THE GROUP OF MOTIONS OF AN EINSTEIN SPACE*

BY

JOHN EIESLAND

Introduction

The question to what extent the general Einstein space is determined by its group of motions seems to be of interest from a physical as well as a geometric standpoint.

In what follows we have discussed the problem of determining the group of motions in a given Riemannian n-space and its converse (Killing's equations). The assumption is then made that an Einstein space whose linear element is

$$(a) \qquad -ds^2 = \sum_1^3 g_{ik}\, dx_i\, dx_k - \sum_0^3 g_{i0}\, dx_i\, dx_0, \qquad x_0 = t,$$

shall admit the group of "rotations"

$$G_3: \qquad x^i \frac{\partial f}{\partial x_k} - x_k \frac{\partial f}{\partial x_i} \qquad (i, k = 1, 2, 3)$$

and the following theorem is proved:

A necessary and sufficient condition that the space (a) *shall be reducible to the form*

$$(b) \qquad -ds^2 = \varphi_2\, dr^2 + \varphi_3 (d\theta^2 + \sin^2\theta\, d\varphi^2) - \varphi_1\, dt^2,$$

φ_1, φ_2 *and* φ_3 *being arbitrary functions of* r *and* t, *is that it shall admit the group* G_3 *as a complete group of motions.*†

It may further be required that (b) admit a one-parameter group

$$\xi_0 \frac{\partial f}{\partial t} + \xi_1 \frac{\partial f}{\partial r},$$

where ξ_0 and ξ_1 are functions of r and t. The necessary and sufficient conditions that this shall be the case are found to be

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† This theorem has generally been taken for granted by writers on relativity.

$$\varphi_1 \varphi_3 G_1^0 = \Psi'(\varphi_3) \frac{\partial \varphi_3}{\partial t} \frac{\partial \varphi_3}{\partial r},$$

$$\varphi_1 \varphi_2 \varphi_3 (G_1^1 - G_0^0) = +2 \Psi'(\varphi_3) \left[\varphi_1 \frac{\partial \varphi_3}{\partial r} + \varphi_2 \frac{\partial \varphi_3}{\partial t}\right],$$

(Ψ an arbitrary function of φ_3 or a constant). It is then shown that if these conditions are satisfied, the space (b) may be reduced to the static form.

Special forms of static spaces are then considered with special reference to their group properties and the principal curvature of their sub-spaces

$$S_3\colon\ t = 0; \qquad S_3^1\colon\ \varphi = 0; \qquad S_3^2\colon\ \theta = 0.$$

The question of the class of the quadratic form (b) is then taken up, and it is proved that a necessary and sufficient condition that (b) shall be of class 1 is

$$(02, 02)\,(13, 13) = (01, 01)\,(23, 23) + (12, 02)\,(13, 03). \tag{c}$$

The general space (b) can therefore be immersed in a flat 6-space, and, if (c) is satisfied, in a flat 5-space.

It is also proved that if a general space (a) admits any one of the abelian groups

$$\frac{\partial f}{\partial x_0}; \quad \frac{\partial f}{\partial x_0}, \frac{\partial f}{\partial x_3}; \quad \frac{\partial f}{\partial x_0}, \frac{\partial f}{\partial x_3}, \frac{\partial f}{\partial x_2},$$

as complete group of motions, it is of the fifth, third, and second class respectively. Among these spaces is found Weyl's static and cylindrical space admitting an abelian G_2.

1. **The general differential quadratic form.** Let there be given a general differential quadratic form

$$ds^2 = \sum_1^n a_{ik}\, dx_i\, dx_k \tag{1}$$

which may be interpreted as the linear element of a curved space S_n of n dimensions. This space is said to admit of a group of rigid motions, if there exists a group of transformations

$$x_i' = f_i(x_1, x_2, \cdots, x_n, a_1, a_2, \cdots, a_r) \qquad (i = 1, 2, \cdots, n), \tag{2}$$

which will carry the form (1) into the form

$$ds^2 = \sum_1^n a_{ik}'\, dx_i'\, dx_k'$$

such that the coefficients a'_{ik} are the same functions of x'_i as the a_{ik}'s are of x_i. If the coefficients a_{ik} are perfectly general, no such group exists.

In order that a given form (1) shall admit a group*

$$Uf = \xi_1 \frac{\partial f}{\partial x_1} + \xi_2 \frac{\partial f}{\partial x_2} + \cdots + \xi_n \frac{\partial f}{\partial x_n},$$

the ξ's must satisfy the so-called Killing's equations,

$$(3) \qquad \sum_1^n \xi_\lambda \frac{\partial a_{ik}}{\partial x_\lambda} + \sum_1^n a_{i\lambda} \frac{\partial \xi_\lambda}{\partial x_k} + \sum_1^n a_{k\lambda} \frac{\partial \xi_\lambda}{\partial x_i} = 0 \quad (i, k = 1, 2, \cdots, n),$$

the integration of which will determine the ξ's as functions of x_i and r constants of integration. The maximum group has $r = n(n+1)/2$ parameters, in which case the space S_n has a constant Riemannian curvature. Bianchi† gives Killing's equations another form,

$$(4) \quad \frac{\partial \eta_i}{\partial x_k} + \frac{\partial \eta_k}{\partial x_i} = 2 \sum_\lambda^{1\cdots n} \begin{Bmatrix} ik \\ \lambda \end{Bmatrix} \eta_\lambda, \quad \eta_i = \sum_\lambda^{1\cdots n} a_{i\lambda}\, \xi_\lambda \qquad (i, k = 1, 2, \cdots, n),$$

where $\begin{Bmatrix} ik \\ \lambda \end{Bmatrix}$ are the usual Christoffel symbols. All the second derivatives obtained from these equations can be expressed linearly and homogeneously in terms of the η's and their first derivatives. We thus obtain the system

$$(5) \qquad \frac{\partial^2 \eta_i}{\partial x_k \partial x_l} = \frac{\partial}{\partial x_l} \sum_\lambda^{1\cdots n} \begin{Bmatrix} ik \\ \lambda \end{Bmatrix} \eta_\lambda + \frac{\partial}{\partial x_k} \sum_\lambda^{1\cdots n} \begin{Bmatrix} il \\ \lambda \end{Bmatrix} \eta_\lambda - \frac{\partial}{\partial x_i} \sum_\lambda^{1\cdots n} \begin{Bmatrix} kl \\ \lambda \end{Bmatrix} \eta_\lambda$$
$$(i, k, l = 1, 2, \cdots, n).$$

If the systems (4) and (5) are completely integrable, the group has $r = n(n+1)/2$ parameters. If $r < n(n+1)/2$, the system is not complete. If therefore we form the conditions of integrability, we find new relations between the η's and their first derivatives which must be added to the system (4). Continuing in this way we shall eventually arrive at the complete Lie-Mayer system defining the group.

2. Let us suppose that a space S_n admits at least a one-parameter group G_1. By proper choice of variables this group may always be reduced

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* It is clear that if (1) is invariant under the ∞^r finite transformations of the group (2) it is also invariant under the corresponding r infinitesimal transformations of the group. For proof of the converse see L. Bianchi, *Lezioni sulla Teoria dei Gruppi Continui di Transformazioni,* Pisa, 1918, pp. 493–495.

† L. Bianchi, loc. cit., pp. 502–503.

to the form $\partial f/\partial x_1$. If therefore we put $\xi_1 = 1$, $\xi_2 = \xi_3 = \cdots = \xi_n = 0$ in Killing's equations (3), we find

$$\frac{\partial a_{ik}}{\partial x_1} = 0$$

which means that the coefficients a_{ik} do not contain x_1. Hence, *if a space S_n admits a one-parameter group of motions, its linear element can always be put in the form*

$$ds^2 = \sum_0^3 a_{ik}\, dx_i\, dx_k, \tag{6}$$

*where the coefficients a_{ik} do not contain x_1.**

Suppose further that the group G_1 is such that the infinitesimal motion at every point of S_n has a constant amplitude. A motion of this kind corresponds to a translation in ordinary euclidean space (Schiebung). Since we have $\delta x_i = \xi_i\, \delta t$, the condition to be satisfied, in addition to those of equations (3), is

$$\frac{\delta s^2}{\delta t^2} = \sum_0^3 a_{ik}\, \xi_i\, \xi_k = \text{const.}; \tag{7}$$

the ξ's are therefore the constants of direction at any point in S_n. If we reduce ds^2 to the form (6) and apply (7) we find $a_{11} =$ const. But the condition $a_{11} =$ const. is the condition that the line x_1 shall be a geodesic in S_n.† We have therefore the

THEOREM I. *An infinitesimal motion is a translation if, and only if, the trajectories of the group G_1 generated by it are geodesic lines in S_n.*

Any finite translation carries all the points of space the same geodesic distance from their original positions.

We shall state the following proposition,‡ the proof of which we shall omit:

If the space S_n admits a translation, any spread formed by ∞^1 trajectories of the motion is of zero curvature.

3. The space of a four-dimensional metric field. After these preliminaries which are largely restatements of well known theorems we shall proceed to study the four-dimensional metric field of Einstein's relativity theory, with a special view to its group-theoretical properties.

* The converse is also true: If the linear element of S_n can be put in the form (6), the space admits at least a one-parameter group of motions.

† L. Bianchi, loc. cit., p. 500.

‡ Loc. cit., p. 500.

Consider the quadratic form

$$ds^2 = \sum_0^3 g_{ik}\, dx_i\, dx_k, \tag{8}$$

which in Einstein's relativity theory contains the metrical relations of time and physical space. Let $x_0 = t$, t being interpreted as time, and let x_1, x_2, x_3 be the coördinates of a space such that its linear element

$$ds_0^2 = -(ds^2)_{dx_0=0}.$$

We may therefore put

$$ds_0^2 = -\sum_1^3 g_{ik}\, dx_i\, dx_k = \sum_1^3 a_{ik}\, dx_i\, dx_k, \tag{9}$$

and we shall assume moreover that this form is positive and definite. The general quadratic form (9) may therefore be written

$$ds^2 = g_{00}\, dt^2 + \sum_0^3 g_{0i}\, dx_i\, dt - \sum_1^3 a_{ik}\, dx_i\, dx_k \tag{10}$$

which is indefinite, the index of inertia being 3. g_{00} may be interpreted as a velocity; for, if t only varies, we have $ds^2/dt^2 = g_{00} = V^2$ so that $\sqrt{g_{00}} = V$ has the dimension of velocity.

Let us assume that the coefficients g_{00}, g_{i0} and a_{ik} do not contain t. By Theorem I this means that (10) admits at least a one-parameter group of motions, namely

$$Uf = \frac{\partial f}{\partial t}, \tag{11}$$

the invariant spreads of which are the 3-spreads $t =$ const. A space of this kind we shall call with Levi-Civita a *stationary space*, so that we have the

THEOREM II. *A necessary and sufficient condition that a general Einstein space* (10) *shall be stationary is that it shall admit the group* (11). *This motion is a "translation" if, and only if,* g_{00} *is a constant* (*Theorem* I).

