A GENERAL THEORY OF LINEAR SETS*

BY

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Introduction

Section I of the following paper, though using the postulational method, is motivated by the consideration of classes of vectors, on a finite range $P^n = (1, 2, \cdots, n)$, whose elements belong to a general division algebra or, as we shall say, number system.

Section II deals only with vectors on a finite range.

Section I is also of use as giving a general basis preliminary to the more intensive study of

(a) classes of vectors on a general range,

(b) number systems over a division number system; that is, to the initiation of a theory analogous to that of an "algebra over a field," where the field is replaced by an associative division number system.

Notation. Throughout the paper certain logical notations† will be used as follows:

$=$	logical identity
$\neq$	logical diversity
$\equiv$	definitional identity
$:\equiv:$	definitional identity between statements
$.).$	implies
$.\sim.$	is equivalent to
$.\ni.$	such that
$\exists$	there exists
$\vert$	is unique, used before the element which is unique: thus, $\vert a$ means a is unique.
$.$	and
$.\cup.$	or
—	not
. : .: :: etc.	punctuation signs; the principal implication of a sentence has its sign accompanied by the largest number of periods, thus $a :): b .). c$ is a statement that a implies that (b implies c) whereas $a .). b :): c$ states that the implication a implies b, implies the fact c. We may also use punctuation to show continued implication, thus $a .). b .). c$ means $a .). b$ and $b .). c$.

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† These signs are mostly taken from E. H. Moore's *Introduction to a Form of General Analysis*. Yale University Press, 1910, p. 150.

$\sim$	corresponds to
$*(a)$	the statement $*$ holds for every a
$[\ ]$	class of elements. A non-vacuous class we call a set.
$\bigcap[P]$	the greatest common subclass of the classes P of the class $[P]$ of classes
$\bigcup[P]$	the least common superclass of the classes P of the class $[P]$ of classes
$\supset$	inclusion. In speaking of classes M and N, $M \supset N$ means M includes N, in the sense that every element of N is an element of M. This may also be written $N \subset M$.

The principal results will frequently be stated both in logical notation and in the written form; proofs, however, will as far as practical be given in logical notation only.

In dealing with subsets of the fundamental classes $\mathfrak{A}$, U, V, etc.,

$$\mathfrak{A}_0 \equiv [a_0], \quad \mathfrak{A}_1 \equiv [a_1], \quad \cdots, \quad U_0 \equiv [u_0], \quad \cdots, \quad V_0 \equiv [v_0], \quad \cdots \quad \text{etc.}$$

as subsets of $\mathfrak{A}, U, V$ etc. respectively.

We shall use exponents to denote properties of an entity a; for example $\mathfrak{A}^A$ denotes that $\mathfrak{A}$ is of type A. When we use the notation for a class as an exponent of an element we shall mean that the element belongs to the given class; thus u_0^U means that u_0 is a member of the class U.

If we have a single valued function, f, of n independent variables whose values belong to the ranges $P_1, \cdots, P_n$, and the functional values of f belong to a class M then we say that the function f is on $P_1 \cdots P_n$ to M, that is $f^{\text{on } P_1 \cdots P_n \text{ to } M}$.

Number systems of type A. We will consider a number system which is a generalization of a "division algebra." We will define what we mean by a number system, $\mathfrak{A}$, being of type A, in such a way that whenever multiplication between every two elements of $\mathfrak{A}$ is commutative, it follows that $\mathfrak{A}$ is a field. If $\mathfrak{A}$ is a field we will say $\mathfrak{A}^F$. However as we do not assume that multiplication is commutative, there is introduced both a right and left distributive law.

We say that a system $\mathfrak{A}$ is a number system of *type A* or symbolically $\mathfrak{A}^A$ if it is of the following type:†

† This definition of a number system of type A is based directly on that used by E. H. Moore in his course in General Analysis. A number system of type A has properties 1–11 as given for real numbers by D. Hilbert but does not necessarily fulfill conditions 12–17. See D. Hilbert, *Grundlagen der Geometrie,* p. 35, 3d edition, 1909, or the translation by E. J. Townsend, *The Foundations of Geometry,* Open Court Publishing Company, 1902, p. 37.

$\mathfrak{A}$ contains at least two distinct elements;

There is an addition function, $+$, on $\mathfrak{A}\mathfrak{A}$ to $\mathfrak{A}$, which forms a commutative group, with identity of addition $\equiv 0$;

There is a multiplication function, $\times$, on $\mathfrak{A}\mathfrak{A}$ to $\mathfrak{A}$ which obeys the following restrictions:

(1) $0\times a = 0 = a\times 0 \quad (a)$;

(2) $\times$ on $\mathfrak{A}$ except 0 forms a group (not supposed commutative);

(3) $a_1\times a_2 = 0 :): a_1 = 0 .\cup. a_2 = 0$;

(4) The identity of multiplication $\equiv 1$;

(5) $a_1\times(a_2+a_3) = (a_1\times a_2)+(a_1\times a_3) \quad (a_1, a_2, a_3)$,

$(a_2+a_3)\times a_1 = (a_2\times a_1)+(a_3\times a_1) \quad (a_1, a_2, a_3)$.

We will call such a system an associative division number system.

For simplicity of notation we will write $a_1\times a_2 = a_1 a_2$. We will use the exponential method of denoting reciprocals thus:

$$a_1 a_2 = a_2 a_1 = 1 :\equiv: a_2^{-1} = a_1 \,.\, a_1^{-1} = a_2.$$

For sake of clarity a few examples of number systems of type A will be given.

Ex. 1. All real rational numbers.

Ex. 2. The system, R, of all real numbers.

Ex. 3. The system, C, of all complex numbers.

Ex. 4. The system, Q, of all real quaternions.

Ex. 5. Any Galois field $GF[p_n]$;

e. g. for $n = 1$ the rational numbers modulo p.

Ex. 6.* The Hilbert example of a non-Archimedian Veronesean number system.

Consider $P\equiv(\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots)$ and a number system $\mathfrak{A}^A$.

Consider $F\equiv$ [all single valued functions $f^{\cdot \text{on } P \text{ to } \mathfrak{A}} ::\ni:: f .:):. \exists\, n_f :\ni: n \leqq n_f .). f(n) = 0$].

For a definiton of addition we have

$$f = f_1+f_2 :\equiv: f(n) = f_1(n)+f_2(n) \quad (n).$$

For a definition of multiplication we have

$$f = f_1 f_2 :\equiv: f(n) = \sum_{jk}^{j+k=n} f_1(j) f_2(k) \quad (n).$$

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* This example is developed by E. H. Moore in his course in General Analysis. For $\mathfrak{A} = R$ this is Hilbert's example of a non-Archimedian Veronesean number system. Loc. cit., p. 31, or trans., p. 34.

The identity of addition is $[f_0(n) = 0 \quad (n)]$.

The identity of multiplication is $[f_1(0) = 1 . f_1(n) = 0 \ (n \neq 0)]$.

F is of type A.

Ex. 7.* Consider $P = (\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots)$ and a number system $\mathfrak{A}$ of type A.

Consider $F[\text{all } f^{\text{on } PP \text{ to } \mathfrak{A}} .::\ni::. \exists\, n_f .:\ni:. n_1 \leqq n_f . n_2 .). f(n_1 n_2) = 0 :: n .:):. \exists\, n_{fn} :\ni: n_3 \leqq n_{fn} .). f(n n_3) = 0]$.

For a definition of addition we have

$$f = f_1 + f_2 :\equiv: f(n_1, n_2) = f_1(n_1, n_2) + f_2(n_1, n_2) \quad (n_1, n_2).$$

For a definition of multiplication we have

$$f = f_1 f_2 :\equiv: f(n_1 n_2) = \sum_{k_1 k_2 m_1 m_2}^{\substack{k_1 + m_1 = n_1 \\ k_2 + m_2 = n_2}} a^{k_1 m_2} f_1(k_1, k_2) f_2(m_1, m_2) \quad (n_1, n_2),$$

where a is a number of $\mathfrak{A}$.

Our identity of addition is $[f_0(n_1, n_2) = 0 \ (n_1, n_2)]$.

Our identity of multiplication is $[f_1(0,0) = 1 . f_1(n_1, n_2) = 0 (n_1 \neq 0 .\cup. n_2 \neq 0)]$.

F is of type A.†

Theorem 1. *If $\mathfrak{A}$ is of type A, and $\mathfrak{A}_0$ is a subset of A, then the totality $\mathfrak{A}_{0c}$, of numbers of $\mathfrak{A}$ which are commutative as to multiplication with all the numbers of $\mathfrak{A}_0$, forms with the original addition and multiplication a number system of type A*:

$$\text{Th. 1.} \quad \mathfrak{A}^A . \mathfrak{A}_0 = [a_0] . \mathfrak{A}_{0c} = [\text{all } a :\ni: a_0 .). a a_0 = a_0 a] :): \mathfrak{A}_{0c}^A.$$

The proof is evident when we note that $a^{\mathfrak{A}_{0c}} .). (a^{-1})^{\mathfrak{A}_{0c}}$ for $a a_0 = a_0 a .). a_0 = a^{-1} a_0 a .). a_0 a^{-1} = a^{-1} a_0 \ (a_0)$.

Corollary. *If $\mathfrak{A}$ is of type A, then the totality $\mathfrak{A}'$ of numbers of $\mathfrak{A}$ which are commutative as to multiplication with all the numbers of $\mathfrak{A}$ forms, with the original addition and multiplication, a field*:

$$\text{Cor.} \qquad \mathfrak{A}^A . \mathfrak{A}' = [\text{all } a :\ni: a_1 .). a a_1 = a_1 a] :): \mathfrak{A}'^F.$$

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* A special case of this example is given by Hilbert, loc. cit., p. 100, or trans., p. 103. J. H. M. Wedderburn brought this example to my attention in a more general form than that of Hilbert.

† The proof of this directly follows Hilbert's proof. It should be noted that in both Ex. 6 and Ex. 7 Hilbert uses formal power series in one or two symbolic parameters respectively, rather than the functional notation used here.

It should be noted that $\mathfrak{A}'$ is not necessarily a maximal field in $\mathfrak{A}$. If, for example, $\mathfrak{A} = Q$ then $\mathfrak{A}' =$ scalars but all the quaternions such that the coefficients of j and k are 0 form a field isomorphic with C and containing the scalars.

$\mathfrak{A}$ may be of infinite rank in respect to $\mathfrak{A}'$ as is shown by Example 7. Thus a number system of finite rank as regards a number system $\mathfrak{A}$ might be of infinite rank as regards the field $\mathfrak{A}'$ of which $\mathfrak{A}$ is an extension.

I. The theory of linear sets

Contents

1. **General postulational basis.** In Section I we consider a system

$$\Sigma \equiv (\mathfrak{A}, U \equiv [u], \oplus, \odot_r, \odot_l),$$

viz., a number system $\mathfrak{A}$ of type A, a class U of (abstract vectors or) elements u and three processes or functions $\oplus$, $\odot_r$, $\odot_l$, serving to connect numbers a and elements, u; as follows:

1. U has at least two distinct elements.
2. $\oplus$ is a function on UU to U which forms a commutative group with identity element 0_U; $u = u_1 \oplus u_2$, u is the sum of u_1 and u_2.
3. $\odot_r$ is a function on $U\mathfrak{A}$ to U; $u = u_1 \odot_r a$; u is the product of u_1 by a (on the right).
4. $\odot_l$ is a function on $\mathfrak{A}U$ to U; $u = a \odot_l u_1$; u is the product by a (on the left) of u_1.
5. $a\,.).\, a \odot_l 0_U = 0_U = 0_U \odot_r a$.
6. $u\,.).\, 0 \odot_l u = 0_U = u \odot_r 0$.
7. $u\,.).\, u \odot_r 1 = u = 1 \odot_l u$.
8. Associative law of addition,

$$u_1\, u_2\, u_3\,.).\, (u_1 \oplus u_2) \oplus u_3 = u_1 \oplus (u_2 \oplus u_3).$$

9. Distributive law,

$$\begin{aligned} u_1\, u_2\, a_1\, a_2\,.).\, \text{(a)}\ & (u_1 \odot_r a_1) \oplus (u_1 \odot_r a_2) = u_1 \odot_r (a_1 + a_2), \\ \text{(b)}\ & (a_1 \odot_l u_1) \oplus (a_2 \odot_l u_1) = (a_1 + a_2) \odot_l u_1, \\ \text{(c)}\ & (u_1 \odot_r a_1) \oplus (u_2 \odot_r a_1) = (u_1 \oplus u_2) \odot_r a_1, \\ \text{(d)}\ & (a_1 \odot_l u_1) \oplus (a_1 \odot_l u_2) = a_1 \odot_l (u_1 \oplus u_2). \end{aligned}$$

10. Associative law of multiplication,

$$u\, a_1\, a_2\, .).\ \text{(a)}\ (u \odot_r a_1) \odot_r a_2 = u \odot_r (a_1\, a_2),$$
$$\text{(b)}\ a_1 \odot_l (a_2 \odot_l u) = (a_1\, a_2) \odot_l u,$$
$$\text{(c)}\ (a_1 \odot_l u) \odot_r a_2 = a_1 \odot_l (u \odot_r a_2).$$

From 9 and 10 the general distributive and associative laws follow. As it will not lead to ambiguity we will simplify the notations as follows:

$$\begin{array}{rcll} u_1 \oplus u_2 & \equiv & u_1 + u_2 & (u_1\, u_2); \\ u \odot_r a & \equiv & ua & (u\, a); \\ a \odot_l u & \equiv & au & (u\, a); \\ (ua_1)\, a_2 & = & u\,(a_1\, a_2) \equiv u a_1\, a_2 & (u\, a_1\, a_2); \\ a_1\,(a_2\, u) & = & (a_1\, a_2)\, u \equiv a_1\, a_2\, u & (u\, a_1\, a_2); \\ a_1\,(ua_2) & = & (a_1\, u)\, a_2 \equiv a_1\, u a_2 & (u\, a_1\, a_2). \end{array}$$

There follow two examples of a system Σ.

