SOME PROPERTIES OF LIMITED CONTINUA, IRREDUCIBLE BETWEEN TWO POINTS*

BY

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1. In reading papers on the theory of point aggregates one frequently meets with the continuum C whose points are defined as follows: when $-1 \leqq x < 0$ or $0 < x \leqq 1$, $y = \sin(1/x)$; when $x = 0$, $-1 \leqq y \leqq 1$. This is perhaps the most commonly cited example of a limited continuum irreducible between two points and having a continuum of condensation. If we regard the concept of connectedness im kleinen as the analogue for continua of continuity in functions, we notice that the properties of this continuum resemble those of pointwise discontinuous functions. The points where the continuum is connected im kleinen form a set of the secondary category with respect to C, while those of the second genre† form a set of the first category.

The question at once arises as to whether this similarity is of a general character or is merely due to the nature of the example cited, and suggests that a study of limited continua irreducible between two points with special reference to the oscillation at the various points would be of interest. This problem has been discussed in a paper by H. Hahn,‡ who has shown that such a continuum is the sum of a set of sub-continua known as "prime parts," no two of which have common points. However, the fact that in many cases a prime part itself can be subdivided indicates that the subject has not been exhausted and it is the purpose of this paper to present some further results along this line.

The first half of the paper (§§4–15) is devoted to the general properties of the oscillation of a limited irreducible continuum. In the second half the properties such continua have when the points of the first genre are everywhere dense are treated, and in particular it is shown that in this case there is a correspondence between the points of such a continuum and those of a linear segment analogous to that between the variables y and x when $y = f(x)$ is a pointwise discontinuous function of a certain type.

* Presented to the Society, September 10, 1925; received by the editors in October, 1925.

† A point of a continuum is of the first or second genre according as the oscillation of the continuum at the point is zero or not. See S. Mazurkiewicz, *Sur les lignes de Jordan*, Fundamenta Mathematicae, vol. 1, p. 170.

‡ H. Hahn, *Über irreduzible Kontinua*, Wiener Sitzungsberichte, vol. 130, pp. 217–250.

2. In a previous paper,* of which this is in one sense a continuation, two concepts were introduced which will be used throughout this work. The first was an auxiliary function $\tau(x)$, suggested as a definition of oscillation alternative to that of Mazurkiewicz.† This is defined as follows. Let c be a point of the continuum A, let $V(c)$ denote those points of A whose distance from c is less than δ and let C_δ be any sub-continuum of A containing $V(c)$. Then the lower bound of the diameters of all such sub-continua C_δ for all values of $\delta > 0$ is denoted by $\tau(c)$. The function $\tau(c)$ will be called the oscillation of A at c.

The second concept was that of *oscillatory sets*. Let $\delta_1 > \delta_2 > \cdots$, $\delta_i \to 0$, and let $\{C_{\delta_i}\}$ be a monotone decreasing sequence of sub-continua of A irreducible about $V_{\delta_i}(c)$. Then $C(c) = Dv[C_{\delta_i}]$ is called an oscillatory set of A about c. It is obvious that, if A is limited, each oscillatory set is a continuum.

Among the general properties of these concepts derived in the paper referred to, the following will be used.

(*a*) For limited continua there exist one or more oscillatory sets at each point and $\tau(c)$ is the lower bound of the diameters of the oscillatory sets of A about c.

(*b*) If $\tau(c) = 0$, the point c is one of the oscillatory sets of A about c.

(*c*) The function $\tau(c)$ is related to Mazurkiewicz' oscillatory function $\sigma(c)$ by the inequalities $\sigma(c) \leqq \tau(c) \leqq 2\sigma(c)$.

3. **Notation.** The ordinary notation of the theory of aggregates is employed, with the following modifications.

The notation $A \subset B$ means that A is a *real* part of B. If A is a part of B and may be identical with B, we write $A \subseteq B$.

If A is the common part of the system of aggregates $\{C\}$, we write $A = Dv[C]$.

If A contains every element of each of the system of aggregates $\{C\}$ and no other elements, we write $A = U[C]$.

When the notation $A = B + C$ is used, it will be understood unless expressly stated to the contrary that $B \cdot C = 0$.

The phrase "irreducible continuum ab" means a continuum which is irreducible between the points a and b.

The symbol $S_\delta(c)$ denotes an open sphere (or hypersphere) of center c and radius δ. If c is a point of the aggregate A, the symbol $V_\delta(c)$ denotes the the subset of points of A each of which has a distance from c less than δ.

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* W. A. Wilson, *On the oscillation of a continuum*, these Transactions, vol. 27, pp. 429–440.

† Loc. cit.

Then $V_\delta(c)=A\cdot S_\delta(c)$ and $V\ (c)$ is a region relative to A. In like manner, if E is a closed subset of A, the symbol $V_\delta(E)$ denotes the subset of points of A each of which has a distance from E less than δ.

4. In the following chain of six theorems, of which the first three are essentially lemmas, certain general properties of a limited irreducible continuum ab are developed. It is shown that ab has about each point a unique oscillatory set of diameter $\tau(c)$, that at no point x of an oscillatory set of diameter greater than zero is $\tau(x)=0$, and that $\tau(c)$ is identical with Mazurkiewicz' oscillatory function $\sigma(c)$.