The path-curves of the transformation (11) are not in general geodesics in S_4. Only when g_{00} is a constant will this be the case, and (10) may be reduced to the geodesic form

$$ds^2 = c^2\, dt^2 - \sum_1^3 a_{ik}\, dx_i\, dx_k, \tag{12}$$

in which the coefficients g_{i0} are absent and the new coefficients a_{ik} do not contain t as before. Since ∞^1 paths-curves of the "translation" will form a spread of zero curvature, the space (12) may be described as "cylindrical".*

We shall, however, assume that the space (10) is general, the coefficients g_{ik} being functions of x_i, $i = 0, 1, 2, 3$. By means of the transformation

$$x'_0 = x_0, \quad x'_i = x'_i\,(x_0, x_1, x_2, x_3) \qquad (i = 1, 2, 3)$$

we may remove the coefficients g_{i0} in (10); in fact, it will be necessary and sufficient that the functions x'_i shall be solutions of the differential equation

$$\nabla(x_0, \theta) = \sum_0^3 g^{i0} \frac{\partial \theta}{\partial x_i} = 0,$$

$\nabla(x_0, x'_i) = 0$ being the conditions that the space x'_1, x'_2, x'_3 shall be orthogonal to the coördinate line x'_0. The space (10) has now the form

$$ds^2 = g_{00}\, dx_0^2 - \sum_1^3 a_{ik}\, dx_i\, dx_k. \tag{13}$$

Let us suppose that this space admits a group G and let the general nature of this group be left arbitrary for the time being, except that it does not operate on $x_0 = t$, i. e., it is a group of the sub-space $x_0 =$ const. We write then

$$Uf = \xi_1 \frac{\partial f}{\partial x_1} + \xi_2 \frac{\partial f}{\partial x_2} + \xi_3 \frac{\partial f}{\partial x_3}. \tag{14}$$

The equations (3) are

$$\begin{gathered}
\sum_1^3 \xi_\lambda \frac{\partial g_{00}}{\partial x_\lambda} + 2\sum_1^3 g_{0\lambda} \frac{\partial \xi_\lambda}{\partial x_0} = 0, \qquad \sum_1^3 \xi_\lambda \frac{\partial a_{11}}{\partial x_\lambda} + 2\sum_1^3 a_{1\lambda} \frac{\partial \xi_\lambda}{\partial x_1} = 0, \\
\sum_1^3 \xi_\lambda \frac{\partial a_{22}}{\partial x_\lambda} + 2\sum_1^3 a_{2\lambda} \frac{\partial \xi_\lambda}{\partial x_2} = 0, \qquad \sum_1^3 \xi_\lambda \frac{\partial a_{33}}{\partial x_\lambda} + 2\sum_1^3 a_{3\lambda} \frac{\partial \xi_\lambda}{\partial x_3} = 0, \\
\sum_1^3 \xi_\lambda \frac{\partial a_{ik}}{\partial x_\lambda} + \sum_1^3 a_{i\lambda} \frac{\partial \xi_\lambda}{\partial x_\lambda} + \sum_1^3 a_{k\lambda} \frac{\partial \xi_\lambda}{\partial x_\lambda} = 0, \\
\sum_1^3 a_{1\lambda} \frac{\partial \xi_\lambda}{\partial x_0} = 0, \qquad \sum_1^3 a_{2\lambda} \frac{\partial \xi_\lambda}{\partial x_0} = 0, \qquad \sum_1^3 a_{3\lambda} \frac{\partial \xi_\lambda}{\partial x_0} = 0.
\end{gathered} \tag{15}$$

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* The term "static" instead of "stationary" has been used by G. D. Birkhoff in a recent publication, ***Relativity and Modern Physics*** (Cambridge, Harvard University Press, 1923). If we consider the hydrodynamic analogy, it would seem that the term "stationary" is a better term. We do not speak of a "static" motion in hydrodynamics, when a stationary or permanent motion is meant. The term "static" field is used by T. Levi-Civita to denote a stationary field in which the coefficients g_{i0} are absent. See T. Levi-Civita, *La Teoria di Einstein e il Principio di Fermat,* Il Nuovo Cimento, ser. 6, vol. 16 (1918), pp. 105–114.

Since the determinant $|a_{ik}|$ cannot vanish, the last three equations show that the ξ's are independent of x_0. The first of equations (15) becomes

$$\sum_1^3 \xi_\lambda \frac{\partial g_{00}}{\partial x_\lambda} = 0,$$

which means that g_{00} is of the form $g_{00}(\varphi, x_0)$ where φ is an invariant of the group G, or else a function of x_0 alone, in which case g_{00} may be reduced to a constant. In the first case, since G does not involve x_0, g_{00} is itself an invariant of the group.

(a) *g_{00} an invariant of G.* G must be an intransitive group considered as belonging to S_4. But since G does not contain x_0, nor operate on x_0, it must be a group of motions in S_3; this is also clear when we consider that the remaining equations in the system (15) are Killing's equations corresponding to the space S_3. It should be noted that this does not prevent G from being a subgroup of a transitive group $\bar{G}$ of motions in S_3, but G will not belong to S_4 unless g_{00} is a function of x_0 alone, or a constant.

(b) g_{00} = const. In this case G may be any group of motions in S_3, transitive or intransitive; it may even be the maximum group G_6 in which case S_3 is a space of constant positive or zero curvature. If G is a transitive group in S_3, it can belong to S_4 if, and only if, g_{00} is a function of x_0 alone or a constant. We shall state these results in the following

THEOREM III. *If the space whose linear element is*

$$ds^2 = g_{00}\, dx_0^2 - \sum_1^3 a_{ik}\, dx_i\, dx_k$$

admits a group of the form

$$Uf = \xi_1 \frac{\partial f}{\partial x_1} + \xi_2 \frac{\partial f}{\partial x_2} + \xi_3 \frac{\partial f}{\partial x_3},$$

the ξ's are independent of x_0, and the group belongs also to the subspace S_3. g_{00} is either an invariant of the group, or a function of x_0 alone. In the first case, the group is intransitive. In the second case, g_{00} may by a transformation be reduced to a constant, and the group is either transitive or intransitive. A transitive group in S_3 belongs to S_4 if, and only if, g_{00} is a constant.

4. **The group of "rotations" in S_3.*** We shall suppose that S_4 admits the intransitive group of "rotations" about the origin in S_3, viz.

$$(16) \qquad x_1 \frac{\partial f}{\partial x_2} - x_2 \frac{\partial f}{\partial x_1}, \quad x_2 \frac{\partial f}{\partial x_3} - x_3 \frac{\partial f}{\partial x_2}, \quad x_3 \frac{\partial f}{\partial x_1} - x_1 \frac{\partial f}{\partial x_3}.$$

* By a "group of rotations" we mean here a 3-parameter group in the variables x_1, x_2, x_3 having the invariant $x_1^2 + x_2^2 + x_3^2$; x_1, x_2, x_3 are not cartesian coördinates.

The sub-space S_3 is then said to be *centro-symmetric.* If in (15) we introduce in succession the following values of the ξ's,

$$\begin{aligned}
&\xi_1 = -x_2, \qquad && \xi_2 = x_1, \qquad && \xi_3 = 0,\\
&\xi_1 = 0, && \xi_2 = -x_3, && \xi_3 = x_2,\\
&\xi_1 = x_3, && \xi_2 = 0, && \xi_3 = -x_1,
\end{aligned}$$

we obtain a system of equations for determining the quantities g_{00} and a_{ik}:

$$(a)\qquad \begin{gathered}
x_1 \frac{\partial g_{00}}{\partial x_2} - x_2 \frac{\partial g_{00}}{\partial x_1} = 0, \qquad x_2 \frac{\partial g_{00}}{\partial x_3} - x_3 \frac{\partial g_{00}}{\partial x_2} = 0,\\
x_3 \frac{\partial g_{00}}{\partial x_1} - x_1 \frac{\partial g_{00}}{\partial x_3} = 0;
\end{gathered}$$

$$(17)\quad (b)\qquad \begin{gathered}
x_1 \frac{\partial a_{11}}{\partial x_2} - x_2 \frac{\partial a_{11}}{\partial x_1} + 2a_{12} = 0, \qquad x_2 \frac{\partial a_{11}}{\partial x_3} - x_3 \frac{\partial a_{11}}{\partial x_2} = 0,\\
x_3 \frac{\partial a_{11}}{\partial x_1} - x_1 \frac{\partial a_{11}}{\partial x_3} - 2a_{13} = 0,\\
x_1 \frac{\partial a_{22}}{\partial x_2} - x_2 \frac{\partial a_{22}}{\partial x_1} - 2a_{12} = 0, \qquad x_2 \frac{\partial a_{22}}{\partial x_3} - x_3 \frac{\partial a_{22}}{\partial x_2} + 2a_{23} = 0,\\
x_3 \frac{\partial a_{22}}{\partial x_1} - x_1 \frac{\partial a_{22}}{\partial x_3} = 0,\\
x_1 \frac{\partial a_{33}}{\partial x_2} - x_2 \frac{\partial a_{33}}{\partial x_1} = 0, \qquad x_2 \frac{\partial a_{33}}{\partial x_3} - x_3 \frac{\partial a_{33}}{\partial x_2} - 2a_{23} = 0,\\
x_3 \frac{\partial a_{33}}{\partial x_1} - x_1 \frac{\partial a_{33}}{\partial x_3} + 2a_{13} = 0,\\
x_1 \frac{\partial a_{12}}{\partial x_2} - x_2 \frac{\partial a_{12}}{\partial x_1} - a_{11} + a_{22} = 0, \qquad x_2 \frac{\partial a_{12}}{\partial x_3} - x_3 \frac{\partial a_{12}}{\partial x_2} + a_{13} = 0,\\
x_3 \frac{\partial a_{12}}{\partial x_1} - x_1 \frac{\partial a_{12}}{\partial x_3} - a_{23} = 0,\\
x_1 \frac{\partial a_{13}}{\partial x_2} - x_2 \frac{\partial a_{13}}{\partial x_1} + a_{23} = 0, \qquad x_2 \frac{\partial a_{13}}{\partial x_3} - x_3 \frac{\partial a_{13}}{\partial x_2} + a_{12} = 0,\\
x_3 \frac{\partial a_{13}}{\partial x_1} - x_1 \frac{\partial a_{13}}{\partial x_3} - a_{33} + a_{11} = 0,\\
x_1 \frac{\partial a_{23}}{\partial x_2} - x_2 \frac{\partial a_{23}}{\partial x_1} - a_{13} = 0, \qquad x_2 \frac{\partial a_{23}}{\partial x_3} - x_3 \frac{\partial a_{23}}{\partial x_2} + a_{22} - a_{33} = 0,\\
x_3 \frac{\partial a_{23}}{\partial x_1} - x_1 \frac{\partial a_{23}}{\partial x_3} + a_{12} = 0.
\end{gathered}$$

If we suppose that g_{00} is not a function of x_0 alone, the equations (a) express the fact that g_{00} is an invariant of the group G_3, so that we may put

$$g_{00} = \varphi_1(\sqrt{x_1^2+x_2^2+x_3^2},\ x_0).$$

We proceed now to integrate (b). By elimination we easily find the following relations:

$$(18) \qquad a_{13} = \frac{x_3}{x_2}a_{12}, \qquad a_{23} = \frac{x_2}{x_1}a_{12}.$$

The equations involving a_{12}, a_{13}, a_{23} give, on integrating, keeping account of (18),

$$(19) \qquad a_{12} = x_1 x_2 \varphi_2, \qquad a_{13} = x_1 x_3 \varphi_2, \qquad a_{23} = x_2 x_3 \varphi_2,$$

φ_2 being an arbitrary function of $\sqrt{x_1^2+x_2^2+x_3^2}$ and x_0. We also find the relations

$$(20) \qquad a_{11} - a_{22} = (x_1^2 - x_2^2)\varphi_2, \qquad a_{22} - a_{33} = (x_2^2 - x_3^2)\varphi_2,$$
$$a_{33} - a_{11} = (x_3^2 - x_1^2)\varphi_2.$$

Integrating the equations in a_{11}, a_{22} and a_{33} we have

$$a_{11} = \varphi_3 + x_1^2\varphi_2, \qquad a_{22} = \varphi_3 + x_2^2\varphi_2, \qquad a_{33} = \varphi_3 + x_3^2\varphi_2.$$

We have thus obtained the following quadratic form,

$$ds^2 = \varphi_1\, dx_0^2 - R^2\varphi_2\left[\frac{x_1\,dx_1 + x_2\,dx_2 + x_3\,dx_3}{R}\right]^2 - \varphi_3[dx_1^2 + dx_2^2 + dx_3^2],$$

where $R = \sqrt{x_1^2+x_2^2+x_3^2}$. Introducing spherical coördinates

$$x_1 = R\sin\theta\cos\varphi, \qquad x_2 = R\sin\theta\sin\varphi, \qquad x_3 = R\cos\theta,$$

we have, remembering that φ_2 and φ_3 are arbitrary functions of R and x_0,

$$(21) \qquad ds^2 = \varphi_1\,dx_0^2 - (\varphi_3+\varphi_2)\,dR^2 - R^2\varphi_3(d\theta^2 + \sin^2\theta\,d\varphi^2).$$

The group G_3 becomes, on introducing the new variables,

$$(22) \qquad \begin{aligned} U_1 &= \sin\varphi\frac{\partial f}{\partial\theta} + \cot\theta\cos\varphi\frac{\partial f}{\partial\varphi}, \\ U_2 &= \cos\varphi\frac{\partial f}{\partial\theta} - \cot\theta\sin\varphi\frac{\partial f}{\partial\varphi}, \qquad U_3 = \frac{\partial f}{\partial\varphi}. \end{aligned}$$

This group is transitive considered as a group in the variables θ and φ, and the variable R does not appear, as was to be expected according to a theorem by Fubini.* We have thus proved the

THEOREM IV. *A necessary and sufficient condition that the space* (13) *shall be reducible to the form* (21) *is that it shall admit the group* G_3 *as a complete group of motions.*

For the purpose of further specialization we shall consider a few invariants that play an important rôle in the classification of 3- and 4-spaces and also in the general relativity theory.

