REPRESENTATIONS OF A COMPLEX POINT BY PAIRS OF ORDERED REAL POINTS*

BY

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1. **Introduction.** A complex point z in a complex space of n dimensions, since it depends on $2n$ real parameters, can be represented by a pair of real points, X and Y, of the space. It is reasonable to restrict the choice of X and Y by the following conditions:

A. The points X and Y representing z shall be ordered: $X \to Y$.

B. If z is a real point, X and Y shall coincide in z.

C. If z is imaginary, X and Y shall be distinct; if $X \to Y$ represents z, $Y \to X$ shall represent $\bar{z}$, the point conjugate-imaginary to z.

D. The coördinates of X and Y, referred to a rectangular coördinate system, shall be analytic functions, without singularities in the finite domain, of the coördinates of z and $\bar{z}$, referred to this system.

E. The representation of z by $X \to Y$ shall be invariant under a chosen group of real point transformations; that is, the ordered points $X' \to Y'$ into which the ordered points $X \to Y$ are carried by an arbitrary transformation of the group shall represent the point z' into which z is carried by the transformation.

If $X \to Y$ is a representation of z which satisfies these conditions, $Y \to X$ is also a representation of z satisfying them. The two representations shall be called *inverse*.

There are two pairs of inverse representations satisfying the prescribed conditions which are well known: those of Laguerre in the plane, whereby z is represented by the real points of intersection, taken in the one or the other order, of the minimal lines through z and $\bar{z}$; and those of Marie in the plane and in space, in which the real points of the representation lie on the line $z\bar{z}$, have the same midpoint as z and $\bar{z}$, and are at a distance apart whose square is the negative of the square of the distance $z\bar{z}$.† The two Marie representations are invariant under the group of affine transformations, whereas those of Laguerre are unchanged by the direct circular transformations of the plane.

This paper deals with the determination of all the representations $X \to Y$

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† For a review of the literature concerning these representations, see E. Study, *Vorlesungen über ausgewählte Gegenstände der Geometrie, erstes Heft*; *Ebene analytische Kurven und zu ihnen gehörige Abbildungen*, Leipzig, 1911; J. L. Coolidge, *Geometry in the Complex Domain* (in press).

of z which satisfy conditions A to D, and are invariant, in compliance with E, under a given group. In § 2 the problem is formulated analytically for an arbitrary group. In the subsequent sections, 3–7, it is solved for various subgroups of the projective group, for the projective group itself, and for the group of direct circular transformations of the plane. Only in the case of the projective group are no representations obtained.

Given a group, G, of direct transformations and the set, S, of corresponding indirect transformations, it is of importance to determine, among the representations invariant under G, those which are invariant also under S, and secondly, those which are inverted, i.e., carried into their inverses, by S. It turns out that all the representations invariant under G are invariant also under S, except when G is the group of direct circular transformations, or when G is either the group of motions or of direct transformations of similarity in the plane.* In the latter cases certain representations of the totality invariant under G are unchanged by S, whereas certain others are inverted by S (§ 8). Moreover, it is only in these same cases that the representations are of exceptional nature in comparison with those obtained for the same group applied in a space of different dimensionality.

2. **Analytic formulation of the problem.** Let $(z_1, z_2, \cdots, z_n)$, where

$$z_i = x_i + iy_i \qquad (i = 1, 2, \cdots, n),$$

be the coördinates, referred to a rectangular cartesian system of axes, of the point z, and let the coördinates of X and Y, referred to the same system, be respectively $(X_1, X_2, \cdots, X_n)$ and $(Y_1, Y_2, \cdots, Y_n)$.

