

# RULED SURFACES WITH GENERATORS IN ONE-TO-ONE CORRESPONDENCE\*

BY

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## 1. INTRODUCTION

The configuration composed of two ruled surfaces whose generators are in one-to-one correspondence occurs frequently in geometry. If we assume that corresponding generators are not coplanar, we find that we can base a projective theory of this configuration on a system of four ordinary linear first order differential equations in four dependent variables. If we choose the fundamental curves of reference on the two surfaces suitably, we are able to reduce this system of equations to a relatively simple canonical form.

The curves which are fundamental for our canonical form we have called intersector curves. They are in some respects similar to the curved asymptotics on a ruled surface. We also define other curves which are analogous to flecnode curves and have some of their properties.

As an application of our theory, we have employed it to investigate Green-reciprocal ruled surfaces. The method proves to be a fruitful one. We are able to generalize some well known theorems concerning ruled surfaces of congruences  $I$  and  $I'$  which are reciprocal in the sense of G. M. Green. And we discover a geometrical characterization of the directrix congruences of a surface in terms of simple concepts.

## 2. THE DIFFERENTIAL EQUATIONS

Let the four homogeneous coördinates  $y^{(1)}, \dots, y^{(4)}$  of an arbitrary point  $P_y$  on a curve  $C_y$  be given as analytic functions of a single independent variable  $x$ . And in like manner let the four coördinates of each of three other points  $P_z, P_\rho, P_\sigma$  be given as functions of the same variable  $x$ . Let us join  $P_y$  and  $P_z$  by a straight line  $l_{yz}$ ; let us also join  $P_\rho$  and  $P_\sigma$  by a line  $l_{\rho\sigma}$ . Then, as  $x$  varies, the locus of  $l_{yz}$  is a ruled surface  $R_{yz}$ , and the locus of  $l_{\rho\sigma}$  is a ruled surface  $R_{\rho\sigma}$ . It is these ruled surfaces that we wish to study.

We shall suppose that  $P_y, P_z, P_\rho, P_\sigma$  are not coplanar, so that corresponding generators  $l_{yz}$  and  $l_{\rho\sigma}$  do not intersect. Then the determinant

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$$\begin{vmatrix} y^{(1)} & y^{(2)} & y^{(3)} & y^{(4)} \\ z^{(1)} & z^{(2)} & z^{(3)} & z^{(4)} \\ \varrho^{(1)} & \varrho^{(2)} & \varrho^{(3)} & \varrho^{(4)} \\ \sigma^{(1)} & \sigma^{(2)} & \sigma^{(3)} & \sigma^{(4)} \end{vmatrix}$$

is different from zero, and it is possible to determine the coefficients in the system of differential equations

$$(1) \quad \begin{aligned} y' &= c_{11} y + c_{12} z + a_{11} \varrho + a_{12} \sigma, \\ z' &= c_{21} y + c_{22} z + a_{21} \varrho + a_{22} \sigma, \\ \varrho' &= b_{11} y + b_{12} z + d_{11} \varrho + d_{12} \sigma, \\ \sigma' &= b_{21} y + b_{22} z + d_{21} \varrho + d_{22} \sigma \end{aligned}$$

so that  $(y^{(i)}, z^{(i)}, \varrho^{(i)}, \sigma^{(i)})$  ( $i = 1, 2, 3, 4$ ) will be four sets of solutions. For example, we may substitute each of these four sets in turn in the first equation of system (1) and then solve the resulting four equations for the coefficients of the first of equations (1). Similarly we may determine the coefficients of each of the other three equations.

System (1) is not uniquely determined when  $R_{yz}$  and  $R_{\rho\sigma}$  are given. If we introduce new reference curves  $C_{\bar{y}}$  and  $C_{\bar{z}}$  on  $R_{yz}$  by the transformation

$$\begin{aligned} y &= a\bar{y} + b\bar{z}, \\ z &= c\bar{y} + d\bar{z}, \quad D \equiv ad - bc \neq 0, \end{aligned}$$

and new reference curves  $C_{\bar{\rho}}$  and  $C_{\bar{\sigma}}$  on  $R_{\rho\sigma}$  by the transformation

$$\begin{aligned} \varrho &= \alpha\bar{\varrho} + \beta\bar{\sigma}, \\ \sigma &= \gamma\bar{\varrho} + \delta\bar{\sigma}, \quad \Delta \equiv \alpha\delta - \beta\gamma \neq 0, \end{aligned}$$

we do not change the surfaces  $R_{yz}$  and  $R_{\rho\sigma}$ ; but system (1) goes over into another system of the same form whose coefficients, indicated by dashes, are given by the following equations:

$$D\bar{c}_{11} = d(-a' + c_{11}a + c_{12}c) + b(c' - c_{21}a - c_{22}c),$$

$$D\bar{c}_{12} = d(-b' + c_{11}b + c_{12}d) + b(d' - c_{21}b - c_{22}d),$$

$$D\bar{a}_{11} = d(a_{11}\alpha + a_{12}\gamma) - b(a_{21}\alpha + a_{22}\gamma),$$

$$D\bar{a}_{12} = d(a_{11}\beta + a_{12}\delta) - b(a_{21}\beta + a_{22}\delta),$$