5. The total curvature of S_4 (curvature scalar) is given by the formula

$$R = \sum_{h,i}^{0\cdots 3} g^{hi} R_{hi}, \qquad R_{hi} = \sum_{p}^{0\cdots 3} \{hp, ip\}. \tag{23}$$

If instead of the symbols $\{hp, ip\}$ we introduce the Riemannian symbols (hp, iq), we have

$$\{hp, ip\} = \sum_{p,q}^{0\cdots 3} g^{pq}(hp, iq) \qquad R = \sum_{p,q}^{0\cdots 3} g^{pq}(hp, iq),$$

where†

$$(hp, iq) = \frac{1}{2}\left[\frac{\partial^2 g_{hq}}{\partial x_i \partial x_p} + \frac{\partial^2 g_{ip}}{\partial x_h \partial x_q} - \frac{\partial^2 g_{hi}}{\partial x_p \partial x_q} - \frac{\partial^2 g_{pq}}{\partial x_h \partial x_i}\right]$$
$$+ \sum_{l,m}^{0\cdots 3} g^{lm}\left(\begin{bmatrix} h\ q \\ m \end{bmatrix}\begin{bmatrix} i\ p \\ l \end{bmatrix} - \begin{bmatrix} h\ i \\ m \end{bmatrix}\begin{bmatrix} p\ q \\ l \end{bmatrix}\right),$$

and the quantities g^{pq} are the co-factors of g_{pq} divided by g. We now define the following expressions:

$$G_{ih} = \frac{1}{2} g_{ih} R - R_{ih}, \tag{24}$$

and, introducing the mixed forms G_i^h, we put

$$G_i^h = \sum_j g^{hj} G_{ji}. \tag{25}$$

We shall also recall here that for the empty space in an Einstein solar field we must have $R_{ik} = 0$, or, what is the same thing, $G_i^h = 0$. Calculating the curvature tensors R_{ik} for the space (21) which we write in the form

$$-ds^2 = \varphi_2\, dR^2 + \varphi_3\,(d\theta^2 + \sin^2\theta\, d\varphi^2) - \varphi_1\, dt^2 \tag{21'}$$

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* L. Bianchi, loc. cit., pp. 517–518. See also Fubini's memoir in vol. 3 of Annali di Matematica.

† The non-vanishing Riemannian symbols (hp, iq) are given on p. 238, equations (58).

we have

$$g_{00} = -\varphi_1, \quad g_{11} = \varphi_2, \quad g_{22} = \varphi_3, \quad g_{33} = \varphi_3 \sin^2\theta,$$

$$g^{11} = \frac{1}{\varphi_2}, \quad g^{22} = \frac{1}{\varphi_3}, \quad g^{33} = \frac{1}{\varphi_3 \sin^2\theta}, \quad g^{00} = -\frac{1}{\varphi_1};$$

$$\begin{aligned} &R_{12} = g^{33}(13,23) + g^{00}(10,20), &&R_{13} = g^{22}(12,32) + g^{00}(10,30),\\ (26a)\quad &R_{23} = g^{11}(21,31) + g^{00}(20,30), &&R_{10} = g^{22}(12,02) + g^{33}(13,03),\\ &R_{20} = g^{11}(21,01) + g^{33}(23,03), &&R_{30} = g^{11}(31,01) + g^{22}(32,02);\end{aligned}$$

$$(26b)\quad \begin{aligned} R_{11} &= g^{22}(12,\,12) + g^{33}(13,\,13) + g^{00}(10,\,10),\\ R_{22} &= g^{11}(21,\,21) + g^{33}(23,\,23) + g^{00}(10,\,10),\\ R_{33} &= g^{11}(31,\,31) + g^{22}(32,\,32) + g^{00}(30,\,30),\\ R_{00} &= g^{11}(01,\,01) + g^{22}(02,\,02) + g^{33}(30,\,30).\end{aligned}$$

Calculating the Riemannian symbols (hp, iq) and substituting in these equations we find

$$R_{12} = 0, \quad R_{13} = 0, \quad R_{23} = 0, \quad R_{20} = 0, \quad R_{30} = 0,$$

$$R_{10} = -\frac{1}{\varphi_3}\frac{\partial^2\varphi_3}{\partial r\,\partial t} + \frac{1}{2\varphi_2\varphi_3}\frac{\partial\varphi_3}{\partial r}\frac{\partial\varphi_2}{\partial t} + \frac{1}{2\varphi_3^2}\frac{\partial\varphi_3}{\partial r}\frac{\partial\varphi_3}{\partial t} + \frac{1}{2\varphi_1\varphi_3}\frac{\partial\varphi_1}{\partial r}\frac{\partial\varphi_3}{\partial t},$$

$$\begin{aligned} R_{22} = \frac{R_{33}}{\sin^2\theta} = &-\frac{1}{2\varphi_2}\frac{\partial^2\varphi_3}{\partial r^2} + \frac{1}{4\varphi_2^2}\frac{\partial\varphi_2}{\partial r}\frac{\partial\varphi_3}{\partial r} - \frac{1}{4\varphi_1^2}\frac{\partial\varphi_3}{\partial t}\frac{\partial\varphi_1}{\partial t} + \frac{1}{2\varphi_1}\frac{\partial^2\varphi_3}{\partial t^2}\\ &+ \frac{1}{4\varphi_1\varphi_2}\left[\frac{\partial\varphi_2}{\partial t}\frac{\partial\varphi_3}{\partial t} - \frac{\partial\varphi_1}{\partial r}\frac{\partial\varphi_3}{\partial r}\right] + 1,\end{aligned}$$

$$(27)\quad \begin{aligned} R_{11} = &-\frac{1}{\varphi_3}\frac{\partial^2\varphi_3}{\partial r^2} + \frac{1}{2\varphi_1}\frac{\partial^2\varphi_2}{\partial t^2} - \frac{1}{2\varphi_1}\frac{\partial^2\varphi_1}{\partial r^2} + \frac{1}{2\varphi_2\varphi_3}\frac{\partial\varphi_2}{\partial r}\frac{\partial\varphi_3}{\partial r}\\ &+ \frac{1}{2\varphi_3^2}\left[\frac{\partial\varphi_3}{\partial r}\right]^2 + \frac{1}{2\varphi_1\varphi_3}\frac{\partial\varphi_2}{\partial t}\frac{\partial\varphi_3}{\partial t} + \frac{1}{4\varphi_1\varphi_2}\left[\frac{\partial\varphi_2}{\partial t}\right]^2 - \frac{1}{4\varphi_1^2}\left[\frac{\partial\varphi_1}{\partial r}\right]^2\\ &+ \frac{1}{4\varphi_2^2}\frac{\partial\varphi_2}{\partial r}\frac{\partial\varphi_1}{\partial t} - \frac{1}{4\varphi_1\varphi_2}\frac{\partial\varphi_2}{\partial r}\frac{\partial\varphi_1}{\partial t},\end{aligned}$$

$$\begin{aligned} R_{00} = &-\frac{1}{\varphi_3}\frac{\partial^2\varphi_3}{\partial t^2} - \frac{1}{2\varphi_2}\frac{\partial^2\varphi_2}{\partial t^2} + \frac{1}{2\varphi_2}\frac{\partial^2\varphi_1}{\partial r^2} + \frac{1}{2\varphi_1\varphi_3}\frac{\partial\varphi_3}{\partial t}\frac{\partial\varphi_1}{\partial t}\\ &+ \frac{1}{2\varphi_3^2}\left[\frac{\partial\varphi_3}{\partial t}\right]^2 + \frac{1}{4\varphi_2^2}\left[\frac{\partial\varphi_2}{\partial t}\right]^2 + \frac{1}{2\varphi_2\varphi_3}\frac{\partial\varphi_3}{\partial r}\frac{\partial\varphi_1}{\partial r} + \frac{1}{4\varphi_1\varphi_2}\frac{\partial\varphi_2}{\partial t}\frac{\partial\varphi_1}{\partial t}\\ &- \frac{1}{\varphi_2^2}\frac{\partial\varphi_2}{\partial r}\frac{\partial\varphi_1}{\partial r} - \frac{1}{4\varphi_1\varphi_2}\left[\frac{\partial\varphi_1}{\partial r}\right]^2;\end{aligned}$$

$$R = g^{11}R_{11} + g^{22}R_{22} + g^{33}R_{33} + g^{00}R_{00}, \qquad G_{ii} = \frac{1}{2} g_{ii} R - R_{ii},$$

$$G_i^i = g^{ii}\, G_{ii} = \frac{1}{2} R - g^{ii} R_{ii}, \qquad G_1^0 = g^{00} G_{01} = - g^{00} R_{10},$$

$$G_i^h = 0, \qquad i, h \neq 1, 0;$$

$$(28)\quad \begin{aligned} G_1^1 &= g^{22}g^{33}(23,\, 23) + g^{00}g^{22}(20,\, 20) + g^{33}g^{00}(03,\, 03),\\ G_2^2 &= g^{11}g^{33}(13,\, 13) + g^{11}g^{00}(10,\, 10) + g^{33}g^{00}(03,\, 03),\\ G_3^3 &= g^{11}g^{22}(12,\, 12) + g^{11}g^{00}(10,\, 10) + g^{22}g^{00}(02,\, 02),\\ G_0^0 &= g^{11}g^{22}(12,\, 12) + g^{11}g^{33}(13,\, 13) + g^{22}g^{33}(23,\, 23); \end{aligned}$$

