locus of the one sextic—an octavic surface with the sextic as a triple curve—meets the other sextic in $8 \times 6 - 14 \times 3 = 6$ points, whence six trisecants of each curve are secants of the other and these two sets of trisecants are a double six of the unique cubic surface on both sextics—the double six of the mapping system. The rôles of the two plane curves are interchanged by the plane Cremona transformation of order 5 with double F-points at the six points r. We observe that the pair of space sextics is the complete intersection of a cubic and a quartic surface.

The number of absolute constants is 4 for the points r and 8 more for each of the plane curves, or 20 in all. Hence in space such a set of 14 points has $20 + 15 = 35$ projective constants. A space sextic of genus three has $15 + 9 = 24$ projective constants so that on a given sextic there are ∞^{11} such sets of 14 points which lie in a linear series I_{11}^{14}. From this there follows that at most 11 of the 14 points can be chosen at random in space. For such sets from P_8^3 to P_{11}^3 we have

THEOREM 6. *The three-dimensional sets* P_8^3, P_9^3, P_{10}^3, *and* P_{11}^3 *can be mapped upon their associated sets* Q_8^3, Q_9^4, Q_{10}^5, *and* Q_{11}^6 *by the linear system of quadrics on a supplemental set* P_6^3, P_5^3, P_4^3, *and* P_3^3 *respectively, which with the given set makes up the* 14 *points of intersection of two space sextics of genus three.*

The mapping system of this theorem is more general than is needful for the purpose. Consider for example the set P_8^3. It lies on a unique elliptic quartic E^4, the intersection of quadrics Q_1, Q_2. Let C be a cubic surface on P_8^3 which cuts E^4 in a residual set P_4. Let two other points in general position be a set P_2. The totality of cubic surfaces on the 12 points $P_8^3 + P_4$ is made up of $C + \pi Q_1 + \pi' Q_2$ where π, π' are arbitrary planes. In this system of ∞^8 surfaces there is a system of dimension 6 on $P_8^3 + P_4 + P_2$, whence quadrics on $P_4 + P_2$ map P_8^3 upon its associated set Q_8^3. This mapping is however a degenerate case of Theorem 6, since E^4 and a bisecant of E^4 from each point of P_2 make up a degenerate sextic of genus three.

The simplest transition from P_8^3 to Q_8^3 is obtained by taking P_8^3 on an E^4 with canonical parameter u (i.e., such that the coplanar condition is $u_1 + u_2 + u_3 + u_4 \equiv 0 \bmod. \omega_1, \omega_2$) for which the parameters of the points of P_8^3 are $u_1, \cdots, u_8$, where $\Sigma_1^8 u = \sigma$. If now we set $u_i + v_i = \sigma/4$ $(i = 1, \cdots, 8)$ then $v_1 + \cdots + v_4 \equiv \sigma - (u_1 + \cdots + u_4) \equiv u_5 + \cdots + u_8$. Hence the four points v are on a plane if the complementary four points u are on a plane, or the set v is associated to the set u. The lines joining u_i, v_i are generators of a regulus on E^4. For given P_8^3 the $\sigma/4$ has 16 determinations, whence

THEOREM 7. *For a given set* P_8^3 *there are* 16 *reguli on the* E^4 *through* P_8^3 *such that the generators of a regulus on the points of* P_8^3 *meet the* E^4 *again in the points of an associated* Q_8^3.

Again let the set P_9^3 be on a quadric with generators t, τ and let $(a\tau)^2(\alpha t)^3$

$= 0$ and $(b\tau)^3(\beta t)^2 = 0$ be two quintics of genus two of different kinds on P_9^3 and Q. These quintics meet in four other points P_4 on Q. Let P_1 be a point in general position. Then if C_1, C_1' are cubic surfaces on the first quintic, C_2, C_2' cubic surfaces on the second quintic, and π is an arbitrary plane we have in $\lambda_1 C_1 + \lambda_2 C_1' + \lambda_3 C_2 + \lambda_4 C_2' + \pi Q$ a system of ∞^7 cubic surfaces on P_9^3 and P_4. Hence there will be a system of dimension 6 on P_9^3, P_4, P_1, or the system of quadrics on $P_4 + P_1$ will map P_9^3 upon Q_9^4. This again is a special case of Theorem 6 since a bisecant to the one quintic from P_1 makes up with the quintic a degenerate sextic of genus three and the two sextics thus made up have 14 common points. We shall however find in § 4 a different mode of transition from P_9^3 to Q_9^4 which exhibits more effectively their mutual relations.

There appears to be no point in S_4 from which a general set Q_9^4 can be projected into its associated set. If Q_9^4 is on an elliptic quintic E^5 (two conditions) a quadric on Q_9^4 will cut E^5 in a tenth point from which the desired projection can be made (§ 4, Theorem 11). However, no general sets except planar sets have been found which are the projections of their associated sets. On the other hand no proof of the impossibility of such a projection has been found.

We complete the mapping of sets P_n^3 upon their associated sets by means of an apparatus derived from the elliptic curves. Let E_k^m be an elliptic curve of order $m > k$ in an S_k. It is the projection of the normal E_{m-1}^m from an S_{m-k-2}. The E_{m-1}^m has one absolute constant and the S_{m-k-2} in S_{m-1} has $(m-k-1)(k+1)$ further constants, so that the projection has $(m-k-1)(k+1)+1$ absolute constants. This number added to the $(k+1)^2 - 1$ constants of a projectivity in S_k furnishes $m(k+1)$. Hence the elliptic m-ic in S_k, E_k^m, has $m(k+1)$ constants and can be passed through $[m(k+1)/(k-1)]$ points in S_k, where the bracket indicates the largest integer equal to or less than the number within it.

