A UNIQUENESS THEOREM FOR THE LEGENDRE AND HERMITE POLYNOMIALS*

BY

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1. If we replace y in the expansion of $(1+y)^{-\nu}$ by $2xz+z^2$, the coefficient of z^n will, when x is replaced by $-x$, be the generalized polynomial $L_n^\nu(x)$ of Legendre. It is also easy to show that the Hermitian polynomial $H_n(x)$, usually defined by

$$e^{x^2}\frac{d^n}{dx^n}e^{-x^2} = H_n(x),$$

is the coefficient of $z^n/n!$ in the series obtained on replacing y in the expansion of e^{-y} by the same expression $2xz+z^2$. Furthermore, there is a simple recursion formula between three successive Legrendre polynomials and between three successive Hermitian polynomials. These facts suggest the following problem.

Let

$$\varphi(y) = a_0 + a_1 y + \frac{a_2}{2!}y^2 + \frac{a_3}{3!}y^3 + \cdots$$

and put

$$\varphi(2xz+z^2) = P_0 + P_1(x)z + P_2(x)z^2 + \cdots.$$

To what extent is the generating function $\varphi(y)$ determined if it is known that a simple recursion relation exists between three of the successive polynomials P_0, $P_1(x)$, $P_2(x)$, $\cdots$? We shall find that the generalized Legendre polynomials and those of Hermite possess a certain uniqueness in this regard.

2. We have

$$P_n(x) = \frac{1}{n!}\frac{d^n}{dz^n}\varphi(2xz+z^2)\Big|_{z=0}.$$

When we make use of the formula for the nth derivative of a function of a function given by Faà de Bruno,† we find without difficulty

$$P_n(x) = \sum\frac{a_{n-j}}{i!\,j!}(2x)^i,$$

* Presented to the Society, October 25, 1924.

† Quarterly Journal of Mathematics, vol. 1, p. 359.

where the summation extends to all values of i and j subject to the relation

$$i+2j = n.$$

When developed, the expression is

$$P_n(x) = \frac{a_n}{n!}(2x)^n + \frac{a_{n-1}}{(n-2)!}(2x)^{n-2} + \frac{a_{n-2}}{(n-4)!\,2!}(2x)^{n-4} + \cdots.$$

It is seen that while P_n is an even or an odd function, the coefficients of the generating function that enter into it form a certain consecutive group, a fact which has important consequences.

3. Let us denote by A_n^m the term in $P_n(x)$ that is of degree m in x. We see that

$$A_n^{n-2j} = \frac{a_{n-j}}{(n-2j)!\,j!}(2x)^{n-2j},$$

$$A_{n+1}^{n-2j-1} = \frac{a_{n-j}}{(n-2j-1)!\,(j+1)!}(2x)^{n-2j-1},$$

$$A_{n+2}^{n-2j} = \frac{a_{n-j+1}}{(n-2j)!\,(j+1)!}(2x)^{n-2j},$$

the expressions being valid for $j = -1, 0, 1, 2, \cdots$ if we agree that $A_n^m = 0$, when $m > n$. The notable fact is that A_n^{n-2j}, A_{n+1}^{n-2j-1} both contain a_{n-j}, but A_{n+2}^{n-2j} contains a_{n-j+1}.

Let k and l be multipliers, which we shall assume to be polynomials in n to be determined; then

$$2xlA_{n+1}^{n-2j-1} + kA_n^{n-2j} = [ln + (k-2l)j + k]\frac{a_{n-j}}{a_{n-j+1}}A_{n+2}^{n-2j},$$

a formula valid for $j = 0, 1, 2, \cdots$. Let

$$\psi(j) = ln + (k-2l)j + k.$$

We see that

$$\psi(-1) = (n+2)l,$$

and when n is even, that

$$\psi\left(\frac{n}{2}\right) = \left(\frac{n}{2}+1\right)k.$$

This shows that, h being another polynomial in n to be determined,

$$h P_{n+2} - 2xlP_{n+1} - kP_n = \sum_{j=-1}^{n'} \left\{ h - \psi(j) \frac{a_{n-j}}{a_{n-j+1}} \right\} A_{n+2}^{n-2j},$$

where $n' = n/2$, if n is even, and $n' = (n-1)/2$ if n is odd.