$$(29)\quad \begin{aligned} G_1^1 &= \frac{1}{\varphi_3} - \frac{1}{4\varphi_2\varphi_3^2}\left[\frac{\partial \varphi_3}{\partial r}\right]^2 - \frac{1}{4\varphi_1\varphi_3^2}\left[\frac{\partial \varphi_3}{\partial t}\right]^2 - \frac{1}{2\varphi_3\varphi_1^2}\frac{\partial \varphi_3}{\partial t}\frac{\partial \varphi_1}{\partial t}\\ &\quad - \frac{1}{2\varphi_1\varphi_2\varphi_3}\frac{\partial \varphi_3}{\partial r}\frac{\partial \varphi_1}{\partial r} + \frac{1}{\varphi_1\varphi_3}\frac{\partial^2 \varphi_3}{\partial t^2},\\ G_0^0 &= \frac{1}{\varphi_3} + \frac{1}{4\varphi_2\varphi_3^2}\left[\frac{\partial \varphi_3}{\partial r}\right]^2 + \frac{1}{4\varphi_1\varphi_3^2}\left[\frac{\partial \varphi_3}{\partial t}\right]^2 + \frac{1}{2\varphi_2^2\varphi_3}\frac{\partial \varphi_2}{\partial r}\frac{\partial \varphi_3}{\partial r}\\ &\quad + \frac{1}{2\varphi_1\varphi_2\varphi_3}\frac{\partial \varphi_2}{\partial t}\frac{\partial \varphi_3}{\partial t} - \frac{1}{\varphi_2\varphi_3}\frac{\partial^2 \varphi_3}{\partial r^2},\\ G_2^2 = G_3^3 &= \frac{1}{2\varphi_1\varphi_2}\frac{\partial^2 \varphi_2}{\partial t^2} - \frac{1}{2\varphi_1\varphi_2}\frac{\partial^2 \varphi_1}{\partial r^2} - \frac{1}{2\varphi_3\varphi_2}\frac{\partial^2 \varphi_3}{\partial r^2}\\ &\quad + \frac{1}{4\varphi_2\varphi_3^2}\left[\frac{\partial \varphi_3}{\partial r}\right]^2 - \frac{1}{4\varphi_1\varphi_2^2}\left[\frac{\partial \varphi_2}{\partial t}\right]^2 + \frac{1}{4\varphi_2\varphi_1^2}\left[\frac{\partial \varphi_1}{\partial r}\right]^2 - \frac{1}{4\varphi_1\varphi_3^2}\left[\frac{\partial \varphi_3}{\partial t}\right]^2\\ &\quad + \frac{1}{4\varphi_3\varphi_2^2}\frac{\partial \varphi_2}{\partial r}\frac{\partial \varphi_3}{\partial r} - \frac{1}{4\varphi_1^2\varphi_2}\frac{\partial \varphi_2}{\partial t}\frac{\partial \varphi_1}{\partial t} - \frac{1}{4\varphi_1\varphi_2\varphi_3}\frac{\partial \varphi_3}{\partial r}\frac{\partial \varphi_1}{\partial r}\\ &\quad - \frac{1}{4\varphi_1^2\varphi_3}\frac{\partial \varphi_3}{\partial t}\frac{\partial \varphi_1}{\partial t} + \frac{1}{4\varphi_1\varphi_2\varphi_3}\frac{\partial \varphi_2}{\partial t}\frac{\partial \varphi_3}{\partial t} + \frac{1}{4\varphi_1\varphi_2^2}\frac{\partial \varphi_2}{\partial r}\frac{\partial \varphi_1}{\partial r}\\ &\quad + \frac{1}{2\varphi_1\varphi_3}\frac{\partial^2 \varphi_3}{\partial t^2},\\ G_1^0 &= \frac{1}{\varphi_1}\left[\frac{1}{\varphi_3}\frac{\partial^2 \varphi_3}{\partial r\,\partial t} - \frac{1}{2\varphi_2\varphi_3}\frac{\partial \varphi_2}{\partial t}\frac{\partial \varphi_3}{\partial r} - \frac{1}{2\varphi_3^2}\frac{\partial \varphi_3}{\partial r}\frac{\partial \varphi_3}{\partial t} - \frac{1}{2\varphi_1\varphi_3}\frac{\partial \varphi_1}{\partial r}\frac{\partial \varphi_3}{\partial t}\right]. \end{aligned}$$

It is significant that these mixed tensors do not contain any of the variables φ, θ, while R_{33} contains θ.

6. We shall now suppose that the space (21′) admits a one-parameter group whose infinitesimal symbol is of the form

$$G_1: \qquad Uf = \xi_0 \frac{\partial f}{\partial t} + \xi_1 \frac{\partial f}{\partial r},$$

where ξ_0 and ξ_1 are functions of r and t. In order that this shall be the case, the functions φ_1, φ_2 and φ_3 must satisfy certain conditions which we shall now proceed to find. The Killing equations (4) are in this case

$$(30)\qquad \begin{aligned} &\xi_0 \frac{\partial \varphi_2}{\partial t} + \xi_1 \frac{\partial \varphi_2}{\partial r} + 2\varphi_2 \frac{\partial \xi_1}{\partial r} = 0, \qquad \xi_0 \frac{\partial \varphi_1}{\partial t} + \xi_1 \frac{\partial \varphi_1}{\partial r} + 2\varphi_1 \frac{\partial \xi_0}{\partial t} = 0, \\ &\xi_0 \frac{\partial \varphi_3}{\partial t} + \xi_1 \frac{\partial \varphi_3}{\partial r} = 0, \qquad \varphi_2 \frac{\partial \xi_1}{\partial t} - \varphi_1 \frac{\partial \xi_0}{\partial r} = 0. \end{aligned}$$

The third equation shows that φ_3 must be an invariant of the group or a constant. If φ_3 is not a constant, we put

$$(31)\qquad \xi_0 = -\lambda \frac{\partial \varphi_3}{\partial r}, \qquad \xi_1 = \lambda \frac{\partial \varphi_3}{\partial t}.$$

Substituting the values of ξ_0, ξ_1, and their derivatives obtained from (31) in (30) we have

$$(32)\qquad \begin{aligned} -2\frac{\partial \varrho}{\partial r} &= \frac{\dfrac{\partial \varphi_2}{\partial r}\dfrac{\partial \varphi_3}{\partial t} - \dfrac{\partial \varphi_2}{\partial t}\dfrac{\partial \varphi_3}{\partial r} + 2\varphi_2 \dfrac{\partial^2 \varphi_3}{\partial r\,\partial t}}{\varphi_2 \dfrac{\partial \varphi_3}{\partial t}}, \\ -2\frac{\partial \varrho}{\partial t} &= \frac{\dfrac{\partial \varphi_1}{\partial t}\dfrac{\partial \varphi_3}{\partial r} - \dfrac{\partial \varphi_1}{\partial r}\dfrac{\partial \varphi_3}{\partial t} + 2\varphi_1 \dfrac{\partial^2 \varphi_3}{\partial r\,\partial t}}{\varphi_1 \dfrac{\partial \varphi_3}{\partial r}}, \end{aligned}$$

$$(33)\qquad -\frac{\partial \varrho}{\partial t}\varphi_2 \frac{\partial \varphi_3}{\partial t} - \frac{\partial \varrho}{\partial r}\varphi_1 \frac{\partial \varphi_3}{\partial r} = \varphi_1 \frac{\partial^2 \varphi_3}{\partial r^2} + \varphi_2 \frac{\partial^2 \varphi_3}{\partial t^2},$$

where $\varrho = \log \lambda$. Taking account of (29), these equations may be written

$$(32')\qquad \begin{aligned} -2\frac{\partial \varrho}{\partial r} &= \frac{\partial}{\partial r}\log \varphi_1 \varphi_2 \varphi_3 + \frac{2\varphi_1 \varphi_3 G_1^0}{\dfrac{\partial \varphi_3}{\partial t}}, \\ -2\frac{\partial \varrho}{\partial t} &= \frac{\partial}{\partial t}\log \varphi_1 \varphi_2 \varphi_3 + \frac{2\varphi_1 \varphi_3 G_1^0}{\dfrac{\partial \varphi_3}{\partial r}}, \end{aligned}$$

$$(33')\qquad \varphi_1 \varphi_2 \varphi_3 (G_1^1 - G_0^0) - \left[\frac{\varphi_1 \varphi_3 G_1^0}{\dfrac{\partial \varphi_3}{\partial t}} \cdot \varphi_1 \frac{\partial \varphi_3}{\partial r} + \frac{\varphi_1 \varphi_3 G_1^0}{\dfrac{\partial \varphi_3}{\partial r}} \cdot \varphi_2 \frac{\partial \varphi_3}{\partial t}\right] = 0.$$

The conditions which must be satisfied by the functions φ_1, φ_2 and φ_3 are therefore, besides (33′), the following:

$$(34) \qquad \frac{\partial}{\partial t}\left[\frac{\varphi_1\,\varphi_3\,G_1^0}{\frac{\partial \varphi_3}{\partial t}}\right] = \frac{\partial}{\partial r}\left[\frac{\varphi_1\,\varphi_3\,G_1^0}{\frac{\partial \varphi_3}{\partial r}}\right].$$

We now put

$$(34\ ) \qquad \frac{\varphi_1\,\varphi_3\,G_1^0}{\frac{\partial \varphi_3}{\partial t}} = \frac{\partial \Psi}{\partial r}, \qquad \frac{\varphi_1\,\varphi_3\,G_1^0}{\frac{\partial \varphi_3}{\partial r}} = \frac{\partial \Psi}{\partial t},$$

from which we derive the differential equation for Ψ,

$$\frac{\partial \varphi_3}{\partial t}\,\frac{\partial \Psi}{\partial r} - \frac{\partial \varphi_3}{\partial r}\,\frac{\partial \Psi}{\partial t} = 0.$$

Hence, Ψ must be an arbitrary function of φ_3 or else a constant. The conditions (34) and (33′) may now be written

$$(35) \qquad \varphi_1\,\varphi_3\,G_1^0 = \Psi'(\varphi_3)\,\frac{\partial \varphi_3}{\partial r}\,\frac{\partial \varphi_3}{\partial t},$$

$$(35') \qquad \varphi_1\,\varphi_2\,\varphi_3\,(G_1^1 - G_0^0) = \Psi'(\varphi_3)\left[\varphi_1\left(\frac{\partial \varphi_3}{\partial r}\right)^2 + \varphi_2\left(\frac{\partial \varphi_3}{\partial t}\right)^2\right].$$

These conditions being satisfied, the corresponding G_1 has the form

$$Uf = -\,\frac{\frac{\partial \varphi_3}{\partial r}}{e^{\Psi}\sqrt{\varphi_1\,\varphi_2\,\varphi_3}}\cdot\frac{\partial f}{\partial t} + \frac{\frac{\partial \varphi_3}{\partial t}}{e^{\Psi}\sqrt{\varphi_1\,\varphi_2\,\varphi_3}}\cdot\frac{\partial f}{\partial r}.$$

We have then

The necessary and sufficient conditions that a centro-symmetric space with linear element

$$(21') \qquad -ds^2 = \varphi_2\,dr^2 + \varphi_3\,(d\theta^2 + \sin^2\theta\,d\varphi^2) - \varphi_1\,dt^2,$$

φ_1, φ_2 and φ_3 being arbitrary functions of r and t and φ_3 not a constant, shall admit a one-parameter group of motions G_1 are

$$(35) \qquad \varphi_1\,\varphi_3\,G_1^0 = \Psi'\cdot\frac{\partial \varphi_3}{\partial r}\,\frac{\partial \varphi_3}{\partial t},$$

$$(35') \qquad \varphi_1\,\varphi_2\,\varphi_3\,(G_1^1 - G_0^0) = +\,\Psi'\left[\varphi_1\left(\frac{\partial \varphi_3}{\partial r}\right)^2 + \varphi_2\left(\frac{\partial \varphi_3}{\partial t}\right)^2\right],$$

where Ψ is an arbitrary function of φ_3, or a constant.

But we know that by a proper choice of variables this group may always be reduced to the form $\partial f/\partial t$ (Theorem II). The transformation $T = T(r, t)$, $R = R(r, t)$, where R and T satisfy the two partial differential equations

$$(36) \qquad \xi_0 \frac{\partial T}{\partial t} + \xi_1 \frac{\partial T}{\partial r} = 1, \qquad \xi_0 \frac{\partial R}{\partial t} + \xi_1 \frac{\partial R}{\partial r} = 0$$

(solved by two quadratures), will carry (21′) into the form

$$(37) \qquad -ds^2 = \bar{\varphi}_2\, dr^2 + \bar{\varphi}_3 (d\theta^2 + \sin^2\theta\, d\varphi^2) - \bar{\varphi}_1\, dt^2,$$

where $\bar{\varphi}_1$, $\bar{\varphi}_2$ and $\bar{\varphi}_3$ are functions of r alone. It should be noted that the transformation may always be so chosen as to preserve the orthogonality of the space $S(r, \theta, \varphi)$ to the time-axis. We have therefore

THEOREM V. *The necessary and sufficient conditions that a centro-symmetric space* (21′) *shall be reducible to the static form are given by the equations* (35) *and* (35′).

