CONGRUENCES WITH CONSTANT ABSOLUTE INVARIANTS*

BY

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I. Introduction

It is well known that any congruence can be regarded as the aggregate of lines tangent to two surfaces, or, as some authors prefer to say, the double tangent lines to a surface of two sheets called the focal surface. The present discussion will deal only with a portion of a congruence in which each line touches the surface in two distinct points. We shall make the further assumption that neither sheet of the focal surface is developable or degenerates into a curve.

Wilczynski has shown that the homogeneous coördinates, y_1, y_2, y_3, y_4 and z_1, z_2, z_3, z_4, respectively, of the points in which a line of the congruence touches the two sheets, S_y and S_z, of the focal surface, may be taken as four linearly independent solutions of a system of partial differential equations of the form

$$(D) \qquad \begin{aligned} y_v &= mz, \qquad z_u = ny, \\ y_{uu} &= ay + bz + cy_u + dz_v, \\ z_{vv} &= a'y + b'z + c'y_u + d'z_v. \end{aligned}$$

Since, when four linearly independent solutions, $y_i, z_i (i = 1, 2, 3, 4)$, are known, any four linearly independent linear combinations of these (with constant coefficients) can be taken as a fundamental system, the system of differential equations (D) can be regarded as representing the totality of congruences projective to a given one.

In order to obtain a system of equations of the form (D) it is necessary, besides taking the loci of y and z to be the two sheets of the focal surface, to choose the independent variables, u and v, so that if u be taken constant the variable line yz will in every case generate a developable having its cuspidal edge on S_y and if v be taken constant the line yz will in every case generate a developable having its cuspidal edge on S_z.

In order that this system of differential equations may have four linearly independent solutions (y, z) certain restrictions must be placed upon the coefficients; in the first place we must have $c_v = d'_u$; in other words, there must exist a function f such that

$$(1) \qquad c = f_u, \qquad d' = f_v.$$

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The following further conditions must be satisfied:

$$
(2)\qquad
\begin{aligned}
&b = -d_v - d f_v, \qquad a' = -c'_u - c' f_u,\\
&mn - c' d = f_{uv} = W,\\
&m_{uu} + d_{vv} + d f_{vv} + d_v f_v - f_u m_u = ma + db',\\
&n_{vv} + c'_{uu} + c' f_{uu} + c'_u f_u - f_v n_v = c' a + nb',\\
&2 m_u n + m n_u = a_v + f_u mn + a' d,\\
&m_v n + 2 m n_v = b'_u + f_v mn + bc'.
\end{aligned}
$$

It is geometrically evident that the most general transformation under which the above-mentioned geometrical properties are preserved is

$$
(3)\qquad y = \lambda(u, v)\bar{y}, \qquad z = \mu(u, v)\bar{z},
$$

$$
(4)\qquad \bar{u} = \alpha(u), \qquad \bar{v} = \beta(v).
$$

The transformed equations will have the particular form (D) if and only if λ is a function of u only and μ is a function of v only,

$$
(5)\qquad y = \lambda(u)\bar{y}, \qquad z = \mu(v)\bar{z}.
$$

Under transformations of the types (4) and (5), certain functions of the coefficients and their derivatives are unchanged except perhaps for multiplication by functions of α, β, λ, and μ and their derivatives; such functions are called relative invariants. An invariant which is absolutely unchanged by transformations of the types (4) and (5) is called an absolute invariant. A fundamental set of relative invariants, i. e., a set having the property that every relative invariant is expressible in terms of these and their derivatives, consists of m, n, c', d, and

$$
(6)\qquad
\begin{aligned}
\overset{(y)}{\mathfrak{B}} &= \frac{f_u}{4} - \frac{1}{8}\frac{\partial}{\partial u}\log(d m^3),\\
\overset{(z)}{\mathfrak{B}} &= \frac{f_u}{4} + \frac{1}{8}\frac{\partial}{\partial u}\log(c'^3 n),\\
\overset{(y)}{\mathfrak{C}}{}'' &= \frac{f_v}{4} + \frac{1}{8}\frac{\partial}{\partial v}\log(d^3 m),\\
\overset{(z)}{\mathfrak{C}}{}'' &= \frac{f_v}{4} - \frac{1}{8}\frac{\partial}{\partial v}\log(c' n^3).
\end{aligned}
$$

In defining these invariants (6) we assume, evidently, that m, n, c', and d are different from 0 for the values of u and v considered. It is the purpose of this thesis to study the properties of those congruences of which the absolute invariants are constants. Under the transformations (4) and (5) the relative invariants mentioned above become*

$$(7)\qquad \begin{aligned} &\bar{m} = \frac{\mu}{\lambda\beta_v} m, \quad \bar{n} = \frac{\lambda}{\mu\alpha_u} n, \quad \bar{c}' = \frac{\lambda\alpha_u}{\mu\beta_v^2} c', \quad \bar{d} = \frac{\mu\beta_v}{\lambda\alpha_u^2} d, \\ &\overline{\overset{(y)}{\mathfrak{B}}} = \frac{1}{\alpha_u}\overset{(y)}{\mathfrak{B}}, \quad \overline{\overset{(z)}{\mathfrak{B}}} = \frac{1}{\alpha_u}\overset{(z)}{\mathfrak{B}}, \quad \overline{\overset{(y)}{\mathfrak{C}}}'' = \frac{1}{\beta_v}\overset{(y)}{\mathfrak{C}}'', \quad \overline{\overset{(z)}{\mathfrak{C}}}'' = \frac{1}{\beta_v}\overset{(z)}{\mathfrak{C}}''. \end{aligned}$$

We shall need some further material from Wilczynski's Brussells paper: the differential equations of the two sheets of the focal surface are, respectively,

$$(8)\qquad \begin{aligned} m y_{uu} - d y_{vv} &= a m y + c m y_u + \left(b - \frac{d m_v}{m}\right) y_v, \\ y_{uv} &= m n y + \frac{m_u}{m} y_v, \end{aligned}$$

and

$$(9)\qquad \begin{aligned} -c' z_{uu} + n z_{vv} &= b' n z + \left(a' - \frac{c' n_u}{n}\right) z_u + d' n z_v, \\ z_{uv} &= m n z + \frac{n_v}{n} z_u.\dagger \end{aligned}$$

The differential equations of the 1st and (-1)st Laplace transforms are

$$(10)\qquad \begin{aligned} y_v^{(1)} &= m^{(1)} z^{(1)}, \qquad z_u^{(1)} = n^{(1)} y^{(1)}, \\ y_{uu}^{(1)} &= a^{(1)} y^{(1)} + b^{(1)} z^{(1)} + c^{(1)} y_u^{(1)} + d^{(1)} z_v^{(1)}, \\ z_{vv}^{(1)} &= a'^{(1)} y^{(1)} + b'^{(1)} z^{(1)} + c'^{(1)} y_u^{(1)} + d'^{(1)} z_v^{(1)}, \end{aligned}$$

and a similar system with the superfix (-1) instead of (1), where

$$(11)\qquad \begin{aligned} m^{(1)} &= m\left(m n - \frac{\partial^2}{\partial u\, \partial v}\log m\right), \\ n^{(1)} &= \frac{1}{m}, \end{aligned}$$

* Wilczynski, *Sur la théorie générale des congruences,* pp. 9-20 (cited hereafter as Brussells paper).

† Brussells paper, Section 7.

$$
(11)\quad
\begin{aligned}
a^{(1)} &= a + f_{uu} - 2\frac{\partial^2}{\partial u^2}\log m + \left(\frac{d_u}{d} - \frac{m_u}{m}\right)\left(\frac{m_u}{m} - f_u\right),\\
b^{(1)} &= m\left(a_u + bn + af_u + dn_v - \frac{\partial^3}{\partial u^3}\log m\right)\\
&\quad + \frac{m_v}{m}\left(b_u + bf_u + dmn - \frac{bm_u}{m}\right)\\
&\quad + m_u\left(f_{uu} + f_u^2 - f_u\frac{m_u}{m} - 2\frac{\partial^2}{\partial u^2}\log m\right)\\
&\quad + \left(f_u + \frac{d_u}{d} - \frac{m_u}{m}\right)\left(m_{uu} - f_u m_u - \frac{bm_v}{m} - am\right),\\
c^{(1)} &= f_u + \frac{d_u}{d} - \frac{m_u}{m},\\
d^{(1)} &= d\left(c'd - \frac{\partial^2}{\partial u\,\partial v}\log d\right),\\
a'^{(1)} &= \frac{1}{d}\left(\frac{m_u}{m} - f_u\right),\\
b'^{(1)} &= -\frac{\partial^2}{\partial v^2}\log m + \frac{1}{d}\left(m_{uu} - am - f_u m_u - \frac{bm_v}{m}\right),\\
c'^{(1)} &= \frac{1}{d},\\
d'^{(1)} &= f_v + \frac{d_v}{d} - \frac{m_v}{m},
\end{aligned}
$$