Ex. 1.* Any algebra over a field.

It should be noted that our system Σ is more general in that we have not limited $\mathfrak{A}$ to be a field and we have not required the existence of a multiplication process between the elements of U ($\odot$ on UU to U).

Ex. 2. If we are given a general range P and a number system $\mathfrak{A}$ of type A, then we may take as U the set F of all vectors f (single valued functions), on P to $\mathfrak{A}$,—

$$\begin{array}{lll} F \equiv [\text{all } f^{\text{on } P \text{ to } \mathfrak{A}}], \text{ with} & & \\ f = f_1 + f_2 & :\equiv: f(p) = f_1(p) + f_2(p) & (p), \\ f = af_1 & :\equiv: f(p) = af_1(p) & (p), \\ f = f_1 a & :\equiv: f(p) = f_1(p)\, a & (p). \end{array}$$

The symmetry between $\odot_r$ and $\odot_l$, and the symmetry between right and left multiplication in $\mathfrak{A}$ should be noted. Each theorem will carry with it a theorem by parity (not always different). As a convention, theorems involving only one type of multiplication will be stated in terms of right hand multiplication.

From Postulate 2 we know that

$$u :) : \exists \,|\, u_1 \,.\!\ni\!.\, u + u_1 = u_1 + u = 0_U;$$

* See L. E. Dickson, *Algebras and their Arithmetics*, University of Chicago Press, 1923, p. 9.

this uniquely existing u_1 we designate the *negative* of u, in notation, $-u$, so that

$$u + u_1 = 0_U \;:=:\; u_1 = -u.$$

THEOREM 1. *The negative of any element of U is that element multiplied on either right or left by the number* -1:

TH. 1. $$u\,.).\,(-1)u = -u = u(-1).$$

Proof. $u = 1u$; then by Postulates 7, 8 and 9,

$$u + (-1)u = 1u + (-1)u = (1-1)u = 0u = 0_U.$$

THEOREM 2. *If the product of an element u of U by a number a of $\mathfrak{A}$ is 0_U then either a is 0 or u is 0_U or both*:

TH. 2. $$au = 0_U :):\, a = 0 \,.\cup.\, u = 0_U.$$

Proof. $a \neq 0 :):\, au = 0_U \,.).\, u = a^{-1}au = 0_U$.

THEOREM 3. *Relative to a subset U_0 of U, the totality $\mathfrak{A}_0$ of numbers a of $\mathfrak{A}$ which are commutative as to multiplication with all the elements of U_0 forms, with the original addition and multiplication, a number system A_0 of type A*:

TH. 3. $$U_0 :):\, (\mathfrak{A}_0 = [\text{all } a :\ni: u_0 \,.).\, au_0 = u_0 a])^A.$$

Proof.

(1) $a_1 u = ua_1 \,.\, a_2 u = ua_2 :):\, (a_1 + a_2)u = u(a_1 + a_2) \,.\, a_1 a_2 u = ua_1 a_2$,

for

$$(a_1 + a_2)u = a_1 u + a_2 u = ua_1 + ua_2 = u(a_1 + a_2),$$

$$(a_1 a_2)u = a_1(a_2 u) = a_1(ua_2) = (a_1 u)a_2 = (ua_1)a_2 = u(a_1 a_2).$$

(2) $$au = ua \,.).\, a^{-1}u = ua^{-1}$$

for

$$u = a^{-1}ua \text{ and therefore } a^{-1}u = ua^{-1}.$$

(3) The set $\mathfrak{A}_0$ contains at least two numbers, for it contains 0 and 1.

Other properties of $\mathfrak{A}^A$ may be readily checked.

Definition. A set U_0 is said to be commutative, U_0^c, in case every number a is commutative with every element u_0 of U_0:

Def. $$U_0^c :=: au_0 \,.).\, au_0 = u_0 a.$$

2. Sum and intersection of two sets, supplementary sets, additive sets, linear sets.

Definition of the *sum* of two sets U_1, U_2.

$$U_1 + U_2 \equiv [\text{all } u :\ni: \exists\, u_1 u_2 .\ni.\, u = u_1 + u_2].$$

Definition of the *intersection* of two sets U_1, U_2. The intersection of $U_1 U_2$, $\cap\, [U_1 U_2]$, is the greatest common subset of U_1 and U_2.

Definition of *supplementary sets.* Two sets U_1 and U_2 are supplementary, $(U_1 U_2)^{\text{sup}}$, if their sum is U and their intersection is the set whose only element is 0_U.

Definitions* of *right linear* (rl), *left linear* (ll),† *properly linear* (l), and *additive*‡ (ad) sets.

$$U_0^{rl} :\equiv: a_1\, a_2\, u_{01}\, u_{02}\, .).\, (u_{01}\, a_1 + u_{02}\, a_2) \text{ belongs to } U_0;$$

$$U_0^{ll} :\equiv: a_1\, a_2\, u_{01}\, u_{02}\, .).\, (a_1\, u_{01} + a_2\, u_{02}) \text{ belongs to } U_0;$$

$$U_0^{l} :\equiv: U_0^{rl.\,ll};$$

$$U_0^{ad} :\equiv: u_{01}\, u_{02}\, .).\, (u_{01} + u_{02}) \text{ belongs to } U_0 \text{ and } u_0\, .).\, (-u_0) \text{ belongs to } U_0.$$

It should be noted that any properly linear subset U_0 of U, other than the set consisting of the single element 0_U, together with the number system $\mathfrak{A}$ and the original definition of addition and multiplication forms a system Σ satisfying the postulates of § 1. In the sequel we shall often make use of this fact by applying theorems stated in terms of U to a properly linear subset U_0 of U.

* We have not included in the text a definition of a linear set. A satisfactory definition of linearity would be such that any right (left) linear set is a linear set, and in case U is commutative should reduce to the definition of right (left) linearity. In arriving at such a definition we make use of the number system $\mathfrak{A}_c$ consisting of all numbers a of $\mathfrak{A}$ which are commutative with every element u of U. $\mathfrak{A}_c$ is a field contained in $\mathfrak{A}'$, for by § 1, Theorem 3, $\mathfrak{A}_c$ is of type $\mathfrak{A}$, and

$$a_c\, a\, u\, .).\, (a_c\, a)\, u = a_c\, (au) = (au)\, a_c = a\, (u a_c) = a\, (a_c u) = (a a_c)\, u\, .).\, a_c\, a = a a_c.$$

We will say that a subset U_0 of U is linear in case for every pair of numbers a_{c1} and a_{c2} in $\mathfrak{A}_c$ and every pair of elements u_{01} and u_{02} of U_0 the element $u_{01}\, a_{c1} + u_{02}\, a_{c2}$ belongs to U_0.

† Note the symmetry of right and left linearity. When a theorem concerns only one of the two we will state it in terms of right linearity and omit the parity theorem in terms of left linearity.

‡ This is a strong form of the definition. One might use the first condition alone as a weaker form.

THEOREM 1. *U is properly linear.*

THEOREM 2. *A right (or left) linear set is additive.*

THEOREM 3. *Every additive set contains 0_U; hence, the intersection of two additive sets is non-vacuous, and each of two additive sets is contained in their sum.*

THEOREM 4. *If two sets U_1 and U_2 are additive (right linear, left linear or properly linear) then their sum and intersection are additive (right linear, left linear or properly linear).*

THEOREM 5. *If U_1 and U_2 are additive sets and their intersection is the set whose only element is 0_U, then any element in their sum can be expressed in one and only one way as the sum of one element of U_1 and one element of U_2:*

$$\text{TH. 5.}\qquad U_1^{ad} \,.\, U_2^{ad} \,.\, \cap\,[U_1\, U_2] = \text{the class } 0_U$$
$$.:):.\; u_{11}^{U_1} \,.\, u_{12}^{U_1} \,.\, u_{21}^{U_2} \,.\, u_{22}^{U_2} \,.\, u_{11} + u_{21} = u_{12} + u_{22}$$
$$:):\; u_{11} = u_{12} \,.\, u_{21} = u_{22}\,.$$

Proof. $u_{11} - u_{12} = u_{22} - u_{21}$. Then, since U_1 and U_2 are additive, $u_{11} - u_{12}$ belongs to U_1 and $u_{22} - u_{21}$ belongs to U_2, hence $u_{11} - u_{12}$ belongs to $\cap\,[U_1\, U_2]$. Therefore $u_{11} - u_{12} = 0_U$ and $u_{11} = u_{12}$ and hence $u_{21} = u_{22}$.

THEOREM 6. *Relative to a subset $\mathfrak{A}_0$ of $\mathfrak{A}$, the class U_0 of all elements u_0 of U which are commutative with every number a_0 of $\mathfrak{A}_0$ is additive and is properly linear in respect to the set $\mathfrak{A}_{0c}$ of all numbers of $\mathfrak{A}$ which are commutative as to multiplication with every number of $\mathfrak{A}_0$.*

Proof. From the distributive and associative laws it follows that

$$u_1\, a_0 = a_0\, u_1 \,.\, u_2\, a_0 = a_0\, u_2 \,.\, a_0\, a_1 = a_1\, a_0 \,.\, a_0\, a_2 = a_2\, a_0$$
$$:):\; (u_1\, a_1 + u_2\, a_2)\, a_0 = a_0 (u_1\, a_1 + u_2\, a_2)\,.$$

Hence, since $\mathfrak{A}_{0c}$ contains the numbers 1 and -1, U_0 is additive and is properly linear in respect to $\mathfrak{A}_{0c}$.

In his *Introduction to a Form of General Analysis*,* E. H. Moore introduces the notion of the extensional attainability of a property P, defined for the subclasses M_0 of M. Considering a class M and a property P defined for the subclasses M_0 of M, we say that the property P is *extensionally attainable* in case "for every subclass M_0 of M there exists a class M_{0P} containing M_0 and contained in M, having the property P and such that every subclass of M which contains M_0 and has the property P contains M_{0P},"

* Loc. cit., p. 54.

or, what is equivalent, (1) M has the property P, and (2) for every class M_0 the greatest common subclass of all classes containing M_0 and having the property P has the property P. For such a property P, the *P-extension of* M_0 is the class M_{0P} of the first definition and also the greatest common subclass etc., of the second definition: In notation

$$M_{0P} = \bigcap [\text{all } M_1^P .\ni. M_1 \supset M_0].$$

It is important to note that $M_{0P} \supset M_1 \supset M_0$:): $M_{0P} = M_{1P}$.

THEOREM 7. *The properties of additivity, right linearity, left linearity and proper linearity are extensionally attainable in* U.

Proof. (The proof is given for right linearity only.)

(1) U is right linear by Theorem 1.

(2) $U_0 \,.\, U_{0r} = \bigcap [\text{all } U_1^{rl} .\ni. U_1 \supset U_0]$:): U_{0r}^{rl} by definition of right linearity.

Accordingly we introduce notations, for the various extensions of U_0 as follows:

$AdU_0 \equiv$ the additive extension of U_0;
$L_r U_0 \equiv$ the right linear extension of U_0;
$L_l U_0 \equiv$ the left linear extension of U_0;
$LU_0 \equiv$ the properly linear extension of U_0.