Since r-ic spreads cut the E_k^m in an I_{mr-1}^{mr}, an r-ic spread on mr general points of E_k^m contains it completely. Hence there are $\infty^{\binom{r+k}{k}-mr-1}$ r-ic spreads on E_k^m and there are $\infty^{\binom{r+k}{k}-mr}$ r-ic spreads on the mr points cut out on E_k^m by a definite r-ic spread.

Beginning then with a set P_{2j}^3 we can pass an E_3^j through its points. Let an r-ic surface on P_{2j}^3 meet E_3^j in $j(r-2)$ further points $P_{j(r-2)}$. Then there are $\infty^{\binom{r+3}{3}-jr}$ r-ic surfaces on $P_{2j}^3 + P_{j(r-2)}$. If we suppose that these surfaces are subject to $\alpha \geqq 0$ further linear conditions, say to pass through a set of points P_α, we have a linear system of $\infty^{\binom{r+3}{3}-jr-\alpha}$ r-ic surfaces on the base $P_{2j}^3 + P_{j(r-2)} + P_\alpha$. If 4 points of P_{2j}^3 are on a plane and if $\binom{r+3}{3} - jr - \alpha = \binom{r+2}{2} - 4$, then an r-ic surface of the linear system can be determined which contains this plane as a factor leaving an $(r-1)$-ic surface on $P_{j(r-2)} + P_\alpha$

which passes through the remaining $2j - 4$ points of P^3_{2j}. This condition becomes

(2) $$\binom{r+2}{3} - jr + 4 = \alpha.$$

Since $\alpha \geqq 0$, then, for given j, r is defined by the inequality

(3) $$\binom{r+2}{3} + 4 \geqq jr.$$

The modification for an odd set P^3_{2j-1} is readily made and we state at once

THEOREM 8. *Through a given set $P^3_{2j}\{P^3_{2j-1}\}$ pass an E^j_3 and cut it by an r-ic surface on $P^3_{2j}\{P^3_{2j-1}\}$ which meets E^j_3 again in a set $P_{j(r-2)}\{P_{j(r-2)+1}\}$ where r is the smallest integer defined by* (3). *The linear system of surfaces of order $r - 1$ on this residual set and on a further general set P_α, where α is defined by* (2), *maps S_3 upon a 3-way in $S_{2j-5}\{S_{2j-6}\}$ in such a way that the set $P^3_{2j}\{P^3_{2j-1}\}$ is mapped upon its associated $Q^{2j-5}_{2j}\{Q^{2j-6}_{2j-1}\}$.*

For the sets P^3_9 and P^3_{10} the numbers j, r, α are 5, 4, 4; for P^3_{11} and P^3_{12}, 6, 4, 0; for P^3_{13} and P^3_{14}, 7, 5, 4; etc.

4. Particular and special sets of points

It is the aim in the present section to consider in more detail the relation of particular sets P^k_n for values of n from 8 to 10 to their associated sets both for cases when the n points of the set are in general position and for cases when they are subject to certain conditions. A question naturally arises as to what types of conditions would be most interesting and as to what types of configurations connected with the associated sets would best exhibit the relations sought. In answer to this inquiry we recall the noteworthy theorem in regard to associated sets (P. S., II (16), p. 361), which states that if P^k_n and P'^k_n are congruent under regular Cremona transformation in S_k their associated sets Q^{n-k-2}_n and Q'^{n-k-2}_n are also congruent under regular Cremona transformation in S_{n-k-2}. More specifically, if P^k_n is congruent to P'^k_n under the Cremona involution $x'_i = 1/x_i$ $(i = 1, \cdots, k+1)$ with its $k+1$ F-points at points of P^k_n, then Q^{n-k-2}_n is congruent to Q'^{n-k-2}_n under the involution $x'_i = 1/x_i$ $(i = 1, \cdots, n-k-1)$ with its $n-k-1$ F-points at the complementary $n-k-1$ points of Q^{n-k-2}_n. The regular Cremona group is generated by this one Cremona involution and projectivities.

We shall seek therefore to express the desired relations in terms of such loci or in terms of such properties of these loci as are invariant under regular Cremona transformation. Thus a rational curve, or an elliptic curve, of order $k+1$ on the points of P^k_n is transformed by regular transformation into a curve of the same order on the points of the congruent set. The same is true of multiples of such curves, i.e., curves of orders $l(k+1)$ with l-fold points at the points of P^k_n, if such curves exist. This property of invariance is shared by a certain type of surface—the rational M^r_2 in S_{r+1}. We shall first derive some facts concerning this surface for later use.

If $r = 2l + 1$ $[2l]$ the system of rational plane curves of order $l + 1$ on the base O^l $[O^l, \sigma]$ has the dimension $r + 1$ and maps the plane upon a 2-way of order r, M_2^r, in S_{r+1}. Each of these surfaces is the projection of the one of next higher order from one of its points. This is evidently the case in passing from the base O^l to the base O^l, σ. But also the base O^l, σ, σ' can be reduced by quadratic transformation to the base O^{l-1}. Thus the series of surfaces M_2^r constitute the progenitors of the quadric M_2^2 in S_3. Lines on the point O map into the ∞^1 "generators" of the surface.

In case r is odd directions at O map into a unique "directrix," a rational norm curve of order l; while the lines of the plane map into ∞^2 "directors," rational norm curves of order $l + 1$. Since S_r's on the directrix are mapped by sets of $l + 1$ lines on O, and S_r's on a given director by sets of l lines on O and a given line, the directrix and a director are in skew S_l, S_{l+1}, and the generators are lines joining corresponding points of these two rational curves. Included, however, among the ∞^2 directors are the ∞^1 which consist of the fixed directrix and a variable generator.

In case r is even there are ∞^1 directrices, the maps of lines on σ, which are rational norm curves of order l. Included in this system is one curve which is the map of directions at O. As before the ∞^1 generators are the maps of lines on O but this system includes the one line which is the map of directions at σ. The line $\overline{O\sigma}$ is mapped into directions on the surface about the point where the generator σ meets the directrix O. If π, ρ are two lines on σ the mapping system can be expressed in the form $\pi\Sigma_1 + \rho\Sigma_2$ where Σ_1, Σ_2 each is the system of l lines on O. Hence any two of the directrices lie in skew S_l's and the generators are lines joining corresponding points on the two.