and

$$
(12)\quad
\begin{aligned}
m^{(-1)} &= \frac{1}{n},\\
n^{(-1)} &= n\left(mn - \frac{\partial^2}{\partial u\,\partial v}\log n\right),\\
a^{(-1)} &= -\frac{\partial^2}{\partial u^2}\log n + \frac{1}{c'}\left(n_{vv} - b'n - f_v n_v - \frac{a'n_u}{n}\right),\\
b^{(-1)} &= \frac{1}{c'}\left(\frac{n_v}{n} - f_v\right),\\
c^{(-1)} &= \frac{c'_u}{c'} - \frac{n_u}{n} + f_u,\\
d^{(-1)} &= \frac{1}{c'},
\end{aligned}
$$

$$
\begin{aligned}
a'^{(-1)} &= n\left(b'_v + a'm + b'f_v + c'm_u - \frac{\partial^3}{\partial v^3}\log n\right)\\
&\quad + \frac{n_u}{n}\left(a'_v + a'f_v + c'mn - \frac{a'n_v}{n}\right)\\
&\quad + n_v\left(f_{vv} + f_v^2 - f_v\frac{n_v}{n} - 2\frac{\partial^2}{\partial v^2}\log n\right)\\
&\quad + \left(f_v + \frac{c'_v}{c'} - \frac{n_v}{n}\right)\left(n_{vv} - f_v n_v - \frac{a'n_u}{n} - b'n\right),\\
b'^{(-1)} &= b' + f_{vv} - 2\frac{\partial^2}{\partial v^2}\log n + \left(\frac{c'_v}{c'} - \frac{n_v}{n}\right)\left(\frac{n_v}{n} - f_v\right),\\
c'^{(-1)} &= c'\left(c'd - \frac{\partial^2}{\partial u\,\partial v}\log c'\right),\\
d'^{(-1)} &= f_v + \frac{c'_v}{c'} - \frac{n_v}{n}.^*
\end{aligned}
\tag{12}
$$

II. Case 1, $\overset{(y)}{\mathfrak{B}} \not\equiv \overset{(z)}{\mathfrak{B}}$, $\overset{(y)}{\mathfrak{C}}{}'' \not\equiv \overset{(z)}{\mathfrak{C}}{}''$

1. Under these conditions it is convenient to use the following absolute invariants, which, as stated above, are assumed to be constants:

$$
\begin{aligned}
i_1 &= \frac{c'd}{mn},\\
i_2 &= \frac{\overset{(y)}{\mathfrak{B}}}{\overset{(z)}{\mathfrak{B}} - \overset{(y)}{\mathfrak{B}}} = \frac{2f_u - \dfrac{\partial}{\partial u}\log(dm^3)}{\dfrac{\partial}{\partial u}\log(c'^3dm^3n)},\\
i_3 &= \frac{\overset{(z)}{\mathfrak{C}}{}''}{\overset{(y)}{\mathfrak{C}}{}'' - \overset{(z)}{\mathfrak{C}}{}''} = \frac{2f_v - \dfrac{\partial}{\partial v}\log(c'n^3)}{\dfrac{\partial}{\partial v}\log(c'd^3mn^3)},\\
i_4 &= \frac{\overset{(z)}{\mathfrak{B}} - \overset{(y)}{\mathfrak{B}}}{m^{1/4}n^{1/2}d^{1/4}} = \frac{\dfrac{\partial}{\partial u}\log(c'^3dm^3n)}{8m^{1/4}n^{1/2}d^{1/4}},\\
i_5 &= \frac{\overset{(y)}{\mathfrak{C}}{}'' - \overset{(z)}{\mathfrak{C}}{}''}{m^{1/2}n^{1/4}c'^{1/4}} = \frac{\dfrac{\partial}{\partial v}\log(c'd^3mn^3)}{8m^{1/2}n^{1/4}c'^{1/4}}.
\end{aligned}
\tag{1}
$$

* Brussells paper, Section 10.

In consequence of the fact that i_1 is a constant, the fourth and fifth of equations (1) can be written

$$
(2)\qquad
\begin{aligned}
i_4 &= \frac{\dfrac{\partial}{\partial u}\log\,(m^{1/2}\,n^{1/4}\,c'^{1/4})}{m^{1/4}\,n^{1/2}\,d^{1/4}}\,,\\
i_5 &= \frac{\dfrac{\partial}{\partial v}\log\,(m^{1/4}\,n^{1/2}\,d^{1/4})}{m^{1/2}\,n^{1/4}\,c'^{1/4}}\,.
\end{aligned}
$$

From (2) we find

$$
(3)\qquad
\begin{aligned}
\frac{\partial^2}{\partial u\,\partial v}\log\,(m^{1/2}\,n^{1/4}\,c'^{1/4}) &= i_4\,\frac{\partial}{\partial v}\,(m^{1/4}\,n^{1/2}\,d^{1/4}) = i_4\,i_5\,m^{3/4}\,n^{3/4}\,c'^{1/4}\,d^{1/4},\\
\frac{\partial^2}{\partial u\,\partial v}\log\,(m^{1/4}\,n^{1/2}\,d^{1/4}) &= i_5\,\frac{\partial}{\partial u}\,(m^{1/2}\,n^{1/4}\,c'^{1/4}) = i_4\,i_5\,m^{3/4}\,n^{3/4}\,c'^{1/4}\,d^{1/4}.
\end{aligned}
$$

Hence

$$
(4)\qquad \frac{\partial^2}{\partial u\,\partial v}\log\left(\frac{mc'}{nd}\right) = 0,
$$

and mc'/nd is the product of a function of u only by a function of v only. Hence, according to equations (I, 7) we can find a transformation of u and v which will make the transform of this expression identically equal to unity. Let us assume that this transformation has already been applied, so that

$$
(5)\qquad \frac{mc'}{nd} \equiv 1.
$$

From the second and third of equations (1)

$$
(6)\qquad
\begin{aligned}
2f_u &= \frac{\partial}{\partial u}\log\,(c'^{3i_2}d^{i_2+1}\,m^{3i_2+3}\,n^{i_2}),\\
2f_v &= \frac{\partial}{\partial v}\log\,(c'^{i_3+1}d^{3i_3}\,m^{i_3}\,n^{3i_3+3}),
\end{aligned}
$$

whence

$$
(7)\qquad \frac{\partial^2}{\partial u\,\partial v}\log\left\{\frac{c'^{3i_2-i_3-1}\,m^{3i_2-i_3+3}}{d^{3i_3-i_2-1}\,n^{3i_3-i_2+3}}\right\} = 0,
$$

and

$$
\frac{c'^{3i_2-i_3-1}\,m^{3i_2-i_3+3}}{d^{3i_3-i_2-1}\,n^{3i_3-i_2+3}}
$$

is the product of a function of u only by a function of v only. From equations (I,7) we see that it is possible to find a transformation of y

and z only (under which (5) is preserved) which will make the transform of the expression in the bracket in (7) identically equal to unity. Let us assume that this transformation has already been applied, so that

$$\frac{c'^{3i_2-i_3-1}\, m^{3i_2-i_3+3}}{d^{3i_3-i_2-1}\, n^{3i_3-i_2+3}} \equiv 1 . \tag{8}$$

From the first of equations (II,1) and equations (5) and (8)

$$\begin{aligned} \frac{c'}{n} &= \frac{d}{m} = i_1^{1/2}, \\ m^{i_2-i_3+1}\, n^{i_2-i_3-1} &= i_1^{-(i_2-i_3)/2}, \\ d^{i_2-i_3+1}\, n^{i_2-i_3-1} &= i_1^{1/2}. \end{aligned} \tag{9}$$

Substituting equations (9) in the last two of equations (1), we obtain

$$i_4 = \frac{\dfrac{\partial (mn)}{\partial u}}{2\, i_1^{1/8} (mn)^{3/2}}, \qquad i_5 = \frac{\dfrac{\partial (mn)}{\partial v}}{2\, i_1^{1/8} (mn)^{3/2}}, \tag{10}$$

whence

$$mn = \frac{1}{i_1^{1/4} (i_4 u + i_5 v)^2}. \tag{11}$$

From equations (9) and (11) we find

$$\begin{aligned} m &= \frac{(i_4 u + i_5 v)^{i_2-i_3-1}}{i_1^{(i_2-i_3+1)/8}}, \\ n &= \frac{i_1^{(i_2-i_3-1)/8}}{(i_4 u + i_5 v)^{i_2-i_3+1}}, \\ c' &= \frac{i_1^{(i_2-i_3+3)/8}}{(i_4 u + i_5 v)^{i_2-i_3+1}}, \\ d &= \frac{(i_4 u + i_5 v)^{i_2-i_3-1}}{i_1^{(i_2-i_3-3)/8}}, \end{aligned} \tag{12}$$

and from equations (6) and (12)

$$\begin{aligned} f_u &= \frac{-2\, i_4 (i_2 + i_3 + 1)}{i_4 u + i_5 v}, \\ f_v &= \frac{-2\, i_5 (i_2 + i_3 + 1)}{i_4 u + i_5 v}. \end{aligned} \tag{13}$$