THEOREM. *Let ab be a limited irreducible continuum. Let A and B be true sub-continua of ab containing a and b respectively and let $A\cdot B=0$. Then there is one and only one sub-continuum K of ab which is irreducible between A and B, and $K=\overline{ab-(A+B)}$.*

Proof. By a continuum K irreducible between A and B we mean a continuum containing points of both A and B such that no true sub-continuum contains points of both these sets. Let $L=ab-(A+B)$ and consider $\bar{L}$. Since ab is a continuum,

$$ab = A+\bar{L}+B, \qquad A\cdot\bar{L}\neq 0, \qquad \bar{L}\cdot B\neq 0.$$

Now there is a partition of L into two closed sets C and D such that $C\cdot D=C\cdot B=D\cdot A=0$, $A\cdot C\neq 0$, and $B\cdot D\neq 0$; or there is not. In the former case, $ab=(A+C)+(B+D)$ and $(A+C)\ (B+D)=0$, which is contrary to the fact that ab is a continuum.

In the latter case an easy generalization of two theorems of Janiszewski* shows that $\bar{L}$ contains a continuum K joining A and B. Since ab is irreducible, $K\supseteq ab-(A+B)=L$, whence $K\supseteq\bar{L}$. Thus $\bar{L}$ is a continuum joining A and B. Finally, any sub-continuum of ab irreducible between A and B must contain L and consequently $\bar{L}$, since ab is irreducible. Hence the theorem.†

COROLLARY. *Let ab be a limited irreducible continuum and A be a true sub-continuum containing a but not b. Then there is one and only one sub-continuum K of ab which is irreducible between A and b, and $K=\overline{ab-A}$.*

5. THEOREM. *Let C be a limited continuum. Let a and c be points of C and let $\delta<$Dist (a, c). Let A be the saturated sub-continuum of C containing a, but no point of $V_\delta(c)$. Then at least one point of A is a limiting point of $V_\delta(c)$.*

* Z. Janiszewski, *Sur les continus irreductibles entre deux points*, Journal de l'Ecole Polytechnique, ser. 2, vol. 16 (1912), pp. 109–111.

† The proof here given was suggested by the referee in lieu of a longer one in the paper as originally submitted.

Proof. Let S_δ denote the interior of a sphere of center c and radius δ. Let Z denote the whole of the n-way euclidean space containing C, and G_δ be the complement of S_δ. Then G_δ is closed and

$$Z = G_\delta + S_\delta.$$

Let $\sigma < \delta$ and let V denote those points of Z whose distance from the closed set A is not greater than σ. Then V is closed and does not contain c. Also

$$V = V \cdot G_\delta + V \cdot S_\delta.$$

By a well known theorem of Janiszewski* there is a sub-continuum K of C joining a point z of A to a point y of Front V and lying wholly in V. If $K \subseteq V \cdot G_\delta$ then K would contain no point of S_δ, while $K \cdot A \neq 0$. This would make $K \subseteq A$ contrary to the fact that y is not a point of A. Thus K contains at least one point in $V \cdot S_\delta$. But $K \subset V$. Hence at least one point of C in S_δ has a distance from A not greater than σ.

As this is true for every σ, we have a sequence of points of $V_\delta(c) = C \cdot S_\delta$ whose distances from A converge to zero. These therefore have at least one limiting point on A.

6. Theorem. *Let ab be a limited irreducible continuum and c be a point of ab. Then there is a unique sub-continuum of ab irreducible about any $V_\delta(c)$.*

Proof. Let C_δ be any sub-continuum of ab irreducible about $V_\delta(c)$. There are several cases to consider.

I. If $V_\delta(c)$ contains both a and b, C_δ obviously equals ab.

II. If $V_\delta(c)$ contains b but not a, let A be the saturated sub-continuum of ab containing a but no point of $V_\delta(c)$ and let $\bar{L} = \overline{ab - A}$.

Since $C_\delta \supseteq V_\delta(c)$, $C_\delta \supseteq \bar{V}_\delta(c)$. Then, by §5, $A \cdot C_\delta \neq 0$. As C_δ contains b and a point of A, we have by §4, Corollary,

$$C_\delta \supseteq \bar{L}. \tag{1}$$

But

$$\bar{L} \supset ab - A \supseteq V_\delta(c). \tag{2}$$

Since C_δ is irreducible about $V_\delta(c)$, relations (1) and (2) give

$$C_\delta = \bar{L}.$$

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* See reference under § 4, p. 100.

III. If $V_\delta(c)$ contains a but not b, we have essentially the same situation as in Case II.

IV. If $V_\delta(c)$ contains neither a nor b, let A be defined as in Case II and let B have the same relation to the point b, while $\bar{L}=\overline{ab-(A+B)}$. As in Case II, $A\cdot C_\delta\neq 0$ and $B\cdot C_\delta\neq 0$. Hence, by §4,

$$C_\delta \supseteq \bar{L}. \tag{3}$$

But

$$\bar{L}\supset ab - (A+B)\supseteq V_\delta(c). \tag{4}$$

Then relations (3) and (4) give

$$C_\delta = \bar{L}.$$

7. THEOREM. *Let ab be a limited irreducible continuum and let c be a point of ab. Then the oscillatory set of ab about c is unique and its diameter is $\tau(c)$, and, if $\tau(c)=0$, it is the point c itself.*

Proof. Let $0<\delta<\eta$, and let C_δ and C_η be the sub-continua of ab irreducible about $V_\delta(c)$ and $V_\eta(c)$ respectively. Obviously $V_\delta(c)\subseteq V_\eta(c)$ and hence $V_\delta(c)\subseteq C_\eta$.

Then C_η has a sub-continuum D irreducible about $V_\delta(c)$. Since $D\subset C_\eta$,

$$D\subseteq ab. \tag{1}$$

But C_δ is the only sub-continuum of ab irreducible about $V_\delta(c)$. Hence (1) gives $C_\delta=D$ and $C_\delta\subseteq C_\eta$.