In either case by estimating the number of constants involved in the choice of the skew spaces; in the choice of the rational curve in each; in the projectivity between the two curves set up by the generators; and by allowing for the freedom in the choice of the skew spaces for given surface, we find that the number of projective constants of the M_2^r is $(r + 2)^2 - 7$, whence the M_2^r admits a 6-parameter collineation group. This group for r odd is the map of the 6-parameter collineation group of the plane with fixed point O; for r even it is the map of the 6-parameter quadratic group with fixed F-points at O, σ.

Since it is $r - 1$ conditions that an M_2^r in S_{r+1} be on a point, we see that there are ∞^2 M_2^r's on $r + 5$ points in general position. Thus on 8 points in S_4 there are ∞^2 M_2^3's, or on 9 points a finite number; on 9 points in S_5 there are ∞^2 M_2^4's which fill up a spread, whence for 10 points in S_5 there is a single condition invariant under regular Cremona transformation which expresses that the 10 points lie on an M_2^4.

The system of plane rational curves of order l on the base O^{l-1} $[O^{l-1}, \sigma]$

has the dimension $r - 1$. Let C_i $(i = 1, \cdots, r)$ be linearly independent in this system and let π, ρ be two lines on O. If then we set $m_i = \pi C_i$, $n_i = \rho C_i$, where m_i, n_i are the linear forms in S_{r+1} which cut M_2^r in the maps of the given plane curves, we find that the equation of M_2^r is

$$\left\| \begin{matrix} m_1 & m_2 & \cdots & m_r \\ n_1 & n_2 & \cdots & n_r \end{matrix} \right\| = 0. \tag{4}$$

Conversely a manifold in S_{r+1} defined by such a matrix is in general an M_2^r mapped as above.

In the case $r = 2l$ a parametric equation of M_2^r is

$$x_0 = (\alpha_0 t)(a_0 \tau)^l, \quad x_1 = (\alpha_1 t)(a_1 \tau)^l, \quad \cdots, \quad x_{r+1} = (\alpha_{r+1} t)(a_{r+1} \tau)^l. \tag{5}$$

For given τ we have one of the ∞^1 generators; for given t one of the ∞^1 directrices. In the case $r = 2l + 1$ the parametric equation is

$$x_0 = (\alpha_0 t)(a_0 \tau)^{l+1}, \qquad x_1 = (\alpha_1 t)(a_1 \tau)^{l+1}, \qquad \cdots, \qquad x_{r+1} = (\alpha_{r+1} t)(a_{r+1} \tau)^{l+1}, \tag{6}$$

where

$$(\alpha_i t')(a_i \tau')^{l+1} = 0 \qquad (i = 0, \cdots, r+1).$$

This is in fact the projection of (5) for $r = 2l + 2$ from a point t', τ' upon it.

Special cases of these rational surfaces occur. Thus cubic curves on the base O^2, o, o' map the plane upon an M_2^3 in S_4. This mapping system can be reduced to conics on the base O by quadratic transformation with F-points at O, o, o' unless o, o' coincide with O in two distinct directions. Thus cubics with node at O and fixed nodal tangents determine an M_2^3 in S_4 which is more properly the projection of an M_2^5 in S_6 from two points on its directrix conic. This special M_2^3 is obtained in S_4 by joining a point directrix to a cubic curve director. Unless expressly mentioned special M_2^r's of such types will not be considered.

We shall now prove

THEOREM 9. *An M_2^r in S_{r+1} is transformed by the Cremona involution $x_i' = 1/x_i$ $(i = 1, \cdots, r+2)$ with $r + 2$ F-points on the M_2^r into an $M_2'^r$.*

The space x' is mapped in the involution upon the space x by the system of spreads of order $r + 1$ with r-fold points at the F-points which are the maps from the plane of the points $p_1, \cdots, p_{r+2}$. Then, for $r = 2l + 1$, the transform of M_2^r is mapped from the plane by curves of order $2(l + 1)^2$ with a $2l(l + 1)$-fold point at O and $(2l + 1)$-fold points at $p_1, \cdots, p_{2l+3}$; for $r = 2l$, by curves of order $(l + 1)(2l + 1)$ with an $l(2l + 1)$-fold point at O, a $(2l + 1)$-fold point at σ; and $2l$-fold points at $p_1, \cdots, p_{2l+2}$. We have merely to show that the two latter mapping systems can be transformed by ternary Cremona transformation into systems of order $l + 1$ on the bases O^l or O^l, σ respectively. For odd r this transformation is effected by using first the Jonquière transformation J^{l+1} of order $l + 1$ with l-fold point (*center*) at O and simple F-points at $p_1, \cdots, p_{2l}$, then a quadratic transformation with F-points at $p_{2l+1}, p_{2l+2}, p_{2l+3}$, and finally the transformation J^{l+1} again. For even r we use first a quadratic transformation with F-points at O, σ, p_1, then a J^l with center at O and simple F-points at $p_2, \cdots, p_{2l-1}$, then the quadratic transformation with F-points at $p_{2l}, p_{2l+1}, p_{2l+2}$, and finally J^l again. It is easily verified that these transformations effect the required change in the mapping system and the proof is complete.

The three theorems which follow relate to special sets of points when, for given k, n is sufficiently large.