Then from the integrability conditions (I,2) it follows that, if $W \neq 0$ (i. e., if $\dot{i}_1 \neq 1$), the coefficients of the differential equations (D) can be written

$$
(14)\qquad
\begin{aligned}
m &= m_0\,(\dot{i}_4 u+\dot{i}_5 v)^{j-2}, & n &= n_0\,(\dot{i}_4 u+\dot{i}_5 v)^{-j},\\
a &= a_0\,(\dot{i}_4 u+\dot{i}_5 v)^{-2}, & b &= b_0\,(\dot{i}_4 u+\dot{i}_5 v)^{j-3},\\
c &= c_0\,(\dot{i}_4 u+\dot{i}_5 v)^{-1}, & d &= d_0\,(\dot{i}_4 u+\dot{i}_5 v)^{j-2},\\
a' &= a_0'\,(\dot{i}_4 u+\dot{i}_5 v)^{-j-1}, & b' &= b_0'\,(\dot{i}_4 u+\dot{i}_5 v)^{-2},\\
c' &= c_0'\,(\dot{i}_4 u+\dot{i}_5 v)^{-j}, & d' &= d_0'\,(\dot{i}_4 u+\dot{i}_5 v)^{-1},
\end{aligned}
$$

where the letters with subscripts 0 represent constants, and where

$$
(15)\qquad j = \dot{i}_2-\dot{i}_3+1.
$$

Substituting equations (12) and (14) in (I, 2), we obtain

$$
(16)\qquad
\begin{aligned}
b_0 &= \frac{\dot{i}_5(\dot{i}_2+3\dot{i}_3+3)}{\dot{i}_1^{(i_2-i_3-3)/8}},\\
a_0' &= \frac{\dot{i}_4(3\dot{i}_2+\dot{i}_3+3)}{\dot{i}_1^{(i_3-i_2-3)/8}},\\
\dot{i}_2+\dot{i}_3+1 &= \frac{\dot{i}_1^{-1/4}-\dot{i}_1^{3/4}}{2\dot{i}_4\dot{i}_5},
\end{aligned}
$$

$$
\begin{aligned}
(\dot{i}_2-\dot{i}_3-1)(3\dot{i}_2+\dot{i}_3)\dot{i}_4^2-\dot{i}_1^{1/2}(\dot{i}_2-\dot{i}_3-2)(\dot{i}_2+3\dot{i}_3+3)\dot{i}_5^2 &= a_0+\dot{i}_1^{1/2}b_0',\\
(\dot{i}_2-\dot{i}_3+2)(3\dot{i}_2+\dot{i}_3+3)\dot{i}_4^2-\dot{i}_1^{-1/2}(\dot{i}_2-\dot{i}_3+1)(\dot{i}_2+3\dot{i}_3)\dot{i}_5^2 &= a_0+\dot{i}_1^{-1/2}b_0',\\
\dot{i}_1^{3/4}(3\dot{i}_2+\dot{i}_3+3)\dot{i}_4-\dot{i}_1^{-1/4}(3\dot{i}_2+\dot{i}_3-1)\dot{i}_4 &= 2\dot{i}_5 a_0,\\
\dot{i}_1^{3/4}(\dot{i}_2+3\dot{i}_3+3)\dot{i}_5-\dot{i}_1^{-1/4}(\dot{i}_2+3\dot{i}_3-1)\dot{i}_5 &= 2\dot{i}_4 b_0'.
\end{aligned}
$$

From the fourth and fifth of (16) we find

$$
(17)\qquad
\begin{aligned}
(\dot{i}_1-1)a_0 &= [\dot{i}_1(\dot{i}_2-\dot{i}_3+2)(3\dot{i}_2+\dot{i}_3+3)-(\dot{i}_2-\dot{i}_3-1)(3\dot{i}_2+\dot{i}_3)]\dot{i}_4^2\\
&\quad -\dot{i}_1^{1/2}(12\dot{i}_3+6)\dot{i}_5^2,\\
(\dot{i}_1-1)b_0' &= -\dot{i}_1^{1/2}(12\dot{i}_2+6)\dot{i}_4^2-[\dot{i}_1(\dot{i}_2-\dot{i}_3-2)(\dot{i}_2+3\dot{i}_3+3)\\
&\quad -(\dot{i}_2-\dot{i}_3+1)(\dot{i}_2+3\dot{i}_3)]\dot{i}_5^2.
\end{aligned}
$$

But from the third, sixth, and seventh of (16) we have

$$
(18)\qquad
\begin{aligned}
(\dot{i}_1-1)a_0 &= -(\dot{i}_2+\dot{i}_3+1)[\dot{i}_1(3\dot{i}_2+\dot{i}_3+3)-(3\dot{i}_2+\dot{i}_3-1)]\dot{i}_4^2,\\
(\dot{i}_1-1)b_0' &= -(\dot{i}_2+\dot{i}_3+1)[\dot{i}_1(\dot{i}_2+3\dot{i}_3+3)-(\dot{i}_2+3\dot{i}_3-1)]\dot{i}_5^2,
\end{aligned}
$$

and from (17) and (18)

$$
(19)\quad \begin{aligned}
&[i_1(2i_2+3)(3i_2+i_3+3)-(6i_2^2+2i_2i_3-i_2-i_3-1)]i_4^2 \\
&\qquad\qquad\qquad\qquad -i_1^{1/2}(12i_3+6)i_5^2 = 0, \\
&-i_1^{1/2}(12i_2+6)i_4^2+[i_1(2i_3+3)(i_2+3i_3+3) \\
&\qquad\qquad\qquad\qquad -(2i_2i_3+6i_3^2-i_2-i_3-1)]i_5^2 = 0.
\end{aligned}
$$

It can now be easily seen that if i_1, i_2, i_3, i_4, and i_5 are constants satisfying the third of equations (16) and equations (19) and if i_1, i_4, and i_5 are all different from zero and i_1 is different from 1, a congruence exists having the absolute invariants i_1, i_2, i_3, i_4, and i_5, as defined by equations (1), equal to the specified constants. If, further, a definite choice be made of the values of the fractional powers of i_1 in equations (12) and the first and second of equations (16), the congruence is uniquely determined except for projective transformations. For under these conditions equations (17) determine a_0 and b_0', which satisfy also equations (18) and the last four of equations (16). The first two of equations (16) determine a_0' and b_0. Then a, a', b, and b' are determined by equations (14), and c, c', d, d', m, and n are determined by equations (12) and (13). Then, according to the general theory, the congruence is determined except for projective transformations.

Since neither i_4 nor i_5 is zero, equations (19) imply

$$
(20)\quad \begin{aligned}
&i_2^2(3i_2+i_3+3)(i_2+3i_3+3)(2i_2+3)(2i_3+3)-i_1[24i_2^3i_3+80i_2^2i_3^2 \\
&\quad +24i_2i_3^3+12i_2^3+116i_2^2i_3+116i_2i_3^2+12i_3^3+30i_2^2+140i_2i_3+30i_3^2 \\
&\quad +36i_2+36i_3+18]+[12i_2^3i_3+40i_2^2i_3^2+12i_2i_3^3-6i_2^3-10i_2^2i_3 \\
&\quad -10i_2i_3^2-6i_3^3-5i_2^2-2i_2i_3-5i_3^2+2i_2+2i_3+1] = 0.
\end{aligned}
$$

2. The differential equations of the two sheets of the focal surface are, by equations (I, 8) and (I,9)

$$
(21)\quad \begin{aligned}
m_0y_{uu}-d_0y_{vv} &= \frac{a_0m_0}{(i_4u+i_5v)^2}y+\frac{c_0m_0}{(i_4u+i_5v)}y_u+\frac{b_0-i_5(j-2)d_0}{(i_4u+i_5v)}y_v, \\
y_{uv} &= \frac{m_0n_0}{(i_4u+i_5v)^2}y+\frac{i_4(j-2)}{(i_4u+i_5v)}y_v,
\end{aligned}
$$

and

$$
(22)\quad \begin{aligned}
-c_0'z_{uu}+n_0z_{vv} &= \frac{b_0'n_0}{(i_4u+i_5v)^2}z+\frac{a_0'+i_4jc_0'}{(i_4u+i_5v)}z_u+\frac{d_0'n_0}{(i_4u+i_5v)}z_v, \\
z_{uv} &= \frac{m_0n_0}{(i_4u+i_5v)^2}z-\frac{i_5j}{(i_4u+i_5v)}z_u.
\end{aligned}
$$