This result shows that if $\{\delta_i\}$ and $\{\eta_i\}$ are monotone decreasing sequences converging to zero, $Dv[C_{\delta_i}]=Dv[C_{\eta_i}]$. Hence the oscillatory set is unique. That its diameter is $\tau(c)$ follows from the fact that for continua in general $\tau(c)$ is equal to the minimum diameter of all the oscillatory sets about c. (See §2.) Then if $\tau(c)=0$, the oscillatory set can contain no other point than c itself.

For continua in general the oscillatory sets are not unique, as is shown in an example given in the paper referred to in §2. The reader should note the analogy of the oscillatory set to the aggregate of limiting values of a one-valued function of a real variable $f(x)$ as $x\to a$. It is to the above theorem that the oscillatory sets of a limited irreducible continuum owe most of their value. A point of the second genre lies on a continuum of condensation and in simple cases there is a greatest such continuum, but this is not always the case. On the other hand the oscillatory set of each point is unique.

8. THEOREM. *Let ab be a limited irreducible continuum. Then for each point c of ab,* $\tau(c)=\sigma(c)$.

Proof. It has been shown elsewhere (see §2) that

$$\sigma(c) \leqq \tau(c). \tag{1}$$

Let $c\neq a$, $c\neq b$. Then by definition* of $\sigma(c)$ *any* two points x and y of ab lying in a closed sphere $\bar{S}_\delta(c)$ whose center is c and whose radius δ is sufficiently small can be joined by a sub-continuum $C(x, y)$ of ab such that

$$\operatorname{Diam} C(x,y) < \sigma(c) + \epsilon, \tag{2}$$

where ϵ is any positive quantity.

Now let $S_\delta(c)$ denote the interior of $\bar{S}_\delta(c)$, and let δ be so small that $\bar{S}_\delta(c)$ contains neither a nor b. Let A and B be the saturated sub-continua of ab containing a and b respectively, but no point of $V_\delta(c)=ab\cdot S_\delta(c)$. By §5, A contains a point x, and B a point y, on Front $\bar{S}_\delta(c)$, and these points satisfy relation (2). Then

$$ab = A + C(x,y) + B, \qquad A\cdot C(x,y) \neq 0, \qquad B\cdot C(x,y)\neq 0.$$

Since $A\cdot V_\delta(c)=B\cdot V_\delta(c)=0$, we have

$$V_\delta(c) \subseteq C(x,y).$$

Then relation (2) gives

$$\tau(c) \leqq \sigma(c) + \epsilon.$$

This with relation (1) gives the theorem for the case under consideration. Similar reasoning applies to the cases that $c=a$ or $c=b$.

9. THEOREM. *Let ab be a limited irreducible continuum. Let c be a point of the second genre of ab and let C(c) be the oscillatory set of ab about c. Then no point of C(c) is of the first genre.*

Proof. Assume that there is a point x of $C(c)$ at which $\tau(x)=0$, and let $\eta=\text{Dist }(x, c)$. Then for any positive $\epsilon<\eta$ there is a $\delta>0$, such that there is a sub-continuum D of ab having the property

$$V_\delta(x) \subseteq D \subseteq V_\epsilon(x). \tag{1}$$

Let c be different from a and b and let δ be so small that $V_\delta(x)$ contains neither a nor b. Let A and B be the saturated sub-continua of ab containing a and b respectively, but no point of $V_\delta(x)$. Then by §5 and relation (1)

$$ab = A + D + B, \qquad A\cdot D \neq 0, \qquad B\cdot D\neq 0.$$

* See S. Mazurkiewicz, *Sur les lignes de Jordan*, Fundamenta Mathematicae, vol. 1, p. 170.

Since $\epsilon<\eta$, c is not a point of D; suppose that c is a point of A. Then A has a sub-continuum irreducible about any $V_\sigma(c)$, if σ is so small that $V_\sigma(c)$ contains no point of D or of B. Since $A \subset ab$, this is the sub-continuum of ab irreducible about $V_\sigma(c)$ by § 6. Hence we have $C(c) \subseteq A$. This is a contradiction, since x is not a point of A.

The cases where c coincides with a or b are treated in a like manner.

Remark. The above theorem shows that the oscillatory set about a point c is a part of the prime part containing c. That it need not be identical with the prime part is seen from the example suggested in § 23.

10. It would be natural to surmise from § 9 that, if $C(c)$ is the oscillatory set of ab about c, then $C(c)$ is the oscillatory set about each point z on $C(c)$ and that $\tau(z)=\tau(c)$. This, however, is not true, as the following example shows. Let ab be the continuum whose points are defined as follows: for $x=0$, $-1\leqq y\leqq 1$; for $0<x\leqq 1$, $y=\sin^2(1/x)$; for $-1\leqq x<0, y=-\sin^2(1/x)$. The oscillatory set of ab about the point $(0, 0)$ is the segment of the y-axis between $(0, -1)$ and $(0, 1)$ and the oscillation is 2. For the point $(0, \frac{1}{2})$ the oscillatory set is the segment joining $(0, 0)$ and $(0, 1)$ and the oscillation is 1.

In order to investigate this question more fully it is necessary to consider the point set remaining when the oscillatory set about the point c is subtracted from the continuum ab. The inference that the remainder is two semi-continua, one containing a and the other b (or one semi-continuum if $C(c)$ contains either a or b) is seen to be false for the point $(0, \frac{1}{2})$ in the example just given. The difficulty is that $(0, \frac{1}{2})$ is a limiting point of only one of the semi-continua containing the end points of ab. The inference is borne out for $(0, 0)$, however, as this point is a limiting point of both semi-continua.