$$D\bar{c}_{21} = -c(-a' + c_{11}a + c_{12}c) - a(c' - c_{21}a - c_{22}c),$$

$$D\bar{c}_{22} = -c(-b' + c_{11}b + c_{12}d) - a(d' - c_{21}b - c_{22}d),$$

$$D\bar{a}_{21} = -c(a_{11}\alpha + a_{12}\gamma) + a(a_{21}\alpha + a_{22}\gamma),$$

$$D\bar{a}_{22} = -c(a_{11}\beta + a_{12}\delta) + a(a_{21}\beta + a_{22}\delta),$$

$$\Delta\bar{b}_{11} = \delta(b_{11}a + b_{12}c) - \beta(b_{21}a + b_{22}c),$$

$$\Delta\bar{b}_{12} = \delta(b_{11}b + b_{12}d) - \beta(b_{21}b + b_{22}d),$$

$$\Delta\bar{d}_{11} = \delta(-\alpha' + d_{11}\alpha + d_{12}\gamma) + \beta(\gamma' - d_{21}\alpha - d_{22}\gamma),$$

$$\Delta\bar{d}_{12} = \delta(-\beta' + d_{11}\beta + d_{12}\delta) + \beta(\delta' - d_{21}\beta - d_{22}\delta),$$

$$\Delta\bar{b}_{21} = -\gamma(b_{11}a + b_{12}c) + \alpha(b_{21}a + b_{22}c),$$

$$\Delta\bar{b}_{22} = -\gamma(b_{11}b + b_{12}d) + \alpha(b_{21}b + b_{22}d),$$

$$\Delta\bar{d}_{21} = -\gamma(-\alpha' + d_{11}\alpha + d_{12}\gamma) - \alpha(\gamma' - d_{21}\alpha - d_{22}\gamma),$$

$$\Delta\bar{d}_{22} = -\gamma(-\beta' + d_{11}\beta + d_{12}\delta) - \alpha(\delta' - d_{21}\beta - d_{22}\delta).$$

By a suitable choice of reference curves, the geometrical characterization of which will be furnished later, we are able to reduce system (1) to a simple canonical form. To this end we observe that if we choose  $a$  and  $c$  as a pair of solutions of the simultaneous differential equations

$$a' = c_{11}a + c_{12}c, \quad c' = c_{21}a + c_{22}c,$$

we shall then have  $\bar{c}_{11} = \bar{c}_{21} = 0$ . Likewise we can make  $\bar{c}_{12} = \bar{c}_{22} = 0$  by choosing  $b$  and  $d$  as solutions of

$$b' = c_{11}b + c_{12}d, \quad d' = c_{21}b + c_{22}d.$$

We can make  $\bar{d}_{11} = \bar{d}_{21}$  by choosing  $\alpha$  and  $\gamma$  as solutions of

$$\alpha' = d_{11} \alpha + d_{12} \gamma, \quad \gamma' = d_{21} \alpha + d_{22} \gamma,$$

and we can make  $\bar{d}_{12} = \bar{d}_{22} = 0$  by choosing  $\beta$  and  $\delta$  as solutions of

$$\beta' = d_{11} \beta + d_{12} \delta, \quad \delta' = d_{21} \beta + d_{22} \delta.$$

When these reductions have been made, system (1) takes the canonical form

$$(2) \quad \begin{aligned} y' &= a_{11} \varrho + a_{12} \sigma, & \varrho' &= b_{11} y + b_{12} z, \\ z' &= a_{21} \varrho + a_{22} \sigma, & \sigma' &= b_{21} y + b_{22} z. \end{aligned}$$

Wilczynski has developed the theory of a single ruled surface, for which his fundamental system of equations has the form\*

$$(A) \quad \begin{aligned} y'' + p_{11} y' + p_{12} z' + q_{11} y + q_{12} z &= 0, \\ z'' + p_{21} y' + p_{22} z' + q_{21} y + q_{22} z &= 0. \end{aligned}$$

In order to obtain system (A) from system (1) it is sufficient to differentiate the first two equations of system (1) and then eliminate  $\varrho$ ,  $\sigma$ ,  $\varrho'$ ,  $\sigma'$ . We prefer however to start from our canonical system (2). We find then that the coefficients of system (A), for our surface  $R_{yz}$ , are given by the following formulas in terms of the coefficients of system (2):

$$(3) \quad \begin{aligned} p_{11} &= (a_{21} a'_{12} - a_{22} a'_{11}) / (a_{11} a_{22} - a_{21} a_{12}), & q_{11} &= -(a_{11} b_{11} + a_{12} b_{21}), \\ p_{12} &= (a_{12} a'_{11} - a_{11} a'_{12}) / (a_{11} a_{22} - a_{21} a_{12}), & q_{12} &= -(a_{12} b_{12} + a_{13} b_{22}), \\ p_{21} &= (a_{21} a'_{22} - a_{22} a'_{21}) / (a_{11} a_{22} - a_{21} a_{12}), & q_{21} &= -(a_{21} b_{11} + a_{22} b_{21}), \\ p_{22} &= (a_{12} a'_{21} - a_{11} a'_{22}) / (a_{11} a_{22} - a_{21} a_{12}), & q_{22} &= -(a_{21} b_{12} + a_{22} b_{22}). \end{aligned}$$

We are thus enabled to avail ourselves of the classical theory of a single ruled surface.