THEOREM 8. *The right linear extension of any subset* U_0 *of* U *is the totality of all right linear combinations of elements of* U_0:

$$\text{TH. 8.}\quad U_0 \,.\, U_{0r} = \left[\text{all } u :\ni: \exists\, n_u\, a_1, \cdots, a_{n_u} u_{01}, \cdots, u_{0n_u} .\ni. u = \sum_i^{1,n_u} u_{0i}\, a_i\right]$$
$$:):\ U_{0r} = L_r U_0.$$

Proof. Obviously $L_r U_0 \supset U_{0r}$, and $U_{0r} \supset U_0$ and is right linear, and therefore $U_0 \supset L_r U_0$.

Due to the symmetry between right and left linearity we will in general state theorems involving only L_r or L_l in terms of L_r and omit the theorem that follows by parity.

3. **Interrelation of certain extensions of U_0.** In this section we consider the iteration of the four processes Ad; L_r; L_l; L and a new process L_0; where $L_0 U_0$ is defined as the (provably existent) maximal properly linear subset of the intersection of the right and left linear extensions of U_0. Moreover, it is shown that these processes along with the iterations $L_0 L_r$ and $L_0 L_l$ are closed under further iteration.

We also consider how far the seven sets AdU_0, $L_r U_0$ etc. are determined from a knowledge of certain of them.

THEOREM 1. *The properly linear extension of any subset U_0 of U is the right linear extension of the left linear extension of U_0 and by parity the left linear extension of the right linear extension of U_0:*

TH. 1. $$U_0 .). L_r L_l U_0 = LU_0 = L_l L_r U_0.$$

Proof. (1) $LU_0 \supset L_r L_l U_0$, for $LU_0 \supset L_l U_0 \supset U_0$ and therefore $LU_0 = LL_l U_0 \supset L_r L_l U_0$.

(2) $L_r L_l U_0$ is properly linear, for by definition $U_1 .). (L_r U_1)^{rl}$ and $L_r L_l U_0$ is left linear, for according to § 2, Theorem 8, every element of $L_r L_l U_0$ is of the form

$$\sum_i^{1,n_0} \left(\sum_j^{1,n_i} a_{ji} u_{0ji} \right) a_i ,^*$$

where n_0 and n_i (i) are positive integers and u_{0ji} belongs to U_0 $(i\,j)$ and conversely, and the distributive and associative laws are holding.

Therefore $L_r L_l U_0 = LU_0$ and similarly $L_l L_r U_0 = LU_0$.

THEOREM 2. $U_1 U_2 :): (1)\ L_r \mathbf{U} [U_1 U_2] = L_r U_1 + L_r U_2;$
$(2)\ L\mathbf{U} [U_1 U_2] = LU_1 + LU_2;$
$(3)\ Ad\mathbf{U} [U_1 U_2] = AdU_1 + AdU_2.$

The proof follows directly from § 2, Theorem 8 and Theorem 1.

THEOREM 3. $U_1^{ad} U_2^{ad} :): (1)\ L_r [U_1 + U_2] = L_r U_1 + L_r U_2;$
$(2)\ L[U_1 + U_2] = LU_1 + LU_2;$
$(3)\ Ad[U_1 + U_2] = AdU_1 + AdU_2.$

THEOREM 4. *Relative to an additive subset U_0 of U, there exists a unique maximal properly linear subset U_{00} of U_0 such that all properly linear subsets of U_0 are contained in U_{00}:*

TH. 4. $$U_0^{ad} .:):. \exists\, |U_{00}{}^l \subset U_0 :\ni: U_{01}{}^l \subset U_0 .). U_{00} \supset U_{01}.$$

Proof. $U_1 \equiv [\text{all } u \ .\ni. Lu \subset U_0]$ is effective as U_{00}; for
(1) U_1 contains the properly linear class $[0_U]$;
(2) $U_{01}^l \subset U_0 .). U_1 \supset U_{01}$;

---

* This may also be written in the form

$$\sum_i^{1,n} a_{1i} u_{0i} a_{2i}.$$

In this case we may not assume as above that the elements u_{0i} are distinct.

(3) U_1 is properly linear, for a right or left linear combination of elements of U_1 is the sum of elements of U_1 and therefore $LU_1 = AdU_1$ and $U_0 \supset LU_1$, hence by (2) $U_1 \supset LU_1$ and is properly linear.

Since relative to a subset U_0 of U, $\cap [L_r U_0 \, L_l U_0]$ is additive, it follows from Theorem 4 that the following definition has content:

Definition of $L_0 U_0$. Relative to a subset U_0 of U we define $L_0 U_0$ as the maximal properly linear subset contained in the intersection of the right and left linear extensions of U_0.

THEOREM 5. *Relative to a right linear subset U_0 of U the maximal properly linear subset U_{00} of U_0 is $L_0 U_0$.*

Proof. $U_0{}^{rl}$.). $L_l U_0 \supset L_r U_0 = U_0$.

THEOREM 6. U_0 .). $L_0 U_0 = \cap [L_0 L_r U_0, L_0 L_l U_0]$.

The following table shows the sets generated from a set U_0 by iterated processes of the types Ad, L_r, L_l, L_0. For example, column 3, row 6 shows us that U_0 .). $L_0 L_r (L_l U_0) = L U_0$.

TABLE I

	Ad	L_r	L_l	L	L_0	$L_0 L_r$	$L_0 L_l$
Ad	Ad	L_r	L_l	L	L_0	$L_0 L_r$	$L_0 L_l$
L_r	L_r	L_r	L	L	L_0	$L_0 L_r$	$L_0 L_l$
L_l	L_l	L	L_l	L	L_0	$L_0 L_r$	$L_0 L_l$
L	L	L	L	L	L_0	$L_0 L_r$	$L_0 L_l$
L_0	L_0	$L_0 L_r$	$L_0 L_l$	L	L_0	$L_0 L_r$	$L_0 L_l$
$L_0 L_r$	$L_0 L_r$	$L_0 L_r$	L	L	L_0	$L_0 L_r$	$L_0 L_l$
$L_0 L_l$	$L_0 L_l$	L	$L_0 L_l$	L	L_0	$L_0 L_r$	$L_0 L_l$

The proof of the above table is readily obtained when we bear in mind that U_0 .). $\cap [L_r U_0 \, L_l U_0] \supset Ad U_0 \supset U_0$.

The fact that relative to a given U_0 the seven sets $Ad U_0$ etc. may be distinct is shown by an example following Table I in II, § 2.

From an examination of Table I it is seen that the iteration of the seven processes Ad, etc., is closed and associative. Moreover, the iteration of the four processes Ad, L_r, L_l, and L is closed.

Table II shows which of the seven sets $Ad U_0$ etc., previously introduced, are determined when any particular combination of them is given. We do not list all the $2^7 - 1$ different combinations, but we give certain combinations, into which all of the $2^7 - 1$ can be decomposed, and such that no combination will determine, in general, more than could be determined from the component parts as listed in the table.

TABLE II

Combinations of sets given	Sets determined uniquely
Ad	$Ad, L_r, L_l, L, L_0, L_0 L_r, L_0 L_l$
L_r	$L_r, L, L_0 L_r$
L_l	$L_l, L, L_0 L_l$
L	L
L_0	L_0
$L_0 L_r$	$L_0 L_r$
$L_0 L_l$	$L_0 L_l$
L_r, L_l	$L_r, L_l, L, L_0, L_0 L_r, L_0 L_l$
$L_r, L_0 L_l$	$L_r, L_0 L_l, L, L_0 L_r, L_0$
$L_l, L_0 L_r$	$L_l, L_0 L_r, L, L_0 L_l, L_0$
$L_0 L_r, L_0 L_l$	$L_0 L_r, L_0 L_l, L_0$

The proof that the table is correct follows at once from Table I and Theorem 6.

The following considerations and examples show that Table II is complete:

Whenever the determination of certain sets determines certain others uniquely it follows that no extra knowledge is gained by adding these others to the original sets in the first column of Table II. Thus, since the determination of $L_r U_0$ determines $L_0 L_r U_0$ it follows that we need not add L_r, $L_0 L_r$ to the combinations in the first column.

In Table III we give examples showing the completeness of Table II. In each of these examples we consider a finite range P^n, where n is a positive integer, $\mathfrak{A} \equiv Q$ (quaternions) and $U \equiv V$ the class of all vectors v on P to Q. We have a system satisfying the postulates of § 1 when multiplication and addition are defined as in § 1, Ex. 2 of a system Σ. We will display the vectors v as rows of ordered elements thus: $(a_i (i = 1, \cdots, 5)) \equiv (a_1\ a_2\ a_3\ a_4\ a_5)$.

In these examples we display subsets V_1 and V_2 of such a nature that the sets arising from V_1 by processes listed in column 2 of Table III are equal respectively to those arising from V_2 by the same processes; but the sets arising from V_1 differ respectively from those arising from V_2 except for sets as shown by Table II to be uniquely determinable from those we have assumed to be equal. Thus, in Example 1, $L_r V_1$, $L V_1$, $L_0 L_r V_1$ are respectively equal to $L_r V_2$, $L V_2$, $L_0 L_r V_2$, but $Ad V_1$, $L_l V_1$, $L_0 V_1$, $L_0 L_l V_1$ differ respectively from $Ad V_2$, $L_l V_2$, $L_0 V_2$, $L_0 L_l V_2$.

When an example in Table III is given showing the completeness of Table II for any particular combination of sets $Ad V_0$ etc. the example for the same sets with left and right interchanged is immediately securable by parity.

TABLE III

Ex. No.	Example shows completeness of Table II for	P^n n	V_1	V_2
1.	$L_r\ (L_l)$	2	$(1\ 0)\ (0\ 1)$	$(1\ i)\ (j\ -k)$
2.	L	2	$(1\ 0)\ (0\ 1)$	$(1\ i)$
3.	L_0	3	$(0\ 0\ 0)$	$(1\ i\ j)\ (k\ -j\ -i)$
4.	$L_0 L_r\quad (L_0 L_l)$	5	$(1\ i\ 0\ 0\ j)$ $(0\ 0\ 1\ 0\ 0)$ $(0\ 0\ 0\ 1\ 0)$	$(1\ j\ 0\ 0\ 0)$ $(0\ 0\ 1\ i\ 0)$ $(0\ 0\ j\ -k\ 0)$
5.	$L_r,\ L_l$	1	(1)	(i)
6.	$L_r,\ L_0\ (L_l,\ L_0)$	2	$(1\ i)\ (j\ k)$	$(1\ i)$
7.	$L_r,\ L_0 L_l$ $(L_l,\ L_0 L_r)$	2	$(1\ i)\ (j\ -k)$	$(1\ j)\ (i\ k)$
8.	$L,\ L_0$	2	$(1\ i)\ (j\ k)$	$(1\ j)\ (i\ k)$
9.	$L,\ L_0 L_r$ $(L,\ L_0 L_l)$	4	$(1\ 0\ 0\ 0)$ $(0\ 1\ 0\ 0)$ $(0\ 0\ 1\ i)$	$(1\ i\ 0\ 0)$ $(j\ -k\ 0\ 0)$ $(0\ 0\ 1\ j)$
10.	$L_0,\ L_0 L_r$ $(L_0,\ L_0 L_l)$	3	$(1\ i\ 0)\ (j\ k\ 0)$	$(1\ 0\ i)$
11.	$L_0 L_r,\ L_0 L_l$	3	$(1\ i\ i)$	$(1\ i\ j)$
12.	$L,\ L_0,\ L_0 L_r$ $(L,\ L_0,\ L_0 L_l)$	2	$(1\ i)\ (j\ k)$	$(1\ j)$
13.	$L,\ L_0 L_r,\ L_0 L_l$	2	$(1\ i)$	$(1\ j)$

4. **Right (left) linear independence, normal order, basis, rank, supplementary and difference sets.** In this section we consider a system Σ such that the set U is normally ordered. In this case we show that a right linear subset U_0 of U contains a right linearly independent base, and that every such base has the same cardinal number, which we call the right rank of U_0. Moreover we show that relative to a right linear subset U_0 of U there exists a supplementary right linear set; and that relative to a properly linear subset U_0 of U all properly linear supplementary sets are isomorphic with the difference set $U - U_0$ which in definition is analogous to a difference algebra.

We say a subset U_0 of U is *right linearly independent** as to $\mathfrak{A}_0$ in case

$$U_0\ \mathfrak{A}_0\ .:\ni:.\ n\ (u_{01}\cdots u_{0n})^{\text{distinct}}\ a_{01}\cdots a_{0n}$$
$$:):\ \sum_i^{1,n} u_{0i}\, a_{0i} = 0_U\ .).\ a_{0i} = 0 \qquad (i = 1, \cdots, n).$$

In case U_0 is right linearly independent as to $\mathfrak{A}$ we say U_0 is right linearly independent (U_0^{rli}). If U_0 is right linearly dependent we use the notation U_0^{rld}. We say U_0 is right linearly independent as to U_1 in case $L_r\, U_1$ does not contain any members of U_0. Definitions of *left linear independence* as to $\mathfrak{A}_0$ etc. follow at once by parity.