If we choose the arbitrary function $\Psi = \text{const.}$, we have, as a corollary of the above theorem,

If $G_1^1 = G_0^0$ *and* $G_1^0 = 0$, *the centro-symmetric space* (21′) *is reducible to the static form.*

Since for an Einstein solar field we have $G_{ik} = 0$ it follows that *the Einstein solar field is necessarily static.**

For a class of spaces of importance in relativity theory the generalized Einstein equations hold: $G_{ik} = \lambda g_{ik}$, where λ is constant. We have therefore $G_1^0 = 0$, $G_1^1 - G_0^0 = 0$. Hence, *these spaces are also necessarily static.* Examples are De Sitter's cosmological space and the gravitational field of an electron.†

The case $\varphi_3 = \text{const.}$ needs special treatement, since

$$\frac{\partial \varphi_3}{\partial r} = \frac{\partial \varphi_3}{\partial t} = 0.$$

It will be convenient to transform (21′) into the form

$$(21'') \qquad -ds^2 = \varphi_1 (dr^2 - dt^2) + \varphi_3 (d\theta^2 + \sin^2\theta\, d\varphi^2).$$

Integrating equations (30) on the hypotheses $\varphi_1 = \varphi_2$ and $\varphi_3 = \text{const.}$ we find

$$\frac{\partial \xi_1}{\partial r} = \frac{\partial \xi_0}{\partial t}, \qquad \frac{\partial \xi_1}{\partial t} = \frac{\partial \xi_0}{\partial r}, \qquad \varphi_1 = \varphi_2 = \varphi(r^2 - t^2);$$

* For a different proof of this proposition see G. D. Birkhoff, loc. cit., pp. 253–256.

† A. S. Eddington, *The Mathematical Theory of Relativity,* Cambridge University Press, 1923, p. 100 and p. 185.

hence

$$\xi_0 = \Phi(r+t) + \Psi(r-t), \qquad \xi_1 = \Phi(r+t) - \Psi(r-t).$$

Without loss of generality we may put $\xi_1 = t$, $\xi_0 = r$, and the group is

$$Uf = r\frac{\partial f}{\partial t} + t\frac{\partial f}{\partial r}.$$

The integration of (36) will yield the transformation

$$r + t = R \cdot e^T, \qquad r - t = R \cdot e^{-T},$$

which will carry (21″) into the static form. We also find

$$G_1^0 = 0, \qquad G_1^1 - G_0^0 = \frac{1}{\varphi_3} - \frac{1}{\varphi_3} = 0.$$

To this class belongs the space discussed by Levi-Civita (p. 238, footnote). This space is also necessarily static.

The case where φ_3 is a function of r alone, say $\varphi_3 = r^2$, is not special, since a transformation

$$R^2 = \varphi_3, \qquad T = \Phi(r, t)$$

will carry (21′) into

(21‴) $$-ds^2 = -\bar{\varphi}_1\, dT^2 + \bar{\varphi}_2\, dR^2 + R^2(d\theta^2 + \sin^2\theta\, d\varphi^2).$$

$G_1^0 = 0$, since $\partial\varphi_3/\partial T = 0$ and (34′) shows that Ψ is a function of R alone. The condition (35′) becomes

(35″) $$\bar{\varphi}_2(G_1^1 - G_0^0) = 4\Psi'(R).$$

7. We shall suppose that the linear element (21′) admits the groups G_1 and G_3, all the transformations of which form a group G_4, and that it has been reduced to the form (37). If φ_3 is not co^r tant, the transformation

$$R = \sqrt{\varphi_3}, \qquad T = t, \qquad \varphi_2 = 1 + \bar{\varphi}_2$$

will reduce (37) to the form

(38) $$ds^2 = \varphi_1\, dt^2 - (1 + \varphi_2)\, dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

admitting the group of motions G_4.

We have now $R_{ik} = 0$, $i \neq k$, and the mixed tensors G_i^i, which we shall substitute for the tensors R_{ii}, become (equations (29))

$$
\begin{aligned}
&G_1^1 = \frac{1}{r^2(1+\varphi_2)}\left[\varphi_2 - \frac{r\varphi_1'}{\varphi_1}\right], \quad G_0^0 = \frac{1}{r^2(1+\varphi_2)}\left[\varphi_2 + \frac{r\varphi_2'}{1+\varphi_2}\right],\\
(39)\quad &G_2^2 = G_3^3 = \frac{-\varphi_1''}{2\varphi_1(1+\varphi_2)} + \frac{\varphi_2'\varphi_1'}{4\varphi_1(1+\varphi_2)^2} + \frac{\varphi_2'}{2(1+\varphi_2)^2 r}\\
&\qquad + \frac{\varphi_1'^2}{4(1+\varphi_2)\varphi_1^2} - \frac{\varphi_1'}{2\varphi_1(1+\varphi_2)r}.
\end{aligned}
$$

It appears from these equations that while $G_1^0 \equiv 0$, the same is not true of the difference $G_1^1 - G_0^0$. We find

$$
(40)\qquad G_1^1 - G_0^0 = -\left[\varphi_1' + \frac{\varphi_1}{1+\varphi_2}\varphi_2'\right]\frac{1}{r(1+\varphi_2)\varphi_1}
$$

which vanishes if, and only if, $\varphi_1(1+\varphi_2) = c$. This ist he "Leithypothese" of Kottler* which may be stated in the form of an equation as follows:

$$
(41)\quad g = \begin{vmatrix} g_{11} & g_{12} & g_{13} & 0 \\ g_{12} & g_{22} & g_{23} & 0 \\ g_{13} & g_{23} & g_{33} & 0 \\ 0 & 0 & 0 & -g_{00} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} & 0 \\ a_{12} & a_{22} & a_{23} & 0 \\ a_{13} & a_{23} & a_{33} & 0 \\ 0 & 0 & 0 & -c^2 \end{vmatrix} = |a_{ik}|(-c^2),
$$

where the a's are the coefficients of the linear element of a euclidean space and c is the velocity of light in "empty" space. In fact, we have

$$
g = -\varphi_1(1+\varphi_2)r^4\sin^2\theta = r^4\sin^2\theta(-c^2),
$$

or

$$
\varphi_1(1+\varphi_2) = c^2.
$$

Consider the three sub-spaces $t = c_1$, $\varphi = c_2$, $\theta = c_3$, having the respective linear elements

$$
\begin{aligned}
&S_3: && ds^2 = (1+\varphi_2)\,dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2),\\
(42)\quad &S_3^1: && ds^2 = (1+\varphi_2)\,dr^2 + r^2\,d\theta^2 - \varphi_1\,dt^2,\\
&S_3^2: && ds^2 = (1+\varphi_2)\,dr^2 + r^2\sin^2\theta\,d\varphi^2 - \varphi_1\,dt^2.
\end{aligned}
$$

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* F. Kottler, *Über die physikalischen Grundlagen der Einsteinschen Gravitationstheorie,* Annalen der Physik, vol. 56, pp. 401–462. Kottler considers this hypothesis as a substitute in general relativity for the hypothesis of the constancy of the velocity of light in Minkowski's mechanics. The hypothesis holds for an Einstein solar field, but not necessarily for spaces with a different mass-distribution. It does not hold for Einstein's cosmological space, although it does for that of De Sitter.

We shall calculate the principal Riemannian curvatures for each of them. We find*

$$
\begin{aligned}
&S_3: && K_1 = \frac{\varphi_2}{r^2(1+\varphi_2)}, \qquad K_2 = K_3 = \frac{\varphi_2'}{2r(1+\varphi_2)^2},\\
&S_3^1: && K_1^1 = \frac{-\varphi_1''}{2\varphi_1(1+\varphi_2)} + \frac{\varphi_2'\varphi_1'}{4\varphi_1(1+\varphi_2)^2} + \frac{\varphi_1'^2}{4\varphi_1^2(1+\varphi_2)},\\
(43)\qquad & && K_2^1 = \frac{\varphi_1'}{4\varphi_1^2(1+\varphi_2)}, \qquad K_3^1 = \frac{\varphi_2'}{2r(1+\varphi_2)^2},\\
&S_3^2: && K_1^2 = K_1, \quad K_2^2 = K_2^1, \quad K_3^2 = K_3^1 = K_2 = K_3.
\end{aligned}
$$

Comparing these results with (40) we have, at once,

$$
(44)\qquad
\begin{aligned}
G_0^0 &= K_1 + K_2 + K_3,\\
G_2^2 &= G_3^3 = K_1^1 + K_2^1 + K_3^1 = K_1^2 + K_2^2 + K_3^2.
\end{aligned}
$$

Since in S_3 we have $K_2 = K_3$, the principal trihedron with respect to any point P has one direction completely determined, namely the direction of the tangent to the r-line, while the other two directions through P, normal to it and to each other, may be chosen arbitrarily in the tangent plane to the surface $r =$ const. The space S_3 can therefore rotate freely about the geodesic r.†

This is not the case with the spaces S_3^1 and S_3^2, since the three principal curvatures are in general unequal. If, however, $K_2^1 = K_3^1$, S_3^1, and also S_3^2 will have the same property as S_3: At a point P the principal direction along the r-line will be determinate, while the other two are arbitrary in the tangent planes to the surface $r =$ const. The condition $K_2^1 = K_3^1$ imposes therefore a certain symmetry on the time space S_4, namely

The three subspaces S_3, S_3^1, S_3^2 possess, at any generic point P, "rotational mobility" about the geodesic r-line which passes through the point.

It should be noted that, in the case of the subspaces S_3^1 and S_3^2, a "rotation" about the r-line means a "hyperbolic" rotation, since these spaces have an indefinite quadratic form as line-element. If with Minkowsky we put $it = \bar{t}$, the hyperbolic rotation becomes ordinary euclidean.

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* L. Bianchi, *Lezione di Geometria Differenziale*, vol. 1, pp. 365–358, and also *Lezioni sulla Teoria dei Gruppi*, pp. 546–547 by the same author.

† It should be observed in this connection that when we say "rotate freely about the geodesic r" this does not imply that the r-line is an axis. It would be a geodesic axis only if $K_2 = K_3 = 0$, that is, if φ_2 is a constant. The transformation $\bar{r} = \int \sqrt{1 + \varphi_r}\, dr$ will make $\bar{r}$ a geodesic axis.

Since $K_2^1 = K_3^1$ is equivalent to the condition $G_1^1 = G_0^0$, or to the condition

$$(45) \qquad \varphi_1(1+\varphi_2) = \text{const.},$$

we see that *Kottler's "Leithypothese" is equivalent to the assumption of free mobility of the subspaces S_3, S_3^1, S_3^2 about the r-line.*

With this assumption the linear element takes the form

$$(46) \qquad -ds^2 = (1+\varphi_2)\,dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2) - \frac{c}{1+\varphi_2}\,dt^2$$

and we also have, from (40),

$$(47)\ G_1^1 = G_0^0 = K_1 + K_2 + K_3,\ G_2^2 = G_3^3 = K_1^1 + K_2^1 + K_3^1 = K_1^2 + K_2^2 + K_3^2.$$

8. We shall specialize further by assuming various values for the sum of the three principal curvatures of S_3, which is the space part of the time-space S_4.

(a) $K_1 + K_2 + K_3 = 0$. We have from (40)

$$\varphi_2\varphi_1 - r\varphi_1' = 0,$$

or, since $\varphi_2 = c^2/\varphi_1 - 1$,

$$r\varphi_1 = c^2 r + C.$$

Putting $C = -\alpha c^2$ we have

$$\varphi_1 = c^2\left(1 - \frac{\alpha}{r}\right), \qquad 1+\varphi_2 = \frac{1}{1 - \dfrac{\alpha}{r}},$$

which is Schwarzschild's solution for the line-element of a static and stationary space with centro-symmetric mass-distribution, viz.:

$$(48) \qquad -ds^2 = \frac{dr^2}{1 - \dfrac{\alpha}{r}} + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2) - c^2\left(1 - \frac{\alpha}{r}\right)dt^2.$$

This space is characterized by the following three geometrical properties:

1. It admits the group of motions G_4 (Assumption L).
2. The three subspaces S_1, S_2, S_3 possess rotational mobility about the r-line (Assumption M).
3. The sum of the three principal Riemannian curvatures of the space S_3 is zero (Assumption N).

