

# ON APPROXIMATION BY RATIONAL FUNCTIONS TO AN ARBITRARY FUNCTION OF A COMPLEX VARIABLE\*

BY

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1. Introduction. In his classical paper† on approximation to analytic functions of a complex variable by means of polynomials and other rational functions, Runge proved two important theorems:

**THEOREM I.** *If the function  $f(z)$  is analytic in a closed Jordan region of the  $z$ -plane, then in that closed region  $f(z)$  can be approximated as closely as desired by a polynomial in  $z$ .*

**THEOREM II.** *If the function  $f(z)$  is analytic in a closed region  $R$  of the  $z$ -plane which is bounded by a finite number  $n$  of Jordan curves of which no two have a common point, then in that closed region  $f(z)$  can be approximated as closely as desired by a rational function of  $z$ .*

Cauchy's Integral for the function and region (if the Jordan curves bounding  $R$  are rectifiable) in Theorem II splits up the function  $f(z)$  into  $n$  functions. If Theorem I is applied to each of these in turn (a simple transformation  $w = 1/(z - \alpha_i)$  is first necessary for  $n - 1$  of them), we find Theorem II immediately.

If the Jordan curves bounding  $R$  are not rectifiable, we readily obtain the same result. Consider an auxiliary variable region  $R'$  of the same connectivity as  $R$ , which lies within  $R$ , which is bounded by  $n$  rectifiable Jordan curves, and which approaches  $R$  as the bounding curves change continuously. For the region  $R'$ , Cauchy's Integral for  $f(z)$  splits that function into  $n$  functions *which do not change as  $R'$  varies*. Each of these  $n$  functions is, when properly defined, analytic not merely in  $R$  but in a region bounded by a single Jordan curve and which contains  $R$ . Theorem II then follows as before from Theorem I.

On the possibility of approximation of arbitrary functions by polynomials, no deeper lying results than Theorems I and II, except for approximation by certain *restricted* types of polynomials, appear to have been found until

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† Acta Mathematica, vol. 6 (1885), pp. 229-244.

recently, when the problem was attacked by use of the modern theory of conformal mapping. We mention generalizations of Theorems I and II:\*

**THEOREM III.** *If the function  $f(z)$  is analytic interior to a Jordan curve and continuous in the corresponding closed region, then in that closed region  $f(z)$  can be approximated as closely as desired by a polynomial in  $z$ .*

**THEOREM IV.** *If the function  $f(z)$  is analytic interior to a region which is bounded by a finite number of Jordan curves of which no two have a common point, and is continuous in the corresponding closed region, then in that closed region  $f(z)$  can be approximated as closely as desired by a rational function of  $z$ .*

These theorems refer to functions which are analytic in some region, but there are similar results which hold for functions known merely to be continuous:†

**THEOREM V.** *If the function  $f(z)$  is continuous on a Jordan arc, then on that arc  $f(z)$  can be approximated as closely as desired by a polynomial in  $z$ .*

**THEOREM VI.** *If the function  $f(z)$  is continuous on a Jordan curve in whose interior the origin lies, then on that curve  $f(z)$  can be approximated as closely as desired by a polynomial in  $z$  and  $1/z$ .*

Theorem V is a direct generalization of Weierstrass's classical theorem on the possibility of approximation by a polynomial to a real continuous function in a given interval of the axis of reals, and Theorem VI is a generalization of Weierstrass's theorem on the possibility of approximation to a real continuous function in an interval of the axis of reals by trigonometric polynomials.‡ For if the Jordan curve of Theorem VI is the unit circle, we have

$$\begin{aligned} z^n &= \cos n\theta + i \sin n\theta, & z^{-n} &= \cos n\theta - i \sin n\theta, \\ \sin n\theta &= \frac{z^n - z^{-n}}{2i}, & \cos n\theta &= \frac{z^n + z^{-n}}{2}, \end{aligned}$$

so that every trigonometric polynomial is a polynomial in  $z$  and  $1/z$ , and conversely.

2. **Principal results to be obtained.** In the present paper we shall establish a result which is of great generality and includes all the results

\* Walsh, *Mathematische Annalen*, vol. 96 (1926), pp. 430-436.

† Walsh, *Mathematische Annalen*, vol. 96 (1926), pp. 437-450.

‡ To be sure, Weierstrass considers merely real functions and real approximations, but there is no difficulty in proceeding from a complex approximation to a real function to a real approximation to that function and vice versa.

just mentioned. The proof is based primarily on results already attained, and uses in addition only a few elementary transformations of the plane.