THEOREM 1. *If a subset U_0 of U is commutative and right linearly independent as to $\mathfrak{A}'$, U_0 is right linearly independent.*

Proof. 1. U_0 does not contain 0_U.

2. If the theorem were not true we would have $\sum_i u_{0i}\, a_i = 0_U$, where the elements u_{0i} are distinct and the numbers a_i are $\neq 0$. Then

$$\sum_i u_{0i}\, a_i\, a_n^{-1} + u_{0n} = 0_U; \tag{1}$$

moreover since $u_{01}, \cdots, u_{0n}$ are right linearly independent as to $\mathfrak{A}'$ $\exists\ (j\ a \neq 0)\ .\ni.\ (aa_j\, a_n^{-1}\, a^{-1} - a_j\, a_n^{-1}) \neq 0$ and since U_0 is commutative

$$\sum_i u_{0i}\, aa_i\, a_n^{-1}\, a^{-1} + u_{0n} = 0_U; \tag{2}$$

hence from (1) and (2) it follows that

$$\sum_i u_{0i}\,(aa_i\, a_n^{-1}\, a^{-1} - a_i\, a_n^{-1}) = 0_U,$$

and therefore

$$\exists\ n_1 :\ni: n > n_1 > 0 \qquad (u_{01_1}\cdots u_{0n_1})^{\text{distinct}\,.\,rld}.$$

By repetition of the above reasoning we conclude that

$$\exists\ (u_0 \neq 0_U\ a \neq 0)\ .\ni.\ u_0\, a = 0_U,$$

and since this is impossible our theorem is valid.

In the remainder of this section we will make use of the notion of normal order. "Normally ordered" is here used synonymously with well ordered

* This could more exactly be called finite right linear independence but as we do not consider the infinite case in this paper we shall use the shorter term.

(wohlgeordnet). Thus we say a subset U_0 of U is normally ordered (U_0^{no}) in case U_0 is linearly ordered in such a way that every subset U_{01} has a first element. We shall use the notation U_{0u}, where u is an element of a normally ordered set containing U_0, to denote all the elements of U_0 which precede u.

We recognize that many accept the Zermelo principle of selection or the multiplicative axiom and therefore feel that the Zermelo demonstration of the normal orderability of any aggregate warrants the assumption of such normal orderability without further stipulation; yet as many do not agree with this point of view we will introduce normal order explicitly whenever we wish to make use of it.

Definition. We say that U_1 is a *right base* for U_0 in case $L_r\, U_1 = U_0$. If, besides, U_1 is right linearly independent we say that U_1 is a *proper right* base for U_0. The definitions of a *left base* and a *proper left base* follow at once by parity.

THEOREM 2. *If U is normally ordered and if U_* is the set of all elements u of U which are right linearly independent of the preceding elements then U_* is a proper right base for U.*

Proof. 1. U_* is non-vacuous, for U has a first element different from 0_U.

2. $L_r\, U_* = U$. We use the indirect method of proof for this. If it is not so, then $\exists\, u \;.\ni.\; L_r\, U_*$ does not contain u, and hence there exists a first such u, say u_0. Since u_0 is not a member of U_* it is a right linear combination of elements of U_{u_0}, all of which are themselves right linear combinations of elements of U_*. Hence u_0 is a right linear combination of elements of U_*, in contradiction to our hypothesis that $L_r\, U_*$ did not contain u_0.

3. U_* is right linearly independent, for, if not,

$$\exists\, ((u_1 \cdots u_n)^{\text{distinct } U_*}\ a_1 \neq 0 \cdots a_n \neq 0) \;.\ni.\; \sum_i u_i\, a_i = 0_U,$$

which is in contradiction to the definition of U_*.

THEOREM 3. *If there exists a normally ordered right base U_* for U, then there exists a normally ordered proper right base U_{0*} for every right linear subset $U_0 \neq [0_U]$ of U.*

Proof. By reasoning analogous to that used in the proof of Theorem 2 we see that [all $u_* \;.\ni.\; u_*$ is right linearly independent as to U_{*u_*}] $\equiv U'$ is a proper right base for U.

From § 2, Theorem 5, it follows that every u is expressible uniquely in the form $u = \sum_i u_i'\, a_i$, where $a_i \neq 0\ (i)$ and $i < j\ .).\ u_i'$ precedes u_j'.

Consider $U_1 =$ [all $u_0 \;.\ni.\; a_1$ (for u_0) $= 1$]; obviously $L_r\, U_1 = U_0$.

Consider the class $[U_p']$ of all sets U_p' consisting of a finite number of elements of U'. We can normally order this class as follows:

U_1' precedes U_2' if

(1) U_1' has fewer members than U_2';

(2) U_1' has the same number of elements as U_2', but the first member of U_1' not in U_2' precedes the first member of U_2' not in U_1'.

Corresponding to every set U_p' there exists a class U_{1p} consisting of all elements of U_1 which are right linear combinations with non-zero coefficients of all elements of U_p', and no other elements. The sets U_{1p} (i. e., the non-vacuous classes U_{1p}) are in 1-1 correspondence with the sets U_p' from which they arise. Every element u_1 of U_1 falls in one and only one such set. We say that the set U_{11} precedes U_{12} provided U_1' precedes U_2'.

No element u_0' is right linearly independent of the elements in the sets preceding the set to which it belongs unless it is the only element in this set. For consider a set U_{11} containing two distinct elements u_{111} and u_{112} where

$$(1) \qquad u_{111} = u_{11} + \sum_{i}^{n} u_{1i}\, a_{1i},$$

$$(2) \qquad u_{112} = u_{11} + \sum_{i}^{n} u_{1i}\, a_{2i},$$

where n is the number of elements in U' and no multiplier a_{1i} or a_{2i} is zero. Hence $L_r[u_{111}, u_{112}]$ contains $u_{111} - u_{112} \equiv u_{03}$ and $u_{03} \neq 0$, and hence there exists a number a such that $u_{03}\, a$ belongs to U_1 and is in a set which precedes U_{11}. Moreover there exists a number a_1 such that $u_{111} - u_{03}\, a_1 \equiv u_{04}$ is in a set preceding U_{11}. However, it is evident that $L_r[u_{03}\, a, u_{04}]$ contains both u_{111} and u_{112}.

Denote by U_2 the totality of elements of U_1 which are the only members of the sets to which they respectively belong. These elements may be normally ordered as were the sets to which they belonged and constitute a right base for U_0, and hence by reasoning analogous to that used in the proof of Theorem 2 it can be readily shown that [all u_2 :∍: u_2 is right linearly independent as to U_{2u_2}] is a normally ordered proper right base for U_0.

THEOREM 4. *If a subset U_0 of U is right linear and if U_1 and U_2 are normally ordered proper right bases for U_0, then U_1 and U_2 have the same cardinal number.*

Proof. For every u_1 consider

$U_{3u_1} \equiv$ [all u_2 right linearly independent as to $\mathbf{U}\,[U_{2u_2}\, U_{1u_1}]$],

$U_{4u_1} \equiv$ [all u_2 right linearly independent as to $\mathbf{U}\,[U_{2u_2}\, U_{1u_1}\, u_1]$];

obviously, for every u_1, $U_{3u_1} \supset U_{4u_1}$ and $L_r \cup [U_{1u_1} U_{3u_1}] = U_0$, but $L_r U_{1u_1}$ does not contain the element u_1. Hence every element u_1 of U_1 is a right linear combination with non-zero coefficients of at least one element of U_{3u_1} and elements of U_{1u_1}. Therefore U_{3u_1} contains at least one element not in U_{4u_1}. Let u_1 correspond to the first u_2 belonging to U_{3u_1} but not in U_{4u_1}.

Moreover, it is obvious that

$$u_{11} \sim u_{21} \cdot u_{12} \sim u_{22} \cdot u_{11} \neq u_{12} :): u_{21} \neq u_{22},$$

and hence there exists a one to one correspondence between U_1 and a part of U_2 and similarly between U_2 and a part of U_1.

THEOREM 5. *If a right linear subset U_0 of U has a normally ordered proper right base U_1, then any other proper right base U_2 for U_0 can be normally ordered.*

Proof. Consider the class $[U_{1p}]$ of all finite sets of elements of U_1. This class may be normally ordered. Corresponding to every such set U_{1p} consider the totality U_{2p} of elements of U_2 which are right linear combinations with non-zero coefficients of all the elements of U_{1p}. Every element of U_2 falls in one and only one such set. Since U_2 is right linearly independent the number of elements in any set U_{2p} can not exceed the number of elements in U_{1p}. Hence U_2 consists of the members of a normally orderable class of finite sets of elements and is therefore normally orderable.

Definitions of *right* (*left*) *rank.* If a subset U_0 of U is right linear and there exists a normally ordered proper right base for U_0 we say that the cardinal number of such a right base is the right rank of U_0: $(rk_r(U_0))$.

THEOREM 6. *If the subsets U_1 and U_2 of U are right linear, have normally ordered right bases, and $U_1 \supset U_2$, then*

$$rk_r(U_1) \geqq rk_r(U_2).$$

COROLLARY. *Relative to a subset U_0 of U of such a nature that there exists a normally ordered right base for LU_0*

$$rk_r(LU_0) \geqq rk_r(L_r U_0) \geqq rk_r(L_0 L_r U_0) \geqq rk_r(L_0 U_0).$$

Note: According to Theorem 3 all of these ranks exist.

THEOREM 7. *If there exists a normally ordered right base U_* for U, then relative to a right linear subset U_0 of U there exists a right linear subset U_1 of U such that U_0 and U_1 are supplementary.*

Proof. Case 1. $U_0 = U$. In this case $[0_U]$ is effective as U_1 of the theorem.

Case 2. $U_0 \neq U$. Consider $U_{1*} \equiv$ [all $u_* \ni . u_*$ is right linearly independent as to $[U_{*u_*} U_0]$]; $L_r U_{1*}$ is effective as U_1 of the theorem.

1. By proof analogous to Theorem 2 and by § 3, Theorem 3, we see that $U \doteq L_r \mathbf{U} [U_{1*} U_0] = L_r U_{1*} + L_r U_0 = L_r U_{1*} + U_0$;

2. $\mathbf{\cap} [U_0 L_r, U_{1*}]$ contains only 0_U, for if it contained an element u_0 of U_0 such that $u_0 \neq 0_U$, then u_0 would be a right linear combination with non-zero coefficients of elements of U_{1*} and hence there would exist an element of U_{1*} not right linearly independent as to U_0 and the preceding elements of U_{1*}.

Definition of a ***difference set.*** † Relative to a properly linear subset U_0 of U we define the *difference* of U and U_0 $(U - U_0)$ as follows:

$$U - U_0 = [\text{all } U_1 :\ni: u \text{ belongs to } U_1 .). [u] + U_0 = U_1 \equiv \{u\}].$$

It is seen that $U - U_0$ is not itself a set of elements of U but a class of sets of elements such that the members of any one set differ from each other by an element of U_0 and all elements of U which differ by an element of U_0 belong to the same set.

LEMMA 1.

$$U_0^l :): \{u_1\} = \{u_2\} .). (1)\ \{u_1 a\} = \{u_2 a\} = \{u_1\} a \quad (a);$$
$$(2)\ \{a u_1\} = \{a u_2\} = a\{u_1\} \quad (a).$$

LEMMA 2.

$$U_0^l .:):. \{u_1\} = \{u_2\} . \{u_3\} = \{u_4\}$$
$$:): \{u_1 + u_3\} = \{u_2 + u_4\} = \{u_1\} + \{u_3\}.$$

From the lemmas and definitions above it follows at once that relative to a properly linear subset U_0 of U where $U_0 \neq U$ the difference, $U - U_0$, together with the number system $\mathfrak{A}$ and with addition defined as in Lemma 2 and multiplication as in Lemma 1, forms a system Σ satisfying the postulates of I, § 1.

THEOREM 8. *If two properly linear subsets U_0 and U_1 of U are supplementary then U_1 is isomorphic with $U - U_0$ under the correspondence $u_1 \sim \{u_1\}$ (u_1).*

Proof. 1. $u_{11} \neq u_{12}$.). $\{u_{11}\} \neq \{u_{12}\}$, for if not $(u_{11} - u_{12})$ would belong both to U_0 and U_1, which is impossible since U_0 and U_1 are supplementary.