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\* Wilczynski, *Projective Differential Geometry*, Leipzig, 1906, p. 126.

## 3. GREEN-RECIPROCAL RULED SURFACES

G. M. Green's reciprocal relation  $R$  plays a significant part in projective differential geometry. This relation may be briefly formulated as follows.\* Consider an arbitrary non-developable surface  $S$ . At an arbitrary point on this surface there is a tangent plane and an osculating quadric. A line  $l$  which lies in the tangent plane but does not pass through the point of contact, and a line  $l'$ , which passes through the point of contact but does not lie in the tangent plane, are said to be reciprocal to each other in case they are reciprocal polars with respect to the osculating quadric. The totality of lines  $l$  forms a congruence, called a  $\Gamma$  congruence, and the reciprocal lines  $l'$  form the reciprocal  $\Gamma'$  congruence.

Let us draw a curve  $C$  on our fundamental surface  $S$ . Corresponding to every point of this curve we have a pair of reciprocal lines  $l$  and  $l'$ . All these lines  $l'$  form a ruled surface, which intersects  $S$  in  $C$ , and the reciprocal lines  $l$  also form a ruled surface. Such ruled surfaces we shall call Green-reciprocal ruled surfaces.

The generators of Green-reciprocal ruled surfaces are in a one-to-one correspondence and are skew to each other. It will therefore be possible to apply our general theory of pairs of ruled surfaces to Green-reciprocal ruled surfaces. For this purpose we shall need to formulate the relation  $R$  analytically.

Let the four homogeneous coordinates  $y^{(1)}, \dots, y^{(4)}$  of an arbitrary point  $P_y$  on a surface  $S_y$  be given as analytic functions of two independent variables  $u$  and  $v$ . Let us suppose that  $S_y$  is non-degenerate and non-developable, and is referred to its asymptotic net. Then the four functions  $y$  are a fundamental system of solutions of a completely integrable system of partial differential equations which may be reduced to the form†

$$(4) \quad y_{uu} + 2by_v + fy = 0, \quad y_{vv} + 2a'y_u + gy = 0.$$

The points  $P_\rho$  and  $P_\sigma$  defined by

$$(5) \quad \rho = y_u - \beta y, \quad \sigma = y_v - \alpha y,$$

where  $\alpha$  and  $\beta$  are functions of  $u$  and  $v$ , lie in the tangent plane of  $S_y$  at  $P_y$ . More precisely,  $P_\rho$  lies on the line tangent at  $P_y$  to the asymptotic curve

\* Green, Proceedings of the National Academy of Sciences, vol. 3 (1917) pp. 587-592.

† Wilczynski, *Projective differential geometry of curved surfaces*, First Memoir, these Transactions, vol. 8 (1907), p. 246.

$v = \text{const.}$  through  $P_y$ , and  $P_\sigma$  lies on the line tangent to the asymptotic  $u = \text{const.}$  at  $P_y$ . The line  $l_{\rho\sigma}$  which joins  $P_\rho$  and  $P_\sigma$  lies in the tangent plane of  $S_y$  at  $P_y$ , and does not pass through  $P_y$ . The line  $l_{yz}$  reciprocal to  $l_{\rho\sigma}$  joins  $P_y$  to the point  $P_z$  defined by\*

$$(6) \quad z = y_{uv} - \alpha y_u - \beta y_v.$$

An arbitrary curve  $C_y$  on  $S_y$  (except the curves  $u = \text{const.}$ ) may be defined by expressing  $v$  as a function of  $u$ , in the form  $v = v(u)$ . As  $u$  varies,  $P_y$  moves along the curve  $C_y$ , and the lines  $l_{yz}$  and  $l_{\rho\sigma}$  generate two Green-reciprocal ruled surfaces,  $R_{yz}$  and  $R_{\rho\sigma}$ .

We are now ready to set up the system (1) for Green-reciprocal ruled surfaces. We first calculate the partial derivatives of  $z$ ,  $\rho$  and  $\sigma$  with respect to  $u$ , and with respect to  $v$ , obtaining

$$(7) \quad \begin{aligned} z_u &= Py - \beta z + A\rho - F'\sigma, \\ z_v &= Qy - \alpha z - G'\rho + B\sigma, \end{aligned}$$

where we have placed

$$(8) \quad \begin{aligned} A &= 4a'b - \alpha_u - \alpha\beta, & F' &= f + \beta^2 + \beta_u - 2b\alpha + 2b_v, \\ B &= 4a'b - \beta_v - \alpha\beta, & G' &= g + \alpha^2 + \alpha_v - 2a'\beta + 2a_u, \\ P &= A\beta - F'\alpha + 2bg - f_v + f\alpha, \\ Q &= B\alpha - G'\beta + 2a'f - g_u + g\beta; \end{aligned}$$

and obtaining

$$(9) \quad \begin{aligned} \rho_u &= -Fy - \beta\rho - 2b\sigma, & \rho_v &= (\alpha\beta - \beta_v)y + z + \alpha\rho, \\ \sigma_u &= -Gy - 2a'\rho - \alpha\sigma, & \sigma_v &= (\alpha\beta - \alpha_u)y + z + \beta\sigma, \end{aligned}$$

where we have placed

$$(10) \quad F = f + \beta^2 + \beta_u + 2b\alpha, \quad G = g + \alpha^2 + \alpha_v + 2a'\beta.$$

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\* Green, *Memoir on the general theory of surfaces and rectilinear congruences*, these Transactions, vol. 20 (1919), p. 87.