Since the latter sets of coördinates are, by condition D, real analytic functions of $z_1, z_2, \cdots, z_n, \bar{z}_1, \bar{z}_2, \cdots, \bar{z}_n$, they are also real analytic functions of the real variables $x_1, y_1, x_2, y_2, \cdots, x_n, y_n$. It is convenient, in light of B, to write these functions as follows:

$$X_i = x_i + F_i(x_1, y_1, \cdots, x_n, y_n),$$
$$Y_i = x_i + G_i(x_1, y_1, \cdots, x_n, y_n) \qquad (i = 1, 2, \cdots, n).$$

By C,

$$G_i(x_1, y_1, \cdots, x_n, y_n) = F_i(x_1, -y_1, \cdots, x_n, -y_n) \qquad (i = 1, 2, \cdots, n).$$

Conditions A to D are thus covered by demanding that X_i and Y_i be of the form

$$\begin{aligned} X_i &= x_i + F_i(x_1, y_1, \cdots, x_n, y_n), \\ Y_i &= x_i + F_i(x_1, -y_1, \cdots, x_n, -y_n) \qquad (i = 1, 2, \cdots, n), \end{aligned} \tag{1}$$

where $F_1, F_2, \cdots, F_n$ are real functions of the real variables $x_1, y_1, \cdots, x_n, y_n$, which are analytic for all finite sets of values, vanish identically when

$$y_1 = y_2 = \cdots = y_n = 0,$$

* In this summary of exceptions, one relatively unimportant case has been omitted.

and are furthermore such that the system of equations

$$F_i(x_1, y_1, \cdots, x_n, y_n) = F_i(x_1, -y_1, \cdots, x_n, -y_n)$$

has no real solutions in $y_1, y_2, \cdots, y_n$ except the solution $0, 0, \cdots, 0$.

Let the equations

$$z_i' = T_i(z_1, z_2, \cdots, z_n) \qquad (i = 1, 2, \cdots, n) \tag{2}$$

represent an arbitrary group of real point transformations. In order that the representation $X \to Y$ of the point z be invariant under this group, it is necessary and sufficient that

$$\begin{aligned} &(a)\ x_i' + F_i(x_1', y_1', \cdots, x_n', y_n') \\ &\qquad = T_i(x_1 + F_1(x_1, y_1, \cdots, x_n, y_n), \cdots, \\ &\qquad\qquad x_n + F_n(x_1, y_1, \cdots, x_n, y_n)), \\ &(b)\ x_i' + F_i(x_1', -y_1', \cdots, x_n', -y_n') \\ &\qquad = T_i(x_1 + F_1(x_1, -y_1, \cdots, x_n, -y_n), \cdots, \\ &\qquad\qquad x_n + F_n(x_1, -y_1, \cdots, x_n, -y_n)) \quad (i = 1, 2, \cdots, n). \end{aligned} \tag{3}$$

Thus our problem, finally formulated, consists in solving the system of functional equations (3) for the functions $F_1, F_2, \cdots, F_n$, subject to the conditions stated under (1).

If the transformations (2) are linear, as will frequently be the case,

$$x_i' = T_i(x_1, x_2, \cdots, x_n), \qquad y_i' = T_i^0(y_1, y_2, \cdots, y_n) \qquad (i = 1, 2, \cdots, n),$$

where T_i^0 is T_i minus its absolute term. Changing the signs of $y_1, y_2, \cdots, y_n$ then changes the signs of $y_1', y_2', \cdots, y_n'$. Consequently, the two systems of equations in (3) are, in this case, each equivalent to the single system

$$\begin{aligned} F_i(x_1', y_1', \cdots, x_n', y_n') = T_i^0(&F_1(x_1, y_1, \cdots, x_n, y_n), \\ &\cdots, F_n(x_1, y_1, \cdots, x_n, y_n)) \quad (i = 1, 2, \cdots, n). \end{aligned} \tag{4}$$

3. **Representations invariant under the group of translations.** Inasmuch as the group of real translations is a subgroup of each of the other groups under which the invariance of representations is to be considered, it is advantageous to deal with it first. Equations (4) then become

$$F_i(x_1 + a_1, y_1, \cdots, x_n + a_n, y_n) = F_i(x_1, y_1, \cdots, x_n, y_n) \qquad (i = 1, 2, \cdots, n).$$

Since $a_1, a_2, \cdots, a_n$, the parameters of the translation, are arbitrary, the system is satisfied if and only if $F_1, F_2, \cdots, F_n$ are functions of $y_1, y_2, \cdots, y_n$ alone.