THEOREM 10. *If P_n^k is on a rational norm curve N^k in S_k, then its associated Q_n^{n-k-2} is on a rational norm curve N^{n-k-2} in S_{n-k-2}. The n parameters of the two sets on their respective norm curves are projective. If $(n-k-2)-k = l+1 > 0$, the set Q is projected upon the set P from any one of the ∞^{l+1} spaces L_l which are $(l+1)$-secant to N^{n-k-2}.*

THEOREM 11. *If P_n^k is on an elliptic norm curve E^{k+1} in S_k, then its associated Q_n^{n-k-2} is on an elliptic norm curve E^{n-k-1} in S_{n-k-2}. If $(n-k-2)-k = l+1 > 0$, the set Q is projected upon the set P from any one of the ∞^l spaces L_l which are $(l+1)$-secant to E^{n-k-1} at the $l+1$ points cut out by a quadric on Q.*

THEOREM 12. *If P_n^k is on a rational norm surface M_2^{k-1} in S_k, then its associated Q_n^{n-k-2} is on a rational norm surface N_2^{n-k-3} in S_{n-k-2}. Then parameters of the two sets of generators on the points are projective.*

In Theorem 10 let the norm curves in S_k and S_{n-k-2} have the respective parametric equations

$$x_0 = 1, \quad x_1 = t, \quad \cdots, \quad x_k = t^k;$$
$$x_0 = 1, \quad x_1 = t, \quad \cdots, \quad x_{n-k-2} = t^{n-k-2};$$

and let the sets P_n, Q_n be determined on these curves by the parameters $t_1, \cdots, t_n$. If $\lambda_1, \cdots, \lambda_n$ are determined by the $n-1$ equations

$$\lambda_1 t_1^i + \lambda_2 t_2^i + \cdots + \lambda_n t_n^i = 0 \qquad (i = 0, 1, \cdots, n-2),$$

then the points of the one set, affected respectively by factors of proportionality $\lambda_1, \cdots, \lambda_n$, satisfy with the points of the other the bilinear relations requisite for association. We observe that here P_n^2 is obtained by projection of Q_n^{n-4} from ∞^{n-6} spaces L_l rather than ∞^{n-7} spaces as in the general case of Theorem 3.

In Theorem 11 let the canonical parameters of P_n^k on E^{k+1} be $u_1, \cdots, u_n$ where $u_1 + \cdots + u_n + b \equiv 0$. Choose then a mapping system on a base B such that the members meet E^{k+1} in $n-k-1$ variable points and also in a certain number of fixed points whose parameters sum up to b. Then, if $k+1$ points of P_n^k are on an S_{k-1}, $u_1 + \cdots + u_{k+1} \equiv 0$ and $u_{k+2} + \cdots + u_n + b \equiv 0$, whence the complementary $n-k-1$ points of P_n^k are on a member of the mapping system or the $n-k-1$ points of Q_n^{n-k-2}, mapped from P_n^k, are on an S_{n-k-3}. Thus E^{k+1} is mapped upon E^{n-k-1} and P_n^k is mapped upon its associated Q_n^{n-k-2}. In this way we find upon each of the associated sets P_7^2, Q_7^3, ∞^2 elliptic norm curves, upon each of the associated sets P_8^2, Q_8^4, ∞^1 elliptic norm curves, and upon each of the associated sets P_9^2, Q_9^5, a unique elliptic norm curve.

For Theorem 12 we give the details of the proof only for the case $k = 2l + 1$. Then M_2^{k-1} is the map of the plane by curves of order $l + 1$ on the base O^l, σ and P_n^k is the map of a set π_n^2 in the plane. If $n = 2m + 1$ then O is the center and σ, π_n^2 the simple F-points of a J^{m+2} whose inverse center and F-points are O', σ', $\pi_n'^2$. Curves of order $m - l - 1$ on the base O'^{m-l-2} map the plane on an M_2^{n-k-3} and map $\pi_n'^2$ upon a set Q_n^{n-k-2} which is associated to P_n^k. For if $k + 1$ points $p_1, \cdots, p_{2l+2}$ of P_n^k are on an S_{k-1} there is a curve of order $l + 1$ with l-fold point at O and simple points at σ, π_1, $\cdots$, π_{2l+2}. This curve is transformed by J^{m+2} into a curve of order $m - l - 1$ with $(m - l - 2)$-fold point at O' and simple points at π'_{2l+3}, $\cdots$, π'_{2m+1}. Hence the points $q_{2l+3}, \cdots, q_n$ are on an S_{n-k-3} in S_{n-k-2}. If, however, n is even we take σ, σ' to be a pair of ordinary corresponding points for a $J^{1+\frac{n}{2}}$.

It should be observed however that an M_2^2 in S_3, an ordinary quadric, counts in two ways as a ruled normal surface. It is mapped from the plane by conics on O, σ and as the points are interchanged in the above proof two normal surfaces in S_{n-k-2} are obtained. Hence

THEOREM 13. *If P_n^3 is on a quadric surface which is not a cone, its associated Q_n^{n-5} is on two normal M_2^{n-6}'s in S_{n-5}.*

A simple statement of the relations among the F-points of a Jonquière transformation can be given in terms of associated sets.

THEOREM 14. *Given the Jonquière transformation J^{n+1} with center at p and simple F-points at P_{2n}^2, then curves of order $n - 2$ with an $(n - 3)$-fold point at p map the plane upon an M_2^{2n-5} in S_{2n-4} and map the set P_{2n}^2 upon a set R_{2n}^{2n-4} which is associated to the set Q_{2n}^2 of simple F-points of the inverse transformation.*

The proof of this is immediate by the foregoing methods.