Assumptions L, M, and N* are equivalent to the physical assumption that in the gravitational field outside the mass the tensors T_{ik} all vanish. In fact Einstein's equations are

$$(49) \qquad G_{ik} = \varkappa T_{ik} = 0, \qquad \varkappa = \frac{8\pi k}{c^4},$$

where k is the Newtonian gravitation constant. But from (40) we have

$$G_i^k = g^{ik} G_{ik} = 0, \qquad i \neq k,$$

and since $\sum_1^3 K_i = 0$ and $K_2^1 = K_3^1$, it follows that $G_1^1 = G_0^0 = 0$. If the values of φ_1 and $1+\varphi_2$ are substituted in the expression for G_2^2 and G_3^3 we find that they vanish also.

It will be noted that the principal Riemannian curvatures of the three subspaces S_3, S_3^1, S_3^2 are

$$K_1 = K_1^1 = K_1^2 = \frac{\alpha}{r^3}, \qquad K_2 = K_2^1 = K_2^2 = K_3 = K_3^1 = K_3^2 = -\frac{\alpha}{2r^3}.$$

If, as has been suggested by Cesàro,† we put

$$K = \frac{1}{3}\sum K_i, \qquad K^1 = \frac{1}{3}\sum K_i^1, \qquad K^2 = \frac{1}{3}\sum K_i^2,$$

and define K, K^1, K^2 as the mean curvatures of the respective spaces, we have

THEOREM VI. *The subspaces S_3, S_3^1 and S_3^2 of a static time-space* (48) *with centro-symmetric mass distribution have their mean curvatures equal to zero, and the three principal curvatures of any subspace are respectively equal to the corresponding principal curvatures of any other subspace.*

(b) $K_1+K_2+K_3 = \text{const.} = 3/R^2$. We have from (43)

$$\frac{\varphi_2\varphi_1 - r\varphi_1'}{r^2c^2} = \frac{3}{R^2},$$

or

$$c^2 - \varphi_1 - r\varphi_1' = \frac{3c^2r^2}{R^2};$$

* L, M and N are also equivalent to postulates I—V of Eisenhart's paper *The permanent gravitational field in the Einstein theory*, Annals of Mathematics, ser. 2, vol. 22, No. 2; December, 1920.

† Ernesto Cesàro, *Lezioni di Geometria Intrinsica*, Napoli, 1896, p. 223.

integrating, we have

$$\varphi_1 = \frac{C}{r} + c^2\left(1 - \frac{r^2}{R^2}\right). \tag{50}$$

Let $C = 0$; the linear element of S_4 is

$$-ds^2 = \frac{dr^2}{1 - \frac{r^2}{R^2}} + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2) - c^2\left(1 - \frac{r^2}{R^2}\right)dt^2. \tag{51}$$

This solution applies to the space inside of a homogeneous sphere of matter provided the inertial density (Trägheitsdichte) is assumed to be zero.* If we calculate the Riemannian symbols (ik, rh), we find

$$(ik, ik) = \frac{1}{R^2}\, a_{ii} a_{kk}, \quad (ik, rh) = 0, \quad i \neq r, \quad k \neq h,$$

which means that the curvature $K_0 = 1/R^2$ (L. Bianchi, *Lezioni*, vol. I, p. 344). The group of S_4 is therefore the maximum G_{10} of non-euclidean rotations and translations, and that of S_3 is the corresponding G_6. G_6 does not belong to S_4, while the group G_4 remains as a subgroup of G_{10}. The space (51) is usually referred to as *De Sitter's space.*

(*c*) If in (50) we let the constant C differ from zero, we may put $C = c^2 b^3/R^2$ and we have

$$\varphi_1 = c^2\left[1 - \frac{r^2}{R^2} + \frac{b^3}{R^2}\cdot\frac{1}{r}\right], \quad 1 + \varphi_2 = \frac{1}{1 - \frac{r^2}{R^2} + \frac{b^3}{R^2}\cdot\frac{1}{r}}. \tag{52}$$

The space to which this solution applies is that of a shell of thickness $a - b$. R is supposed to vary between the limits $b > R < a$.† Since the shell acts on the region outside of it like a Newtonian masspoint m determined by the relation

$$\frac{2km}{c^2} = \frac{a^3 - b^3}{R^2},$$

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* H. Weyl, *Raum, Zeit und Materie*, 4th edition, p. 232; F. Kottler, loc. cit., pp. 438–439. Kottler assumes that the cohesion pressure equals the entire energy of the mass, i. e. $\mu = \varepsilon - p = 0$ where ε is the energy and p the cohesion pressure. See also H. Weyl, loc. cit., p. 254, and p. 256.

† F. Kottler, loc. cit., p. 493. He obtains this solution by assuming that the space has no inertial mass, i. e. the cohesion pressure $p = \varepsilon$. Kottler's solution has not been accepted by Einstein, who prefers the one obtained by Schwarzschild (equation 55).

we have, for $R > a$,

$$\varphi_1 = c^2\left[1 - \frac{a^3 - b^3}{R^2}\,\frac{1}{r}\right], \quad 1 + \varphi_2 = \frac{1}{1 - \dfrac{a^3 - b^3}{R^2}\,\dfrac{1}{r}}.$$

(*d*) We shall consider the case where $K_1 + K_2 + K_3$ is inversely proportional to the fourth power of the "distance" r from the mass-center. We have

$$K = \frac{\varphi_2 \varphi_1 - r\varphi_1}{r^2 c^2} = \frac{\alpha}{r^4}, \tag{53}$$

or, since we are still working under assumption M,

$$c^2 - \varphi_1 - r\varphi_1' = \frac{c^2 \alpha}{r^2}.$$

Integrating we have

$$\varphi_1 = c^2\left(1 - \frac{\beta}{r} + \frac{\alpha}{r^2}\right) \quad \text{and} \quad 1 + \varphi_2 = \frac{1}{1 - \dfrac{\beta}{r} + \dfrac{\alpha}{r^2}}.$$

S_4 has therefore the linear element

$$-ds_2 = \frac{dr^2}{1 - \dfrac{\beta}{r} + \dfrac{\alpha}{r^2}} + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2) - c^2\left(1 - \frac{\beta}{r} + \frac{\alpha}{r^2}\right) dt^2, \tag{54}$$

which has been obtained by Weyl.* He considers a sphere having a Newtonian mass m and a static charge e. If we put $\beta = 2a$ and $\alpha = ke^2/c^4$, where $a = km/c^2$, we get the identical form due to Weyl. The group of the space (54) is G_4, assumptions L and M hold, while for N is substituted (53). This space, and the two preceding ones, belong to a class of spaces characterized by the property of having the sum of the three principal curvatures of S_3 equal to a function of the distance from the mass-center, assumptions L and M being valid. We find for the value of φ_1

$$\varphi_1 = c^2\left[1 - \frac{1}{r}\int F(r)\, dr\right].$$

(*e*) Let us retain assumption L. Instead of M, i.e. $K_2^1 = K_3^1$, we put

M': $$K_1^1 = K_2^1,$$

* H. Weyl, loc. cit., pp. 236–237.

and let, as before in case (*b*),

N':
$$K_1 + K_2 + K_3 = \frac{3}{R^2}.$$

φ_1 and φ_2 must now satisfy the following differential equations obtained from (42):

$$\frac{\varphi_2}{r^2(1+\varphi_2)} + \frac{\varphi_2'}{r(1+\varphi_2)^2} = \frac{3}{R^2},$$
$$\frac{\varphi_1'}{r\varphi_1} + \frac{\varphi_1'^2}{2\varphi_1^2} + \frac{\varphi_2'\varphi_1'}{2\varphi_1(1+\varphi_2)} - \frac{\varphi_1''}{\varphi_1} = 0.$$

Integrating the first equation, we find

$$1+\varphi_2 = \frac{1}{1-\frac{C}{r}-\frac{r^2}{R^2}};$$

we let C be equal to zero and substitute the value of φ_2 in the second equation and integrate; the result is

$$\sqrt{\varphi_1} = c\left[\alpha - \frac{\sqrt{1-\frac{r^2}{R^2}}}{2}\right],$$

where c and α are integration constants. If we determine the initial value of φ_1 in such a way that for $r = R$ we have

$$\sqrt{\varphi_1} = \frac{3}{2}c\sqrt{1-\frac{a^2}{R^2}},$$

we get

$$\sqrt{\varphi_1} = c\left[\frac{3\sqrt{1-\frac{a^2}{R^2}}-\sqrt{1-\frac{r^2}{R^2}}}{2}\right],$$

and the linear element of the space S_4 is

$$(55)\quad -ds^2 = \frac{dr^2}{1-\frac{r^2}{R^2}} + r^2(d\theta^2+\sin^2\theta\, d\varphi^2)$$
$$-c^2\left[\frac{3\sqrt{1-\frac{a^2}{R^2}}-\sqrt{1-\frac{r^2}{R^2}}}{2}\right]^2 dt^2,$$

which is Schwarzschild's solution for the gravitational space within a liquid sphere of radius a. The group of S_3 is obviously G_6, *but* G_6 *does not belong to* S_4. The condition $C=0$ is equivalent to

$$P: \qquad K_1 = K_2 = K_3 = \frac{1}{R^2}.$$

S_3 is therefore a space of constant curvature. Assumptions L, M', and P determine completely the Schwarzschild solution (55).

9. **The case** $g_{00} =$ const. If g_{00} is constant, the equations (43) become

$$K_1^1 = K_2^1 = 0, \quad K_1^2 = K_2^2 = 0, \quad K_2 = K_3 = K_3^1 = \frac{\varphi_2'}{2r(1+\varphi_2)^2},$$

$$K_1 = \frac{\varphi_2}{r^2(1+\varphi_2)}.$$

The group of S_4 is the systatic G_4, and, since g_{00} is constant, the transformation $\partial f/\partial t$ is a translation (Theorem II). The space S_4 can therefore rotate freely about the t-line as geodesic axis. We shall consider the following cases:

(a) $K_1+K_2+K_3 = 0$. We have

$$\varphi_2(1+\varphi_2)+r\varphi_2' = 0.$$

Integrating we find

$$1+\varphi_2 = \frac{1}{1-\dfrac{\alpha}{r}},$$

where α is the integration constant. If $\alpha = 0$, the space is euclidean. $\alpha \neq 0$ does not correspond to any physical space, since the velocity of light is constant.

(b) $K_1+K_2+K_4 = 3/R^2$. This case gives us the Einstein-Schrödinger solution for a closed space with incoherent mass in static equilibrium,*

$$(56) \qquad -ds^2 = \frac{dr^2}{1-\dfrac{r^2}{R^2}} + r^2(d\theta^2+\sin^2\theta\, d\varphi^2) - g_{00}\,dt^2.$$

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* E. Schrödinger, *Über die Lösungssysteme der allgemeinen kovarianten Gravitationsgleichungen*, Physikalische Zeitschrift, vol. 19 (1918), p. 20. See also F. Kottler, loc. cit., p. 483, and H. Weyl, loc. cit., pp. 252-253.

S_3 is a space of constant curvature $1/R^2$. The group of S_4 is a G_7, namely $\partial f/\partial t$, and G_6 which belongs also to S_3. G_7 is systatic, the systatic spreads being the geodesic t-lines.

A general time-space (21′) which admits the group G_7 can be reduced to the form (56); for since S_4 admits the group G_6 which is a transitive group in S_3, φ_1 must be constant (Theorem III). Hence, the group G_7 completely characterizes the Einstein-Schrödinger solution (56).

