THE LINEAR COMPLEX OF CONICS*

BY

EARL E. LIBMAN

1. Although the complex of first degree curves, that is, the rectilinear complex, has been thoroughly investigated, very little exists in the literature concerning the space complex of curves of the second degree. It is the purpose of this paper to discuss the linear complex of such curves.

Just what would be a linear complex of curves depends, largely, upon the algebra used to investigate them. In the following, where the algebra of quaternions is used, I have called a complex of curves linear when the essential constants of the curve are each a nonhomogeneous linear function of three independent parameters. This may not, perhaps, be what would be designated as the most general linear system under another algebra.

Any conic in space may be represented as the intersection of a right circular cone and a plane. Such a conic is specified by eight essential scalar constants, three for the plane of the conic, and five for the conic upon it. If these are all functions of three independent parameters, we shall call the system a complex of conics. If the eight constants are linear functions of the three parameters, we shall call the system a linear complex of conics. In the following discussion of the linear complex, the algebra used is quaternions with its usual Hamiltonian symbols.

2. If ω is the vector from the origin to the vertex of the right circular cone, $U\alpha$ the direction of the axis of the cone, $\cos^{-1}e$ the vertical semi-angle, γ the normal vector from the origin upon the plane of the conic, and ϱ a variable vector from the origin to any point on the conic, then the conic is represented by the equations

$$(1)\qquad \begin{aligned} S^2\alpha(\varrho-\omega)-e^2\alpha^2(\varrho-\omega)^2 &= 0 \qquad \text{(the generating cone)},\\ S\gamma\varrho-1 &= 0 \qquad \text{(the plane of the conic)}.\end{aligned}$$

The eight essential scalar constants of the conic (1) are involved in the three vector constants, ω, γ, and $U\alpha$, three each for the vectors ω and γ and two for the unit vector $U\alpha$. These are to be linear functions of three scalar parameters. Let us take for the parameters the components of ω along any three noncoplanar directions. We may then replace the three

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scalar parameters by the single vector parameter ω. Hence, for a linear complex as defined above,

$$\begin{aligned} \alpha &= \theta\omega + \lambda, \\ \gamma &= \varphi\omega + \eta, \end{aligned} \tag{2}$$

where θ and φ are linear vector operators and λ and η are constant vectors.

3. For a given γ, equations (2) show that both ω and α are determined.

A. *Upon each plane in space there is one and only one conic of the complex.*

When the conic passes through a point ϱ_0, equations (1) give two conditions upon ω. Since ω is equivalent to three scalar parameters,

B. *There is a single infinity of conics of the complex through every point in space.*

If ϱ_0 is a point in space, the values of γ corresponding to the conics through ϱ_0 satisfy

$$\begin{aligned} S^2[\theta\varphi^{-1}(\gamma-\eta)+\lambda]\,[\varrho_0-\varphi^{-1}(\gamma-\eta)] & \\ -e^2[\theta\varphi^{-1}(\gamma-\eta)+\lambda]^2\,[\varrho_0-\varphi^{-1}(\gamma-\eta)]^2 &= 0, \\ S\varrho_0\gamma-1 &= 0, \end{aligned} \tag{3}$$

obtained by substituting for ω in terms of γ in equations (1) and putting $\varrho = \varrho_0$. This is evidently the tangential equation of the cone enveloped by the planes of the conics that pass through the point ϱ_0. It is a cone of class four.

If we eliminate ω from

$$\begin{aligned} S^2(\theta\omega+\lambda)(\varrho_0-\omega)-e^2(\theta\omega+\lambda)^2(\varrho_0-\omega)^2 &= 0, \\ S(\varphi\omega+\eta)\varrho_0-1 &= 0, \\ S^2(\theta\omega+\lambda)(\varrho-\omega)-e^2(\theta\omega+\lambda)^2(\varrho-\omega)^2 &= 0, \\ S(\varphi\omega+\eta)\varrho-1 &= 0, \end{aligned} \tag{4}$$

obtained by substituting (2) in (1), and writing the condition that the conic passes through ϱ_0, we have a surface upon which lie all the conics of the complex that pass through the point ϱ_0.

C. *Associated with every point in space there is a surface and a cone such that the tangent planes to the cone intersect the surface in the conics of the complex that pass through the point.*

The equation of any line in space passing through the point β and parallel to the vector δ is

$$\varrho = \beta + x\delta. \tag{5}$$

If the plane of the conic (1) contains this line, then β is a vector from the origin to a point on this plane, and δ is a vector lying in the plane, and we have

$$S\gamma\beta - 1 = 0, \qquad S\gamma\delta = 0. \tag{6}$$

The line intersects the cone (1) in points corresponding to the roots of

$$(7)\quad \begin{aligned} x^2 S\delta[\alpha S\alpha - e^2\alpha^2]\delta - 2xS(\beta-\omega)\,[\alpha S\alpha - e^2\alpha^2]\delta \\ + S(\beta-\omega)\,[\alpha S\alpha - e^2\alpha^2]\,(\beta-\omega) = 0. \end{aligned}$$

If the line is tangent to the cone the roots of (7) must be equal. Hence

$$(8)\quad \begin{aligned} S^2(\beta-\omega)\,[\alpha S\alpha - e^2\alpha^2]\delta \\ - S\delta[\alpha S\alpha - e^2\alpha^2]\delta S(\beta-\omega)\,[\alpha S\alpha - e^2\alpha^2]\,(\beta-\omega) = 0. \end{aligned}$$

If the line (5) is tangent to the conic then equations (6) and (8) are satisfied and solving gives six values for ω.