The representations $X \to Y$ of z invariant under the group of real translations *are given by*

$$\text{I}\qquad X_i = x_i + F_i(y_1, y_2, \cdots, y_n), \quad Y_i = x_i + F_i(-y_1, -y_2, \cdots, -y_n) \qquad (i = 1, 2, \cdots, n),$$

where $F_1, F_2, \cdots, F_n$ *are arbitrary real analytic functions which vanish for* $y_1 = y_2 = \cdots = y_n = 0$ *and are such that the system of equations*

$$F_i(y_1, y_2, \cdots, y_n) = F_i(-y_1, -y_2, \cdots, -y_n) \quad (i = 1, 2, \cdots, n)$$

has no real solutions other than $0, 0, \cdots, 0$.

In the case of a group of linear transformations containing the group of translations, (4) can now be replaced by the system of equations

$$(5)\qquad \begin{aligned} F_i(y_1', y_2', \cdots, y_n') = T_i^0(F_1&(y_1, y_2, \cdots, y_n), \\ &\cdots, F_n(y_1, y_2, \cdots, y_n)) \quad (i = 1, 2, \cdots, n), \end{aligned}$$

in which $x_1, x_2, \cdots, x_n, x_1', x_2', \cdots, x_n'$ no longer appear.*

4. **The group of homothetic transformations.** The representation $X \to Y$ is invariant under the group of real homothetic transformations

$$z_i' = \rho z_i + a_i, \qquad \rho \neq 0 \qquad (i = 1, 2, \cdots, n),$$

if and only if the system (5), formed for this group, namely

$$F_i(\rho y_1, \rho y_2, \cdots, \rho y_n) = \rho F_i(y_1, y_2, \cdots, y_n) \quad (i = 1, 2, \cdots, n),$$

is satisfied. But then, regardless of whether ρ is unrestricted or takes on only positive values, $F_1, F_2, \cdots, F_n$ must be linear and homogeneous in $y_1, y_2, \cdots, y_n$.

The representations $X \to Y$ *of* z *invariant under the group of either direct or general homothetic transformations are*

$$\text{II}\qquad X_i = x_i + \sum_{j=1}^{n} k_{ij} y_j, \qquad Y_i = x_i - \sum_{j=1}^{n} k_{ij} y_j \quad (i = 1, 2, \cdots, n),$$

where the k_{ij} *are real constants subject only to the restriction that the n-rowed determinant* $|k_{ij}|$ *does not vanish.*

* The equations (5) are also necessary and sufficient conditions that the real analytic point transformation

$$(a)\qquad y_i' = F_i(y_1, y_2, \cdots, y_n) \qquad (i = 1, 2, \cdots, n)$$

be commutative with each transformation of the group of real linear transformations

$$(b)\qquad y_i' = T_i^0(y_1, y_2, \cdots, y_n) \qquad (i = 1, 2, \cdots, n)$$

leaving the origin invariant. Accordingly, as we solve (5) in the subsequent paragraphs for various groups (*b*), we determine at the same time, for each particular group (*b*), the group of transformations (*a*), each of which is commutative with every transformation (*b*). Thus we find, for example, in § 6, that the group of transformations commutative with the group of motions about the origin in the plane is

$$y_1' = y_1 R_1(r^2) + y_2 R_2(r^2), \qquad y_2' = -y_1 R_2(r^2) + y_2 R_1(r^2),$$

where $r^2 = y_1^2 + y_2^2$ and R_1, R_2 are analytic functions of r^2, not both zero.

The condition $|k_{ij}| \neq 0$ is necessary and sufficient, not merely that X and Y coincide only when z is real, but also that the $2n$ equations II can be solved uniquely for $x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n$ and hence the coördinates of z found in terms of those of X and Y. In other words, the correspondence between the ordered point pairs $X \to Y$ and the points z under a representation II is one-to-one without exception in the finite domain.