In order to obtain a system of differential equations representing S_y referred to its asymptotic curves, we make the transformation

$$\bar{u} = \sqrt{-d_0}\, u + \sqrt{m_0}\, v, \qquad \bar{v} = \sqrt{-d_0}\, u - \sqrt{m_0}\, v,^*$$

under which equations (21) become

$$
\begin{aligned}
y_{\bar{u}\bar{u}} &= \frac{2 n_0 \sqrt{-d_0 m_0} - a_0}{4 d_0 \nu} y \\
&- \frac{c_0 m_0 \sqrt{-d_0} + \sqrt{m_0}(b_0 - i_5 (j-2) d_0) - 2 i_4 (j-2) m_0 \sqrt{-d_0}}{4 d_0 m_0 \nu} y_{\bar{u}} \\
&- \frac{c_0 m_0 \sqrt{-d_0} + \sqrt{m_0}(b_0 - i_5 (j-2) d_0) + 2 i_4 (j-2) m_0 \sqrt{-d_0}}{4 d_0 m_0 \nu} y_{\bar{v}}, \\
y_{\bar{v}\bar{v}} &= \frac{-2 n_0 \sqrt{-d_0 m_0} - a_0}{4 d_0 \nu} y \\
&- \frac{c_0 m_0 \sqrt{-d_0} + \sqrt{m_0}(b_0 - i_5 (j-2) d_0) + 2 i_4 (j-2) m_0 \sqrt{-d_0}}{4 d_0 m_0 \nu} y_{\bar{u}} \\
&- \frac{c_0 m_0 \sqrt{-d_0} + \sqrt{m_0}(b_0 - i_5 (j-2) d_0) - 2 i_4 (j-2) m_0 \sqrt{-d_0}}{4 d_0 m_0 \nu} y_{\bar{v}},
\end{aligned} \tag{23}
$$

where

$$\nu = i_4 u + i_5 v = \frac{(i_4 \sqrt{m_0} + i_5 \sqrt{-d_0})\bar{u} + (i_4 \sqrt{m_0} - i_5 \sqrt{-d_0})\bar{v}}{2\sqrt{-m_0 d_0}}. \tag{24}$$

If we write equations (23), for brevity, in the form

$$
\begin{aligned}
y_{\bar{u}\bar{u}} + 2\alpha\, y_{\bar{u}} + 2\beta\, y_{\bar{v}} + \gamma\, y &= 0, \\
y_{\bar{v}\bar{v}} + 2\alpha' y_{\bar{u}} + 2\beta' y_{\bar{v}} + \gamma' y &= 0,
\end{aligned} \tag{25}
$$

we see that the fundamental seminvariants a', b, f, g† are of degrees -1, -1, -2, -2 respectively in ν. Hence the relative invariants a', b, h, k are of degrees -1, -1, -4, -4 respectively. Since they are of weights $(-1, 2)$, $(2, -1)$, $(6, -2)$, $(-2, 6)$ respectively, the exponents

* Wilczynski, *Projective differential geometry of curved surfaces*, first memoir, these Transactions, vol. 8 (1907), pp. 233-260, equation (22); this memoir is cited hereafter as *Curved surfaces*.

† *Curved surfaces*, equation (38).

p, q, r, s in an absolute invariant of the form $a'^p\, b^q\, h^r\, k^s$ must satisfy the relations

$$(26)\qquad \begin{aligned} -p+2q+6r-2s &= 0, \\ 2p-\ \ q-2r+6s &= 0. \end{aligned}$$

Hence

$$(27)\qquad p+q+4r+4s = 0;$$

i. e., any absolute invariant of this type is constant.

From the invariants a', b, h, k, four others,

$$(28)\qquad A = a'b^2, \qquad B = a'^2 b, \qquad H = a'h, \qquad K = bk,$$

are derived, which have weights (3,0), (0,3), (5,0), (0,5) and degrees -3, -3, -5, -5 respectively. From these, all others can be obtained by means of the operations

$$U \equiv \alpha' \frac{\partial}{\partial \bar{u}}, \qquad V \equiv \beta \frac{\partial}{\partial \bar{v}}$$

and the Wronskian operation*; the U operation is applied only to invariants of zero $\bar{u}$-weight, and adds 2 to the $\bar{v}$-weight and -2 to the degree; the V operation is applied only to invariants of zero $\bar{v}$-weight, and adds 2 to the $\bar{u}$-weight and -2 to the degree. Since for each of the invariants thus far mentioned the sum of the two weights and the degree is zero, any invariant obtained from them by means of the U and V operations must have the same property. Hence, if the weights are both zero, the degree must be zero. Since the Wronskian operation is essentially partial differentiation of an absolute invariant, it can, under our hypothesis, yield only invariants which are identically zero. Hence all the absolute invariants of the y sheet of the focal surface are constants; the same proposition can evidently be proved in regard to the z sheet. It is also readily seen that both sheets are of the type discussed by Wilczynski in his paper *On a certain class of self-projective surfaces.*†

By successive differentiation of equations (21) and elimination of all terms involving differentiation with respect to v, we obtain for the curves $v =$ constant on S_y a differential equation of the form

$$(29)\qquad \begin{aligned} p_0\, y_{uuuu} &+ \frac{4p_1}{(i_4 u + i_5 v)}\, y_{uuu} + \frac{6p_2}{(i_4 u + i_5 v)^2}\, y_{uu} \\ &+ \frac{4p_3}{(i_4 u + i_5 v)^3}\, y_u + \frac{p_4 y}{(i_4 u + i_5 v)^4} = 0, \end{aligned}$$

* *Curved surfaces,* Section 7.

† These Transactions, vol. 14 (1913), pp. 421-443.

where p_0, p_1, p_2, p_3, and p_4 are constants. Since the absolute invariants of this equation are constants (independent of both u and v), the curves $v =$ constant on S_y are anharmonic curves, all projective to one another. In the same manner each of the families of curves $u =$ constant and $v =$ constant on S_y and S_z can be shown to consist of projectively equivalent anharmonic curves.*

If we write the differential equations of the 1st and (—1)st Laplace transforms according to formulas (I, 10), (I, 11), and (I, 12), we find that they have constant absolute invariants, $\overset{(y)}{\mathfrak{B}} \neq \overset{(z)}{\mathfrak{B}}$, and $\overset{(y)}{\mathfrak{C}}{}'' \neq \overset{(z)}{\mathfrak{C}}{}''$, but they are not projective to the original congruence.

3. In the special case, $i_1 = 1$, excluded from the above discussion subsequent to equation (13), the sixth and seventh of the integrability conditions can not be solved for a and b', and hence equations (14) are not valid. In this case equations (12) still hold (with $i_1 = 1$), but equations (13), together with the assumption that $f_{uv} \equiv 0$, become

$$f_u \equiv f_v \equiv 0, \tag{30}$$

whence

$$i_2 + i_3 + 1 = 0. \tag{31}$$

The integrability conditions then take the form

$$\begin{aligned} c = d' = 0, \qquad b &= -d_v, \qquad a' = -c'_u, \\ m_{uu} + \quad d_{vv} &= m a + d b', \\ n_{vv} + \quad c'_{uu} &= c' a + n b', \\ 2 m_u n + \ m n_u &= a_v \ + a' d, \\ m_v n + 2 m n_v &= b_u \ + b c'. \end{aligned} \tag{32}$$

Substituting equations (12) in the last four of (32), we obtain

$$\begin{gathered} (i_2 - i_3 - 1)(i_2 - i_3 - 2)(i_4^2 + i_5^2) = (a + b')(i_4 u + i_5 v)^2, \\ (i_2 - i_3 + 1)(i_2 - i_3 + 2)(i_4^2 + i_5^2) = (a + b')(i_4 u + i_5 v)^2, \\ a_v = \frac{-4 i_4}{(i_4 u + i_5 v)^3}, \qquad b'_u = \frac{-4 i_5}{(i_4 u + i_5 v)^3}. \end{gathered} \tag{33}$$

* Wilczynski, *Projective Differential Geometry of Curves and Ruled Surfaces*, pp. 243, 279 (cited hereafter as *Projective Differential Geometry*).

From the last two of equations (33)

$$a = \frac{2 i_4}{i_5 (i_4 u + i_5 v)^2} + (\text{a function of } u \text{ only}),$$
$$b' = \frac{2 i_5}{i_4 (i_4 u + i_5 v)^2} + (\text{a function of } v \text{ only}).$$

Substituting these expressions in the first two of equations (33) we see that

$$\begin{aligned} a &= \frac{2 i_4}{i_5 (i_4 u + i_5 v)^2} + k, \\ b' &= \frac{2 i_5}{i_4 (i_4 u + i_5 v)^2} - k, \end{aligned} \tag{34}$$

where k is a constant independent of the invariants, and that

$$\begin{aligned} (i_2 - i_3 - 1)(i_2 - i_3 - 2)(i_4^2 + i_5^2) &= (i_2 - i_3 + 1)(i_2 - i_3 + 2)(i_4^2 + i_5^2) \\ &= \frac{2 (i_4^2 + i_5^2)}{i_4 i_5}. \end{aligned} \tag{35}$$

From the first two members of this equation it follows that

$$(i_2 - i_3)(i_4^2 + i_5^2) = 0.$$

From equations (12) we find

$$\begin{aligned} m &= d = (i_4 u + i_5 v)^{i_2 - i_3 - 1}, \\ n &= c' = (i_4 u + i_5 v)^{-i_2 + i_3 - 1}. \end{aligned} \tag{36}$$

As in (II, 1) it can be shown that if $i_1 (= 1)$ and i_2, i_3, i_4, i_5, and k be given as constants satisfying equations (35) and if neither i_4 nor i_5 is zero, the congruence is determined except for projective transformations.