We next calculate the total derivatives of  $y$ ,  $z$ ,  $\varrho$  and  $\sigma$  with respect to  $u$ , using the formula  $y' = y_u + v'y_v$ , where  $v'$  is obtained by differentiating the equation  $v = v(u)$  defining  $R_{yz}$  and  $R_{\rho\sigma}$ . We obtain in this way a system of equations of the form (1), whose coefficients have the following values:

$$\begin{aligned}
 c_{11} &= \beta + v'\alpha, & c_{12} &= 0, & a_{11} &= 1, & a_{12} &= v', \\
 c_{21} &= P + v'Q, & c_{22} &= -\beta - v'\alpha, & a_{21} &= A - v'G, & a_{22} &= -F' + v'B, \\
 (11) & & & & & & & \\
 d_{11} &= -\beta + v'\alpha, & d_{12} &= -2b, & b_{11} &= -F + v'(\alpha\beta - \beta v), & b_{12} &= v', \\
 d_{21} &= -2v'a', & d_{22} &= \beta - v'\alpha, & b_{21} &= \alpha\beta - \alpha_u - v'G, & b_{22} &= 1.
 \end{aligned}$$

We shall make use of this system of equations whenever we wish to apply our general theory to Green-reciprocal ruled surfaces.

#### 4. INTERSECTOR CURVES

A curve on  $R_{yz}$  will be called an *intersector curve* (with respect to  $R_{\rho\sigma}$ ) in case the tangent at each point of the curve intersects the line  $l_{\rho\sigma}$  which corresponds to the generator  $l_{yz}$  that passes through the point. A similar definition may be made for intersector curves on  $R_{\rho\sigma}$ .

We shall now determine the intersector curves on  $R_{yz}$ . Any point  $P_\varphi$  on a generator  $l_{yz}$  (except  $P_z$ ) may be defined by setting  $\varphi = y + \lambda z$ , where  $\lambda$  is a function of  $x$ . As  $x$  varies, the locus of  $P_\varphi$  is a curve  $C_\varphi$  on  $R_{yz}$ . We wish to determine  $\lambda$  so that  $C_\varphi$  will be an intersector curve. The point  $\varphi'$ , where  $\varphi' = y' + \lambda z' + \lambda'z$ , is a point on the tangent of  $C_\varphi$ . Substituting from the first two equations of system (1), we find

$$(12) \quad \varphi' = (c_{11} + \lambda c_{21})y + (\lambda' + c_{12} + \lambda c_{22})z + (a_{11} + \lambda a_{21})\varrho + (a_{12} + \lambda a_{22})\sigma.$$

Then an arbitrary point on the tangent of  $C_\varphi$  is given by an expression of the form  $\varphi' + k\varphi$ . If this tangent intersects  $l_{\rho\sigma}$ , there must exist a value of  $k$  so that the expression  $\varphi' + k\varphi$  is a linear combination of  $\varrho$  and  $\sigma$  only. Therefore  $k$  and  $\lambda$  must satisfy the two conditions

$$c_{11} + \lambda c_{21} + k = 0, \quad \lambda' + c_{12} + \lambda c_{22} + k\lambda = 0.$$

Eliminating  $k$ , we obtain the differential equation of the intersector curves on  $R_{yz}$ ,

$$(13) \quad \lambda' = -c_{12} + (c_{11} - c_{22})\lambda + c_{21}\lambda^2.$$

The corresponding equation for the intersector curves on  $R_{\rho\sigma}$  may be deduced in the same way. If we denote an arbitrary point on  $l_{\rho\sigma}$  by  $P_\psi$ , where  $\psi = \varrho + \mu\sigma$ , we find

$$(14) \quad \psi' = (b_{11} + \mu b_{21})y + (b_{12} + \mu b_{22})z + (d_{11} + \mu d_{21})\varrho + (\mu' + d_{12} + \mu d_{22})\sigma.$$

And the differential equation for the intersector curves on  $R_{\rho\sigma}$  is

$$(15) \quad \mu' = -d_{12} + (d_{11} - d_{22})\mu + d_{21}\mu^2.$$

We observe that if our fundamental equations are written in the canonical form (2), then the curves  $\lambda = \text{const.}$  are the intersector curves on  $R_{yz}$ , and the curves  $\mu = \text{const.}$  are the intersector curves on  $R_{\rho\sigma}$ . In fact, inspection of system (1) will show that  $C_y$  is an intersector curve if  $c_{12} = 0$ , and  $C_z$  is an intersector curve if  $c_{21} = 0$ . Similarly,  $C_\rho$  is an intersector curve if  $d_{12} = 0$ , and  $C_\sigma$  is an intersector curve if  $d_{21} = 0$ .