4. We see from the last expression what must be the character of the recursion relation,* and that for it to exist we must have

$$a_{n+1} = \varphi(n)\, a_n,$$

where $\varphi(n)$ is a polynomial in n. In order that the summation on the right vanish, it is necessary that

$$\psi(j) = \varphi(n-j)\, \theta(n),$$

$\theta(n)$ being a polynomial in n. The polynomial $h(n)$ is then given at once by

$$h(n) = \theta(n).$$

It is easy to determine l and k, so that $\psi(j)$ will have the desired form. Since $\varphi(n-j)$ is of the same degree in j that $\varphi(n)$ is in n, and since $\psi(j)$ is linear in j, we see that $\varphi(n)$ must be linear in n.

Put

$$\varphi(n) = \alpha n + \beta.$$

Then

$$ln + (k - 2l)j + k = (\alpha n - \alpha j + \beta)\, \theta(n).$$

This is to be an identity in both n and j. Put $j = -1$, and we find

$$(n+2)\, l = (\alpha n + \alpha + \beta)\, \theta(n).$$

Since α and β are arbitrary it follows that $\theta(n)$ must contain $n+2$ as a factor, and

$$l = (\alpha n + \alpha + \beta) \frac{\theta(n)}{n+2}.$$

* It is evident that a linear recursion relation will not exist unless the factor $2x$ is introduced as in the middle term above.

It follows then at once that

$$k = (\alpha n + 2\beta) \frac{\theta(n)}{n+2}.$$

No loss of generality results from putting

$$h = \theta(n) = (n+2), \qquad l = (\alpha n + \alpha + \beta), \qquad k = (\alpha n + 2\beta).$$

The polynomials will therefore have the recursion relation

$$(n+2) P_{n+2}(x) - 2x(\alpha n + \alpha + \beta) P_{n+1}(x) - (\alpha n + 2\beta) P_n(x) = 0,$$

if

$$a_{n+1} = (\alpha n + \beta) a_n.$$

Taking $a_0 = 1$, we have for generating function

$$\varphi(y) = F\left(a, \frac{\beta}{\alpha}, a, \alpha y\right) = (1 - \alpha y)^{-\beta/\alpha}, \text{ if } \alpha \neq 0,$$

where F represents the hypergeometric function, and

$$\varphi(y) = e^{\beta y}, \text{ if } \alpha = 0.$$

These then are the only types of generating function that will give a recursion relation, with the conditions that h, l, and k are polynomials in n.*

5. A further remark might be made about the case $\alpha \neq 0$.

We have

$$2\varphi'(2xz + z^2) = P_1'(x) + P_2'(x) z + P_3'(x) z^2 + \cdots.$$

Also we find

$$\varphi'(y)(1 - \alpha y) = \beta \cdot \varphi(y),$$

and can then deduce

$$P_{n+2}'(x) - 2\alpha x P_{n+1}'(x) - \alpha P_n'(x) = 2\beta P_{n+1}(x).$$

When this is combined with the recursion formula we have

$$x P_{n+1}'(x) + P_n'(x) = (n+1) P_{n+1}(x),$$

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* It would evidently be no more general to take h, l, k rational in n.

a relation independent of α and β. From this and the recursion relation we can obtain

$$(1+\alpha x^2) P_n'' + (\alpha + 2\beta) x P_n' - n(\alpha n + 2\beta) P_n = 0.$$

Now the differential equation

$$(1+\alpha x^2)\frac{d^2 y}{dx^2} + (\alpha+2\beta) x \frac{dy}{dx} - n(\alpha n + 2\beta) y = 0$$

is changed into

$$(u^2-1)\frac{d^2 y}{du^2} + (1+2\gamma) u \frac{dy}{du} - n(n+2\gamma) y = 0$$

by putting $u = \sqrt{-\alpha}\cdot x$, $\gamma = \beta/\alpha$. But this is the differential equation satisfied by the generalized Legendre polynomials.

6. It is evident that we can now state the following theorem:

Let

$$\varphi(y) = a_0 + a_1 y + \frac{a_2}{2!} y^2 + \cdots,$$

and put

$$\varphi(2xz+z^2) = P_0 + P_1(x) z + P_2(x) z^2 + \cdots.$$

The only cases in which there will be a recursion relation of the form

$$h(n) P_{n+2}(x) - 2 l(n) x P_{n+1}(x) - k(n) P_n(x) = 0,$$

where $h(n)$, $l(n)$, and $k(n)$ are polynomials, are essentially where we have the generalized polynomials of Legendre, and the polynomials of Hermite.

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