ON THE SECOND DERIVATIVES OF AN EXTREMAL-INTEGRAL WITH AN APPLICATION TO A PROBLEM WITH VARIABLE END POINTS

(SUPPLEMENTARY PAPER)

BY

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In a paper with the above title published in these Transactions* there appeared a statement that "$B(x)$ becomes infinite as x approaches x_1'". In a letter, dated October 18, 1921, Professor Hans Hahn called attention to the fact that the proof of this statement, as implied by the context, although not explicitly mentioned, was insufficient, because it depended upon the non-vanishing of the quadratic form $\sum_{ij} \beta_{ij}\, c_i c_j$, which had been proved to be definite by the aid of the fact that $Z(x) \neq 0$, and this condition does not hold at x_1'. It is the purpose of this note to supply a proof of the statement quoted above, upon which the validity of the concluding theorem of the earlier paper depends. I acknowledge my indebtedness to Professor Hahn for having pointed out the need for this supplementary proof.

Starting with the definition of $B(x)$ as given on page 435 and putting $\tau_{ij}(x) = \sum_k R_{ik}(x_1)\, Z^{(kj)}(x)$, we have $B(x) = \sum_{ij} \tau_{ij}\, c_i c_j / Z$. We shall show first that not every τ_{ij} can vanish at x_1' of the same order as $Z(x)$. For, suppose that $Z(x) = (x - x_1')^s Z_1(x)$, $Z_1(x_1') \neq 0$. If now *every* $\tau_{ij}(x)$ had the factor $(x - x_1')^s$, then, since $|R_{ik}(x_1)| \neq 0$† it would follow that *every* $Z^{(jk)}(x)$ had it also. From the definition of the functions $Z^{(jk)}(x)$, we derive the following system of equations:

$$
(1) \qquad
\begin{aligned}
Z^{(kj)}(x) &= \sum_l \zeta_{kl}(x)\, Z_{jl}(x), \\
\delta_{kj}\, Z(x) &= \sum_l \zeta_{n+k,l}(x)\, Z_{jl}(x),
\end{aligned}
$$

in which Z_{jl} designates the cofactor of $\zeta_{n+j,l}$ in the determinant Z, and δ_{kj} is the familiar Kronecker symbol. For every fixed j, we have here a system

* Vol. 17 (1916), p. 436.

† Loc. cit., p. 434, equation (22); also Bolza, *Vorlesungen über Variationsrechnung*, p. 609.

of $2n$ equations. If now every $Z^{(kj)}(x_1') = 0$, we can form from (1) a set of $\binom{2n}{n}$ systems of n linear homogeneous equations in $Z_{jl}(x_1')$, $l = 1, \ldots, n$, for every value of j from 1 to n. The determinants of these systems would include every nth order determinant formed from the matrix

$$(2) \qquad \left\| \begin{matrix} \zeta_{1,1}(x), \ldots, \zeta_{2n,1}(x) \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \zeta_{1,3n}(x), \ldots, \zeta_{2n,n}(x) \end{matrix} \right\|$$

for the value x_1'. It would therefore follow that either *every* $Z_{jl}(x_1') = 0$ or the matrix (2) has rank less than n. In the former case, equations (1) could be divided through by $x - x_1'$ and the argument repeated. This may be continued until we reach the conclusion that either every $Z_{jl}(x)$ has the factor $(x - x_1')^s$ or else the rank of the matrix is less than n. The former of these alternatives contradicts the hypothesis that $Z(x)$ has the factor $(x - x_1')^s$ but not the factor $(x - x_1')^{s+l}$. The latter alternative would carry with it the consequence that the $2n$-rowed determinant obtained by adjoining to the matrix (2) n rows consisting of the derivatives of the functions appearing in (2) would vanish at x_1'. But since this determinant is the determinant of a fundamental system of solutions of Jacobi's equations,* it does not vanish. We conclude that not every $\tau_{ij}(x)$ has the factor $(x - x_1')^s$.

Next we observe that the matrix T of the quadratic form $\sum_{ij} \tau_{ij} c_i c_j$ is equal to $|R_{ik}(x_1)| Z_0(x) Y(x)$, where $Z_0(x)$ is the determinant formed from the first n columns of (2) and $Y(x)$ is the adjoint of $Z(x)$. Moreover, a k-rowed minor of T is equal to a sum of k-rowed minors of Y, each multiplied by polynomials in $R_{ik}(x_1)$ and the elements of $Z_0(x)$. If now $Z(x_1')$ is of rank $n-1$, then $Y(x_1')$ is of rank 0 or 1 and consequently $T(x_1')$ is of rank 0 or 1.† If $T(x_1')$ were of rank 0, every $\tau_{ij}(x_1')$ would vanish, which would contradict the result obtained above. Hence $T(x_1')$ is of rank 1 and hence $\sum_{ij} \tau_{ij}(x_1') c_i c_j$ is different from 0, since moreover $\sum_i c_i^2 \neq 0$.‡ For the case that the rank of $Z(x_1')$ is $n - 1$ our statement has therefore been proved.

Suppose now that the rank of $Z(x_1')$ is $n - s\, (s > 1)$. The rank of every $Z_{jl}(x_1')$ is then at most $n - s$. In this case $Z(x) = (x - x_1')^s Z_1(x)$, $Z_{lj}(x) = (x - x_1')^{s-1} Z_{lj1}(x)$, where not *every* $Z_{lj1}(x_1')$ is equal to 0. We have then $B(x) = \sum_{ij} \tau_{ij1}(x) c_i c_j / Z_1(x)(x - x_1')$, where $\tau_{ij1}(x) = \sum_{kl} R_{ik}(x_1) \zeta_{kl}(x) Z_{jl1}(x)$ and $\tau_{ij}(x) = (x - x_1')^{s-1} \tau_{ij1}(x)$ and there-

* Loc. cit., p. 435.

† See Kowalewski, *Determinantentheorie,* p. 81.

‡ Loc. cit., p. 434.

fore not every $\tau_{ij1}(x_1') = 0$. Now we can set up a determinant of which $Z_{jl1}(x)$ are the cofactors. The elements of this determinant are given by the formula $z_{jl}(x) = \xi_{jl}(x)/|Z_{jl1}(x)|^{\frac{n-2}{n-1}}$, where ξ_{jl} is the cofactor of Z_{jl1} in $|Z_{jl1}|$. We find now that $|z_{jl}(x)| = Z_1(x)(x-x_1')^{\frac{n-s}{n-1}}$, so that $|z_{jl}(x_1')| = 0$. Consequently the adjoint of this determinant, viz. $|Z_{jl1}(x_1')|$, is of rank 0 or 1. Furthermore the matrix T_1 of the quadratic form $\sum_{lj}\tau_{ij1}c_i c_j$ is equal to $|R_{ik}(x_1)|\cdot Z_0(x)\cdot|Z_{jl1}(x)|$ and is therefore of rank 0 or 1 at x_1'. We proceed now exactly as in the case $s=1$, treated above, and conclude that $\sum_{ij}\tau_{ij1}(x_1')c_i c_j$ is different from 0. Hence we have also in this case the result that $\lim_{x\to x_1}B(x) = \infty$.

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