The following notation will now be used. If x is a point of ab, the oscillatory set of ab about x will be denoted by $X(x)$, or simply X. The saturated semi-continua of $ab-X$ containing a and b respectively will be denoted by X_a and X_b. If b is a point of X, obviously $X_b=0$; if a is a point of X, $X_a=0$. It is evident that

$$(1) \qquad X_a + X + X_b \subseteq ab \quad \text{and} \quad X_a \cdot X_b = 0.$$

By a theorem of Janiszewski* we know that $X\cdot\overline{X}_a\neq 0$ and $X\cdot\overline{X}_b\neq 0$; hence

$$(2) \qquad ab = \overline{X}_a + X + \overline{X}_b, \qquad X\cdot\overline{X}_a \neq 0, \qquad X\cdot\overline{X}_b \neq 0.$$

Thus $\overline{X}_a+X$ and $X+\overline{X}_b$ are continua. Analogous theorems hold, of course, for the cases that $X_a=0$ and $X_b=0$.

* See reference under § 4, p. 123.

It is also easy to sho" that

$$\bar{X}_a \cdot X_b = 0 \quad \text{and} \quad X_a \cdot \bar{X}_b = 0. \tag{3}$$

For, let A and B be the saturated sub-continua of ab containing a and b but no points of $V_\delta(x)$. Then $A \cdot X_b = 0$ and $B \cdot X_a = 0$, since ab is irreducible. But

$$ab = A + X_\delta(x) + B, \qquad A \cdot X_\delta(x) \neq 0, \qquad B \cdot X_\delta(x) \neq 0.$$

Then

$$X_a \subseteq A + X_\delta(x),$$

and

$$\bar{X}_a \cdot X_b \subseteq A \cdot X_b + X_b \cdot X_\delta(x) = X_b \cdot X_\delta(x).$$

Hence

$$\bar{X}_a \cdot X_b \subseteq Dv[X_b \cdot X_\delta(x)] = X \cdot X_b = 0.$$

11. THEOREM. *Let C be the oscillatory set of the limited irreducible continuum ab about one of its points c and let C contain neither a nor b. Let c be a common limiting point of C_a and C_b. Then $ab = C_a + C + C_b$.*

Proof. Owing to the hypothesis regarding c any sub-continuum $C_\delta(c)$ of ab irreducible about $V_\delta(c)$ contains points of C_a and C_b. Hence

$$ab = C_a + C_\delta(c) + C_b, \qquad C_a \cdot C_\delta(c) \neq 0, \qquad C_b \cdot C_\delta(c) \neq 0,$$

or

$$C_\delta(c) \supseteq ab - (C_a + C_b).$$

Therefore

$$C \supseteq ab - (C_a + C_b). \tag{1}$$

But

$$C_a + C + C_b \subseteq ab, \qquad C \cdot C_a = 0, \qquad C \cdot C_b = 0,$$

whence

$$C \subseteq ab - (C_a + C_b). \tag{2}$$

Relations (1) and (2) give the theorem.

COROLLARY 1. *Let C be the oscillatory set of the limited irreducible continuum ab about the point c. Let a be a point of C and c be a limiting point of C_b. Then $ab = C + C_b$.*

COROLLARY 2. *Under the conditions of the above theorem the sets $C_a + C$ and $C + C_b$ are continua.*

For $\bar{C}_a C_b = 0$ by § 10, relation (3).

COROLLARY 3. *If* $\tau(c)=0$, $ab=C_a+c+C_b=\bar{C}_a+C_b=C_a+\bar{C}_b$.

For by § 10, relation (2), c is a common limiting point of C_a and C_b.

COROLLARY 4. *Under the conditions of the above theorem, C is a continuum of condensation of ab.*

For c is a point of $\bar{C}_a \cdot \bar{C}_b$. Hence $ab=\bar{C}_a+\bar{C}_b$. Therefore $C \subset \overline{C_a+C_b}$.

Analogous theorems hold for the oscillatory sets of the points a and b.

12. From § 11, Corollary 3, we can easily deduce the following well known properties of irreducible continua.

(*a*) If c is a point of the first genre of the limited irreducible continuum ab, then c divides ab into two unique irreducible continua ac and cb, and $ac \cdot cb = c$.

(*b*) If x and y are two points of the first genre of the limited irreducible continuum ab, then x and y divide ab into three unique irreducible continua; if x is a point of ay, $ab=ax+xy+yb$, and $ax\cdot xy=x$, $xy\cdot yb=y$, and $ax\cdot yb=0$.

(*c*) If c is a point of the first genre of the limited irreducible continuum ab and x is a point of ac, then every irreducible sub-continuum ax of ab is a real part of ac and every xb contains cb as a real part.

It is also evident that analogous theorems can be deduced from the partition $ab=C_a+C+C_b$ of § 11. These we shall not go into, but we need the two theorems immediately following.

13. THEOREM. *Let X be the oscillatory set of the limited irreducible continuum ab about the point x and let X contain neither a nor b. Let c be a common limiting point of X_a and X_b. Then the oscillatory set C about c contains X.*

Proof. Any sub-continuum $C_\delta(c)$ of ab irreducible about $V_\delta(c)$ contains points of both X_a and X_b. Hence

$$C_\delta(c) \supseteq ab - (X_a + X_b). \tag{1}$$

But

$$X_a + X + X_b \subset ab,$$

or

$$X \subseteq ab - (X_a + X_b). \tag{2}$$

Relations (1) and (2) give

$$C_\delta(c) \supseteq X.$$

Hence

$$C \supseteq X.$$

Analogous theorems hold for the oscillatory sets of a and b.