We remark further that equation (13) is a Riccati equation. Therefore *there is a one-parameter family of intersector curves on  $R_{yz}$ , and they are determined by solving a Riccati equation.* Conversely, we can show that, if we have given a ruled surface  $R_{yz}$  and a Riccati equation

$$\lambda' = P + Q\lambda + R\lambda^2,$$

then there exists a ruled surface  $R_{\rho\sigma}$  with respect to which the one-parameter family of curves defined by the given equation are intersector curves. Comparing the given equation with equation (13), let us choose  $c_{11} = 0$ , and set

$$c_{12} = -P, \quad c_{22} = -Q, \quad c_{21} = R.$$

If we introduce these values into the first two equations of system (1), and solve these two equations for  $\varrho$  and  $\sigma$ , we obtain

$$\varrho = Pz + y', \quad \sigma = -Ry + (P + Q)z + y' + z'.$$

These formulas determine the required ruled surface  $R_{\rho\sigma}$ . Therefore we have shown that *the theory of a Riccati equation is equivalent to the theory of the intersector curves on a ruled surface with respect to some associated ruled surface.* It should be noted that our determination of  $R_{\rho\sigma}$  is not unique, since only the difference  $c_{11} - c_{22}$  is defined.

Since the cross ratio of any four particular solutions of a Riccati equation is constant, we see that, *if we select any four intersector curves on  $R_{yz}$ , they will cut each generator  $l_{yz}$  in a set of four points having the same cross ratio on all generators.*

The curved asymptotics on a ruled surface have long been known\* to possess the cross ratio property which we have just established for intersector curves in general. We shall find a Riccati equation defining the curved asymptotics on  $R_{yz}$ , using our canonical system (2). Let  $P_\varphi$ , where  $\varphi = y + \lambda z$ , be an arbitrary point on  $l_{yz}$ . Then differentiating twice and making use of system (2), we find

$$\begin{aligned} \varphi' &= \lambda' z + (a_{11} + \lambda a_{21})\rho + (a_{12} + \lambda a_{22})\sigma, \\ (16) \quad \varphi'' &= (\ )y + (\ )z + [(a_{11} + \lambda a_{21})' + \lambda' a_{21}]\rho + [(a_{12} + \lambda a_{22})' + \lambda' a_{22}]\sigma, \end{aligned}$$

the omitted coefficients being immaterial. The locus  $C_\varphi$  of  $P_\varphi$  will be an asymptotic curve on  $R_{yz}$  if the osculating plane of  $C_\varphi$  at every point coincides with the tangent plane of  $R_{yz}$  at the point. Then the four points  $y, \varphi, \varphi', \varphi''$  will be coplanar. The condition for their coplanarity reduces to

$$\begin{aligned} 2(a_{12} a_{21} - a_{11} a_{22}) \lambda' &= a_{11} a'_{12} - a_{12} a'_{11} \\ (17) \quad &+ (a_{11} a'_{22} - a_{22} a'_{11} + a_{21} a'_{12} - a_{12} a'_{21}) \lambda + (a_{21} a'_{22} - a_{22} a'_{21}) \lambda^2. \end{aligned}$$

This equation defines the curved asymptotics on  $R_{yz}$ .

A second property of the asymptotics may also be extended to intersector curves. Since an asymptotic tangent intersects three consecutive generators on a ruled surface, it follows that the asymptotic tangents of  $R_{yz}$  constructed at points of a fixed generator  $l_{yz}$  form a quadric. This quadric is called the osculating quadric of  $R_{yz}$  at  $l_{yz}$ . Now a tangent of an intersector curve on  $R_{yz}$  intersects two consecutive generators at  $l_{yz}$  and also intersects  $l_{\rho\sigma}$ . Therefore the intersector tangents of  $R_{yz}$  constructed at points of a fixed generator

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\* P. Serret, *Théorie Nouvelle Géométrie et Mécanique des Lignes à Double Courbure*, Paris, 1860.

$l_{yz}$  also form a quadric, which we shall call the *intersector quadric* of  $R_{yz}$  at  $l_{yz}$ . In order to find the equation of this quadric, we observe that an arbitrary point on the tangent of an arbitrary intersector curve is given by  $\varphi' + k\varphi$ , where  $\varphi'$  has the value written in equation (12), and  $\lambda$  satisfies equation (13). If we introduce a local tetrahedron of reference with vertices at the points  $y, z, \rho, \sigma$ , and with suitably chosen unit point, we may write the local coördinates of the point  $\varphi' + k\varphi$  in the form

$$\begin{aligned}x_1 &= c_{11} + \lambda c_{21} + k, & x_3 &= a_{11} + \lambda a_{21}, \\x_2 &= (c_{11} + \lambda c_{21} + k)\lambda, & x_4 &= a_{12} + \lambda a_{22}.\end{aligned}$$

Eliminating  $k$  and  $\lambda$ , and making the result homogeneous in the usual way, we obtain the equation of the intersector quadric of  $R_{yz}$  in the form

$$a_{12} x_1 x_3 - a_{11} x_1 x_4 + a_{22} x_2 x_3 - a_{21} x_2 x_4 = 0.$$

The relation of asymptotics to intersector curves is made clear by the following considerations. Suppose that  $l_{\rho\sigma}$  happens to be a generator of the same set as  $l_{yz}$  on the osculating quadric of  $R_{yz}$ . Then  $R_{\rho\sigma}$  is called a derivative ruled surface\* of  $R_{yz}$ , and the intersector curves are the asymptotics. *The asymptotics on  $R_{yz}$  are intersector curves with respect to an arbitrary derivative ruled surface of  $R_{yz}$ .*

## 5. ANALOGUES OF FLECNODE CURVES

When four skew straight lines are given, there exist two other straight lines each of which intersects all four given lines. Indeed, the flecnodes on a generator of a ruled surface have been defined† as the two points at each of which a line may be drawn intersecting four consecutive generators. We wish to extend this notion of flecnode in two directions.