We now proceed to particular sets beginning with P_8^2, Q_8^4. The ∞^1 elliptic quintics, E^5's, on Q_8^4 are obtained by the mapping of P_8^2 on Q_8^4 by conics on the 9th base point p_9 of the pencil of cubics on P_8^2. This pencil becomes a pencil of E^5's on an M_2^3 on Q_8^4 and the generators of M_2^3, which arise from the lines of the plane on p_9, are bisecants of all these E^5's. However, each of the ∞^1 E^5's on Q_8^4 has ∞^1 M_2^3's on it, whose generators on points v_1, v_2 satisfy the involution $v_1 + v_2 \equiv k$.* That particular M_2^3 common to all the E^5's is determined by the involution cut out on any E^5 by quadrics on Q_8^4. For if, in the plane, $u_1 + u_2 + \cdots + u_8 + u_9 \equiv 0$, $v_1 + \cdots + v_5 + u_9 \equiv 0$, $w_1 + w_2 + u_9 \equiv 0$ represent the sections of a cubic of the pencil by respectively a cubic

* Segre, Mathematische Annalen, vol. 27 (1886).

of the pencil, a mapping conic, and a line on p_9, then, on writing the second relation in the form $(v_1 + \frac{1}{5}u_9) + \cdots + (v_5 + \frac{1}{5}u_9) \equiv 0$ in order to introduce the canonical parameter $v' = v + \frac{1}{5}u_9$ on the mapped E^5, we have for Q_8^4 and the meets of a generator of the unique M_2^3 the relations

$$(u_1 + \tfrac{1}{5}u_9) + \cdots + (u_8 + \tfrac{1}{5}u_9) \equiv \tfrac{3}{5}u_9, \quad (w_1 + \tfrac{1}{5}u_9) + (w_2 + \tfrac{1}{5}u_9) \equiv -\tfrac{3}{5}u_9,$$

whence on E^5 $v_1' + \cdots + v_8' + w_1' + w_2' \equiv 0$ and the ten points are a quadric section.

We may relate Q_8^4 and any one of the ∞^2 M_2^3's on it to P_8^2 in the plane as follows. Let P_8^2, R_8^2 be F-points of a J^5 with centers at p, r where p is any one of the ∞^2 points of the plane. Then if p_1, p_2, p_3 are on a line, the points $r_4, \cdots, r_8, r$ are on a conic. Hence conics on r map the plane upon an M_2^3 in S_4 in such a way that R_8^2 is mapped upon the set Q_8^4 associated to P_8^2.

In addition to the ∞^1 E^5's on Q_8^4 there are ∞^2 rational quintics R^5 on Q_8^4. These are in one-to-one correspondence with the ∞^2 M_2^3's on Q_8^4. For, given an M_2^3 on Q_8^4, of the 7 linearly independent quadrics on Q_8^4 three are on M_2^3 (the three determinants of the matrix (4)) and of the remaining four one is on the directrix of M_2^3 and cuts M_2^3 in a residual R^5 trisecant to the directrix and unisecant to the generators. Conversely, given an R^5 on Q_8^4 it has a unique trisecant (with parameters determined by the canonizant of the binary quintic apolar to all S_3 sections) whose points are in 1–1 correspondence with the points of the curve (the correspondence being determined by making the three points common to the curve and trisecant self-corresponding) and the lines joining corresponding points are generators of an M_2^3 on Q_8^4. The question then arises as to the nature of the spread which is the locus of the ∞^2 R^5's on Q_8^4 or the nature of the condition that a Q_9^4 be on an R^5, and as to the corresponding condition on the associated P_9^3. The two theorems which follow answer these questions.

THEOREM 15. *There are two M_2^3's on a given Q_9^4 which are covariantly related to the set under regular Cremona transformation. They are isolated by the same irrationality as separates the two reguli on the unique quadric on the associated set P_9^3. The parameters of the* 9 *generators of one of the M_2^3's on Q_9^4 are projective to those of the* 9 *generators of one of the reguli on P_9^3. If the set Q_9^4 lies on an R^5* (*a single condition*) *then it lies on but one M_2^3 and its associated P_9^3 lies on a quadric cone.*

Two M_2^3's in S_4 meet in a set Q_9^4. That on Q_9^4 there are two M_2^3's is proved by Theorem 13. That there are only two is proved as follows. The ∞^2 M_2^3's on Q_8^4 are loci of ∞^1 bisecants of the ∞^1 E^5's on Q_8^4. One of these M_2^3's, say m_2^3, is a locus of bisecants of each of the E^5's; the others are each a bisecant locus of only one E^5. If then an M_2^3 is on a 9th point q_9 there is a bisecant of an E^5 on q_9; if two M_2^3's are on q_9 their plane cuts m_2^3 in the 4 meets of two bisecants with their respective E^5's. Hence this plane cuts m_2^3 in one of the director conics on it. A third bisecant on Q_9 would have to be in this plane else there would be two director conics with two intersections, whereas such conics have only one. A director conic meets each of the ∞^1 E^5's in three points and on this conic there is an involution of triads whose joining triangles envelop another conic. Hence on the point q_9 in the plane of this conic there are just two lines of this envelope each belonging to one of the two M_2^3's on Q_9^4.

If Q_9^4 is on an R^5 which must lie on one of the two M_2^3's on Q_9^4 and must cut its directrix in three points and each generator in one point, then in the notation of the proof of Theorem 12 the R^5 must be the map of a rational plane quartic with triple point at O' and on π_8', $\cdots$, π_8' as well as on σ'. But then σ must coincide in some direction with O, and the quadric on P_9^3 mapped by conics on O, σ is a quadric cone.