10. A somewhat interesting type of centro-symmetric spaces is obtained from the form (38) by assuming $\varphi_3 =$ const. Let this constant be R^2. The transformation

$$R\theta = \bar{\theta}, \qquad R\varphi = \bar{\varphi}, \qquad \sqrt{1+\varphi_2}\,dr = d\bar{r}$$

will carry the linear element into the form

$$-ds^2 = d\bar{r}^2 + d\bar{\theta}^2 + \sin^2\frac{\bar{\theta}}{R}\,d\bar{\varphi}^2 - \varphi_1\,dt^2. \tag{57}$$

The principal curvatures of the sub-spaces S_3, S_3^1, S_3^2 are

$$K_1 = \frac{1}{R^2} = G_1^1 = G_0^0;\quad K_2 = K_3 = 0;\quad K_1^1 = K_1^2 = 0;\quad K_3^1 = K_3^2 = 0;$$
$$K_2^1 = K_2^2 = -\frac{1}{2\varphi_1}\frac{\partial^2\varphi_1}{\partial\bar{r}^2} + \frac{1}{4\varphi_1^2}\left(\frac{\partial\varphi_1}{\partial\bar{r}}\right)^2 = G_2^2 = G_3^3.$$

S_3 belongs to a type of L. Bianchi's *normal spaces*, namely that one for which the three principal curvatures are constant. If the mean curvature $K = \sum K_i$ is positive, it is said to be of type B.* This is here the case. The group of the space S_4 is G_4 as before, but S_3 admits a 4-parameter group of motions, namely G_3 and the translation $\partial f/\partial r$, the latter not belonging to S_4 except when φ_1 is constant.

A notable case is where the mean curvature of the space S_4 is zero. We have then

$$M = \sum_0^3 G_i^i = \frac{2}{R^2} - \frac{1}{\varphi_1}\frac{\partial^2\varphi_1}{\partial\bar{r}^2} + \frac{1}{2\varphi_1^2}\left(\frac{\partial\varphi_1}{\partial\bar{r}}\right)^2 = 0,$$

which integrated gives

$$\sqrt{\varphi_1} = c_1 e^{\bar{r}/R} + c_2 e^{-\bar{r}/R}.$$

* L. Bianchi, *Sugli spazi normali a tre dimenzioni colle curvature principali costanti*, Lincei Rendiconti, ser. 5, vol. 25, 1st semester 1916, pp. 59–63.

This space has been obtained by T. Levi-Civita. He assumes that the physical space S_3 is supplied with an electrical potential $\varphi = -Cr$. The intensity of the force is $|C|\sqrt{g^{11}}$ and is directed normal to the surfaces $r = \text{const.}$*

11. **Class of a centro-symmetric space.** It follows from a theorem due to E. Kasner that the special Einstein 4-dimensional spread (48) cannot be immersed in a flat 5-space. In other words, the linear element (48) is at least of the second class.† The necessary and sufficient conditions that a differential form ds^2 in n-space shall be of the first class are

1. It must be possible to find a doubly symmetric system of quantities b_{ik} (coefficients of the second differential form) such that

$$(a) \qquad (rk, ih) = b_{ri}\, b_{kh} - b_{ki}\, b_{rh}.$$

2. The system b_{ik} must satisfy the differential equations

$$(b) \qquad \frac{\partial b_{ri}}{\partial x_h} - \frac{\partial b_{rh}}{\partial x_i} + \sum_t^{1\cdots n} \begin{Bmatrix} r & i \\ & t \end{Bmatrix} b_{ht} - \sum_t^{1\cdots n} \begin{Bmatrix} r & h \\ & t \end{Bmatrix} b_{it} = 0.$$

Equations (a) and (b) are known as the generalized equations of Gauss and Codazzi‡ for a euclidean n-space.

Starting with the general centro-symmetric space (21′) we shall proceed to find the conditions (a). Calculating the Riemannian symbols (rk, ih) we find that they all vanish except the following:

$$\begin{aligned}
&(12, 12) = -\frac{1}{2}\frac{\partial^2 \varphi_3}{\partial r^2} + \frac{1}{4\varphi_2}\frac{\partial \varphi_2}{\partial r}\frac{\partial \varphi_3}{\partial r} + \frac{1}{4\varphi_3}\left(\frac{\partial \varphi_3}{\partial r}\right)^2 + \frac{1}{4\varphi_1}\frac{\partial \varphi_2}{\partial t}\frac{\partial \varphi_3}{\partial t};\\
&(13, 13) = (12, 12)\sin^2\theta;\\
(58)\quad &(10, 10) = -\frac{1}{2}\frac{\partial^2 \varphi_2}{\partial t^2} + \frac{1}{2}\frac{\partial^2 \varphi_1}{\partial r^2} + \frac{1}{4\varphi_2}\left(\frac{\partial \varphi_2}{\partial t}\right)^2 - \frac{1}{4\varphi_2}\frac{\partial \varphi_2}{\partial r}\frac{\partial \varphi_1}{\partial r}\\
&\qquad - \frac{1}{4\varphi_1}\left(\frac{\partial \varphi_1}{\partial r}\right)^2 + \frac{1}{4\varphi_1}\frac{\partial \varphi_2}{\partial t}\frac{\partial \varphi_1}{\partial t};\\
&(23, 23) = \left[\varphi_3 + \frac{1}{4\varphi_1}\left(\frac{\partial \varphi_3}{\partial t}\right)^2 - \frac{1}{4\varphi_2}\left(\frac{\partial \varphi_3}{\partial r}\right)^2\right]\sin^2\theta;
\end{aligned}$$

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* T. Levi-Civita, *Realtà fisica di alcuni spaci normali del Bianchi*, Lincei Rendiconti, ser. 5, vol. 25, 1st semester 1917.

† E. Kasner, *The impossibility of Einstein fields immersed in a flat space of five dimensions*, American Journal of Mathematics, vol. 43, pp. 126–129. For the definition of class see Ricci and Levi-Civita, *Méthodes de calcul différentiel absolu*, Mathematische Annalen, vol. 54, p. 160.

‡ L. Bianchi, *Lezioni di Geometria Differenziali*, 2d edition, vol. 1, p. 462.

$$
(58)\quad
\begin{aligned}
(02,02) &= -\frac{1}{2}\frac{\partial^2\varphi_3}{\partial t^2} + \frac{1}{4\varphi_2}\frac{\partial\varphi_1}{\partial r}\frac{\partial\varphi_3}{\partial r} + \frac{1}{4\varphi_3}\left(\frac{\partial\varphi_3}{\partial t}\right)^2 \\
&\quad + \frac{1}{4\varphi_1}\frac{\partial\varphi_1}{\partial t}\frac{\partial\varphi_3}{\partial t};\\
(03,03) &= (02,02)\sin^2\theta;\\
(12,02) &= -\frac{1}{2}\frac{\partial^2\varphi_3}{\partial r\,\partial t} + \frac{1}{4\varphi_2}\frac{\partial\varphi_2}{\partial t}\frac{\partial\varphi_3}{\partial r} + \frac{1}{4\varphi_3}\frac{\partial\varphi_3}{\partial r}\frac{\partial\varphi_3}{\partial t}\\
&\quad + \frac{1}{4\varphi_1}\frac{\partial\varphi_1}{\partial r}\frac{\partial\varphi_3}{\partial t};\\
(13,03) &= (12,02)\sin^2\theta.
\end{aligned}
$$

Equations (a) are now

$$
(59)\quad
\begin{gathered}
b_{02} = b_{03} = b_{13} = b_{23} = b_{12} = 0, \quad b_{01}b_{22} = (12,02),\\
b_{01}b_{33} = (13,03),\ b_{00}b_{33} = (03,03),\ b_{00}b_{22} = (02,02),\ b_{11}b_{33} = (13,13),\\
b_{22}b_{33} = (23,23), \quad b_{11}b_{22} = (12,12), \quad b_{00}b_{11} - b_{01}^2 = (01,01),
\end{gathered}
$$

from which we derive the relations

$$
\frac{b_{22}}{b_{33}} = \frac{(12,12)}{(13,13)} = \frac{(02,02)}{(03,03)} = \frac{(12,02)}{(13,03)},
$$

that is, we have the following two conditions:

$$
(a_1)\qquad (12,12)(03,03) = (02,02)(13,13),
$$

$$
(a_2)\qquad (12,12)(13,03) = (13,13)(12,02),
$$

which are satisfied by (58). We also find the following values for the non-vanishing b's:

$$
b_{00}^2 = \frac{(02,02)^2}{D},\ b_{11}^2 = \frac{(12,12)^2}{D},\ b_{22}^2 = D,\ b_{33}^2 = D\sin^4\theta,\ b_{10}^2 = \frac{(12,02)^2}{D},
$$

where $D = (23,23)/\sin^2\theta$. Substituting in the last equation of (59), we obtain the condition

$$
(a_3)\qquad (02,02)(13,13) = (01,01)(23,23) + (12,02)(13,03),
$$

which is *not* satisfied for a general space (21′).

Suppose now that the condition (a_3) is satisfied. A rather long, but not difficult, calculation will show that the b's also satisfy the Codazzi equations (b). We have thus proved the

Theorem VII. *A necessary and sufficient condition that the general centro-symmetric space* (21′) *shall be of the first class is*

$$(a_3) \qquad (02, 02)(13, 13) = (01, 01)(23, 23) + (12, 02)(13, 03).$$

Let the space (21′) be static, and reduced to the form

$$-ds^2 = (1+\varphi_2)\,dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2) - \varphi_1\, dt^2. \tag{38}$$

The condition (a_3) becomes

$$\varphi_1'' = \frac{1}{2}\left[\frac{\varphi_1'\varphi_2'}{\varphi_2} + \frac{(\varphi_1')^2}{\varphi_1}\right],$$

which integrated gives

$$\varphi_2 = \frac{k^2}{4}\frac{(\varphi_1')^2}{\varphi_1}. \tag{60}$$

This condition being satisfied, the spread (38) may be represented in a flat 5-space, the coördinates being

$$x_1 = r\sin\theta\sin\varphi, \qquad x_2 = r\sin\theta\cos\varphi, \qquad x_3 = r\cos\theta,$$
$$x_4 = k\sqrt{\varphi_1}\cos\frac{\bar t}{k}, \qquad x_5 = k\sqrt{\varphi_1}\sin\frac{\bar t}{k},$$

where $\bar t = it$. Hence

A necessary and sufficient condition that the static centro-symmetric space (38) *shall be immersible in a flat* 5-*space is*

$$\varphi_2 = \frac{k^2}{4}\frac{(\varphi_1')^2}{\varphi_1}. \tag{60}$$

12. Particular spaces which occur in the theory of relativity and for which the condition (60) is satisfied are the following:

1. De Sitter's space (51); $\varphi_1 = c^2(1-r^2/R^2)$, $1+\varphi_2 = 1/(1-r^2/R^2)$.
2. All spaces for which $g_{00} = \varphi_1$ is constant (Einstein's cosmic space).
3. The Schwarzschild solution for a space inside a liquid sphere of radius a_0 (equation (55)). The coördinates of the 4-spread are

$$x_1 = r\sin\theta\sin\varphi, \qquad x_2 = r\sin\theta\cos\varphi, \qquad x_3 = r\cos\theta,$$

$$x_4 = r\left[3a_0 - \sqrt{1-\frac{r^2}{R^2}}\right]\cos\frac{c\bar t}{2R}, \qquad x_5 = r\left[3a_0 - \sqrt{1-\frac{r^2}{R^2}}\right]\sin\frac{c\bar t}{2R}.$$

This surface is a 4-dimensional torus generated by revolving the 3-dimensional sphere

$$(x_4^2 - 3\,a_0\,r)^2 + x_1^2 + x_2^2 + x_3^2 = r^2$$

in such a way that the center describes the circle $x_4^2 + x_5^2 = (3\,a_0 r)^2$. The equation of the torus is

$$\left[\sum_1^3 x_i^2 - r^2(1+9\,a_0^2)\right]^2 = 36\,a_0^2\,r^2\left[r^2 - \sum_1^3 x_i^2\right].$$

Since $a_0 > \frac{1}{3}$,* we have $3\,a_0\,r > r$ so that the generating sphere does not intersect the x_1, x_2, x_3 and x_4-axes. The space inside the liquid sphere in S_3 is represented by a 3-dimensional spherical cap on the generating sphere defined by the limit $\sum x_i^2 < a_0^2 < r^2(1-a_0^2)$, and the corresponding time-space by a 4-dimensional zone on the torus generated by the revolution of the cap.