In the first place, we recall that an asymptotic tangent of  $R_{yz}$  intersects three consecutive generators  $l_{yz}$ . Therefore there are two points on each generator  $l_{yz}$  which are characterized by the fact that the asymptotic tangent at each of them intersects the corresponding line  $l_{\rho\sigma}$ . If  $l_{\rho\sigma}$  happens to be also a generator of  $R_{yz}$ , and if we let  $l_{\rho\sigma}$  approach  $l_{yz}$  over the surface  $R_{yz}$ , then our two points approach the flecnodes of  $l_{yz}$  as limiting positions.

\* Wilczynski, *Projective Differential Geometry*, p. 147.

† Wilczynski, *Projective Differential Geometry*, p. 149.

We may determine our two points as follows. Let the fundamental equations be written in the canonical form (2), and let any point  $P_\varphi$  on  $l_{yz}$  be defined by  $\varphi = y + \lambda z$ . If the asymptotic tangent at  $P_\varphi$  coincides with the intersector tangent, then equation (17) of the asymptotics and the equation  $\lambda' = 0$  of the intersector curves may be regarded as simultaneous. Therefore we determine the two points at which the asymptotic tangents are also intersector tangents by solving the quadratic

$$(18) \quad a_{11} a'_{12} + a_{12} a'_{11} + (a_{11} a'_{22} - a_{22} a'_{11} + a_{21} a'_{12} - a_{12} a'_{21}) \lambda \\ + (a_{21} a'_{22} - a_{22} a'_{21}) \lambda^2 = 0.$$

The locus of each of these points is a curve analogous to a flecnode curve. At each of its points the asymptotic and intersector tangents coincide. This analogue of the flecnode curve is indeterminate in case the asymptotic and intersector curves coincide, that is, in case  $R_{\rho\sigma}$  is a derivative of  $R_{yz}$ .

We arrive at our second analogue of a flecnode as follows. Since any tangent of  $R_{yz}$  intersects two consecutive generators  $l_{yz}$ , and since any tangent of  $R_{\rho\sigma}$  intersects two consecutive generators  $l_{\rho\sigma}$ , it follows that there are two lines which are tangent to both  $R_{yz}$  and  $R_{\rho\sigma}$ , the points of contact of each line being on corresponding generators  $l_{yz}$  and  $l_{\rho\sigma}$ .

In order to determine the common tangent lines of  $R_{yz}$  and  $R_{\rho\sigma}$ , we first determine the tangent plane of each surface. The tangent plane of  $R_{yz}$  at a point  $P_\varphi$ , where  $\varphi = y + \lambda z$ , is determined by  $P_y$ ,  $P_z$  and  $P_{\varphi'}$ , where  $\varphi'$  is given by equation (12). Therefore this plane is determined by  $l_{yz}$  and the point

$$(19) \quad (a_{11} + \lambda a_{21}) \rho + (a_{12} + \lambda a_{22}) \sigma,$$

where the plane intersects the generator  $l_{\rho\sigma}$  corresponding to  $l_{yz}$ . Similarly, the tangent plane of  $R_{\rho\sigma}$  at a point  $P_\psi$ , where  $\psi = \rho + \mu \sigma$ , is determined by  $P_\rho$ ,  $P_\sigma$ , and  $P_{\psi'}$ , where  $\psi'$  is given by equation (14). Therefore this plane is determined by  $l_{\rho\sigma}$  and the point

$$(20) \quad (b_{11} + \mu b_{21}) y + (b_{12} + \mu b_{22}) z,$$

where the plane intersects the generator  $l_{yz}$  corresponding to  $l_{\rho\sigma}$ .

Now the line of intersection of the tangent planes of  $R_{yz}$  and  $R_{\rho\sigma}$  joins the points (19) and (20). This line is tangent to  $R_{yz}$  if the point (20) coincides

with  $P_\varphi$ , that is, if  $\lambda$  and  $\mu$  satisfy the condition

$$(21) \quad \lambda = (b_{12} + \mu b_{22}) / (b_{11} + \mu b_{21}).$$

Similarly, this line is tangent to  $R_{\rho\sigma}$  if the point (19) coincides with  $P_\psi$ , that is, if  $\lambda$  and  $\mu$  satisfy the condition

$$(22) \quad \mu = (a_{12} + \lambda a_{22}) / (a_{11} + \lambda a_{21}).$$

If we regard equations (21) and (22) as simultaneous and eliminate  $\mu$ , we obtain

$$(23) \quad (a_{21} b_{11} + a_{22} b_{21}) \lambda^2 + (a_{11} b_{11} + a_{12} b_{21} - a_{22} b_{22} - a_{21} b_{12}) \lambda \\ - (a_{11} b_{12} + a_{12} b_{22}) = 0.$$