An E^5 in S_4 is projected from a line into an elliptic plane quintic with five nodes and from a line which meets E^5 into an elliptic plane quartic with two nodes, whence the bisecant locus of E^5 is a quintic spread on which E^5 is a triple curve. The ∞^1 E^5's on a given Q_8^4 can be put into 1, 1 correspondence with a pencil of plane cubics and therefore can be named rationally in terms of a parameter λ. Through a point there pass two bisecants belonging to two of these E^5's, whence the aggregate of these bisecant spreads of the ∞^1 E^5's constitute a quadratic system. The two bisecants isolate the two M_2^3's on Q_8^4 and the given point, whence if they coincide the two M_2^3's coincide and the given point and Q_8^4 are on an R^5. Hence

THEOREM 16. *If $\lambda^2 B_0 + 2\lambda B_1 + B_2 = 0$ is the quadratic system of bisecant spreads of the ∞^1 E^5's on Q_8^4, the spread $B_1^2 - B_0 B_2 = 0$ (a spread of order 10 with 6-fold points at Q_8^4 and a double M_2^3 consisting of the ∞^1 E^5's) is the locus of the ∞^2 rational quintics on Q_8^4, or the locus of points through which there can be drawn but one line bisecant to an E^5 on Q_8^4, or through which there can be passed but one M_2^3 on Q_8^4. Its equation may be obtained by replacing in the condition that a quadric on P_9^3 be nodal (a condition of degree 8 in the coördinates of each point of P_9^3 whose terms consist of products of 18 determinants $|p_i p_j\ p_k p_l|$) each determinant $|p_{i_1} p_{i_2} p_{i_3} p_{i_4}|$ by the complementary determinant $|q_{i_5}q_{i_6}q_{i_7}q_{i_8}q_{i_9}|$ formed for Q_9^4 and allowing the 9th point to vary.*

Here then we have an instance of the actual determination of a covariant of Q_8^4 or an invariant of Q_9^4 under the infinite group of regular Cremona transformations attached to the set.

We complete the discussion of sets of 9 points with the Q_9^5 associated with the set P_9^2. In S_5 the elliptic norm sextic E^6 has one absolute and 36 projective constants; the rational sextic R^6 has three absolute and 38 projective constants; and the M_2^4 has 29 projective constants; whence on Q_9^5 there is a finite number of E^6's, ∞^2 R^6's, and ∞^2 M_2^4's. There is, however, in S_5 a new type of rational 2-way of order 4, the Veronese surface V_2^4, which shares with M_2^4 the property that its projection from one of its points is an M_2^3. The V_2^4 is the map of the plane by the linear system of all conics in the plane. It contains ∞^2 conics, the maps of lines of the plane, and the locus of the ∞^2 planes of these conics is a V_4^3 upon which V_2^4 is a double manifold. Analytically V_4^3 is obtained by setting a 3-row symmetric determinant of linear forms equal to zero and V_2^4 is the locus for which the six first minors vanish. The V_2^4 is unaltered by an 8-parameter collineation group, the map of the ternary group, whence it has $35 - 8 = 27$ projective constants. We should expect, therefore, to find on Q_9^5 a finite number of V_2^4's. The surface V_2^4 shares with M_2^4 also the property expressed by

THEOREM 17. *The Veronese surface V_2^4 is transformed into a Veronese surface $V_2'^4$ by a regular Cremona transformation whose F-points are on V_2^4. If the regular transformation in S_5 is $y_i = 1/x_i$ $(i = 0, \cdots, 5)$ the two V_2^4's are mapped by conics from planes which are in correspondence under the ternary quintic transformation with* 6 *double F-points. The V_4^3 with double V_2^4 is transformed into the $V_4'^3$ with double $V_2'^4$.*

Indeed the given involution maps the $S_4(y)$'s upon a system of quintic spreads with 4-fold points at the 6 F-points on V_2^4. This is the map of a ternary system of 10-ics with 4-fold points at 6 points, which can be transformed by the ternary transformation mentioned into a system of conics. The same involution transforms a cubic spread with nodes at the 6 F-points into a similar spread, whence V_4^3 on V_2^4 passes into $V_4'^3$ on $V_2'^4$.

Upon V_2^4 there is a linear system of ∞^9 E^6's, the maps of cubic curves in the plane. Conversely an E^6 on V_2^4 is cut out by a quadric which meets V_2^4 in a residual conic, whence the corresponding quartic in the plane breaks up into a line and a cubic. Therefore there are no other E^6's on V_2^4. The conics on V_2^4 are trisecant to the E^6's on V_2^4. A canonical elliptic parameter on the plane cubic is mapped into a canonical parameter on E^6 whence the planes of V_4^3 are those which meet E^6 in three points for which $u_1 + u_2 + u_3 \equiv 0$. Obviously any two of these planes lie in an S_4 and meet in a point. But the same thing is true of the three other involutions for which $u_1 + u_2 + u_3 \equiv \omega/2$. Hence on E^6 there are 4 V_2^4's or also there are 4 V_4^3's which contain E^6 doubled. Such a V_4^3 must contain every bisecant of E^6. The locus of bisecants, B_3^9, of E^6 is a 3-way of order 9 which has E^6 as a 4-fold curve, since from a plane E^6 is projected into a plane sextic with 9 nodes, and from a plane which meets E^6 the E^6 is projected into a plane quintic with 5 nodes. Hence the bisecant locus is the complete intersection of two of the four V_4^3's and the four lie in a pencil. A member of this pencil other than a V_4^3 also contains B_3^9. Given then a trisecant plane for which $u_1 + u_2 + u_3 \equiv k$, the above pencil of W_4^3's contains the three bisecants in the plane, whence one member, say W_4^3, contains the plane. Since any plane for which $v_1 + v_2 + v_3 \equiv -k$ meets the above plane in a point, W_4^3 must meet this latter plane in its bisecants and an outside point and therefore must contain it. Hence W_4^3 is the locus of the ∞^2 trisecant planes $v_1 + v_2 + v_3 \equiv -k$ or also of the ∞^2 trisecant planes $u_1 + u_2 + u_3 \equiv k$. For each of the 4 V_4^3's in the pencil of W_4^3's the two systems of generating planes coincide into a single system, since $k \equiv -k$ when $k \equiv \omega/2$. Hence

THEOREM 18. *An E^6 is contained on 4 V_2^4's whose V_4^3's are in the pencil of spreads W_4^3 on the bisecant locus B_3^9 of E^6 for which E^6 is a 4-fold curve. A particular W_4^3 of the pencil with double E^6 has the two systems of ∞^2 generating trisecant planes for which $u_1 + u_2 + u_3 \equiv -k$, k which coincide for the 4 V_4^3's. Under regular Cremona transformation with F-points on E^6 the properties of this pencil are invariant.*

That there are on E^6 four V_2^4's may be seen by the use of an elementary theorem. Isolate one of the V_2^4's as the map of a plane. The V_4^3's of the other three V_2^4's cut the isolated one in E^6 doubled, whence in the plane we have the square of a cubic expressed in three ways as a symmetric 3-row determinant whose elements are conics. But we know that a cubic can be expressed in three ways as a symmetric 3-row determinant of linear forms, since it is the hessian of three cubics and the square of a symmetric determinant is symmetric. Moreover we know that the relation of the hessian to the three cubics involves the three half periods.