5. **The affine and projective groups.** In order that the representation II be invariant under the n transformations

$$z'_q = -z_q, \qquad z'_i = z_i \qquad (q = 1, 2, \cdots, n),$$

where in each transformation i takes on all the integral values from 1 to n except the fixed value q, it is necessary that $k_{ij} = 0, i \neq j$. If the resulting representation is to be unchanged by each of the transformations

$$z'_q = z_r, \qquad z'_r = z_q, \qquad z'_i = z_i \qquad (q \neq r),$$

where in each transformation q and r are fixed and i takes on the remaining values from 1 to n, it follows that $k_{11} = k_{22} = \cdots = k_{nn}$. A necessary condition, therefore, that II be invariant under the affine group is that $k_{ij} = 0$, $i \neq j$, and $k_{ii} = k$ for all i. That this condition is also sufficient can be easily verified.

If, instead of all real affine transformations, merely those for which the determinant of the coefficients is positive had been admitted, the result would have been the same.

The representations $X \to Y$ of z invariant under the group of either direct or general affine transformations are

$$\text{III} \qquad X_i = x_i + ky_i, \qquad Y_i = x_i - ky_i \qquad (i = 1, 2, \cdots, n),$$

where k is an arbitrary constant, $\neq 0$.

The two inverse representations for which $k = \pm 1$ are *the representations of Marie.*

If III is to be invariant under the projective group, it must be unchanged by the transformation

$$z'_1 = \frac{1}{z_1}, \qquad z'_i = \frac{z_i}{z_1} \qquad (i = 2, \cdots, n).$$

But then the first of the equations (3) necessitates that $k^2 = -1$.

There is no representation $X \to Y$ of z which is invariant under the projective group.

6. **The group of motions.** The treatment in this case varies with the dimensionality of the space.

A. $n = 1$. The group of motions is the group of translations. According to I, *the invariant representations are*

$$\text{IVA} \qquad X_1 = x_1 + y_1 \Phi(y_1), \qquad Y_1 = x_1 - y_1 \Phi(y_1),$$

where $\Phi(y_1)$ *is a real analytic function and the equation* $\Phi(y_1) = -\Phi(-y_1)$ *is satisfied by no real value of* y_1 *except perhaps* $y_1 = 0$.

B. $n = 2$. The representation must be of the form

$$X_i = x_i + F_i(y_1, y_2), \qquad Y_i = x_i + F_i(-y_1, -y_2) \qquad (i = 1, 2), \tag{6}$$

and be invariant under the transformations

$$z_1' = \cos\phi\, z_1 - \sin\phi\, z_2 + a_1, \qquad z_2' = \sin\phi\, z_1 + \cos\phi\, z_2 + a_2.$$

That is, $F_1(y_1, y_2)$, $F_2(y_1, y_2)$ must be solutions of the system of equations, obtained from (5),

$$\begin{aligned} F_1(y_1', y_2') &= \cos\phi\, F_1(y_1, y_2) - \sin\phi\, F_2(y_1, y_2), \\ F_2(y_1', y_2') &= \sin\phi\, F_1(y_1, y_2) + \cos\phi\, F_2(y_1, y_2), \end{aligned} \tag{7}$$

where

$$y_1' = \cos\phi\, y_1 - \sin\phi\, y_2, \qquad y_2' = \sin\phi\, y_1 + \cos\phi\, y_2.$$

For $\phi = \pi$, equations (7) become the conditions that F_1, F_2 be odd functions of y_1, y_2. Consequently, if F_i is written in the form

$$F_i(y_1, y_2) = \frac{F_i(y_1, y_2) + F_i(y_1, -y_2)}{2} + \frac{F_i(y_1, y_2) + F_i(-y_1, y_2)}{2} \qquad (i = 1, 2),$$

the first quotient on the right is even in y_2, odd in y_1 and hence identically zero for $y_1 = 0$, whereas the second is even in y_1, odd in y_2 and so identically zero for $y_2 = 0$. Accordingly,

$$F_i(y_1, y_2) = y_1 F_i'(y_1, y_2) + y_2 F_i''(y_1, y_2) \qquad (i = 1, 2), \tag{8}$$

where each of the functions F_1', F_1'', F_2', F_2'' is analytic and is even in y_1, in y_2, and hence in y_1, y_2.