2. In every set $\{u\}$ there exists an element of U_1, for $\exists\, u_0, u_1 \ni . u = u_0 + u_1$ and therefore $\{u\} = \{u_1\}$.

† For an abstract definition of a difference algebra see L. E. Dickson, *Algebras and their Arithmetics*, p. 36 ff. Our definition could be made more general by not requiring that U_0 be properly linear, but many of the most useful properties would not be preserved. We therefore limit ourselves to this case.

3. The preservation of the correspondence under addition and multiplication follows from the fact that U_0 and U_1 are properly linear and from Lemmas 1 and 2.

5. **Systems with a commutative base.** In this section we will consider a system Σ of such a nature that there exists a proper right base U_* of U which is commutative with $\mathfrak{A}$; hence U_* is a proper left base for U. We show that in this case U is isomorphic with the set of all finitely non-zero vectors on a certain range P and that any properly linear subset U_0 of U has a commutative right base and conversely.

If Σ is such that U has a commutative proper right base then we say that Σ is of type 1.

Note. That not all systems Σ are of type 1 is seen from the following example.

Consider both $\mathfrak{A}$ and U as the Hilbert example† of a Veronesean number system. Associate with every number $a \equiv (a(i)\ (i = -\infty \cdots +\infty))$ the number a' where $a' \equiv (a'(2i) = a(i) \,.\, a'(2i+1) = 0\ (i = -\infty \cdots +\infty))$. We define the processes $\oplus$, $\odot_r$ and $\odot_l$ as follows:

$$u_1 u_2 .) . \ u_1 \oplus u_2 \equiv u_1 + u_2,$$

$$a\, u\ .) . \ u \odot_r a \equiv u a,$$

$$a\, u\ .) . \ a \odot_l u \equiv u \odot_r a' = u a',$$

where the addition and multiplication in the right hand members of the above definitions are the ordinary addition and multiplication for numbers of such a system. Then $0_U (= 0)$ is the only element of U which is commutative with every number of $\mathfrak{A}$, for consider $a_1 \equiv (a_1(1) = 1, a_1(i) = 0\ (i \neq 1))$; then $a_1' = (a_1'(2) = 1,\ a_1'(i) = 0\ (i \neq 2))$, and it follows that $u \neq 0_U .) . \ a_1 \odot_l u = u a_1' \neq u a_1 = u \odot_r a_1$. This is also an example of a properly linear set with a right rank different from the left rank, for $u_0 \equiv (u_0(0) = 1 \,.\, u_0(i) = 0\ (i \neq 0))$ is a proper right base for U; but U has left rank 2 for u_0 and $u_1 \equiv (u_1(1) = 1 \,.\, u_1(i) = 0\ (i \neq 1))$ form a proper left base for U, for $a u_0 = u .) . \ u(i) = 0$ for i odd and $a u_1 = u .) . \ u(i) = 0$ for i even.

THEOREM 1. *If Σ is of type 1 and if we consider $P \equiv [p] \equiv U_*$ and $U' \equiv$ [all vectors u' on P to $\mathfrak{A}$ finitely non-zero] and $U_*' \equiv [\delta_p\ (p)]$ where $\delta_p(p) = 1$ and $\delta_p(p_1) = 0$ for every $p_1 \neq p$, then U is isomorphic with U' under the correspondence*

$$u = \sum_i^{1,n} u_{*i} a_i \sim u' = \sum_i^{1,n} \delta_{u_{*i}} a.$$

† Ex. 6 of a system $\mathfrak{A}$ of type A.

Proof. The theorem follows at once from the fact that U_* is commutative.

Relative to a general range P and any number system $\mathfrak{A}$ of type A the class of all vectors u on P to $\mathfrak{A}$ finitely non-zero is a set U belonging to a system Σ of type 1. In the remainder of this article we will therefore consider only systems of finitely non-zero vectors on a range P.

Relative to P^1 and the numbers of $\mathfrak{A}$ as vectors on P^1 to $\mathfrak{A}$ obviously $a \neq 0 .). L(a) = L(1)$.

LEMMA 1. *If P is finite and u is a vector on P to $\mathfrak{A}$ nowhere zero, then either there exist a vector u_1 and a number $a \neq 0$ such that u_1 is commutative and $u_1 a = u$ or there exist in $L(u)$ two vectors u_2 and u_3 and elements p_2 and p_3 of the range P such that $u_2(p_2) = 0$, $u_3(p_3) = 0$ and $L(u) = L[u_2 u_3]$.*

Proof. It is sufficient to prove this for the special case $P = P^2$. Then $u = (a_1, a_2)$ with $a_1 a_2 \neq 0$ and $L(u) = L(u_1)$ where $u_1 \equiv (1, a_2 a_1^{-1})$. If u is not commutative, $\exists\, a \,.\ni.\, a a_2 a_1^{-1} a^{-1} \neq a_2 a_1^{-1}$, and since Lu contains $(1, a a_2 a_1^{-1} a^{-1})$ it also contains $(0, a_2 a_1^{-1} - a a_2 a_1^{-1} a^{-1})$; therefore Lu contains $(0, 1)$ and $(1, 0)$.

LEMMA 2. *Relative to a general range P and the set U of all vectors u on P to $\mathfrak{A}$ finitely non-zero, it is true that u :): $\exists\, U_0^c \,.\ni.\, L U_0 = Lu$.*

Proof. This lemma follows by the repeated application of Lemma 1.

Theorem 2 follows at once from Lemma 2.

THEOREM 2. *Relative to a general range P, U the set of all finitely non-zero vectors on P to $\mathfrak{A}$ and a properly linear subset U_0 of U, there exists a commutative right base U_{0*} for U_0 which therefore is also a left base for U_0.*

THEOREM 3. *Relative to a normally ordered range P, U the set of all finitely non-zero vectors on P to $\mathfrak{A}$ and a properly linear subset U_0 of U, there exists a normally ordered commutative proper right base U_{0*} for U_0 which therefore is also a proper left base for U_0.*

Proof. Since by Theorem 2 U and U_0 are the linear extensions respectively of their commutative subsets U' and U_0', it follows from I, § 4, Theorem 1, and I, § 2, Theorem 6, that we need only prove the theorem relative to the system $\Sigma_1 = (\mathfrak{A}'\, U' \oplus \odot_r \odot_l)$. In this form, however, the theorem is merely a special case of I, § 4, Theorem 3.

Relative to a normally ordered range P, U the set of all finitely non-zero vectors on P to $\mathfrak{A}$ and a properly linear subset U_0 of U, the right rank of U_0 is equal to the left rank of U_0. In this case we will speak of either the right or left rank of U_0 as the *rank* of U_0 $(rk\, U_0)$.

II. Sets of vectors on a finite range

Contents

Introduction. In this section we will consider a system composed of a number system $\mathfrak{A}$ of type A, the totality $V \equiv [v]$ of all vectors on a finite range $P^n \equiv [1, 2, 3, \cdots, n]$ to $\mathfrak{A}$ and addition and multiplication defined as in Example 2 of I, § 1, for a system Σ. Thus we are dealing with a special case of a system Σ of type 1.

We introduce notations as follows for matrices, vectors and their composition:

$$W \equiv [\text{all matrices } w \text{ on } PP \text{ to } \mathfrak{A}]$$

$$w = w(i,j) \qquad (i = 1, \cdots, n,\ j = 1, \cdots, n).$$

We say that a matrix w is commutative, w^c, in case every element of w belongs to $\mathfrak{A}'$.

Composition of matrices: *S notation*:*

$$w_3 = S w_1 w_2 :=: w_3(j,k) = \sum_i w_1(j,i)\, w_2(i,k) \qquad (j,k).$$

Composition of a vector and a matrix.

$$v_1 = S w v :\equiv: v_1(i) = \sum_j^{1,n} w(i,j)\, v(j) \qquad (i).$$

$$v_1 = S v w :\equiv: v_1(i) = \sum_j^{1,n} v(j)\, w(j,i) \qquad (i).$$

Inner product of two vectors:

$$a = S v_1 v_2 :=: a = \sum_i^{1,n} v_1(i)\, v_2(i).$$

We say that a matrix w is non-singular (w^{ns}) in case it has a right and left reciprocal, which is equivalent to the rows of w being left linearly independent and the columns right linearly independent. We will use the notation δ for the identity matrix.

* This notation is that used by E. H. Moore in his course in General Analysis.

1. Normal forms for bases. In this article we define what we mean by the base of a right (left) linear set being in semi-normal or normal form. We show that two right linear sets are equal if and only if the normal forms of their bases are equal. We show also that a right base for a properly linear set which is in semi-normal form is also a left base for the same set and is composed of commutative vectors.

THEOREM 1. *If a right linear subset V_0 of V has right rank r, and σ is a set of distinct elements $p_1, \cdots, p_r$ of the range P such that V as on σ is of right rank r, then their exists one and only one set of vectors $V_{0\sigma} \equiv (v_{01} \cdots v_{0r})$ of such a nature that*

$$(1) \qquad L_r V_{0\sigma} = V_0;$$

$$(2) \qquad v_{0i}(p_i) = 1 \qquad (i = 1, \cdots, r);$$

$$(3) \qquad v_{0i}(p_j) = 0 \qquad (i \neq j,\ i = 1, \cdots, r, j = 1, \cdots, r).$$

Proof. 1. $\exists\, V_1 = (v_1 \cdots v_r) :\ni: V_0 \supset V_1$ and on σ is identical with the set $\delta_{p_1}, \cdots, \delta_{p_r}$. V_1 as on σ has right rank r. Therefore aside from uniqueness V_1 is effective as the $V_{0\sigma}$ of the theorem.

2. $V_{0\sigma}$ is unique, for consider $i \leqq r$ and v_i' in V_0 of such a nature that $v_i'(p_i) = 1$ and $v_i'(p_j) = 0$ for $j \leqq r$ and unequal to i; hence $v_i' = v_{0i}$ since it belongs to V_0 and therefore to $L_r V_{0\sigma}$.

THEOREM 2.

$V_0^{rl} \cdot rk_r V_0 = r ::) :: \exists \mid \sigma_* = (p_1 \cdots p_r) .:\ni:.$

(1) *V_0 as on σ_* is of right rank r;*

(2) *$p_1 < p_2 \cdots < p_r$;*

(3) *$\sigma' = (p_1' < p_2' \cdots < p_r') .\ni. V_0$ as on σ' is of right rank r :): $\sigma_* \langle\langle \sigma'$.*

The proof is obvious.

We say that a right base V_{01} of a right linear subset V_0 of V is in semi-normal form in case there exists a σ satisfying the conditions of Theorem 1 for which V_{01} is effective as the V_0 of the theorem. In such a case we say that V_{01} is in normal form provided σ is effective as σ_* of Theorem 2.

THEOREM 3. *Two right linear sets are equal if and only if the normal forms of their bases are equal.*

THEOREM 4. *Relative to a properly linear subset V_0 of V a right base V_{01} for V_0 in semi-normal form is also a left base for V_0 in semi-normal form and is commutative.*

Proof. Since $rk_r V_0 = rk_l V_0$, V_{01} is a left base for V_0 in semi-normal form. Hence the right hand multiples of the vectors of V_{01} must be equal to the left hand multiples with the same coefficients and therefore V_{01} is commutative.

COROLLARY. *The normal form of the right base of a properly linear subset V_0 of V is equal to the normal form of its left base and is composed of commutative vectors.*

2. **Orthogonal sets.** In this article we give a definition of the right (left) orthogonality of one vector to another in respect to a commutative non-singular matrix w, and in terms of these relations we define the right (left) orthogonal complements $0_{rw} V_0$ $(0_{lw} V_0)$ of a subset V_0 of V in respect to w. We then study the iteration of the processes 0_{rw}, 0_{lw} together with L_r, L_l, etc. and give the iteration table of the resulting twelve distinct processes, one set of whose generators are 0_{rw}, 0_{lw} and L_0; and show that these are closed under further iteration. We then make a generalization of these processes such that the resulting iteration table is abstractly equivalent to that obtained from 0_{rw}, 0_{lw}, and L_0 and applies to the more general situation of Section I.

The statements we make in the remainder of this article will be relative to a commutative symmetric non-singular matrix w.