14. THEOREM. *Let ab be a limited irreducible continuum. Let c be a point of ab and let C be the oscillatory set of ab about c. Let c be a common limiting point of the saturated semi-continua C_a and C_b of $ab-C$. Let x be any point of C and X be the oscillatory set of ab about x. Then $X \subseteq C$.*

Proof. Let $\{\delta_i\}$ and $\{\delta_k\}$ be two decreasing sequences of positive numbers converging to zero. Let C_i be the sub-continuum of ab irreducible about $V_{\delta_i}(c)$, while D_k has the corresponding meaning for $V_{\delta_k}(x)$. Assume that C contains neither a nor b.

Since c is a limiting point of both C_a and C_b, for every i there is a sub-continuum A_i of C_a joining a to a point of $V_{\delta_i}(c)$, and likewise a B_i of analogous properties. Hence

$$ab = A_i + C_i + B_i, \qquad A_i \cdot C_i \neq 0, \qquad C_i \cdot B_i \neq 0 .$$

Since $C \cdot A_i = C \cdot B_i = 0$, $V_{\delta_k}(x)$ contains no point of either A_i or B_i if k is sufficiently great. Thus $V_{\delta_k}(x) \subseteq C_i$ and so for every i there is a k such that

$$D_k \subseteq C_i.$$

Since $X = Dv[D_k]$ and $C = Dv[C_i]$, this gives at once

$$X \subseteq C .$$

The special cases where C contains a or b are treated in like manner.

COROLLARY. *If in the above theorem x is a common limiting point of C_a and C_b, then the oscillatory set X of ab about x is identical with C.*

15. The example of § 10 shows that if a point x lies on an oscillatory set $C(c)$, the oscillatory set $X(x)$ may form only a part of $C(c)$. In the example mentioned the point $(0, \frac{1}{2})$ is an instance of this. This fact, coupled with § 14, suggests setting apart as a separate class the oscillatory sets satisfying the hypotheses of that theorem. Accordingly we have the definition

If $C(c)$ is the oscillatory set of the limited irreducible continuum ab about a point c, we shall call $C(c)$ complete when it contains neither a nor b and c is a common limiting point of the saturated semi-continua C_a and C_b of $ab-C(c)$. Likewise, *if c is identical with a or b and c is a limiting point of C_b or C_a, respectively, we shall call $C(c)$ complete.*

In addition to the properties already derived in §§ 11-14, we have the following general theorem.

Let ab be a limited irreducible continuum and the oscillatory set $C(c)$ be complete. Let x be a point of C_a or C_b. Then the oscillatory set $X(x)$ is a part of C_a or C_b, respectively.

Proof. To fix the ideas let x be a point of C_a. Then it follows from the definition of oscillatory sets that there is a $\delta > 0$ so small that, if C_δ is the sub-continuum of ab irreducible about $V_\delta(c)$, C_δ does not contain x. Since $C(c)$ is complete, there is a sub-continuum E of C_a containing a and a point in $V_\delta(c)$. Then

$$ab = E + C_\delta + C_b \text{ and } E \cdot C_b = 0, \qquad E \cdot C_\delta \neq 0, \qquad C_\delta \cdot C_b \neq 0.$$

Since x is not a point of C_δ or C_b, it is a point of E. Then for some $\eta > 0$, $V_\eta(x) \subset E$. Then by the definition of oscillatory set,

$$X(x) \subseteq E \subset C_a,$$

which was to be proved.

A similar theorem holds when the oscillatory set of a or b is complete.

16. **Simple irreducible continua.** From § 11, Corollary 3, we see that each point of the first genre is a complete oscillatory set. We now turn to a consideration of limited continua, irreducible between two points and containing an everywhere dense set of points of the first genre. To save repetition in the statements of theorems, we shall call such continua *simple irreducible continua.* It may be remarked here that it is evident that the oscillatory sets of such continua about a and b are always complete. Furthermore, if x' and x'' are two points of the first genre of ab and a complete oscillatory set C is contained in the sub-continuum $x'x''$, it follows from § 12 (b) and § 11, Corollary 4, that C is a continuum of condensation of $x'x''$. The following lemmas are convenient in later work.

LEMMA I. *Let ab be a simple irreducible continuum and let C be a complete oscillatory set containing neither a nor b. Then there are sequences of points $\{x_i\}$ and $\{x_j\}$ of the first genre converging to c and lying in C_a and C_b, respectively, and $C = Dv[x_i x_j]$.*

Proof. If y_i is any point of C_a, there is a δ_i so small that $V_{\delta_i}(y_i)$ contains no point of C or C_b. But every $V_\delta(y_i)$ contains points of the first genre. Hence it is evident that any sequence $\{y_i\}$ of points of C_a converging to c can be replaced by a sequence of points of the first genre belonging to C_a and converging to c. Thus the sequences $\{x_i\}$ and $\{x_j\}$ exist.

Now for any η, $V_\eta(c)$ contains an x_i and an x_j. Hence $C \supseteq Dv\{x_i x_j\}$. On the other hand, for any given x_i and x_j there is an η so small that $V_\eta(c) \subseteq x_i x_j$. Thus $C \subseteq x_i x_j$ and hence $C \subseteq Dv[x_i x_j]$. Therefore $C = Dv[x_i x_j]$.

LEMMA II. *Let ab be a simple irreducible continuum and A be the oscillatory set about a. Then there is a sequence of points $\{x_i\}$ of the first genre converging to a and lying in A_b, and $A = Dv[ax_i]$.*

This and the corresponding lemma for the point b are proved in the same way as Lemma I.