THEOREM 19. *On a general set Q_9^5 there is a unique E^6 and four V_2^4's.*

We see at once that an E^6 and an E'^6 on Q_9^5 could not have different absolute invariants. For an E^6 on Q_9^5 is projected from a properly chosen trisecant plane into an E^3 on the associated P_9^2, and E'^6 into an E'^3 on P_9^2, whence, since E^3 and E'^3 cannot coincide, the set P_9^2 is the special set of 9 base points of a pencil of E^3's and Q_9^5 is also a special set. If, however, there were an E^6 and an E'^6 on Q_9^5, then on projecting from q_9 we should have in S_4 an E^5 and E'^5 on R_8^4, members of a pencil on an M_2^3 in S_4. Hence in S_5 there are ∞^1 elliptic quintic 2-way cones with vertex at q_9 and on $q_1, \cdots, q_8$, and with no other points common to any two. A quadric on Q_9^5 and four generators of any one of these cones meets the cone in an E^6 on Q_9^5, whence there is a pencil of such E^6's on Q_9^5 with all values of the absolute invariant and again Q_9^5 is the special set above. This unique E^6 and therefore Q_9^5 also carries four V_2^4's. There are no V_2^4's on Q_9^5 which are not also on E^6, else there would be on such a V_2^4 an E'^6 on Q_9^5.

THEOREM 20. *If P_9^2 is the set of base points of a pencil of E^3's, its associated Q_9^5 is the set of base points of a pencil of E^6's on a V_2^4, the map of the plane by conics.*

This is an immediate consequence of the elementary theorem that if three of the points of such a planar set are on a line the remaining six are on a conic.

We observe that for such a pencil of E^6's on a given V_2^4 each E^6 according to Theorem 18 is contained on three other V_2^4's whence this special Q_9^5 is on ∞^1 V_2^4's one of which is isolated while the others divide into triads which depend rationally on a parameter.

If 8 points of such a special Q_9^5 are given, the locus of the 9th is a 3-way, four of whose points are on any E^6 through the given 8 points. This 3-way is the extension of the Weddle surface and bears the same relation to the hyperelliptic functions of genus three as the Weddle to those of genus two. This relation will be discussed in a forthcoming paper.

If P_9^2 of Theorem 20 is the set of flex points of an E^3, the base points of a syzygetic pencil, then any two are on a line with a third, whence

THEOREM 21. *There exists in S_5 a set of 9 points invariant under a Hesse collineation G_{216} with the property that any two points determine a third such that the remaining six are on an S_4. The configuration contains 12 S_4's, eight on each point.*

This set of 9 points has the unusual property that if six be selected which form a reference 6-point, no other one can be taken to be the unit point, since each of the other three must lie in one of the reference S_4's. Using a proper set of six as reference points the coördinates of the other three are

$$\begin{array}{cccccc} \omega, & \omega^2, & -1, & -\omega^2, & 0, & -\omega; \\ -1, & 1, & \omega^2, & -1, & -\omega^2, & 0; \\ 1, & -1, & \omega, & 0, & -\omega, & -1 \end{array} \qquad (\omega = e^{2\pi i/3}).$$

The problem of obtaining the four surfaces V_2^4 on a given Q_9^5 may be solved through the use of the associated set P_9^2 as follows:

THEOREM 22. *On the E^3 on P_9^2 join the 9th base point of the pencil on $p_1, \cdots, p_8$ to p_9 to meet E^3 again in p'. From p' draw a tangent to E^3 at p'' (4 choices). Construct a set $r_9, r_1, \cdots, r_8$ congruent to $p'', p_1, \cdots, p_8$ under J^5 with centers r_9, p''. Then conics map the set R_9^2 upon the set Q_9^5 associated to P_9^2 and map the plane upon one of the four V_2^4's on Q_9^5.*

We now consider sets of 9 and of 10 points in S_5 with reference to the normal surfaces M_2^4 and the rational sextic curves R^6. We have noted that on Q_9^5 there are ∞^2 M_2^4's and ∞^2 R^6's. Only ∞^1 of the M_2^4's contain the unique E^6 on Q_9^5. For the M_2^4 mapped from the plane by cubic curves on the base O^2, σ contains ∞^8 E^6's which are mapped from quartic curves with nodes at O, σ whence M_2^4 and E^6 on it have 37 constants. But E^6 alone has 36 constants, whence on E^6 there are ∞^1 M_2^4's. These are the bisecants of the ∞^1 involutions $u + u' \equiv k$, since lines on O cut out such an involution on a ternary quartic with node at O. An M_2^4 on Q_9^5 and not containing E^6 can have no other point in common with E^6. For if E^6 were to meet M_2^4 in 10 points, at least four of the quadrics on M_2^4 would contain E^6. But four such quadrics meet in a residual conic. We now prove

THEOREM 23. *The locus of the* ∞^2 M_2^4*'s on* Q_9^5 *is a cubic spread with the* E^6 *on* Q_9^5 *for double curve. A point of this cubic spread forms with* Q_9^5 *a symmetrical set* Q_{10}^5 *which are the meets of two* M_2^4*'s and whose associated set* P_{10}^3 *is on a quadric surface. The cubic spread is that locus of* ∞^2 *trisecant planes of* E^6 *whose meets with* E^6 *lie with* Q_9^5 *on a quadric.*