As before, we see that the successive Laplace transforms are all projectively distinct.

III. Case 2, $\overset{(y)}{\mathfrak{B}} \equiv \overset{(z)}{\mathfrak{B}}$, $\overset{(y)}{\mathfrak{C}}{}'' \equiv \overset{(z)}{\mathfrak{C}}{}''$

1. We shall use the absolute invariant i_1 defined in equations (II, 1) and the absolute invariants

$$(1)\qquad j_4 = \frac{\overset{(y)}{\mathfrak{B}}}{m^{1/4}\, n^{1/2}\, d^{1/4}}, \qquad j_5 = \frac{\overset{(z)}{\mathfrak{C}''}}{m^{1/2}\, n^{1/4}\, c'^{1/4}}.$$

From equations (I, 6) and the hypotheses of this section

$$(2)\qquad c'^3 d m^3 n = \varphi(v), \qquad c' d^3 m n^3 = \psi(u).$$

Since, according to equations (I, 7), under the transformations (I, 4) and (I, 5),

$$(3)\qquad \bar{c}'^3 \bar{d} \bar{m}^3 \bar{n} = \beta_v^{-2} c'^3 d m^3 n, \qquad \bar{c}' \bar{d}^3 \bar{m} \bar{n}^3 = \alpha_u^{-2} c' d^3 m n^3,$$

we can choose α and β so that $\overline{c'^3 d m^3 n} = \overline{c' d^3 m n^3} = 1$. Let us assume that this transformation has already been applied, i. e., that

$$(4)\qquad c'^3 d m^3 n \equiv c' d^3 m n^3 \equiv 1.$$

Hence, according to (I, 7) and (3), we can choose α and β so that

$$(5)\qquad c' m \equiv d n \equiv 1.$$

From equations (II, 1) and (5)

$$(6)\qquad m = i_1^{-1/2} n^{-1}, \qquad c' = i_1^{1/2} n, \qquad d = n^{-1}.$$

From equations (I, 6), (1), and (6)

$$(7)\qquad \frac{f_u}{4} + \frac{n_u}{2n} = i_1^{-1/8} j_4, \qquad \frac{f_v}{4} - \frac{n_v}{2n} = i_i^{-1/8} j_5.$$

Hence

$$(8)\qquad f_{uv} = -2 \frac{\partial^2}{\partial u\, \partial v} \log n = 2 \frac{\partial^2}{\partial u\, \partial v} \log n,$$

and

$$
(9) \qquad \begin{aligned} f_{uv} &= 0, \\ \frac{\partial^2}{\partial u\, \partial v} \log n &= 0. \end{aligned}
$$

Hence n is the product of a function of u only by a function of v only, and can be reduced to unity by a transformation of form (I, 5), which does not disturb equations (5). Let us assume that this has already been done. From the second of equations (I, 2), the definition of i_1, and the first of equations (9)

$$
(10) \qquad i_1 = 1.
$$

Then, since $n \equiv 1$, equations (6) and (10) give

$$
(11) \qquad m = n \equiv c' = d = 1.
$$

Hence from (7) and (11)

$$
(12) \qquad \begin{aligned} f_u &= 4j_4, \\ f_v &= 4j_5. \end{aligned}
$$

The integrability conditions can now be written

$$
(13) \qquad \begin{aligned} c &= 4j_4, \qquad d = 4j_5, \\ b &= -4j_5, \qquad a' = -4j_4, \\ W &\equiv mn - c'd = 0, \\ a + b' &= 0, \\ a_v &= 0, \qquad b'_u = 0. \end{aligned}
$$

From the last three of equations (13)

$$
(14) \qquad a = k, \qquad b' = -k,
$$

where k is a constant independent of the invariants.

The differential equations of the congruence are therefore

$$
(15) \qquad \begin{aligned} y_v &= z, \qquad z_u = y, \\ y_{uu} &= ky - 4j_5 z + 4j_4 y_u + z_v, \\ z_{vv} &= -4j_4 y - kz + y_u + 4j_5 z_v. \end{aligned}
$$

Obviously the congruence is determined, except for projective transformations, by the (constant) values of $i_1 (= 1)$, j_4, j_5, and k.

2. The differential equations of the two sheets of the focal surface are

$$(16) \qquad \begin{aligned} y_{uu} - y_{vv} &= ky + 4j_4 y_u - 4j_5 y_v, \\ y_{uv} &= y, \end{aligned}$$

$$(17) \qquad \begin{aligned} z_{uu} - z_{vv} &= kz + 4j_4 z_u - 4j_5 z_v, \\ z_{uv} &= z. \end{aligned}$$

(See (I, 8) and (I, 9).) The two sheets are obviously projective to one another, with each point on one sheet corresponding to the point on the other which is given by the same pair of values of u and v.

It is easily seen that each of the surfaces S_y and S_z has constant absolute invariants*. The net of curves $u =$ constant and $v =$ constant on each surface is isothermally conjugate and has equal Laplace-Darboux invariants.†

As indicated in § 2, we find the differential equations of the four families of parametric curves to be

$$(18) \qquad y_{uuuu} - 4j_4 y_{uuu} - ky_{uu} + 4j_5 y_u - y = 0,$$

$$(19) \qquad y_{vvvv} - 4j_5 y_{vvv} + ky_{vv} + 4j_4 y_v - y = 0,$$

$$(20) \qquad z_{uuuu} - 4j_4 z_{uuu} - kz_{uu} + 4j_5 z_u - z = 0,$$

$$(21) \qquad z_{vvvv} - 4j_5 z_{vvv} + kz_{vv} + 4j_4 z_v - z = 0.$$

Since each of these differential equations has constant seminvariants, each of the four families of parametric curves consists of projectively equivalent anharmonic curves; in fact, the curves represented by equations (18) and (20) are all projectively equivalent; similarly the curves represented by equations (19) and (21) are all projectively equivalent. Furthermore, since the differential equations, for example, of any curve, $v = c_1$, of the family of curves $v =$ constant on S_y can be transformed into the differential equation of any other curve, $v = c_2$, of the same family, *without change of the independent variable* u,‡ it follows that any two curves of the family $v =$ constant on S_y are projective to one another, any pair of

* *Curved surfaces*, 7.

† Wilczynski, *General theory of congruences*, these Transactions, vol. 16 (1915), pp. 318–322.

‡ *Projective Differential Geometry*, pp. 239, 242.

corresponding points lying on the same curve of the family $u=$ constant on S_y. Similarly it can be seen that any two curves of the family $u=$ constant on S_y are projective to one another, any pair of corresponding points lying on the same curve of the family $v=$ constant on S_y. Evidently the other focal sheet, S_z, has the same property. Furthermore, since each developable surface of the congruence is completely determined by its cuspidal edge and since two developables are projective to one another if (and only if) their cuspidal edges are projective to one another, it follows that any two developables of the congruence belonging to one family are projective to one another, any pair of corresponding generators lying on the same developable of the other family.

A fundamental system of solutions of equations (15) is

$$
\begin{aligned}
y_i &= e^{e_i u + v/e_i}, \\
z_i &= \frac{1}{e_i} e^{e_i u + v/e_i} \qquad (i = 1, 2, 3, 4),
\end{aligned}
\tag{22}
$$

where $e_1, \cdots, e_4$ are the roots (assumed to be distinct) of the equation

$$
e^4 - 4j_4 e^3 - k e^2 + 4 j_5 e - 1 = 0. \tag{23}
$$

The Plücker coördinates of the lines of the congruence are then

$$
\omega_{ij} = \left(\frac{1}{e_j} - \frac{1}{e_i}\right) e^{(e_i + e_j)(u + v/e_i e_j)} \qquad (i, j = 1, \cdots, 4). \tag{24}
$$

Hence the congruence belongs to the tetrahedral complex

$$
\frac{\omega_{12}\,\omega_{34}}{(e_2-e_1)(e_4-e_3)} = \frac{\omega_{13}\,\omega_{42}}{(e_3-e_1)(e_2-e_4)} = \frac{\omega_{14}\,\omega_{23}}{(e_4-e_1)(e_3-e_2)}. \tag{25}
$$

These two equations represent one and the same complex, since from the first, viz.