17. THEOREM. *Let ab be a simple irreducible continuum. Then* (i) *no two complete oscillatory sets have common points*; (ii) *every incomplete oscillatory set Y contains a point whose oscillatory set is complete and contains Y*; (iii) *every point lies on one and only one complete oscillatory set*; *and* (iv) *if two oscillatory sets have common points, one is complete and contains the other, or both are contained in the same complete oscillatory set.*

Proof. (i) Let X be a complete oscillatory set and $ab=X_a+X+X_b$, where either X_a or X_b may be void. Let $Y=Y(y)$ be another complete oscillatory set. If y is a point of X_a or X_b, say the former, § 15 shows that $Y\subset X_a$; hence $X\cdot Y=0$. If y is a point of X, then § 14 shows that

$$Y\subseteq X. \tag{1}$$

In this case we have $ab=Y_a+Y+Y_b$. If x is not a point of Y, then by § 15 X is a part of Y_a or Y_b, which is false, since $X\cdot Y\supset y$. Then x is a point of Y and hence

$$X\subseteq Y. \tag{2}$$

Then either $X\cdot Y=0$, or by relations (1) and (2) $X=Y$.

(ii) Let Y be an oscillatory set which is not complete. If c is a point of the first genre, $Y\subset ac$ or $Y\subset cb$, by § 15. Then c is a point of Y_b or Y_a, respectively. Thus all the points of the first genre are on Y_a and Y_b. As they are everywhere dense in ab, $Y=Y\cdot\overline{Y}_a+\cdot Y\overline{Y}_b$, $Y\cdot\overline{Y}_a\neq 0$, $Y\cdot\overline{Y}_b\neq 0$. Since Y is a continuum, at least one point x of Y is a limiting point of both Y_a and Y_b. Then, by § 13, Y is a part of the oscillatory set X about x.

To show that X is complete, observe first that $X_a\subseteq Y_a$ and $X_b\subseteq Y_b$. Then, as all the points of the first genre are on X_a and X_b, X_a contains all the points of Y_a that are of the first genre. Since by § 10, $Y_a\cdot\overline{Y}_b=0$, for every point z of Y_a any $V_\delta(z)$ contains points of the first genre lying on Y_a. Then, as x is a limiting point of Y_a, it is a limiting point of points of the first genre of Y_a and consequently a limiting point of X_a. Likewise it is a limiting point of X_b. Hence X is complete.

(iii) This follows from (i) and (ii).

(iv) If one of two oscillatory sets with common points is complete, it contains the other by (i) and (ii). If neither is complete, each lies in a complete oscillatory set by (ii). These complete oscillatory sets must be identical by (i).

18. The aggregate of complete oscillatory sets. Let X be any complete oscillatory set of the simple irreducible continuum ab, and let $K=\{X\}$ be the aggregate whose elements are these complete oscillatory sets. From § 17 it follows that $ab=U[X]$ and that no two sets X have common points.

The fact that $ab=X_a+X+X_b$ permits us to order the aggregate K. For, if Y is another element of K, we have seen (§ 17) that either $Y\subset X_a$ or $Y\subset X_b$. In the former case we say that Y precedes X and write $Y<X$; in the latter, that Y follows X and we write $Y>X$. It is easy to show that $X>Y$ if $Y<X$ and $X<Y$ if $Y>X$, and that, if X, Y, and Z are three complete oscillatory sets such that $X<Y$ and $Y<Z$, then $X<Z$. Thus K is a simply ordered aggregate. Furthermore K has a first and a last element, namely the complete oscillatory sets containing a and b, respectively.

We shall now proceed to show that the order type of the set K is similar to that of the aggregate whose elements are the points of a finite closed segment. Then by virtue of this similarity we shall determine a correspondence between the points of ab and those of a linear segment.

19. Theorem. *The aggregate of complete oscillatory sets of a simple irreducible continuum ab has a dense order type.*

Proof. We must prove that, if X and Y are two complete oscillatory sets, then there is a third lying between them. Let $X<Y$. Then

$$ab = X_a + X + X_b, \tag{1}$$

and

$$ab = Y_a + Y + Y_b.$$

Since $X<Y$ it follows that

$$Y + Y_b \subseteq X_b.$$

If $Y+Y_b=X_b$, relation (1) gives

$$ab = X_a + X + Y + Y_b,$$

and

$$(X_a + X)\cdot(Y + Y_b) = 0.$$

This is an obvious contradiction, since ab is a continuum.

Hence X_b contains at least one point z not belonging to $Y+Y_b$. Therefore z is a point of Y_a and the complete oscillatory set Z containing z is a part of Y_a. Thus $Z<Y$. But, as z is also a point of X_b, $Z\subset X_b$ and $Z>X$. This proves the theorem.

20. THEOREM. *Let ab be a simple irreducible continuum. Then the aggregate K of complete oscillatory sets of ab has an open enumerable subset which is dense in itself and also dense in ab.*

Proof. Since the set of points of the first genre is dense in ab, there is an enumerable subset of this which is also dense in ab; call it E. Also, if a or b is of the first genre, we can assume that E contains neither of these.

Now let X and Y be two elements of K and let $X<Y$. By § 19 there is an element Z of K such that

$$X < Z < Y.$$

Then Z contains no points of the continua X_a+X and $Y+Y_b$. Then if z is a point of Z, for δ sufficiently small

$$V_\delta(z) \subset X_b \text{ and } V_\delta(z) \subset Y_a.$$

But E is dense in ab; hence $V_\delta(z)$ contains a point e of E. As $\tau(e)=0$, the complete oscillatory set containing e is e itself. Then the complete oscillatory set of e is a part of both X_b and Y_a. Hence

$$X < e < Y.$$

Therefore E considered as an aggregate of complete oscillatory sets is dense in K and obviously dense in itself. It is also evident that it has no first and no last element.