We say that v_1 is left orthogonal to v_2 and v_2 is right orthogonal to v_1 in respect to w in case $S^2 v_1 w v_2 = 0$. We define the *right* (*left*) *orthogonal complement* of a subset V_0 of V in respect to w, $0_{rw} V_0$ $(0_{lw} V_0)$, and the sets $0_w V_0$ and $0_{0w} V_0$ as follows:

$$
\begin{aligned}
0_{rw} V_0 &\equiv [\text{all } v :\ni: v_0 .).\, S^2 v_0 w v = 0], \\
0_{lw} V_0 &\equiv [\text{all } v :\ni: v_0 .).\, S^2 v w v_0 = 0], \\
0_w V_0 &\equiv 0_{rw} L V_0 = 0_{lw} L V_0 && \text{(see Lemma 3)}, \\
0_{0w} V_0 &\equiv 0_{rw} L_0 V_0 = 0_{lw} L_0 V_0 && \text{(see Lemma 3)}.
\end{aligned}
$$

In case $w = \delta$ the orthogonality condition reduces to the vanishing of the inner product $S v_1 v_2$, and we say v_1 is left orthogonal to v_2 etc., and we use the notations $0_r V_0$ for $0_{r\delta} V_0$ etc.

LEMMA 1. $V_0 .).\ (1)\ \ 0_{lw} V_0 = 0_{lw} L_r V_0,$
$\qquad\qquad\qquad (2)\ \ 0_{rw} V_0 = 0_{rw} L_l V_0.$

LEMMA 2. $V_1 \supset V_2 .).\ (1)\ \ 0_{rw} V_2 \supset 0_{rw} V_1,$
$\qquad\qquad\qquad\qquad (2)\ \ 0_{lw} V_2 \supset 0_{lw} V_1.$

LEMMA 3. $V_0^l .).\ 0_{lw} V_0 = 0_{rw} V_0.$

Proof. There exists a commutative base V_{01} for V_0 and hence $0_{lw} V_0 = 0_{lw} V_{01} = 0_{rw} V_{01} = 0_{rw} V_0.$

LEMMA 4. $V_0 :): (0_{rw} V_0)^{rl} . (0_{lw} V_0)^{ll}.$

LEMMA 5.† $V_0 .).\ (1)\quad 0_{rw} 0_{lw} V_0 = L_r V_0,$

$(2)\quad 0_{lw} 0_{rw} V_0 = L_l V_0.$

Proof: Obviously $0_{rw} 0_{lw} V_0 \supset L_r V_0$, and $0_{rw} 0_{lw} V_0$ is right linear, hence the lemma is true provided $rk_r\, 0_{rw}\, 0_{lw} V_0 = rk_r L_r V_0$. This follows if we show that

$$\text{(I)}\qquad V_0 .).\ (1)\quad rk_l\, 0_{lw} V_0 + rk_r L_r V_0 = n,$$

$$(2)\quad rk_r\, 0_{rw} V_0 + rk_l\, L_l V_0 = n.$$

However, since w is non-singular we need only prove (I) for the special case in which $w = \delta$. Let $r \equiv rk_r L_r V_0$. Consider $(p_1 \cdots p_r) \equiv \sigma$ as the effective σ_* of § 1, Theorem 2, for $L_r V_0$, and $(v_1 \cdots v_r) \equiv$ the normal form of the right base for $L_r V_0$. Let $\sigma' \equiv (p_1' \cdots p_{n-r}')$ be the set of elements of P not in σ. Then consider $V_*' \equiv (v_1' \cdots v_{n-r}')$, where, for every i, $v_i'(p_i') = 1$, $v_i'(p_j') = 0$ $(i \neq j)$, $v_i'(p_k) = -v_k(p_i')$ $(k = 1 \cdots r)$, V_*' is a left base in semi-normal form for the left linear set $V' = L_l V_*'$. which is of left rank $n - r$. Moreover, $0_l V_0 \supset V'$. Hence the $rk_l 0_l V_0 \geqq n - r$. However, if $rk_l 0_l V_0 > n - r$ there would exist a vector v in $0_l V_0$ and an element p_i $(i \leqq r)$ of the range P such that $v(p_i) = 1$ and $v(p_j') = 0$ for every $j \leqq n - r$. Since this is impossible, $rk_l 0_l V_0 = n - r$ and our lemma is proved.

LEMMA 6. $V_0 :):\ (0_w V_0)^l . (0_{0w} V_0)^l.$

This follows directly from Lemmas 3 and 4.

LEMMA 7. $V_0^l .).\ L_r V_0 = L_l V_0 = L V_0 = L_0 V_0 = L_0 L_r V_0 = L_0 L_l V_0.$

LEMMA 8. $V_0 .).\ 0_{rw} L_r V_0 = 0_{rw} L V_0 = 0_{lw} L V_0 = 0_{lw} L_l V_0 = 0_w V_0.$

LEMMA 9. $V_0^{rl} .).\ (0_{rw} V_0)^l$ and $V_0^{ll} .).\ (0_{lw} V_0)^l.$

LEMMA 10. $V_0 :):\ (0_{rw}^2 V_0)^l . (0_{lw}^2 V_0)^l.$

LEMMA 11. $V_0 .).\ L_0\, 0_{rw} V_0 = 0_w V_0 = L_0\, 0_{lw} V_0.$

Proof. $L V_0 \supset L_r V_0$ and hence $0_w V_0 = 0_{lw} L V_0 \subset 0_{lw} L_r V_0 = 0_{lw} V_0$, and since $0_w V_0$ is properly linear it follows that

$$(1)\qquad 0_w V_0 \subset L_0\, 0_{lw} V_0.$$

† This lemma in its equivalent matricial form is due to E. H. Moore and is given in his course in General Analysis. The proof is the writer's. It may be stated in the following form:

$$P' P'' w''' v_1' .:):.\ \exists\ v'' .\ni .\ S'' w''' v'' = v_1' :\sim: S' v' w''' = 0_{V''} .).\ S' v' v_1' = 0.$$

$L_0\, L_l\, V_0 \subset L_l\, V_0$ and hence $0_w\, L_0\, L_l\, V_0 = 0_{rw}\, L_0\, L_l\, V_0 \supset 0_{rw}\, L_l\, V_0 = 0_{rw} V_0$, and since $0_w\, L_0\, L_l V_0$ is properly linear it follows that $0_w\, L_0\, L_l\, V_0 \supset L\, 0_{rw} V_0$, and hence

$$(2) \qquad 0_w^2\, L_0\, L_l\, V_0 = L_0\, L_l\, V_0 \subset 0_w\, L\, 0_{rw}\, V_0;$$

using Lemma 5 and $0_{lw} V_0$ as V_0 in (2) we obtain

$$(3) \qquad L_0\, 0_{lw}\, V_0 \subset 0_w\, L 0_{rw}\, 0_{lw}\, V_0 = 0_w\, L\, L_r\, V_0 = 0_w\, L\, V_0 = 0_w\, V_0,$$

hence by (1) and (3) we have $0_w V_0 = L_0\, 0_{lw} V_0$.

LEMMA 12. V_0 .). $L 0_{rw} V_0 = 0_{0w}\, L_l V_0$.

Proof. Applying Lemmas 11 and 1 to $0_{rw}\, V_0$ for V_0 we obtain $L_0\, L_l\, V_0 = L_0\, 0_{lw}\, 0_{rw}\, V_0 = 0_w\, 0_{rw}\, V_0 = 0_w\, L 0_{rw}\, V_0$ and the lemma follows at once. This may also be stated in the form

$$(12_1) \qquad V_0 \text{ .). } 0_w\, 0_{rw}\, V_0 = L_0\, L_l\, V_0.$$

By use of the above lemmas we derive Table I, which gives the results of iteration of the processes $L_r, L_l, \cdots, 0_{0w}$. Thus we find from row 4, column 7, that V_0 .). $L_0\, 0_{rw}\, V_0 = 0_w V_0$. It should be especially noted that the three processes 0_{rw}, 0_{lw}, and L_0 are generators of the whole table.

TABLE I

	L_r	L_l	L	L_0	$L_0 L_r$	$L_0 L_l$	0_{rw}	0_{lw}	$L0_{rw}$	$L0_{lw}$	0_w	0_{0w}
L_r	L_r	L	L	L_0	$L_0 L_r$	$L_0 L_l$	0_{rw}	$L0_{lw}$	$L0_{rw}$	$L0_{lw}$	0_w	0_{0w}
L_l	L	L_l	L	L_0	$L_0 L_r$	$L_0 L_l$	$L0_{rw}$	0_{lw}	$L0_{rw}$	$L0_{lw}$	0_w	0_{0w}
L	L	L	L	L_0	$L_0 L_r$	$L_0 L_l$	$L0_{rw}$	$L0_{lw}$	$L0_{rw}$	$L0_{lw}$	0_w	0_{0w}
L_0	$L_0 L_r$	$L_0 L_l$	L	L_0	$L_0 L_r$	$L_0 L_l$	0_w	0_w	$L0_{rw}$	$L0_{lw}$	0_w	0_{0w}
$L_0 L_r$	$L_0 L_r$	L	L	L_0	$L_0 L_r$	$L_0 L_l$	0_w	$L0_{lw}$	$L0_{rw}$	$L0_{lw}$	0_w	0_{0w}
$L_0 L_l$	L	$L_0 L$	L	L_0	$L_0 L_r$	$L_0 L_l$	$L0_{rw}$	0_w	$L0_{rw}$	$L0_{lw}$	0_w	0_{0w}
0_{rw}	0_w	0_{rw}	0_w	0_{0w}	$L0_{lw}$	$L0_{rw}$	$L_0 L_l$	L_r	$L_0 L_l$	$L_0 L_r$	L	L_0
0_{lw}	0_{lw}	0_w	0_w	0_{0w}	$L0_{lw}$	$L0_{rw}$	L_l	$L_0 L_r$	$L_0 L_l$	$L_0 L_r$	L	L_0
$L0_{rw}$	0_w	$L0_{rw}$	0_w	0_{0w}	$L0_{lw}$	$L0_{rw}$	$L_0 L_l$	L	$L_0 L_l$	$L_0 L_r$	L	L_0
$L0_{lw}$	$L0_{lw}$	0_w	0_w	0_{0w}	$L0_{lw}$	$L0_{rw}$	L	$L_0 L_r$	$L_0 L_l$	$L_0 L_r$	L	L_0
0_w	0_w	0_w	0_w	0_{0w}	$L0_{lw}$	$L0_{rw}$	$L_0 L_l$	$L_0 L_r$	$L_0 L_l$	$L_0 L_r$	L	L_0
0_{0w}	$L0_{lw}$	$L0_{rw}$	0_w	0_{0w}	$L0_{lw}$	$L0_{rw}$	L	L	$L_0 L_l$	$L_0 L_r$	L	L_0

The proof of Table I may be readily effected by using the lemmas as listed in the following table. Numbers refer to lemmas of this article.

	L_r	L_l	L	L_0	$L_0 L_r$	$L_0 L_l$	0_{rw}	0_{lw}	$L0_{rw}$	$L0_{lw}$	0_w	0_{0w}
L_r	I § 3 Table I						4	4	6, 7, 8, 12			
L_l							4	4				
L							Def.	Def.				
L_0							11	11				
$L_0 L_r$							11	4				
$L_0 L_l$							4	11				
0_{rw}	8	1	$7, 8, 12_1$				$4, 12_1$	5				
0_{lw}	1	8					5	$4, 12_1$				
$L0_{rw}$	6	1					$4, 12_1$	5				
$L0_{lw}$	1	6					5	$4, 12_1$				
0_w	Def.	Def.					12_1	12_1				
0_{0w}	12	12					11	11				

In order to see that there exists a number system $\mathfrak{A}$, a finite range P, a commutative symmetric non-singular matrix w on PP to $\mathfrak{A}$ and a set V_0 of vectors on P to $\mathfrak{A}$ such that the twelve sets $L_r V_0, \cdots, 0_{0w} V_0$ are distinct, consider $\mathfrak{A} = Q$ (real quaternions), the range P^5 and $w \equiv \delta$ and

$$V_0 \equiv \begin{matrix} (1 & 0 & 0 & 0 & 0\ ) \\ (0 & 1 & i & 0 & 0\ ) \\ (0 & j & k & 0 & 0\ ) \\ (0 & 0 & 0 & 1 & i\ ) \\ (0 & 0 & 0 & j & -k). \end{matrix}$$

The twelve sets $L_r V_0, \cdots, 0_{0w} V_0$ are given below with their bases in the normal form:

$$L_r V_0 = L_r \begin{matrix} (1 & 0 & 0 & 0 & 0) \\ (0 & 1 & i & 0 & 0) \\ (0 & 0 & 0 & 1 & 0) \\ (0 & 0 & 0 & 0 & 1) \end{matrix} \qquad L_l V_0 = L_l \begin{matrix} (1 & 0 & 0 & 0 & 0) \\ (0 & 1 & 0 & 0 & 0) \\ (0 & 0 & 1 & 0 & 0) \\ (0 & 0 & 0 & 1 & i) \end{matrix}$$

$$L V = V \qquad L_0 V_0 = L\,(1 \ 0 \ 0 \ 0 \ 0)$$

$$L_0 L_r V_0 = L \begin{matrix} (1 & 0 & 0 & 0 & 0) \\ (0 & 0 & 0 & 1 & 0) \\ (0 & 0 & 0 & 0 & 1) \end{matrix} \qquad L_0 L_l V_0 = L \begin{matrix} (1 & 0 & 0 & 0 & 0) \\ (0 & 1 & 0 & 0 & 0) \\ (0 & 0 & 1 & 0 & 0) \end{matrix}$$

$$0_r V_0 = L_r (0 \ 0 \ 0 \ 1 \ i) \qquad 0_l V_0 = L_l (0 \ 1 \ i \ 0 \ 0)$$

$$L0_r V_0 = L \begin{matrix} (0 & 0 & 0 & 1 & 0) \\ (0 & 0 & 0 & 0 & 1) \end{matrix} \qquad L0_l V_0 = L \begin{matrix} (0 & 1 & 0 & 0 & 0) \\ (0 & 0 & 1 & 0 & 0) \end{matrix}$$

$$0\,V_0 = 0_V = (0\ \ 0\ \ 0\ \ 0\ \ 0) \qquad 0_0\,V_0 = L\begin{matrix}(0 & 1 & 0 & 0 & 0)\\(0 & 0 & 1 & 0 & 0)\\(0 & 0 & 0 & 1 & 0)\\(0 & 0 & 0 & 0 & 1)\end{matrix}$$

The iteration of the processes represented in Table I is associative. This fact can be readily checked by consideration of the generators 0_{rw}, 0_{lw}, L_0.