Solution of this quadratic determines the two points on  $l_{yz}$  at which lines can be drawn tangent to both of  $R_{yz}$  and  $R_{\rho\sigma}$ . The points of contact of these common tangent lines on  $R_{\rho\sigma}$  are determined by eliminating  $\lambda$  from equations (21) and (22) and solving the quadratic

$$(24) \quad (a_{11} b_{21} + a_{21} b_{22}) \mu^2 + (a_{11} b_{11} + a_{21} b_{12} - a_{12} b_{21} - a_{22} b_{22}) \mu \\ - (a_{12} b_{11} + a_{22} b_{12}) = 0.$$

## 6. APPLICATIONS TO GREEN-RECIPROCAL RULED SURFACES

We shall now consider a pair of Green-reciprocal ruled surfaces and apply our general theory to them. For this purpose we employ equations (11). Since  $c_{12} = 0$  in equations (11), it follows that  $C_y$  is an intersector curve on  $R_{yz}$ . Moreover, this fact is geometrically obvious, since the line tangent to  $C_y$  lies in the tangent plane of  $S_y$  and therefore intersects  $l_{\rho\sigma}$ . The equation (13) of the intersector curves on  $R_{yz}$  becomes

$$(25) \quad \lambda' = 2(\beta + v'\alpha)\lambda + (P + v'Q)\lambda^2.$$

Since a particular solution of this equation, namely  $\lambda = 0$ , is known, the intersector curves on  $R_{yz}$  can be determined by quadratures. In fact, if we

place  $\lambda = 1/\nu$ , equation (25) reduces to the linear equation

$$(26) \quad \nu' = -(P + \nu'Q) - 2(\beta + \nu'\alpha)\nu.$$

We might expect  $C_y$  to be also an asymptotic on  $R_{yz}$ . But if  $C_y$  is an asymptotic on  $R_{yz}$ , then the osculating plane of  $C_y$  is tangent to  $R_{yz}$  at  $P_y$  and therefore contains the generator  $l_{yz}$  through  $P_y$ . Then, using the language of Miss Sperry, we may say that *the curve  $C_y$  is an asymptotic on  $R_{yz}$  if, and only if,  $C_y$  is a union curve\* of the congruence  $I'$* . This theorem is a generalization of one of Green's theorems.† He considered an arbitrary conjugate net on  $S_y$  and the axis congruence of this net. Then his theorem asserts that the ruled surface of axes which intersects  $S_y$  in a curve of the fundamental conjugate net has this curve for an asymptotic. To derive Green's theorem from ours, we have only to note that a curve of the fundamental conjugate net is a union curve of the axis congruence of the net.

If we select any particular point on  $l_{yz}$ , the plane of this point and  $l_{\rho\sigma}$  will touch  $R_{\rho\sigma}$  in a definite point. Let us consider the point  $P_y$ . The plane of  $P_y$  and  $l_{\rho\sigma}$ , which is the tangent plane of  $S_y$  at  $P_y$ , touches  $R_{\rho\sigma}$  in a point which is of particular interest. Let this point be  $P_\psi$ , where  $\psi = \varrho + \mu\sigma$ . Substituting from equations (11) into equations (20), we find that the tangent plane of  $R_{\rho\sigma}$  at  $P_\psi$  intersects  $l_{yz}$  in the point

$$(27) \quad [-F + \nu'(\alpha\beta - \beta\nu) + \mu(\alpha\beta - \alpha\nu - \nu'G)]y + (\nu' + \mu)z.$$

This point coincides with  $P_y$  if  $\mu = -\nu'$ . Therefore the tangent plane of  $S_y$  at  $P_y$ , touches  $R_{\rho\sigma}$  at the point  $\varrho - \nu'\sigma$ . But if we recall that the surface  $S_y$  is referred to its asymptotic net, we see that the point  $\varrho - \nu'\sigma$  lies on the conjugate of the tangent to the curve  $C_y$  that corresponds to  $R_{\rho\sigma}$ . Therefore *the point of contact of the tangent plane of  $S_y$  with an arbitrary ruled surface  $R_{\rho\sigma}$  lies on the conjugate of the tangent to the curve on  $S_y$  that corresponds to  $R_{\rho\sigma}$* .

This theorem is a generalization of a theorem which has had an interesting history. It was first proved by Wilczynski for the directrix congruences, and by Sullivan for the scroll directrix congruences. Green proved it for an arbitrary

\* P. Sperry, *Properties of a certain projectively defined two-parameter family of curves on a general surface*, American Journal of Mathematics, vol. 40 (1918), pp. 213-224.

† G. M. Green, Second Memoir, American Journal of Mathematics, vol. 38 (1916), p. 303.

trary pair of reciprocal congruences. His theorem\* amounts to this, that, for a ruled surface  $R_{\rho\sigma}$  which is developable, the line  $l_{\rho\sigma}$  touches the edge of regression of the developable in a point which lies on the conjugate of the tangent to the curve on  $S_y$  which corresponds to the developable. Our generalization is suggested by the fact that the point where  $l_{\rho\sigma}$  touches the edge of regression of  $R_{\rho\sigma}$ , namely the focal point of  $l_{\rho\sigma}$ , is also the point where the tangent plane of  $S_y$  touches  $R_{\rho\sigma}$ . In fact, any plane containing  $l_{\rho\sigma}$  will touch  $R_{\rho\sigma}$  at the focal point of  $l_{\rho\sigma}$ .