21. THEOREM. *Let ab be a simple irreducible continuum, and let K be the aggregate of complete oscillatory sets of ab. Then the order type of K is that of a finite segment.*

Proof. Let $K=P+Q$ be a partition such that every element of P precedes every element of Q. Let X be any element of P and let Y be any element of Q; then $P=\{X\}$ and $Q=\{Y\}$. Also $B=U[X]$ and $C=U[Y]$. Then we have

$$ab = B + C.$$

Suppose now that P has no last and Q no first element. Let Y be *any* element of Q. Then there is an element Y' of Q preceding Y and every element X of P precedes Y'. Then $B=U[X]\subset Y_a'$ and

$$\overline{B} \subseteq Y_a' + Y' \subset Y_a.$$

Hence $\overline{B}\cdot Y=0$ and no point of $\overline{B}$ lies in $C=U[Y]$.

In the same way we show that $B\cdot\overline{C}=0$. Since $\overline{B}\cdot C+B\cdot\overline{C}=0$, we have a contradiction, as ab is a continuum. Hence either P has a last, or Q has a first element. As it has been proved in § 19 that K is dense, this shows that K is continuous.*

If we omit from K the complete oscillatory sets of a and b, the resulting set K' is also continuous and by § 20 K' contains an open enumerable aggregate dense in itself and dense in K'. Therefore the order type of K' is that of the linear continuum† and K is similar to a finite closed segment.

22. The effect of the theorem just proved is to set up a uniform correspondence between the points of a finite segment $(0 \leqq t \leqq 1)$ and the complete oscillatory sets of a simple irreducible continuum $ab = \{x\}$. This defines a correspondence between the points of ab and those of the segment, which need not be uniform. If we denote this by $x = f(t)$, we have a function which is in general multivalued. It is, moreover, but a slight extension of the ordinary notion of continuity if we say that $f(t)$ is continuous at a point t' when $f(t')$ is a single point x' and for every $\epsilon > 0$, the aggregate of images of all points t in some $V_\delta(t')$ lie in $V_\epsilon(x')$.

THEOREM. *Let $ab = \{x\}$ be a simple irreducible continuum. Then there is a correspondence $x = f(t)$ between the points of ab and those of the unit segment $T = (0 \leqq t \leqq 1)$ having the following properties:*

(i) *To each point t corresponds one and only one point of the first genre or one and only one complete oscillatory set.*

(ii) *$f(t)$ is continuous at each point t whose image is a single point.*

(iii) *At a point t_0 whose image is not a single point the corresponding oscillatory set $X_0 = \{x_0\}$, $x_0 = f(t_0)$, is the set of accumulation‡ of the sets $x = f(t)$ as $t \to t_0$.*

(iv) *The images of 0 and 1 are the complete oscillatory sets of a and b, respectively.*

Proof. The correspondence in question is that defined by the similarity of the aggregate of complete oscillatory sets $K = \{X\}$ of ab to any finite closed segment. This gives us (i) and (iv) at once. It remains to prove (ii) and (iii).

Let t_0 be a point whose image is the single point x_0. Then $\tau(x_0) = 0$ and this fact, with § 16, shows that there are two points of the first genre, x' and x'', such that $x' < x_0 < x''$ and the irreducible sub-continuum $x'x''$ is a

* See F. Hausdorff, *Grundzüge der Mengenlehre*, p. 90.

† Ibid., p. 101.

‡ This notion is defined by Janiszewski (loc cit., p. 93) for the case of an enumerable system of sets. The following is a natural extension. Let $\{f(t)\}$ denote a system of sets depending on the parameter t, which ranges over an interval and let t_0 be any point in this interval. Let F be the class of all points $\{x\}$ such that, for every $\epsilon > 0$, $S_\epsilon(x)$ contains a point of some $f(t)$ for at least one t in every $V_\delta(t_0)$, $\delta > 0$. Then F is the aggregate of accumulation of the sets $f(t)$ as $t \to t_0$.

part of $V_\epsilon(x_0)$ for a fixed positive ϵ. Let $x'=f(t')$ and $x''=f(t'')$. Then by similarity, $t'<t_0<t''$. Now for δ sufficiently small $V_\delta(t_0)\subset t't''$. Hence the images of points in $V_\delta(t_0)$ lie in $x'x''$, since $x'<X<x''$, when $t'<t<t''$. But then they lie in $V_\epsilon(x_0)$ and we have continuity. The cases where $t_0=0$ or $t_0=1$ are treated similarly.

Now let t_0 be a point for which the corresponding oscillatory set X_0 contains more than one point. Let x_0 be a point of X_0 which is a common limiting point of the saturated semi-continua X_a and X_b of $ab-X_0$. Then, by § 16, X_a and X_b contain respectively sequences $\{x_i\}$ and $\{x_j\}$ of points of the first genre converging to x_0 and $X_0=Dv[x_ix_j]$. Hence for any $\epsilon>0$ there is an x' and an x'', points of the first genre, such that

$$x' < X_0 < x'' \quad \text{and} \quad X_0 \subset x'x'' \subset V_\epsilon(X_0).$$

Let $x'=f(t')$, $x''=f(t'')$. Then $t'<t_0<t''$. For δ sufficiently small, then, the images of all points in $V_\delta(t_0)$ will lie in $x'x''$ and consequently in $V_\epsilon(X_0)$. Hence, as $t\to t_0$, the points of accumulation of the corresponding points x will be contained in X_0.