$$
\frac{\omega_{12}\,\omega_{34}}{(e_2-e_1)(e_4-e_3)} = \frac{\omega_{13}\,\omega_{42}}{(e_3-e_1)(e_2-e_4)},
$$

together with the fundamental identity

$$
\omega_{12}\,\omega_{34} + \omega_{13}\,\omega_{42} + \omega_{14}\,\omega_{23} \equiv 0,
$$

it follows that

$$\frac{\omega_{14}\,\omega_{23}}{(e_4-e_1)(e_3-e_2)}=\frac{-\omega_{12}\,\omega_{34}-\omega_{13}\,\omega_{42}}{(e_4-e_1)(e_3-e_2)}$$
$$=\left[\frac{-(e_2-e_1)(e_4-e_3)-(e_3-e_1)(e_2-e_4)}{(e_4-e_1)(e_3-e_2)(e_2-e_1)(e_4-e_3)}\right]\omega_{12}\,\omega_{34}$$
$$=\frac{\omega_{12}\,\omega_{34}}{(e_2-e_1)(e_4-e_3)}.$$

In order to obtain another equation of the congruence we replace the subscript j in equation (24) by k and divide the resulting equation by equation (24), obtaining the homogeneous equation

$$\frac{\omega_{ij}}{\omega_{ik}}=\frac{(e_i-e_j)\,e_k}{(e_i-e_k)\,e_j}\,e^{(e_j-e_k)(u-v/e_j e_k)},$$

where i, j, and k are any three distinct numbers of the set 1, 2, 3, 4. If we take the logarithm of each side, set (i, j, k) equal in turn to (1, 2, 3), (2, 3, 4), and (3, 4, 1), and eliminate u and v from the three resulting equations, we obtain, as the equation of another complex to which our congruence belongs,

$$\frac{e_2(e_3-e_1)}{(e_2-e_3)}\log\left[\frac{(e_1-e_3)\,e_2\,\omega_{12}}{(e_1-e_2)\,e_3\,\omega_{13}}\right]+\frac{(e_4\,e_1-e_2\,e_3)}{(e_3-e_4)}\log\left[\frac{(e_4-e_2)\,e_3\,\omega_{23}}{(e_2-e_3)\,e_4\,\omega_{42}}\right]$$
$$+\frac{e_1(e_2-e_4)}{(e_4-e_1)}\log\left[\frac{(e_1-e_3)\,e_4\,\omega_{34}}{(e_3-e_4)\,e_1\,\omega_{23}}\right]=0.$$

If the roots of equation (23) are not all distinct, equations (15) can still be easily solved, and it can be shown in each case that the congruence belongs to a quadratic complex. In each case the congruence belongs to a complex obtained from a tetrahedral complex by making some or all of the faces of the fundamental tetrahedron approach coincidence.

From equations (I, 11) and (I, 12) it is evident that the differential equations of the 1st and (-1)st Laplace transforms of the congruence (15), and hence of all the Laplace transforms, are identical with equations (15). Hence each Laplace transform is projective to the original congruence, corresponding lines being determined by the same pair of values of u and v.

3. Conversely, if a congruence is projective to its 1st and (-1)st Laplace transforms, corresponding lines being given by the same pair of values of u and v, the congruence is of the type discussed in this section.

This hypothesis implies that there exist functions λ and λ' of u only and μ and μ' of v only such that transformations (I,5) will convert the differential equations of the original congruence into those of its 1st Laplace transform if the functions λ, μ are used, and into those of its (-1)st Laplace transform if the functions λ', μ' are used. Hence, in Wilczynski's notation,

$$(26)\qquad \begin{aligned} m_1 &= m\left(mn - \frac{\partial^2}{\partial u\,\partial v}\log m\right) = \frac{\mu m}{\lambda}, \\ n_1 &= \frac{1}{m} = \frac{\lambda n}{\mu}, \\ c_1' &= \frac{1}{d} = \frac{\lambda}{\mu} c', \\ d_1 &= d\left(c'd - \frac{\partial^2}{\partial u\,\partial v}\log d\right) = \frac{\mu d}{\lambda}, \end{aligned}$$

$$(27)\qquad \begin{aligned} m_{-1} &= \frac{1}{n} = \frac{\mu' m}{\lambda'}, \\ n_{-1} &= n\left(mn - \frac{\partial^2}{\partial u\,\partial v}\log n\right) = \frac{\lambda' n}{\mu'}, \\ c'_{-1} &= c'\left(c'd - \frac{\partial^2}{\partial u\,\partial v}\log c'\right) = \frac{\lambda' c'}{\mu'}, \\ d_{-1} &= \frac{1}{c'} = \frac{\mu' d}{\lambda'}. \end{aligned}$$

From equations (26) and (27)

$$(28)\qquad \begin{aligned} mn - \frac{\partial^2}{\partial u\,\partial v}\log m &= mn = \frac{\mu}{\lambda}, \\ mn - \frac{\partial^2}{\partial u\,\partial v}\log n &= mn = \frac{\lambda'}{\mu'}, \\ c'd - \frac{\partial^2}{\partial u\,\partial v}\log c' &= c'd = \frac{\lambda'}{\mu'}, \\ c'd - \frac{\partial^2}{\partial u\,\partial v}\log d &= c'd = \frac{\mu}{\lambda}. \end{aligned}$$

Equations (28) imply that each of the coefficients m, n, c', d is the product of a function of u alone by a function of v alone. Hence it is possible by means of transformations of types (I,4) and (I,5) to reduce m

and n to unity; let us assume that this has already been done, and that equations (26), (27) and (28) apply to the coefficients in this form. Thus

$$
\begin{aligned}
m &\equiv n \equiv 1, \\
\lambda &\equiv \mu, \\
\lambda' &\equiv \mu'.
\end{aligned}
\tag{29}
$$

Hence λ, μ, λ', and μ' are constants.

Next note that

$$
\begin{aligned}
c_1 &= f_u + \frac{\partial}{\partial u} \log\left(\frac{d}{m}\right) = f_u - \frac{2\lambda_u}{\lambda}, \\
d'_1 &= f_v + \frac{\partial}{\partial v} \log\left(\frac{d}{m}\right) = f_v - \frac{2\mu_v}{\mu}, \\
c_{-1} &= f_u + \frac{\partial}{\partial u} \log\left(\frac{c'}{n}\right) = f_u - \frac{2\lambda'_u}{\lambda'}, \\
d'_{-1} &= f_v + \frac{\partial}{\partial v} \log\left(\frac{c'}{n}\right) = f_v - \frac{2\mu'_v}{\mu'}.
\end{aligned}
\tag{30}
$$

From (29), (30) and the fact that λ, μ, λ', and μ' are constants, it follows that c' and d are constants. From (28) and (29)

$$
c' d \equiv 1. \tag{31}
$$

Hence, according to (I, 7) it is possible to reduce c' and d to unity without disturbing m and n. Let us assume that this has already been done, and that in equations (26), (27), (28), (29), (30), and (31)

$$
m \equiv n \equiv c' \equiv d \equiv 1. \tag{32}
$$

Since

$$
\begin{aligned}
a_1 &= a + f_{uu} = a, \\
b'_{-1} &= b' + f_{vv} = b',
\end{aligned}
\tag{33}
$$

and since, from (29) and (31),

$$
f_{uv} = mn - c'd = 0, \tag{34}
$$

it follows that f_u and f_v are constants,

$$
\begin{aligned}
f_u &= 4j_4, \\
f_v &= 4j_5.
\end{aligned}
\tag{35}
$$

From the first and second of the integrability conditions (I, 2)

$$a' = -f_u = -4j_4, \qquad b = -f_v = -4j_5. \tag{36}$$

From the fourth, fifth, sixth, and seventh of the integrability conditions (I, 2)

$$a = k, \qquad b' = -k, \tag{37}$$

where k is a constant.

We can now easily prove the stronger theorem that if a congruence Γ is projective to its 1st Laplace transform Γ_1, each line of the original congruence corresponding to the line of the Laplace transform which touches a focal sheet at the same point, then the congruence has constant absolute invariants. For, in the first place, since the Laplace transform is projectively defined, any congruence projective to the given one is projective to the first Laplace transform of the former. Hence Γ_1 is projective to the second Laplace transform, Γ_2, of the original congruence. Thus Γ_1 is projective to its first Laplace transform Γ_2 and to its (-1)st Laplace transform Γ, with correspondence as described above. Hence, as we have just shown, Γ must be projective to its 1st and (-1)st Laplace transforms, and therefore, by the theorem just proved, Γ has constant absolute invariants, with $\overset{(y)}{\mathfrak{B}} \equiv \overset{(z)}{\mathfrak{B}}$, and $\overset{(y)}{\mathfrak{C}}{}'' \equiv \overset{(z)}{\mathfrak{C}}{}''$.