There are 107 distinct closed sub-tables in Table I. They are listed below by giving the generators, in each case choosing the minimum number necessary. The order chosen is not dependent on the number of generators but on the number of processes generated. This number is shown in Roman numerals. If the two tables include the same number of processes we list first that one whose first process (relative to the order of Table I) not in the second table precedes the first process in the second table but not in the first. We denote by setting two numbers before a list of generators that one may secure by parity a distinct table which therefore is not listed.

I. 1, 2. L_r. 3. L. 4. L_0. 5, 6. $L_0\,L_r$.

II. 7, 8. L_r, L. 9, 10. L_r, $L_0\,L_r$. 11. L, L_0. 12, 13. L, $L_0\,L_r$. 14. 0_w. 15, 16. L_0, $L_0\,L_r$. 17. 0_{0w}. 18. $L_0\,L_r$, $L_0\,L_l$. 19, 20. $L\,0_{lw}$.

III. 21. L_r, L_l. 22, 23. L_r, L, $L_0\,L_r$. 24, 25. L_r, $L_0\,L_l$. 26, 27. L_r, 0_w. 28, 29. L_r, L_0. 30, 31. L_r, $L\,0_{lw}$. 32, 33. L, L_0, $L_0\,L_r$. 34. L, $L_0 L_r$, $L_0\,L_l$. 35. L_0, $L_0\,L_r$, $L_0\,L_l$. 36, 37. 0_{lw}.

IV. 38, 39. L_r, L_l, $L_0\,L_r$. 40. L_r, L_l, 0_w. 41, 42. L_r, L, L_0. 43, 44. L_r, $L_0\,L_r$, $L_0\,L_l$. 45, 46. L_r, 0_{lw}. 47. L, L_0, $L_0\,L_r$, $L_0\,L_l$. 48. L, 0_{0w}. 49, 50. L, $L\,0_{lw}$. 51, 52. L_0, $L\,0_{lw}$. 53. $L_0\,L_r$, $L\,0_{rw}$.

V. 54. L_r, L_l, $L_0\,L_r$, $L_0\,L_l$. 55, 56. L_r, L_0, $L_0\,L_l$. 57, 58. L_r, L, $L\,0_{lw}$. 59, 60. L_r, $L\,0_{rw}$. 61, 62. L_r, 0_{0w}. 63, 64. L, 0_{lw}.

VI. 65. L_r, L_l, L_0. 66, 67. L_r, L_l, $L\,0_{lw}$. 68, 69. L_r, L, 0_{lw}. 70, 71. L_r, 0_{rw}. 72, 73. L, L_0, $L\,0_{lw}$. 74. L, $L_0\,L_r$, $L\,0_{rw}$.

VII. 75, 76. L_r, L_l, 0_{lw}. 77, 78. L_r, L, L_0, $L\,0_{lw}$. 79, 80. L_r, $L_0 L_r$, $L\,0_{rw}$. 81, 82. L_0, 0_{lw}. 83, 84. $L_0\,L_r$, 0_{rw}.

VIII. 85. L_r, L_l, $L_0\,L_r$, $L\,0_{rw}$. 86, 87. L_r, L_0, 0_{lw}. 88, 89. L_r, $L_0\,L_r$, 0_{rw}. 90, 91. L_r, $L_0\,L_l$, 0_{lw}. 92. L, L_0, $L_0\,L_r$, $L\,0_{rw}$.

IX. 93, 94. L_r, L_l, $L_0 L_r$, 0_{rw}. 95, 96. L_r, L_0, $L\,0_{rw}$. 97, 98. L_0, $L_0 L_r$, 0_{rw}.

X. 99. L_r, L_l, L_0, $L\,0_{rw}$. 100. 0_{rw}, 0_{lw}. 101, 102. L_r, L_0, 0_{rw}. 103, 104. L_r, L_0, $L_0\,L_l$, 0_{lw}.

XI. 105, 106. L_r, L_l, L_0, 0_{rw}.

XII. 107. 0_{rw}, 0_{lw}, L_0.

Since relative to a subset V_0 of V we may determine the sets $0_{rw}\,V_0$, $0_{lw}\,V_0$, $L0_{rw}\,V_0$, $L0_{lw}\,V_0$, $0_w\,V_0$ and $0_{0w}\,V_0$ from the sets $L_l\,V_0$, $L_r\,V_0$, $L_0\,L_l\,V_0$, $L_0\,L_r\,V_0$, $L V_0$, $L_0\,V_0$ respectively, and conversely, we see that the Table II of I, § 3, shows us which of the twelve sets are determined, in general, when any combination of a number of the sets is given.

Although we have defined the orthogonal complement of a set V_0 explicitly our table could be arrived at from a postulational point of view.

Consider a system Σ satisfying the postulates of I, § 1, and further two processes T_r and T_l such that corresponding to every subset U_0 of U there exist two subsets $T_r U_0$ and $T_l U_0$ of U and the four following conditions are satisfied:

(1) $$U_0 \;.).\; (a)\; T_r U_0 = T_r L_l U_0, \quad (b)\; T_l U_0 = T_l L_r U_0;$$

(2) $$U_0 \;.).\; (a)\; T_r T_l U_0 = L_r U_0, \quad (b)\; T_l T_r U_0 = L_l U_0;$$

(3) $$U_1 \supset U_2 \;.).\; (a)\; T_r U_1 \subset T_r U_2, \quad (b)\; T_l U_1 \subset T_l U_2;$$

(4) $$U_0^l \;.).\; T_r U_0 = T_l U_0.$$

From the above conditions we see that

$$U_0 :): (T_r U_0)^{rl} . (T_l U_0)^{ll},$$

for $L_r T_r U_0 = T_r T_l L_r T_r U_0 = T_r T_l T_r U_0 = T_r L_l U_0 = T_r U_0$. We define TU_0 and $T_0 U_0$ as $T_r LU_0$ and $T_r L_0 U_0$ respectively. The necessary lemmas for the construction of an iteration table of the T-processes may be readily derived and a table arrived at in terms of the T's of which Table I is a special case.

3. Applications to the case where $\mathfrak{A}$ is real, complex or quaternionic. In case $\mathfrak{A}$ is the real, complex or quaternion number system there exists for every number a its conjugate $\bar{a}$. We define the conjugate of a vector $v = (v(i) \mid i)$ as $\bar{v} = (\overline{v(i)} \mid i)$ and the conjugate of a subset V_0 of V as the totality of the conjugates of the vectors of V_0, in notation $\overline{V_0}$. We readily verify the following statements:

(1) $$V_0 \;.).\; (a)\; L_r \overline{V_0} = \overline{L_l V_0}, \quad (b)\; L\overline{V_0} = \overline{LV_0}, \quad (c)\; L_0 \overline{V_0} = \overline{L_0 V_0}.$$

Since in the case of quaternions every properly linear subset of V has a commutative base it follows that

(2) $$\mathfrak{A} = Q \;.:):.\; V_0^l :\sim: V_0 \supset v \;.).\; V_0 \supset \bar{v}.$$

In such a case the notion of what we shall call conjugate orthogonality proves useful. We define the sets $0_r' V_0$ etc. as follows:

$$
\begin{aligned}
0_r' V_0 &\equiv \overline{0_l V_0} &&= [\text{all } v \ni . S\bar{v}v_0 = 0 \quad (v_0)];\\
0_l' V_0 &\equiv \overline{0_r V_0} &&= [\text{all } v \ni . S v_0 \bar{v} = 0 \quad (v_0)];\\
0' V_0 &\equiv 0_r' L V_0 &&= \overline{0 V_0};\\
0_0' V_0 &\equiv 0_r' L_0 V_0 &&= \overline{0_0 V_0}.
\end{aligned}
$$

Since every properly linear subset V_0 of V has a commutative base it follows that

(3) $\qquad V_0{}^l .). \quad 0_r' V_0 = 0_l' V_0 = 0' V_0;$

(4) $\qquad V_0 .). \text{ (a) } 0_r'^2 V_0 = \overline{0_l \overline{0_l V_0}} = \overline{\overline{0_r 0_l V_0}} = L_r V_0,$

$\qquad\qquad \text{(b) } 0_r' 0_l' V_0 = \overline{0_l \overline{0_r V_0}} = \overline{\overline{0_r 0_r V_0}} = L_0 L_l V_0.$

By use of the above definitions and lemmas in connection with Table I of § 2, we arrive at Table I which gives the iteration of the processes $0_r'$, $0_l'$ and L_0 and the processes which they generate.

TABLE I

	L_r	L_l	L	L_0	$L_0 L_r$	$L_0 L_l$	$0_r'$	$0_l'$	$L0_r'$	$L0_l'$	$0'$	$0_0'$
L_r	L_r	L	L	L_0	$L_0 L_r$	$L_0 L_l$	$0_r'$	$L0_l'$	$L0_r'$	$L0_l'$	$0'$	$0_0'$
L_l	L	L_l	L	L_0	$L_0 L_r$	$L_0 L_l$	$L0_r'$	$0_l'$	$L0_r'$	$L0_l'$	$0'$	$0_0'$
L	L	L	L	L_0	$L_0 L_r$	$L_0 L_l$	$L0_r'$	$L0_l'$	$L0_r'$	$L0_l'$	$0'$	$0_0'$
L_0	$L_0 L_r$	$L_0 L_l$	L	L_0	$L_0 L_r$	$L_0 L_l$	$0'$	$0'$	$L0_r'$	$L0_l'$	$0'$	$0_0'$
$L_0 L_r$	$L_0 L_r$	L	L	L_0	$L_0 L_r$	$L_0 L_l$	$0'$	$L0_l'$	$L0_r'$	$L0_l'$	$0'$	$0_0'$
$L_0 L_l$	L	$L_0 L_l$	L	L_0	$L_0 L_r$	$L_0 L_l$	$L0_r'$	$0'$	$L0_r'$	$L0_l'$	$0'$	$0_0'$
$0_r'$	$0_r'$	$0'$	$0'$	$0_0'$	$L0_r'$	$L0_l'$	L_r	$L_0 L_l$	$L_0 L_r$	$L_0 L_l$	L	L_0
$0_l'$	$0'$	$0_l'$	$0'$	$0_0'$	$L0_r'$	$L0_l'$	$L_0 L_r$	L_l	$L_0 L_r$	$L_0 L_l$	L	L_0
$L0_r'$	$L0_r'$	$0'$	$0'$	$0_0'$	$L0_r'$	$L0_l'$	L	$L_0 L_l$	$L_0 L_r$	$L_0 L_l$	L	L_0
$L0_l'$	$0'$	$L0_l'$	$0'$	$0_0'$	$L0_r'$	$L0_l'$	$L_0 L_r$	L	$L_0 L_r$	$L_0 L_l$	L	L_0
$0'$	$0'$	$0'$	$0'$	$0_0'$	$L0_r'$	$L0_l'$	$L_0 L_r$	$L_0 L_l$	$L_0 L_r$	$L_0 L_l$	L	L_0
$0_0'$	$L0_r'$	$L0_l'$	$0'$	$0_0'$	$L0_r'$	$L0_l'$	L	L	$L_0 L_r$	$L_0 L_l$	L	L_0

The example illustrating the distinctness of the twelve processes of Table I, § 2, may be used to show the fact that the twelve processes of the above table are distinct. In this case,

$$
\begin{aligned}
0_r' V_0 &= L_r\,(0 \;\; 1 \;\; -i \;\; 0 \;\; 0), & 0_l' V_0 &= L_l\,(0 \;\; 0 \;\; 0 \;\; 1 \;\; -i),\\
L0_r' V_0 &= L0_l V_0, & L0_l' V_0 &= L0_r V_0,\\
0' V_0 &= 0 V_0, & 0_0' V_0 &= 0_0 V_0.
\end{aligned}
$$

It should be noted that relative to subset V_0 of V we can determine the sets $0_r' V_0$, $0_l' V_0$, $L0_r' V_0$, $L0_l' V_0$, $0' V_0$ and $0_0' V_0$ from the sets $L_r V_0$, $L_l V_0$, $L_0 L_r V_0$, $L_0 L_l V_0$, $L V_0$ and $L_0 V_0$ respectively, and the converse is true. Thus Table II of I, § 3, shows which of the twelve sets $L_r V_0, \cdots, 0_0' V_0$ are determined, in general, when any combination of these sets is known.