In the equations obtained from (7) by substituting for F_1 and F_2 their values as given by (8), replace ϕ by $-\phi$, y_1 by $-y_1$, and combine by addition and subtraction each of the resulting equations with that from which it arose. The system obtained is

$$\begin{aligned} y_1' F_1'(y_1', y_2') &= y_1 \cos\phi\, F_1'(y_1, y_2) - y_2 \sin\phi\, F_2''(y_1, y_2), \\ y_2' F_2''(y_1', y_2') &= y_1 \sin\phi\, F_1'(y_1, y_2) + y_2 \cos\phi\, F_2''(y_1, y_2), \\ y_1' F_2'(y_1', y_2') &= y_1 \cos\phi\, F_2'(y_1, y_2) + y_2 \sin\phi\, F_1''(y_1, y_2), \\ y_2' F_1''(y_1', y_2') &= -y_1 \sin\phi\, F_2'(y_1, y_2) + y_2 \cos\phi\, F_1''(y_1, y_2). \end{aligned} \tag{9}$$

These equations reduce, for $\phi = \pi/2$, to

$$F_2'(y_1, y_2) = -F_1''(y_2, y_1), \qquad F_2''(y_1, y_2) = F_1'(y_2, y_1). \tag{10}$$

When the values for F_2' and F_2'' given by (10) are substituted in (9) and y_2 is set equal to zero, the four resulting equations yield immediately the identities

$$F_1'(y_1 \cos\phi, y_1 \sin\phi) = F_1'(y_1 \sin\phi, y_1 \cos\phi),$$
$$F_1''(y_1 \cos\phi, y_1 \sin\phi) = F_1''(y_1 \sin\phi, y_1 \cos\phi),$$

in the independent variables y_1 and ϕ. Consequently, $F_1'(y_1, y_2)$ and $F_1''(y_1, y_2)$ are symmetric functions of y_1, y_2, and (10) becomes

$$(11) \qquad F_2'(y_1, y_2) = -F_1''(y_1, y_2), \qquad F_2''(y_1, y_2) = F_1'(y_1, y_2).$$

But the system (9) then reduces to

$$F_1'(y_1', y_2') = F_1'(y_1, y_2), \qquad F_1''(y_1', y_2') = F_1''(y_1, y_2).$$

That is, F_1' and F_1'' are absolute invariants of the group of rotations about the origin, and hence must be of the form

$$(12) \qquad F_1' = R_1(r^2), \qquad F_1'' = R_2(r^2),$$

where

$$r^2 = y_1^2 + y_2^2.$$

The necessary conditions (8), (11), (12) that $F_1(y_1, y_2)$, $F_2(y_1, y_2)$ be solutions of (7) are shown sufficient by direct substitution.

The representations $X \to Y$ of z invariant under the group of motions in the plane are

$$\text{IVB} \quad \begin{array}{ll} X_1 = x_1 + y_1 R_1(r^2) + y_2 R_2(r^2), & Y_1 = x_1 - y_1 R_1(r^2) - y_2 R_2(r^2), \\ X_2 = x_2 - y_1 R_2(r^2) + y_2 R_1(r^2), & Y_2 = x_2 + y_1 R_2(r^2) - y_2 R_1(r^2), \end{array}$$

where $r^2 = y_1^2 + y_2^2$, and R_1, R_2 are real functions of r^2, analytic for all finite values and not vanishing simultaneously except perhaps when $r^2 = 0$.