Now let x be *any* point in X_0. Let $\epsilon>0$. For any $\delta>0$ there are points t_i and t_j in $V_\delta(t_0)$ such that $t_i<t_0<t_j$ and the images of t_i and t_j are points of the first genre. Let $x_i=f(t_i)$ and $x_j=f(t_j)$. By § 16, X_0 is a continuum of condensation of x_ix_j; hence $V_\epsilon(x)$ contains at least one point x' of x_ix_j which is not on X_0. But if $x'=f(t')$, then t' lies in $V_\delta(t_0)$. Thus each point of X_0 is a point of accumulation of the points $x=f(t)$ as $t\to t_0$, by definition.

This, with the preceding paragraph, completes the proof of (iii).

23. The nature of the correspondence involved in the theorem just proved is made clearer by relating it to the function theory. If $x=f(t)$ is a limited one-valued function defined for a set E of values of t everywhere dense in the interval $a\leqq t\leqq b$ and continuous at each point, it is well known that in general there is no function continuous at every point of ab and equal to $f(t)$ at the points of E, for the reason that, at a point c not in E, $\lim_{t\to c}f(t)$ may not exist. If, however, we agree to set $F(t)=f(t)$ at the points of E and to assign to $F(t)$ at $t=c$ the aggregate of limiting values of $f(t)$ as $t\to c$ over all possible sequences in E and for each point c this aggregate is a continuum, we have a multivalued function possessing many of the attributes of continuity. The graph is easily seen to be a continuum and, for any point c, all of the limits of $F(t)$ as $t\to c$ are values of $F(c)$. Such a function is $x=\sin(1/t)$ for $t\neq 0$, $-1\leqq x\leqq 1$ for $t=0$. If $F(t)$ happens to be one-valued in ab, it is an ordinary continuous function; if $G(t)=F(t)$ at points where the latter is one-valued and at other points $G(t)$ has one of the values of $F(t)$, then $G(t)$ is a one-valued pointwise discontinuous function.

By the method of condensation of singularities* it is easy to build up simple irreducible continua which have an everywhere dense set of points of the second genre. The previous theorem therefore applies, although such continua have only one prime part.

We now turn to the converse theorem.

24. THEOREM. *Let $x=f(t)$ denote a correspondence of the set $C=\{x\}$ to the closed segment $T=\{t\}$ having the following properties:*

(i) *To each t corresponds a point or a point set forming a limited continuum.*

(ii) *No point of C corresponds to more than one point of T.*

(iii) *$f(t)$ is one-valued and continuous at a set of points everywhere dense in T.*

(iv) *At points t' where $f(t)$ is many-valued, $f(t')$ is the aggregate of accumulation of $f(t)$ as $t\to t'$.*

Then C is a simple irreducible continuum.

Proof. That C is limited follows from the definition of the correspondence. It is also easy to see that the image in C of a closed set in T is closed, and vice versa.

It will now be shown that the image in C of a sub-continuum of T is a continuum, and vice versa. Let F be a sub-continuum of T and let G be its image in C. If G is not a continuum, there is a partition $G=G_1+G_2$, where both sets are closed and $G_1 \cdot G_2=0$. Suppose that F_1 and F_2 are the images in T of G_1 and G_2 respectively. F_1 and F_2 are closed. They have no common point t, for the image of t is a continuum or a point and hence must lie wholly in G_1 or G_2. Thus $F_1 \cdot F_2=0$, which contradicts the hypothesis that F is a continuum. Hence G is a continuum.

Conversely, let G be a sub-continuum of C and let F be the subset of T to which correspond† points in G. The set F is closed. If F is not a continuum, we have $F=F_1+F_2$, $F_1 \cdot F_2=0$, and F_1 and F_2 are closed. Then $G=G_1+G_2$, where G_1 and G_2 are closed and $G_1 \cdot G_2=0$, contrary to the hypothesis. Hence, if G is a continuum, so is F.

It will now be shown that, if the image of a point t_0 of T is a single point x_0, then x_0 is a point of the first genre. For every $\epsilon>0$ there is a $\delta>0$ such that x is a point of $V_\epsilon(x_0)$ for every t in $V_\delta(t_0)$. Let $t'<t_0$ and $t''>t_0$ be points in $V_\delta(t_0)$ whose images are single points. Then the image in C of $t't''$ is a continuum contained wholly in $V_\epsilon(x_0)$ while the images of the intervals

* See E. W. Hobson, *Theory of Functions of a Real Variable*, p. 618.

† It should be noted that in general G will be only a part of the set of points of C corresponding to points of F.

0, t' and t'', 1 are continua neither of which contains x_0. Hence for $\eta>0$ sufficiently small, $V_\eta(x_0)$ is contained in the image of $t't''$. Thus for any $\epsilon>0$ there is an $\eta>0$ such that there is a sub-continuum of C lying in $V_\epsilon(x_0)$ and containing $V_\eta(x_0)$. Therefore $\tau(x_0)=0$.

Since the points of C are either unique images of points of T or are limiting points of this set, this shows that the points of the first genre are everywhere dense in C.

Now let $a=f(0)$ and $b=f(1)$. If D is a sub-continuum of C which contains a and b, its image in T must be T itself since it is a continuum containing the points 0 and 1. Hence D must contain all the points of the first genre of C. As these are everywhere dense in C, $D=C$. Hence C is irreducible between a and b.

This completes the proof of the theorem.

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