Except for the $0_r'$ and $0_l'$ rows and columns, Table I can be obtained from Table I of § 2 by the substitution of $L0_r'$, $L0_l'$, $0'$ and $0_0'$ for $L0_{lw}$, $L0_{rw}$, 0_w and 0_{0w} respectively. Hence a list of the closed subtables of Table I not involving either $0_r'$ or $0_l'$ may be obtained by the same substitution in the list of the closed subtables of Table I of § 2 which do not involve 0_{rw} and 0_{lw}. Besides these 73 closed subtables we have the following 18 listed by their generators:

II. 1, 2. $0_r'$.
VI. 3, 4. L, $0_r'$.
VII. 5, 6. L_r, $0_l'$.
VIII. 7, 8. L_0, $0_r'$. 9, 10. $L_0 L_r$, $L_0 L_l$, $0_r'$.
IX. 11, 12. L_r, $L_0 L_r$, $0_l'$.
X. 13. $0_r'$, $0_l'$. 14, 15. L_0, $L_0 L_r$, $0_l'$.
XI. 16, 17. L_r, L_0, $0_l'$.
XII. 18. L_0, $0_r'$, $0_l'$.

We can arrive at a generalization of Table I from a postulational point of view. Consider a system Σ satisfying the conditions of I, § 1, and two processes T_r' and T_l' of such a nature that for every subset U_0 of U there exist two subsets $T_r' U_0$, and $T_l' U_0$ and the following conditions are satisfied:

$$(1) \qquad U_0 .). \text{ (a) } \quad T_r' U_0 = T_r' L_r U_0, \qquad \text{(b) } \quad T_l' U_0 = T_l' L_l U_0;$$

$$(2) \qquad U_0 .). \text{ (a) } \quad T_r'^2 U_0 = L_r U_0, \qquad \text{(b) } \quad T_l'^2 U_0 = L_l L_0;$$

$$(3) \qquad U_1 \supset U_2 .). \text{ (a) } \quad T_r' U_2 \supset T_r' U_1, \qquad \text{(b) } \quad T_l' U_2 \supset T_l' U_2;$$

$$(4) \qquad U_0^l .). \; T_r' U_0 = T_l' U_0;$$

and we make the following definitions:

$$T' U_0 = T_r' L U_0, \qquad T_0' U_0 = T_r' L_0 U_0.$$

The necessary lemmas for the proof of Table I in terms of the T's instead of the 0's may be readily derived.

4. **Identity matrices for properly linear sets.** In this article we arrive at a generalization, relative to properly linear subsets V_0 of V and certain commutative non-singular matrices, of the notion of an identity matrix. Moreover we show that for every properly linear subset V_0 of V there exists a commutative symmetric non-singular matrix w which transforms V_0 into a properly linear subset V_1 of V which is supplementary to its orthogonal complement.

Throughout we prove theorems by proving them for the case where $\mathfrak{A}$ is a field, and noting that due to the existence of a commutative base for every properly linear set the theorem follows from the theorem for the special case.

In the case where $\mathfrak{A}$ is a field the six linear processes L_r etc. and the six orthogonal processes 0_{rw} etc. coincide.

LEMMA 1. *If $\mathfrak{A}$ is a field not modulo 2, V_0 is a linear subset of V with rank r greater than zero, and w is an n by n commutative symmetric non-singular matrix such that $\bigcap[0_w V_0, V_0]$ is the set consisting of the single vector 0_V, then there exists a vector v_0 of V_0 such that $S^2 v_0 w v_0 \neq 0$.*

Proof. Let $v_{01} \cdots v_{0r}$ be a base for V_0. If for every $i \leqq r$, $S^2 v_{0i} w v_{0i} = 0$, there exists an i and a j such that $i \neq j$ and $S^2 v_{0i} w v_{0j} \neq 0$. Since $2 = 1 + 1 \neq 0$ it follows that $S^2(v_{0i} + v_{0j})\, w(v_{0i} + v_{0j}) = 2 S^2 v_{0i} w v_{0j} \neq 0$, and hence $v_{0i} + v_{0j}$ is effective as the v_0 of the lemma.

That the lemma need not hold for the case of a field with a modulus 2 is shown by the following example. Consider P^3, $\mathfrak{A}$ = integers modulo 2, $w \equiv \delta$, and

$$V_0 = L \begin{matrix}(1 & 1 & 0)\\(1 & 0 & 1);\end{matrix}$$

the $0 V_0 = L(1\ 1\ 1)$ and V_0 consists of the four vectors $(0\ 0\ 0)$, $(1\ 1\ 0)$, $(1\ 0\ 1)$, and $(0\ 1\ 1)$, but all the elements of V_0 are self orthogonal.

THEOREM 1. *If $\mathfrak{A}'$ is a field not modulo 2, V_0 is a properly linear subset of V and w is an n by n commutative symmetric non-singular matrix such that $\bigcap[0_w V_0, V_0]$ is the set consisting of the single vector 0_V, then there exists one and only one n by n matrix ε of such a nature that*

$$(1) \qquad S\varepsilon v_0 = v_0 = S v_0 \breve{\varepsilon} \qquad (v_0),$$

$$(2) \qquad V_0 \supset S\varepsilon v \qquad (v),$$

$$(3) \qquad 0_w V_0 \supset (v - S\varepsilon v) \qquad (v).$$

Moreover ε is commutative and therefore $S\varepsilon v = S v \breve{\varepsilon}$ for every v.

Proof. Since V_0 has a commutative base it is sufficient to prove the theorem for the case in which $\mathfrak{A}$ is a field. Relative to two vectors v_1 and v_2 we define the dyad $(v_1 v_2)$ as the matrix Φ where $\Phi(i,j) = v_1(i)v_2(j)$ (i,j).

The proof will be divided into two parts, Part I the existence of ε and Part 2 the uniqueness of ε.

Part 1. Existence. *Case* 1. $rk\,V_0 = 0$.

In this case the zero matrix is effective as ε.

Case 2. $rk\,V_0 = r > 0$.

According to Lemma 1 there exists v_{01} such that $S^2 v_{01}\, w v_{01} \neq 0$. Let

$$\varepsilon_1 \equiv \frac{S(v_{01}\, v_{01})w}{S^2 v_{01}\, w v_{01}},$$

and $V_{01} \equiv L(v_{01})\,.\,\varepsilon_1$ is effective as ε for V_{01}. Consider $V_{02} = [v_0 - S\varepsilon_1 v_0\ (v_0)]$. It follows at once that V_{02} is linear and $V_{01} + V_{02} = V_0\,.\,\cap[V_{01}, V_{02}]$ is the set consisting of the single vector 0_V and $0_w V_{02} \supset V_{01}$. Hence if there exists a matrix ε_2 effective as ε for V_{02}, $\varepsilon_1 + \varepsilon_2$ is effective as ε. Thus the existential part of our theorem is true for the case when the rank of V_0 is r in case it is true when the rank of V_0 is $r-1$. Hence, since we have found an effective ε in case $rk\,V_0 = 0$, there exists an effective ε for the case in which $rk\,V_0 = r$.

Part 2. Uniqueness.

Consider ε' and ε'' effective as ε of the theorem:

$$(1) \qquad \begin{aligned} v \cdot v_0 :):\ & S^2(v - S\varepsilon' v) w v_0 = 0\,.\, S^2(v - S\varepsilon'' v)\, w v_0 = 0 \\ :):\ & S^3(\varepsilon' - \varepsilon'')\, v w v_0 = 0; \end{aligned}$$

$$(2) \qquad v\ .).\ S(\varepsilon' - \varepsilon'')\, v \subset V_0.$$

Hence from (1) and (2) and the hypothesis of the theorem it follows that

$$v\ .).\ S(\varepsilon' - \varepsilon'')\, v = 0_V,$$

and hence $\varepsilon' - \varepsilon''$ is the zero matrix and $\varepsilon' = \varepsilon''$.

THEOREM 2. *Relative to a properly linear subset V_0 of V there exists a commutative non-singular n by n matrix Φ such that*

$$w = S\check{\Phi}\Phi\ .).\ \cap\,[0_w V_0, V_0] = [0_V].$$

Proof. Since V_0 has a commutative base it is sufficient to prove the theorem for the case in which $\mathfrak{A}$ is a field.

Case 1. $\mathfrak{A}$ is a field not modulo 2 or 3.

Consider $V_{0*} = [v_1 \cdots v_r]$ the base for V_0 of normal form. It is readily seen since order is not involved that we may assume $\sigma_* = (1 \cdots r)$.

Let $\Phi_1 \equiv \delta$ if $Sv_1 v_1 \neq 0$ and $\Phi_1 \equiv \delta + \delta_{11}$ if $Sv_1 v_1 = 0$ and $w_1 = S\breve{\Phi}_1 \Phi_1$. Then

$$Sv_1 v_1 \neq 0 .).\ Sv_1 w_1 v_1 = Sv_1 v_1,$$
$$Sv_1 v_1 = 0 .).\ Sv_1 w_1 v_1 = 3 \neq 0.$$

Let $V_1 = L(v_1)$ and ε_1 be the identity matrix of Theorem 2 for V_1 in respect to w_1. Then the set $[v_i - S\varepsilon_1 v_i \equiv v_{i1}\ (i = 2, 3, \cdots, r)]$ is a base in semi-normal form for the w_1-orthogonal complement of V_1 in V_0.

This process may be repeated by the general recursion formulas for $j = 1, \cdots, r-1$. (Let $v_{10} = v_1$.)

$V_j = L[v_1, v_{21}, \cdots, v_{j\,j-1}]$;

$\varepsilon_j =$ the identity matrix of Theorem 2 for V_j in respect to w_j.

$[v_i - S\varepsilon_j v_i = v_{ij}\ (i = j+1, \cdots, r)]$ is a base in semi-normal form for the w_j-orthogonal complement of V_j in V_0, and

$$\Phi_{j+1} = \Phi_j \text{ if } Sv_{j+1\,j} w_j v_{j+1\,j} \neq 0,$$

but

$$\Phi_{j+1} = \Phi_j + \delta_{j+1\,j+1} \text{ if } Sv_{j+1\,j} w_j v_{j+1\,j} = 0$$

and

$$w_{j+1} = S\breve{\Phi}_{j+1} \Phi_{j+1}.$$

Hence

$$Sv_{i\,i-1} w_{j+1} v_{i\,i-1} = Sv_{i\,i-1} w_i v_{i\,i-1} \neq 0 \qquad (i = 1, \cdots, j),$$
$$Sv_{i\,i-1} w_{j+1} v_{k\,k-1} = 0 \quad (i \neq k,\ i = 1, \cdots, j,\ k = 1, \cdots, j+1),$$
$$Sv_{j+1\,j} w_{j+1} v_{j+1\,j} \neq 0.$$

Hence Φ_r is effective as the Φ of the theorem.

Case 2. $\mathfrak{A}$ is a field modulo 2 or 3.

In this case we make $\Phi_{j+1} = \Phi_j + \delta_{j+2\,j+1}$ if $Sv_{j+1\,j} w_j v_{j+1\,j} = 0$. Otherwise the proof is analogous to that for Case 1.

We may state Theorem 3 as follows: *Relative to a properly linear subset V_0 of V there exists an n by n commutative non-singular matrix Φ which transforms V_0 into a properly linear subset V_0' of such a nature that $0V_0'$ is supplementary to V_0'.*

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