C. $n \geqq 3$. In case $n = 3$, the functions F_1, F_2, F_3 in the representation

$$(13) \qquad X_i = x_i + F_i(y_1, y_2, y_3), \qquad Y_i = x_i + F_i(-y_1, -y_2, -y_3) \qquad (i = 1, 2, 3)$$

are to be determined from the system of equations

$$(14) \qquad F_i(y_1', y_2', y_3') = \sum_{j=1}^{3} a_{ij} F_j(y_1, y_2, y_3) \qquad (i = 1, 2, 3),$$

where

$$y_i' = \sum_{j=1}^{3} a_{ij} y_j \qquad (i = 1, 2, 3),$$

and the determinant $|a_{ij}|$ is orthogonal and has the value $+1$.

In case of an arbitrary rotation about the axis of z_3, $y_3' = y_3$ and y_1, y_2 are transformed as for $n = 2$. The last equation of (14) reduces to

$$F_3(y_1', y_2', y_3) = F_3(y_1, y_2, y_3),$$

and the first two to the equations (7), with the variable y_3 added in each function. Consequently,

$$
(15) \qquad \begin{aligned}
F_1 &= y_1 F_3'(y_1^2 + y_2^2, y_3) + y_2 F_3''(y_1^2 + y_2^2, y_3),\\
F_2 &= -y_1 F_3''(y_1^2 + y_2^2, y_3) + y_2 F_3'(y_1^2 + y_2^2, y_3),\\
F_3 &= F_3'''(y_1^2 + y_2^2, y_3).
\end{aligned}
$$

Corresponding respectively to the rotations about the axes of z_1 and z_2, there are two other sets of equations like (15) and obtainable from (15) by permuting the subscripts $1, 2, 3$ cyclicly. From the three sets of equations it follows that $F_1'' = F_2'' = F_3'' = 0$; for example, since $F_1 = F_1'''(y_2^2 + y_3^2, y_1)$, F_1 is an even function of y_2, whence, by the first equation of (15), $F_3'' = 0$. Therefore

$$F_i = y_i F_j'(y_i^2 + y_k^2, y_j) = y_i F_k'(y_i^2 + y_j^2, y_k),$$

where i, j, k are to be permuted cyclicly through the values $1, 2, 3$. Thus

$$F_1' = F_2' = F_3' = R(r^2), \qquad \text{where} \qquad r^2 = y_1^2 + y_2^2 + y_3^2,$$

and (13) becomes

$$X_i = x_i + y_i R(r^2), \qquad Y_i = x_i - y_i R(r^2) \qquad (i = 1, 2, 3).$$

Substitution shows that the functions F_i in this representation satisfy (14).

This result for $n = 3$ can be generalized by mathematical induction to hold for the case $n > 3$.

The representations $X \rightarrow Y$ of z invariant under the group of motions in n-dimensional space, where $n \geqq 3$, are

$$\text{IVC} \qquad X_i = x_i + y_i R(r^2), \qquad Y_i = x_i - y_i R(r^2) \quad (i = 1, 2, \cdots, n),$$

where

$$r^2 = y_1^2 + y_2^2 + \cdots + y_n^2,$$

and $R(r^2)$ is a real function analytic for all finite values and not zero except perhaps when $r^2 = 0$.

For the group of motions and reflections the result incorporated in IVC *holds for all values of n.* To establish the truth of this statement for $n \geqq 3$, it is sufficient to show that the representation IVC is invariant under a reflection in one of the coördinate hyperplanes. If $n = 2$, the representation must be of the form IVB and be invariant under $z_1' = z_1$, $z_2' = -z_2$; then $R_2(r^2) = 0$, and conversely. Finally, IVA is invariant under $z_1' = -z_1$ if and only if $\Phi(y_1)$ is an even function of y_1, i.e., is of the form $R(y_1^2)$.