13. We have seen that the space of an Einstein solar field is at least of the second class. We shall prove the following general

THEOREM VIII. *The linear element* (21′) *for which the condition* (a_3) *is not satisfied is of the second class.*

The transformation

$$\varphi_3(r,\,t) = R^2, \qquad \psi(r,\,t) = T$$

will carry (21′) into the form

$$(21''') \qquad -ds^2 = -\bar{\varphi}_1\,dT^2 + \bar{\varphi}_2\,dR^2 + R^2(d\theta^2 + \sin^2\theta\,d\varphi^2)$$

or

$$(61) \qquad \begin{aligned} -ds^2 = d\sigma^2 + ds_0^2 = -\bar{\varphi}_1\,dT^2 &+ (\bar{\varphi}_2 - 1)\,dR^2 \\ &+ (dR^2 + R^2\,d\theta^2 + R^2\sin^2\theta\,d\varphi^2). \end{aligned}$$

Consider the linear element

$$(62) \qquad d\sigma^2 = -\bar{\varphi}_1\,dT^2 + (\bar{\varphi}_2 - 1)\,dR^2.$$

By the general theory of surfaces, if the Gaussian curvature of the 2-spread (62) differs from zero, it is always possible to find three functions x_4, x_5, and x_6 of R and T such that

$$dx_4^2 + dx_5^2 + dx_6^2 = d\sigma^2,$$

* H. Weyl, loc. cit., p. 242; $a_0 = r_0$ and $r = a$ in Weyl's notation.

and the linear element (61) may be put in the form

$$-ds^2 = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 + dx_5^2 + dx_6^2,$$

where

$$x_1 = R\sin\theta\sin\varphi, \qquad x_2 = R\sin\theta\cos\varphi, \qquad x_3 = R\cos\theta.$$

The 2-spread whose coördinates are x_4, x_5, and x_6 we shall call *the auxiliary surface.*

If the Gaussian curvature of the 2-spread (62) is zero, it will be possible to express $d\sigma^2$ in the form $d\sigma^2 = dx_4^2 + dx_5^2$, that is, the linear element (61) is of the first class. We find

$$K = \frac{1}{2\sqrt{\bar\varphi_1 - \bar\varphi_1\bar\varphi_2}}\left\{\frac{\partial}{\partial T}\left(-\frac{1}{\sqrt{\bar\varphi_1 - \bar\varphi_1\bar\varphi_2}}\cdot\frac{\partial\bar\varphi_2}{\partial T}\right) + \frac{\partial}{\partial R}\left(\frac{1}{\sqrt{\bar\varphi_1 - \bar\varphi_1\bar\varphi_2}}\cdot\frac{\partial\bar\varphi_1}{\partial R}\right)\right\},$$

or, putting K equal to zero and simplifying,

$$\text{(63)} \quad \frac{\partial^2\bar\varphi_1}{\partial R^2} - \frac{\partial^2\bar\varphi_2}{\partial T^2} + \frac{1}{2\bar\varphi_1}\frac{\partial\bar\varphi_2}{\partial T}\frac{\partial\bar\varphi_1}{\partial T} - \frac{1}{2\bar\varphi_1}\left(\frac{\partial\bar\varphi_1}{\partial R}\right)^2 + \frac{1}{2(\bar\varphi_2 - 1)}\cdot\left\{\left(\frac{\partial\bar\varphi_2}{\partial T}\right)^2 - \frac{\partial\bar\varphi_2}{\partial R}\frac{\partial\bar\varphi_1}{\partial R}\right\} = 0.$$

But this is precisely the condition (a_3), as is easily seen on calculating the requisite Riemannian symbols for the form (61).

In the exceptional case $\varphi_3 = \text{const.}$ no transformation is necessary; (a_3) reduces to $(10, 10) = 0$, that is, the Gaussian curvature of the 2-space $d\sigma^2 = -\varphi_1 dt^2 + \varphi_2 dr^2$ must be zero. This condition is not satisfied for the space (21′), φ_3 being a constant; it is therefore of the second class. Hence, the space (57) is also of the second class.

A necessary and sufficient condition that a 2-space (u, v) shall be of class zero is that it shall admit an abelian group $\partial f/\partial u$, $\partial f/\partial v$ as a group of motions. This is the group-property which characterizes all surfaces of zero Gaussian curvature. The condition (a_3) reduces, in the case of the transformed element (21‴), to the simpler form

$$(02, 02)(13, 13) = (01, 01)(23, 23),$$

which by (63) is equivalent to the condition that the sub-space $\varphi = \text{const.}$, $\theta = \pm iR + \text{const.}$, or the space (62), shall admit an abelian group G_2

of two parameters as a group of motions. Theorems VII and VIII may therefore be stated thus:

A necessary and sufficient condition that the space (21‴) *shall be of the first class is that the sub-space* (62) *shall admit an abelian group* G_2 *as a group of motions. A* 4*-space which admits the group* G_4, *i.e. the space* (21′), *is at most of the second class.*

A similar statement holds for spaces (21′) for which $\varphi_3 = \text{const.}$

14. Let the space be static and its linear element written in the form

$$-ds^2 = dR^2 + R^2(d\theta^2 + \sin^2\theta\, d\varphi^2) + \varphi_2\, dR^2 + \varphi_1\, dT^2,$$

where $T = it$. The spread has the coördinates

$$(64)\qquad \begin{aligned} x_1 &= R\sin\theta\sin\varphi, \qquad x_2 = R\sin\theta\cos\varphi, \quad x_3 = R\cos\theta,\\ x_4 &= \int\sqrt{\left[\varphi_2 - \frac{k^2}{4}\frac{(\varphi_1')^2}{\varphi_1}\right]}\,dR, \quad x_5 = k\sqrt{\varphi_1}\cos\frac{T}{k}, \quad x_6 = k\sqrt{\varphi_1}\sin\frac{T}{k}.\end{aligned}$$

The auxiliary surface is a surface of revolution generated by revolving the curve

$$x_4 = \int\sqrt{\varphi_2 - \frac{k^2(\varphi_1')^2}{4\varphi_1}}\,dR, \qquad x_5 = k\sqrt{\varphi_1}$$

about the x_4-axis. The surface $x_1 = R\sin\varphi$, $x_2 = R\cos\varphi$, $x_4 = \sqrt{\varphi_2}\,dR$ is a generalization of Flamm's quartic surface.*

If in (64) we put $(\varphi_1')^2 = (4/k^2)\varphi_1\varphi_2$, $x_4 = 0$, and the linear element is of class 1. If we put $\varphi_2 = \alpha/(R-\alpha)$, $\varphi_1 = c^2(1-\alpha/R)$ and $k = 1$, we get the representation of the Einstein solar field in a flat 6-space which was obtained by Kasner in a slightly different form.†

15. Static spaces which are not centro-symmetric are in general of a class higher than 2. Thus *the general static space*

$$(65)\qquad -ds^2 = g_0\,dx_0^2 + g_1\,dx_1^2 + g_2\,dx_2^2 + g_3\,dx_3^2, \quad x_0 = it$$

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* H. Weyl, loc. cit., p. 236.

† E. Kasner, *Finite representation of the solar gravitational field in a flat space of six dimensions*, American Journal of Mathematics, vol. 43, pp. 130–133. It follows as a corollary from Theorem VIII that a general space (21′) can be conformally represented on a flat 5-space, and, when (a_8) is satisfied, on a flat 4-space. The space of an Einstein solar field can therefore not be represented conformally on a euclidean 4-space.

is of the fifth class, or one less than the maximum. To prove this we shall write the element in the form

$$(65') \quad -ds^2 = g_0\left[dx_0^2 + \frac{g_1}{g_0}dx_1^2 + \frac{g_2}{g_0}dx_2^2 + \frac{g_3}{g_0}dx_3^2\right] = g_0(dx_0^2 + d\sigma^2).$$

But a general curved 3-space may be immersed in a euclidean 6-space so that we have

$$-ds^2 = g_0\left[dx_0^2 + \sum_1^6 dy_i^2\right],$$

the y's being functions of x_1, x_2, x_3. The space (65) is thus represented conformally on a euclidean 7-space. *Such a space may be immersed in a euclidean 9-space*; for, write

$$z_1 = \sqrt{g_0}\,x_0, \qquad z_2 = \sqrt{g_0}\,y_1, \; \cdots, \; z_7 = \sqrt{g_0}\,y_6,$$

$$z_8 + z_9 = -\sqrt{g_0}, \qquad z_8 - z_9 = \sqrt{g_0}\left(\sum_1^6 y_i^2 + x_0^2\right),$$

and an easy calculation shows that

$$-ds^2 = g_0\left(dx_0^2 + \sum_1^6 dy_i^2\right) = \sum_1^7 dz_i^2 + dz_8^2 - dz_9^2.$$

Since a 4-space which admits a group G_1 as complete group of motions can always be reduced to the form (65) it follows that *such a space is of the fifth class.*

If $\partial f/\partial x_0$ is a translation, g_0 is const. and (65) is of the third class. If, in (65′), the sub-space $d\sigma^2$ admits the group $\partial f/\partial x_3$, g_1/g_0, g_2/g_0, g_3/g_0 are functions of x_1 and x_2 alone. As will be proved in § 16, a 3-space of this kind is of the first class, i. e. it can be immersed in a euclidean 4-space. (65) is therefore of the third class.

16. Let the space (65) admit the abelian group $\partial f/\partial x_0$, $\partial f/\partial x_3$ as complete group of motions, in which case the g's are functions of x_1 and x_2 alone. We shall prove that (65) is of the third class. We write (65) in the form

$$-ds^2 = d\sigma^2 + g_3\,dx_3^2;$$

$d\sigma^2$ is the line-element of a 3-spread in a euclidean 4-space. To prove this we calculate the Riemannian symbols, of which the following are non-vanishing:

$$(10, 10), \qquad (20, 20), \qquad (12, 12), \qquad (10, 20),$$

and set up the Gaussian equations for the calculation of the b's. We find

$$b_{00}\, b_{11} = (10, 10), \quad b_{00}\, b_{22} = (20, 20), \quad b_{00}\, b_{12} = (10, 20),$$
$$b_{11}\, b_{22} - b_{12}^2 = (12, 12).$$

These equations determine the b's and they also satisfy the four Codazzi equations, which we shall not take the trouble of writing here. The line-element is now

$$-ds^2 = dy_1^2 + dy_2^2 + dy_3^2 + dy_4^2 + g_3\, dx_3^2.$$

If we put

$$y_6 + y_7 = \sqrt{g_3}, \quad y_6 - y_7 = -x_3^2 \sqrt{g_3}, \quad y_5 = \sqrt{g_3}\, x_3,$$

we have the final result

$$-ds^2 = \sum_1^6 dy_i^2 - dy_7^2. \tag{66}$$

If $\partial f/\partial x_3$ is a translation, g_3 is constant and the 4-spread belongs to a euclidean 5-space. Its class is 1.

A notable space of this kind is Weyl's cylindrical and static space

$$-ds^2 = h(dz^2 + dr^2) + \frac{r^2 d\theta^2}{f} - f dt^2,$$

which is of the third class and admits the group $\partial f/\partial t$, $\partial f/\partial \theta$ as a complete group of motions.* If $hf = 1$ we have the static centro-symmetric space of class 2 which admits a G_4.

17. Let the 4-space admit the group $\partial f/\partial x_0$, $\partial f/\partial x_3$, $\partial f/\partial x_2$; it is found that no reduction in class takes place. The space is of the third class as in the preceding case.

It thus appears that the complete group of a 4-space (65) determines its class, at least in the case of the abelian group, the group G_3 of "rotations" and the group of "translations." Whether this is true for all the groups of motions in a 4-space is an open question that might be worth while answering. Fubini's classification of 4-spaces (vols. 8 and 9, Annali di Matematica) would here render a notable service. It should however be noted that the group of certain sub-spaces will also play a rôle in the determination of the class.

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* H. Weyl, Annalen der Physik, vol. 54, pp. 134–137.

West Virginia University,
Morgantown, W. Va.