When  $R_{\rho\sigma}$  is developable the point (27) is indeterminate, and we have

$$(28) \quad \begin{aligned} v' + \mu &= 0, \\ -F + v'(\alpha\beta - \beta_v) + \mu(\alpha\beta - \alpha_u - v'G) &= 0. \end{aligned}$$

If we eliminate  $\mu$  we obtain Green's equation for determining the developables of the  $\Gamma$  congruence. And if we eliminate  $v'$ , we obtain Green's equation for determining the focal sheets of the  $\Gamma$  congruence.†

The locus of the point  $P_\psi$ , where  $\psi = \rho - v'\sigma$ , is a curve on  $R_{\rho\sigma}$ . We shall call this curve the *contact curve* on  $R_{\rho\sigma}$ , because at each of its points the tangent plane of  $S_y$  touches  $R_{\rho\sigma}$ . Let us enquire under what condition the contact curve  $C_\psi$  is also an intersector curve. Since  $C_\psi$  is an intersector curve its tangent at  $P_\psi$  must intersect  $l_{yz}$ , but, since  $C_\psi$  is the contact curve, its tangent must lie in the tangent plane of  $S_y$  at  $P_y$ . Therefore the tangent passes through  $P_y$  and is the conjugate of the tangent of the curve  $C_y$  which corresponds to  $R_{\rho\sigma}$ . The locus of all these conjugate tangents is a developable, called the conjugate developable of  $C_y$ , which, in our case, has  $C_\psi$  for its edge of regression. So at every point  $P_y$  the conjugate of the tangent of  $C_y$  touches the edge of regression of the conjugate developable of  $C_y$  at a point on  $l_{\rho\sigma}$ . Such a curve  $C_y$  has been called an *adjoint union curve*‡ of the  $\Gamma$  congruence. Therefore, *if the contact curve on  $R_{\rho\sigma}$  is also an intersector curve, then the curve  $C_y$  which corresponds to  $R_{\rho\sigma}$  is an adjoint union curve of the  $\Gamma$  congruence.* If we wish to prove this theorem analytically, we may substitute from equation (11) into equation (15) and replace therein  $\mu$  by  $-v'$ . The result is the well known differential equation of the second order for the adjoint union curves on  $S_y$ .

\* Green, *Memoir on the general theory of surfaces*, these Transactions, vol. 20 (1919), p. 94.

† Green, loc. cit., p. 89 and p. 91.

‡ Green, loc. cit., p. 140.

Let us again employ the geometrical argument to demonstrate still another theorem concerning the contact curve on  $R_{\rho\sigma}$ . The tangent planes of  $S_y$ , constructed at points of the curve  $C_y$ , are also the tangent planes of  $R_{\rho\sigma}$  constructed at points of the contact curve. These planes envelope the conjugate developable of  $C_y$ , and, if the contact curve is also an intersector curve, the conjugate developable is the developable of the tangents of the contact curve. Since the planes enveloping a developable are also the osculating planes of the edge of regression of the developable, it follows in our case that at each point of the contact curve the osculating plane of the curve is the tangent plane of  $R_{\rho\sigma}$ . Therefore, *if the contact curve is an intersector curve, then it is an asymptotic*. This theorem may also be proved analytically by direct calculation.

### 7. THE DIRECTRIX CONGRUENCES

One of the most important examples of pairs of congruences reciprocal with respect to a surface is the pair of directrix congruences\* of the surface. Wilczynski defined the directrix congruences in terms of the osculating linear complexes of the asymptotic curves on the surface. We are able to give another geometrical characterization of them. Let us substitute from equations (11) into equation (24). We obtain in this way the equation

$$(29) \quad [4a'b - 2\alpha_u - v'(G + G')] \mu^2 + [F' - F - v'^2(G' - G)] \mu + v'[F + F' - v'(4a'b - 2\beta_v)] = 0,$$

which determines the two points  $l_{\rho\sigma}$  at which tangents to  $R_{\rho\sigma}$  can be drawn which also touch  $R_{yz}$  at points on the corresponding generator  $l_{yz}$ . Now these two points separate  $P_\rho$  and  $P_\sigma$  harmonically if, and only if,

$$F' - F - v'^2(G' - G) = 0.$$

This equation is satisfied for every  $v'$ , that is, for every pair of Green-reciprocal ruled surfaces, if, and only if,

$$F' = F, \quad G' = G.$$

\* Wilczynski, *Curved surfaces*, Second Memoir, these Transactions, vol. 9 (1908), pp. 79-120.

Reference to equations (8) and (10) will show that these equations are equivalent to

$$\alpha = b_v/2b, \quad \beta = a'_v/2a'.$$

Therefore the congruences  $\Gamma$  and  $\Gamma'$  are the directrix congruences.\* Consequently *we may characterize the directrix congruences by saying that they are the only reciprocal congruences which have the property that, for every pair of Green-reciprocal ruled surfaces  $R_{yz}$  and  $R_{\rho\sigma}$ , the two points, on  $l_{\rho\sigma}$ , at which tangents of  $R_{\rho\sigma}$  can be drawn which are also tangent to  $R_{yz}$  at points on  $l_{yz}$ , separate  $P_\rho$  and  $P_\sigma$  harmonically.*

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\* G. M. Green, *Memoir on the general theory of surfaces*, these Transactions, vol. 20 (1919), p. 92.

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