IV. Case 3, $\overset{(y)}{\mathfrak{B}} \equiv \overset{(z)}{\mathfrak{B}}$, $\overset{(y)}{\mathfrak{C}}{}'' \not\equiv \overset{(z)}{\mathfrak{C}}{}''$

1. It is convenient to use the absolute invariants

$$\begin{aligned} i_1 &= \frac{c'd}{mn}, \\ j_4 &= \frac{\overset{(y)}{\mathfrak{B}}}{m^{1/4}\, n^{1/2}\, d^{1/4}}, \\ j_5 &= \frac{\overset{(z)}{\mathfrak{C}}{}''}{m^{1/2}\, n^{1/4}\, c'^{1/4}}, \\ j_6 &= \frac{\overset{(y)}{\mathfrak{C}}{}''}{m^{1/2}\, n^{1/4}\, c'^{1/4}}. \end{aligned} \tag{1}$$

Since

$$8\,(\overset{(z)}{\mathfrak{B}} - \overset{(y)}{\mathfrak{B}}) = \frac{\partial}{\partial u} \log (c'^8 d m^3 n) \equiv 0, \tag{2}$$

we can, by a suitable transformation of v, make

$$(3) \qquad c'^3 dm^3 n = \frac{i_1}{(j_6 - j_5)^8 v^8}.$$

Let us assume that this has already been done. Then from equations (1) and (3)

$$(4) \qquad \frac{\partial}{\partial v} \log (c' d^3 mn^3) = 8\, m^{1/2} n^{1/4} c'^{1/4} (j_6 - j_5) = \frac{8}{v}.$$

Hence, by a transformation of u, we can make

$$(5) \qquad c' d^3 mn^3 = v^8$$

without altering equation (3). Let us assume that this has already been done. From (3), (5), and the first of (1)

$$(6) \qquad \begin{aligned} c' d &= \frac{i_1^{5/8}}{j_6 - j_5}, \\ mn &= \frac{i_1^{-3/8}}{j_6 - j_5}, \\ c' m &= \frac{i_1^{3/8}}{(j_6 - j_5)^3 v^4}, \\ dn &= i_1^{-1/8} (j_6 - j_5) v^4. \end{aligned}$$

From equations (1) and (6)

$$(7) \qquad \begin{aligned} \overset{(y)}{\mathfrak{B}} &= \overset{(z)}{\mathfrak{B}} = i_1^{-1/8} j_4 v, \\ \overset{(y)}{\mathfrak{C}}{}'' &= \frac{j_5}{(j_6 - j_5) v}, \\ \overset{(z)}{\mathfrak{C}}{}'' &= \frac{j_6}{(j_6 - j_5) v}. \end{aligned}$$

From equations (1) and (7)

$$(8) \qquad \frac{\partial^2}{\partial u\, \partial v} \log (dm) = 2 \frac{\partial \overset{(y)}{\mathfrak{C}}{}''}{\partial u} - 2 \frac{\partial \overset{(y)}{\mathfrak{B}}}{\partial v} = \frac{-2 j_4}{i_1^{1/8}}.$$

Hence by a transformation of y and z, which leaves equations (6) and (7) unchanged, we can obtain

$$(9) \qquad dm = v^4 e^{-2kuv},$$

where

$$k = i_1^{-1/8} j_4. \tag{10}$$

From equations (6) and (9)

$$\begin{aligned} m &= \frac{e^{-kuv}}{i_1^{1/8}(j_6 - j_5)}, \\ n &= i_1^{-1/4} e^{kuv}, \\ c' &= \frac{i_1^{1/2} e^{kuv}}{(j_6 - j_5)^2 v^4}, \\ d &= i_1^{1/8}(j_6 - j_5) v^4 e^{-kuv}. \end{aligned} \tag{11}$$

From equations (I, 6), (7) and (11),

$$\begin{aligned} f_u &= 2kv, \\ f_v &= 2ku + \frac{6j_5 - 2j_6}{(j_6 - j_5)v}. \end{aligned} \tag{12}$$

The integrability conditions can be written

$$\begin{aligned} b &= i_1^{1/8}[k(j_5 - j_6)uv^4 - 2(j_6 + j_5)v^3]e^{-kuv}, \\ a' &= \frac{-3i_1^{1/2}ke^{kuv}}{(j_6 - j_5)^2 v^3}, \\ i_1^{-1/4} - i_1^{3/4} &= 2j_4(j_6 - j_5), \\ 3k^2v^2 + 6i_1^{1/4}(j_6^2 - j_5^2)v^2 &+ 2i_1^{1/4}k(j_6 - j_5)(j_6 - 3j_5)uv^3 - i_1^{1/4}k^2(j_6 - j_5)^2u^2v^4 \\ &= a + i_1^{1/4}(j_6 - j_5)^2 v^4 b', \\ -k^2(j_6 - j_5)^2u^2v^4 + 3i_1^{3/4}k^2v^2 &+ 2k(j_6 - j_5)(j_6 - 3j_5)uv^3 \\ &= i_1^{3/4}a + (j_6 - j_5)^2 v^4 b', \\ a_v &= \frac{3(i_1^{5/8} - i_1^{-3/8})kv}{(j_6 - j_5)} = -6k^2v, \\ b'_u &= \frac{(i_1 - 1)ku}{i_1^{3/8}(j_6 - j_5)} + \frac{2[i_1^{5/8}(j_6 + j_5)^2 + i_1^{-3/8}(j_6 - 3j_5)]}{(j_6 - j_5)^2 v}. \end{aligned} \tag{13}$$

If i_1 were equal to 1, j_4, and hence k, would vanish. It would then follow from (11) that $\overset{(y)}{\mathfrak{C}}{}'' \equiv \overset{(z)}{\mathfrak{C}}{}''$, which is contrary to hypothesis. Hence $i_1 \neq 1$ and the fourth and fifth of equations (13) give

$$
(14)\qquad
\begin{aligned}
a &= 3k^2v^2+\frac{6\,i_1^{1/4}(j_6^2-j_5^2)v^2}{(1-i_1)},\\
b' &= \frac{2k(j_6-3j_5)u}{(j_6-j_5)v}-\frac{6\,i_1(j_6+j_5)}{(1-i_1)(j_6-j_5)v^2}-k^2u^2.
\end{aligned}
$$

From the sixth of equations (13) and the first of (14)

$$
(15)\qquad j_6^2-j_5^2=i_1^{-1/4}k^2(i_1-1).
$$

From the seventh of (13) and the second of (14)

$$
(16)\qquad i_1^{3/8}k(j_6-3j_5)(j_6-j_5)-i_1(j_6+j_5)+(3j_5-j_6)=0.
$$

From the third of (13) and (15)

$$
(17)\qquad
\begin{aligned}
j_5 &= \frac{-j_4^3}{i_1^{1/4}}-\frac{(1-i_1)}{4\,i_1^{1/4}j_4},\\
j_6 &= \frac{-j_4^3}{i_1^{1/4}}+\frac{(1-i_1)}{4\,i_1^{1/4}j_4}.
\end{aligned}
$$

From (16) and (17)

$$
(18)\qquad 2j_4^4+1+i_1=0.
$$

From (17) and (18)

$$
(19)\qquad
\begin{aligned}
j_5 &= \frac{1+3i_1}{4\,i_1^{1/4}j_4},\\
j_6 &= \frac{3+i_1}{4\,i_1^{1/4}j_4}.
\end{aligned}
$$

Substituting equations (19) in (11), (12), (13), and (14), we find

$$
\begin{aligned}
m &= \frac{2\,i_1^{1/8}j_4e^{-kuv}}{(1-i_1)},\\
n &= i_1^{-1/4}e^{kuv},\\
a &= -3\,i_1^{-1/4}j_4^2v^2,\\
b &= \left[\frac{(i_1-1)}{2\,i_1^{1/4}}uv^4+\frac{4j_4^3v^3}{i_1^{1/8}}\right]e^{-kuv},
\end{aligned}
$$

$$
(20)\qquad
\begin{aligned}
c &= 2 i_1^{-1/8} j_4 v,\\
d &= \frac{(1-i_1) v^4 e^{-kuv}}{2 i_1^{1/8} j_4},\\
a' &= \frac{-12 i_1^{7/8} j_4^3 e^{kuv}}{(1-i_1)^2 v^3},\\
b' &= \frac{-12(1+i_1)}{(1-i_1)^2 v^2} - \frac{8 i_1^{7/8} j_4 u}{(1-i_1) v} - \frac{j_4^2 u^2}{i_1^{1/4}},\\
c' &= \frac{4 i_1 j_4^2 e^{kuv}}{(1-i_1)^2 v^4},\\
d' &= \frac{2 j_4 u}{i_1^{1/8}} + \frac{8 i_1}{(1-i_1) v}.
\end{aligned}
$$

Evidently, if the absolute invariant j_4 be given equal to a constant different from zero and from $\sqrt[4]{-1}$ and if definite values be assigned to the fractional powers of i_1, the congruence is determined except for projective transformations; for i_1, j_5, j_6, and k are determined by equations (18), (17), and (10), and the coefficients of the differential equations are determined by equations (20).