D. *Every straight line in space is touched by six conics of the complex.*

4. In as much as the vector ω is at once the parameter of the complex and the vertex of the generating cone, and therefore every conic of the complex corresponds to a certain position of that vertex, we shall call the point ω the generating point, and shall speak of the corresponding conic as being generated by the generating point.

If the plane of the conic passes through its generating point, the conic will obviously be a line pair, since the plane contains the vertex of the cone, and this is the only case of such degeneracy. In order that the plane of the conic contain its generating point we must have

$$(9)\qquad S\gamma\omega - 1 = 0.$$

If in this equation we substitute for γ its value in terms of ω from equations (2), we will obtain an expression in ω which is the locus of the centers of all the degenerate conics of the complex. If, on the other hand, we substitute in this equation for ω its value in terms of γ from equations (2), we obtain an expression in γ which is the envelope of the planes of the degenerate conics of the complex. We will call these two surfaces the first and second singular surfaces respectively. Their expressions, as given below, show them to be quadrics:

(10)

$$S\omega\varphi\omega + S\omega\eta - 1 = 0$$ (the first singular quadric; point equation of locus of centers of degenerate conics);

$$S\gamma\varphi^{-1}\gamma - S\gamma\varphi^{-1}\eta - 1 = 0$$ (the second singular quadric; tangential equation of surface enveloped by planes of degenerate conics).

Note that the two singular quadrics have parallel axes.

E. *The centers of the degenerate conics of the complex lie upon a quadric surface, and their planes envelope another quadric surface. These are the first and second singular quadrics.*

5. The equations (1) will represent any conic through the point ϱ_0 that belongs to the complex if ω satisfies

$$
(11) \qquad \begin{aligned} S^2(\theta\omega+\lambda)(\varrho_0-\omega)-e^2(\theta\omega+\lambda)^2(\varrho_0-\omega)^2 &= 0, \\ S(\varphi\omega+\eta)\varrho_0-1 &= 0, \end{aligned}
$$

obtained by substituting (2) in (1) and putting $\varrho=\varrho_0$. It is at once evident that this is a plane quartic curve which is the locus of the generating points corresponding to all conics through the point ϱ_0.

F. *Associated with every point in space there is a plane quartic curve such that when the generating point describes the curve, the corresponding conics pass through the point. We will call this the generating quartic of the point.*

G. *The conics through a given point are in one to one correspondence with the points of the generating quartic of the given point.*

The generating points of the degenerate conics of the complex that pass through a point lie both upon the generating quartic of the point and upon the first singular quadric.

H. *Through every point in space pass eight degenerate conics of the complex.*

6. Let the plane of the generating quartic of the point ϱ_0 be

$$
(12) \qquad S\xi\varrho-1=0.
$$

Then, since ω is a vector to the quartic, we have also $S\xi\omega-1=0$. Writing the equation of the plane (11) in the form

$$
(13) \qquad S\omega\frac{\varphi'\varrho_0}{1-S\eta\varrho_0}-1=0,
$$

where φ' is the conjugate of φ, comparison shows that

$$
(14) \qquad \xi=\frac{\varphi'\varrho_0}{1-S\eta\varrho_0},
$$

from which

$$
(15) \qquad \varrho_0=\frac{\varphi'^{-1}\xi}{1+S\xi\varphi^{-1}\eta}.
$$

Evidently there is a one to one linear correspondence between point (coördinate ϱ_0) and plane (coördinate ξ) and we will call the plane of the generating quartic of a point its polar plane; and the point the pole of the plane of its generating quartic with respect to the complex:

$$(16)\qquad \begin{array}{ll} S\varrho\dfrac{\varphi'\varrho_0}{1-S\eta\varrho_0}-1=0 & \text{(polar plane of point } \varrho_0 \text{ with respect to the complex);} \\ \dfrac{\varphi'^{-1}\xi}{1+S\eta\varphi'^{-1}\xi} & \text{(pole of the plane } S\xi\varrho-1=0 \text{ with respect to the complex).} \end{array}$$

The names "pole" and "polar" as used above for point and plane are justified by the similarity between the behavior of this point and plane and that of the pole and polar of a quadric. By means of (16) it can be readily shown that

I. *When a point describes a line its polar plane with respect to the complex rotates about another line*;

J. *When a plane rotates about a line its pole with respect to the complex describes another line.*

7. Let us now consider what will be the loci of the poles that lie upon their polar planes and also of the planes that contain their poles. If the point ϱ_0 lies upon its polar plane (16) we have, putting $\varrho=\varrho_0$ in the first equation of (16),

$$(17)\qquad S\varrho_0\varphi\varrho_0+S\eta\varrho_0-1=0.$$

Comparison of this with equations (10) shows that it is the first singular quadric.

K. *The locus of the points that lie upon their polar planes is the first singular quadric.*

Again if the plane $S\xi\varrho-1=0$ contains its pole (16), then substituting $\varrho=(\varphi'^{-1}\xi)/(1+S\eta\varphi'^{-1}\xi)$ in this equation we get

$$(18)\qquad S\xi\varphi^{-1}\xi-S\xi\varphi^{-1}\eta-1=0,$$

which is evidently the second singular quadric.

L. *The envelope of the planes that contain their poles is the second singular quadric.*

8. It is to be noted that, although the machinery used in this work is metric, yet the results are essentially projective. Moreover, it is evident that the Theorems A, B, C, D and H are independent of the method used to designate the conic (that is, the circular cone and the plane) while Theorems I, J, K, L are called into prominence by this construction.

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UNIVERSITY OF ILLINOIS,
URBANA, ILL.