7. **The group of direct transformations of similarity.** Necessary and sufficient that the representation $X \rightarrow Y$ be unchanged by the direct transformations of similarity in the cases $n = 1$, $n = 2$, $n \geqq 3$ is the requirement that IVA, IVB, IVC be respectively invariant under an arbitrary stretching from the origin of positive ratio. According to § 4, the arbitrary functions involved in the representations IV are, then, all constants, and conversely.

The representations $X \to Y$ of z invariant under the group of direct transformations of similarity are

$$\text{VA} \qquad X_1 = x_1 + ky_1, \qquad Y_1 = x_1 - ky_1, \qquad k \neq 0, \quad \text{if} \quad n = 1,$$

$$\text{VB} \qquad \begin{aligned} X_1 &= x_1 + ky_1 + ly_2, & Y_1 &= x_1 - ky_1 - ly_2, \\ X_2 &= x_2 - ly_1 + ky_2, & Y_2 &= x_2 + ly_1 - ky_2, \end{aligned} \qquad k^2 + l^2 \neq 0, \quad \text{if} \quad n = 2,$$

$$\text{VC} \qquad X_i = x_i + ky_i, \qquad Y_i = x_i - ky_i, \qquad k \neq 0, \quad i = 1, 2, \cdots, n, \quad \text{if} \quad n \geqq 3.$$

Except in the plane, these representations are identical with the representations III invariant under the affine group.

For the group of direct and indirect transformations of similarity the representations are for all n given by VC *and are identical with those peculiar to the affine group.*

The two inverse representations VB, for which $k = 0$ and $l = \pm 1$, namely

$$\text{VI} \qquad \begin{aligned} X_1 &= x_1 \pm y_2, & Y_1 &= x_1 \mp y_2, \\ X_2 &= x_2 \mp y_1, & Y_2 &= x_2 \pm y_1, \end{aligned}$$

are *the representations of Laguerre.* From their geometrical definition (§ 1), it is evident that these representations are invariant under the direct circular transformations. It can be shown that they are the only representations with this property.

8. **Representations inverted by a set of transformations.** Given a group, G, of direct transformations of a certain type and the set, S, of indirect transformations of the same type, to determine, among the representations invariant under G, those which, though not invariant under S, are merely inverted by the transformations of S.

It is geometrically evident that a Laguerre representation is inverted by an indirect circular transformation. Of the groups G other than the circular group, every one under which invariant representations exist is linear and contains the group of translations. Hence the representation must be of the form I. In order that it be inverted by a linear transformation (2), it is necessary and sufficient that the functions F_i satisfy the system of equations

$$(16) \qquad \begin{aligned} &F_i(-y'_1, -y'_2, \cdots, -y'_n) \\ &\qquad = T_i^0(F_1(y_1, y_2, \cdots, y_n), \cdots, F_n(y_1, y_2, \cdots, y_n)) \\ &\qquad\qquad (i = 1, 2, \cdots, n), \end{aligned}$$

In §§ 4–7 it has been shown that, when G is any one of the following groups of direct transformations: homothetic group, affine group, group of motions

$(n \geqq 3)$, similarity group $(n \neq 2)$, all the representations invariant under G are also invariant under S. It is true also that no one of the representations IVA, invariant under the group of motions for $n = 1$, is inverted by any transformation of S, as can readily be proved by applying (16) with $z_1' = -z_1$ as the transformation used; but in this case, not all the representations are invariant under S (§ 6).

If G is the group of motions (or of direct transformations of similarity) in the plane, the functions F_i to be used in (16) are as in IVB (or VB). It is sufficient to choose as the transformation a reflection in a coördinate axis. Then equations (16) reduce to $R_1 = 0$ (or $k = 0$).

Of the representations IVB *invariant under the group of motions in the plane, only those for which* $R_2 = 0$ *are invariant under reflections, whereas only those for which* $R_1 = 0$ *are inverted by every reflection.*

Of the representations VB *invariant under the direct transformations of similarity, only those for which* $l = 0$ *are invariant under the indirect transformations of similarity, whereas only those for which* $k = 0$ *are inverted by every such transformation.*

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