2. The differential equations of the two sheets of the focal surface are, according to equations (I, 8) and (I, 9),

$$
(21)\qquad
\begin{aligned}
y_{uu} - \frac{(1-i_1)^2 v^4}{4 i_1^{1/4} j_4^2} y_{vv} &= -3 i_1^{-1/4} j_4^2 v^2 y + 2 i_1^{-1/8} j_4 v y_u + \frac{2(1-i_1) j_4^2 v^3}{i_1^{1/4}} y_v,\\
y_{uv} &= \frac{2 j_4}{i_1^{1/8}(1-i_1)} y - i_1^{-1/8} j_4 v y_v;
\end{aligned}
$$

$$
(22)\qquad
\begin{aligned}
\frac{4 i_1^{5/4} j_4^2}{(1-i_1)^2 v^4} z_{uu} - z_{vv} &= \left[\frac{12(1+i_1)}{(1-i_1)^2 v^2} + \frac{8 i_1^{7/8} j_4 u}{(1-i_1) v} + \frac{j_4^2 u^2}{i_1^{1/4}}\right] z\\
&\quad + \frac{16 i_1^{9/8} j_4^3}{(1-i_1)^2 v^3} z_u - \left[\frac{2 j_4 u}{i_1^{1/8}} + \frac{8 i_1}{(1-i_1) v}\right] z_v\\
z_{uv} &= \frac{2 j_4}{i_1^{1/8}(1-i_1)} z + i_1^{-1/8} j_4 u z_u.
\end{aligned}
$$

From equations (21) we find the equations of the parametric curves on the surface S_y to be

$$y_{uuuu} = 0, \tag{23}$$

$$\begin{aligned} y_{vvvv} &+ \frac{4(1-3i_1)}{(1-i_1)v} y_{vvv} - \frac{12 i_1(1-3i_1)}{(1-i_1)^2 v^2} y_{vv} + \frac{8i_1(1-3i_1)(1+i_1)}{(1-i_1)^3 v^3} y_v \\ &- \frac{4i_1(1-3i_1)(1+i_1)}{(1-i_1)^4 v^4} y = 0. \end{aligned} \tag{24}$$

Evidently the curves $v =$ constant on S_y are cubics (obviously all projective to one another).

In order to simplify the computation and the resulting form of the differential equations of the parametric curves on the surface S_z we make the transformation

$$z = e^{kuv}\bar{z},$$

under which equations (22) become

$$\begin{aligned} \frac{4i_1^{5/4} j_4^2}{(1-i_1)^2 v^4} \bar{z}_{uu} - \bar{z}_{vv} &= \frac{6(1+i_1)(2-i_1)}{(1-i_1)^2 v^2} \bar{z} + \frac{8 i_1^{9/8} j_4^3}{(1-i_1)^2 v^3} \bar{z}_u \\ &\quad - \frac{8 i_1}{(1-i_1)v} \bar{z}_v \\ \bar{z}_{uv} &= \frac{2 j_4}{i_1^{1/8}(1-i_1)} \bar{z} - \frac{j_4 v}{i_1^{1/8}} \bar{z}_v . \end{aligned} \tag{25}$$

The differential equations of the parametric curves on S_z are

$$\begin{aligned} \bar{z}_{uuuu} &+ \frac{6(1-i_1) j_4^2 v^2}{i_1^{5/4}} \bar{z}_{uu} - \frac{2(1-i_1)(3+2i_1) v^3}{i_1^{11/8} j_4} \bar{z}_u \\ &- \frac{(1-i_1)(7+3i_1) v^4}{2 i_1^{3/2}} \bar{z} = 0; \end{aligned} \tag{26}$$

$$\begin{aligned} \bar{z}_{vvvv} &+ \frac{8(1-2i_1)}{(1-i_1)v} \bar{z}_{vvv} + \frac{12(2-5i_1+5i_1^2)}{(1-i_1)^2 v^2} \bar{z}_{vv} \\ &+ \frac{4(12-17i_1+8i_1^2-11i_1^3)}{(1-i_1)^3 v^3} \bar{z}_v \\ &+ \frac{4(1+i_1)(6-20i_1+19i_1^2-3i_1^3)}{(1+i_1)^4 v^4} \bar{z} = 0. \end{aligned} \tag{27}$$

It is easily seen that each of the differential equations (23), (24), (26), and (27) has constant absolute invariants, and therefore represents a family of projectively equivalent anharmonic curves. Hence all the developable surfaces of each family are projectively equivalent. Furthermore, equations (24) and (27) have seminvariants independent of u; hence, as in III, since the differential equation of any curve $u =$ constant on either sheet of the focal surface can be transformed into the differential equation of any other curve $u =$ constant on the same sheet by means of a transformation of the type $y = \lambda \overline{y}$ or $z = \lambda \overline{z}$, where λ is a function of u und v, it follows that, in the projective correspondence between any two curves $u =$ constant on the same sheet of the focal surface, any pair of corresponding points lie on the same curve $v =$ constant on the focal sheet in question. Similarly, the curves $v =$ constant on S_y are projectively equivalent, corresponding points lying on the same curve $u =$ constant on S_y.

By means of formulas (I,10), (I,11), and (I,12) it can easily be shown that both the 1st and the (-1)st Laplace transforms of our congruence are of the same type as the original congruence (i. e., have constant absolute invariants), with $\overset{(y)}{\mathfrak{B}} \equiv \overset{(z)}{\mathfrak{B}}$, $\overset{(y)}{\mathfrak{C}''} \not\equiv \overset{(z)}{\mathfrak{C}''}$ but are not projective to it. If $i_1 = 3$ the sheet $S_{y^{(1)}}$ of the focal surface of the 1st Laplace transform degenerates into a curve, and if $i_1 = 1/3$ the sheet $S_{z^{(-1)}}$ of the focal surface of the (-1)st Laplace transform degenerates into a curve.

The above discussion can be applied to congruences having $\overset{(y)}{\mathfrak{B}} \not\equiv \overset{(z)}{\mathfrak{B}}$, $\overset{(y)}{\mathfrak{C}''} \equiv \overset{(z)}{\mathfrak{C}''}$ by interchanging u and v, y and z, and making other corresponding changes.

V. Summary of results

It is assumed throughout this thesis that the absolute invariants of the congruences discussed are constants and that m, n, c', and d are all different from zero. The congruences divide themselves into three main types in which, respectively, neither, both, or only one of the relative invariants $(\overset{(z)}{\mathfrak{B}} - \overset{(y)}{\mathfrak{B}})$, $(\overset{(y)}{\mathfrak{C}''} - \overset{(z)}{\mathfrak{C}''}) \equiv 0$.

In the first case, that in which neither $(\overset{(z)}{\mathfrak{B}} - \overset{(y)}{\mathfrak{B}})$ nor $(\overset{(y)}{\mathfrak{C}''} - \overset{(z)}{\mathfrak{C}''}) \equiv 0$, if the congruences are not W congruences the differential equations can be reduced by a suitable transformation of the variables to a form in which the coefficients are equal to constants (depending only on the absolute invariants) multiplied by powers of $(i_4 u + i_5 v)$. Conversely, any system of differential equations of form (I, D) with coefficients of this form satisfying the integrability conditions represents a family of projectively equivalent congruences with constant absolute invariants. W congruences of this type depend on an additional constant independent of the absolute invariants. Any congruence of the first type has all its Laplace transforms of the same type, but projectively distinct. Both sheets of the focal surface and the cuspidal edges of all the developable surfaces have constant absolute invariants, except in the case of some of the W congruences.

In the second case (i. e., if $(\overset{(z)}{\mathfrak{B}} - \overset{(y)}{\mathfrak{B}}) \equiv (\overset{(y)}{\mathfrak{C}}'' - \overset{(z)}{\mathfrak{C}}'') \equiv 0$) the differential equations can be reduced by a suitable transformation of the variables to a set with constant coefficients depending on the absolute invariants and on one additional arbitrary constant. Conversely, every congruence whose differential equations have constant coefficients satisfying the integrability conditions has constant absolute invariants, with $(\overset{(z)}{\mathfrak{B}} - \overset{(y)}{\mathfrak{B}}) \equiv (\overset{(y)}{\mathfrak{C}}'' - \overset{(z)}{\mathfrak{C}}'') \equiv 0$. Furthermore, every congruence of this type is a W congruence and is projectively equivalent to its 1st and (-1)st Laplace transforms, corresponding lines being tangent to the common focal sheet at the same point; conversely, every congruence which is projective in this way to its 1st or (-1)st Laplace transform has constant absolute invariants, with $(\overset{(z)}{\mathfrak{B}} - \overset{(y)}{\mathfrak{B}}) \equiv (\overset{(y)}{\mathfrak{C}}'' - \overset{(z)}{\mathfrak{C}}'') \equiv 0$. Every congruence of this type belongs to a quadratic complex. Both sheets of the focal surface and the cuspidal edges of all the developable surfaces have constant absolute invariants. The developable surfaces of each family are projective to one another.

A congruence of the third type, having $(\overset{(z)}{\mathfrak{B}} - \overset{(y)}{\mathfrak{B}}) \equiv 0$ but $(\overset{(y)}{\mathfrak{C}}'' - \overset{(z)}{\mathfrak{C}}'') \not\equiv 0$, is determined (except for projective transformations) by one of its absolute invariants, j_4. The developable surfaces of each family are projective to one another. The 1st and (-1)st Laplace transforms are projectively distinct from one another and from the original congruence.

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