For the condition that P_{10}^3 is on a quadric surface is of degree two in each point p_i and therefore is a sum of products of 5 four-row determinants. The corresponding condition on Q_{10}^5 is a sum of products of 5 six-row determinants and therefore is of degree three in each point q_i. Since the condition on P_{10}^3 is invariant under regular Cremona transformation this is likewise true of Q_{10}^5. Hence if q_{10} is variable the cubic spread must have nodes at Q_9^5. According to Theorem 13, Q_{10}^5 is the set of points of intersection of two M_2^4's on Q_9^5. Since the cubic spread contains the ∞^1 M_2^4's on Q_9^5 which contain E^6, it contains the bisecant locus B_3^9 of E^6 and therefore is a member of the pencil of Theorem 18 and contains E^6 as a double curve. To prove the trisecant plane property we observe (and omit the verification) that if a quadric contains M_2^4, a plane on this quadric meets M_2^4 in a point. Given then an M_2^4 and a plane trisecant to E^6 at v_1, v_2, v_3 such that $u_1 + \cdots + u_9 + v_1 + v_2 + v_3 \equiv 0$, of the 6 quadrics on M_2^4 and therefore on Q_9^5 at least four are on v_1, v_2, v_3 and at least one contains the plane $v_1 v_2 v_3$ which therefore meets M_2^4 in a point. As M_2^4 varies in the ∞^2 system on Q_9^5, this point runs over the trisecant plane.

An R^6 on Q_9^5 is on a unique M_2^4 on Q_9^5 and vice versa. For given the M_2^4 mapped by cubics on O^2, σ the R^6's are mapped from ternary quintics with 4-fold point at O and simple point at σ, whence on Q_9^5 there is a unique R^6. These R^6's meet the generators in one point and the directrix conics in four points whose four parameters on conic and on R^6 are projective. Given R^6 on Q_9^5, its quadrisecant planes each carry a unique conic with the projective 4-point property just mentioned and the locus of these conics is the unique M_2^4 on Q_9^5 and R^6.

If in the proof of Theorem 12 the point σ' is on a ternary quintic which maps into an R^6 on Q_9^5, then σ coincides with O in some direction and the set P_{10}^3 is on a nodal quadric. For such a set the two M_2^4's coincide. Hence

THEOREM 24. *The two conditions that* Q_{10}^5 *be on an* R^6 *are that its associated* P_{10}^3 *be on a nodal quadric. On such a* Q_{10}^5 *there is but one* M_2^4.

The ∞^1 M_2^4's on Q_9^5 which contain the E^6 on Q_9^5 are obtained by mapping P_9^2 on Q_9^5 in the ∞^1 ways described in § 1. All of the ∞^2 M_2^4's on Q_9^5 are obtained by the following construction.

THEOREM 25. *For the set* P_9^2 *we choose a center* p *(in* ∞^2 *ways) and, for arbitrarily chosen* p_{10}*, construct a set* $r_1, \cdots, r_9, \sigma, O$ *congruent to* $p_1, \cdots,$

p_9, p_{10}, p under J^6 with centers O, p. Then cubics on $O^2\sigma$ map $r_1, \cdots, r_9$ upon the set Q_9^5 associated to P_9^2, and map the plane upon one of the ∞^2 M_2^4's on Q_9^5.

For if p_1, p_2, p_3 are on a line, then $r_4, \cdots, r_9, \sigma, O^2$ are on a cubic or $q_4, \cdots, q_9$ are on an S_4. That these M_2^4's are all distinct follows from the fact that the ∞^2 line pencils from p to P_9^2 are projectively distinct. We observe that, when p has been chosen and thereby an M_2^4 isolated, the variation of p_{10} implies the variation of the point $\overline{\sigma O}$ of M_2^4 over the M_2^4.

We shall close with an application to the sets of 10 nodes of a rational plane sextic and of a symmetroid quartic surface Σ. These two figures are related as follows. The sextic $S(t)$ has a conjugate rational sextic $R(t)$ in space such that the plane sections of the one are apolar to the line sections of the other. The locus of planes which cut $R(t)$ in catalectic sextics is Σ (as an envelope) and the 10 planes which cut $R(t)$ in cyclic sextics (reducible to a sum of two sixth powers) are the ten double planes of Σ. If such a cyclic sextic is $(p_1 t)^6 + (p_2 t)^6 = 0$, then $(p_1 t)\cdot(p_2 t) = 0$ are the nodal parameters of a double point of $S(t)$. Thus the nodes of Σ and the nodes of $S(t)$ are in correspondence. It is known that there are two projectively distinct rational sextics $S(t)$, $S(\tau)$ which determine the same Σ. I have proved but not yet published the fact that if 6 nodes of $S(t)$ are on a conic then the complementary 4 nodes of Σ are on a plane. Hence conics on the plane map the plane on a V_2^4 in S_5 and the ten nodes of $S(t)$ upon a Q_{10}^5 on V_2^4 which is associated to the P_{10}^3 of nodes of Σ. But also conics of the plane of $S(\tau)$ map this plane on a $V_2'^4$ and the ten nodes of $S(\tau)$ upon the same Q_{10}^5, since this set also is associated to P_{10}^2. From this there follows at once

THEOREM 26. *If two Veronese surfaces V_2^4, $V_2'^4$ meet in 10 points, Q_{10}^5, then this set is associated to the set P_{10}^3 of nodes of a Cayley symmetroid. The spreads V_4^3, $V_4'^3$, with double V_2^4, $V_2'^4$ respectively, each cut the double spread of the other in a 12-ic curve with nodes at Q_{10}^5. These curves are the maps from the plane of the two rational plane sextics associated with the symmetroid.*

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