We notice that uniform approximation with an arbitrarily small error is equivalent to uniform expansion in series, and frequently the latter language is more convenient for us. As a matter of convention, the point at infinity is to be considered adjoined to the plane whenever we are dealing with approximation by general rational functions (i.e., not by polynomials), but is not to be so considered when we are dealing with approximation by polynomials, even though these polynomials may be, for instance, polynomials in  $z$  and  $1/z$ . When we are dealing with rational functions, a Jordan region is thus an arbitrary region *on the sphere* (stereographic projection of the plane) bounded by a simple closed curve. Likewise as a matter of notation, if a region or other open point set is denoted by such a symbol as  $C$  or  $R$ , then the corresponding closed region or closed point set is denoted by  $\bar{C}$  or  $\bar{R}$ .\*

Suppose that on a given closed point set it is desired to approximate by rational functions a given function  $f(z)$  analytic on that closed point set. This point set may be so closely approximated by a region bounded entirely by Jordan curves, of which no two have a common point, that  $f(z)$  is analytic in this new closed region, and so that the given point set is contained in this new region. Theorem II yields directly

**THEOREM VII.** *An arbitrary function  $f(z)$  analytic on a closed point set can be uniformly approximated on that point set as closely as desired, by a rational function of  $z$ .*

If we do not assume analyticity of the function on the closed point set, we shall have to content ourselves with a less general point set:

**THEOREM VIII.** *Let  $M$  be a closed point set whose boundary consists of a finite number of Jordan arcs which do not separate the plane into more than a finite number of regions. Then an arbitrary function  $f(z)$  continuous on  $M$  and analytic in the interior points of  $M$  can be uniformly approximated on  $M$  as closely as desired by a rational function of  $z$ .*

*If  $M$  divides the plane into precisely  $k$  regions none of which belongs to  $M$ , this rational function can be chosen so that its only poles lie in  $k$  preassigned points, one in each of these regions.*

This is the most interesting result of the present paper; most of the remainder of the paper is devoted to its proof. Theorem VIII contains Theorems I-VI as special cases. Another special case worth mentioning is

\* For typographical reasons, if the given region is  $C'$  or  $C_1$ , the corresponding closed region will be denoted by  $\bar{C}'$  or  $\bar{C}_1$ , the bar not extending over the subscript or superscript.

**THEOREM IX.** *If  $M$  is a closed point set composed of a finite number of Jordan arcs which divide the plane into at most a finite number of regions, then an arbitrary function continuous on  $M$  can be uniformly approximated on  $M$  as closely as desired by a rational function of  $z$ .*

We state merely for the sake of reference two other results on approximation which have been previously proved and are to be used in the sequel:

**THEOREM X.** *If on the closed point set  $M$  the function  $f(z)$  can be uniformly approximated as closely as desired by a rational function, and if there be chosen  $n$  arbitrary points of  $M$ , then there exists a rational function which on  $M$  approximates  $f(z)$  as closely as desired and which coincides with  $f(z)$  in these  $n$  points.\**

**THEOREM XI.** *If the point set  $M$  consists of the closed interiors of two Jordan regions with but a single common point, then an arbitrary function  $f(z)$  analytic in the interior points of  $M$  and continuous on  $M$  can be uniformly approximated on  $M$  as closely as desired by a polynomial in  $z$ .†*

We proceed now to prove a sequence of theorems leading up to Theorem VIII.

3. **Regions bounded by two Jordan curves.** We shall prove the following theorem:

**THEOREM XII.** *Let the Jordan curve  $C_2$  contain the origin in its interior and itself lie interior to the Jordan curve  $C_1$ , except that  $C_1$  and  $C_2$  have the single point  $z = \alpha$  in common. Denote by  $R$  the region bounded by  $C_1$  and  $C_2$ . Then an arbitrary function  $f(z)$  analytic in  $R$  and continuous in  $\bar{R}$  can be developed in  $\bar{R}$  uniformly in a series of polynomials in  $z$  and  $1/z$ .*

We shall prove the theorem first in a particular case, but the general case can be reduced to this particular case by a number of intermediate steps which we treat in detail.

**Case I.** *The curves  $C_1$  and  $C_2$  have continuously turning tangents and continuous curvature, and the function  $f(z)$  satisfies a Lipschitz condition‡ on  $C_1$  and  $C_2$ . The function  $f(z)$  can be represented in  $R$  by Cauchy's Integral*

\* Walsh, these Transactions, vol. 30 (1928), pp. 307-332; Theorem X. The theorem is there proved for the case of approximation by polynomials, but the proof holds, after a suitable transformation  $w = 1/(z - \alpha)$  if  $M$  contains the point at infinity, also in the present case, even if  $f(z)$  becomes infinite on  $M$ . We notice, as an essential part of the proof, that the theorem remains true if  $M$  is the entire plane, for in that case  $f(z)$  must itself be a rational function.

† Walsh, these Transactions, vol. 30 (1928), pp. 472-482; Theorem III.

‡ Here may be taken in the form  $|f(z_1) - f(z_2)| \leq M|z_1 - z_2|$ ,  $M$  fixed,  $z_1$  and  $z_2$  on  $C_1$  or on  $C_2$ .

$$(1) \quad f(z) = f_1(z) - f_2(z),$$

$$f_1(z) = \frac{1}{2\pi i} \int_{C_1} \frac{f(t)dt}{t-z}, \quad f_2(z) = \frac{1}{2\pi i} \int_{C_2} \frac{f(t)dt}{t-z},$$

where both integrals are taken in the positive (i.e., counterclockwise) sense on the curves. But with the present restrictions on the curves and on the function  $f(z)$ , it follows from a theorem due to Plemelj\* that the function  $f_1(z)$  just defined is analytic interior to  $C_1$  and continuous in the corresponding closed region (when properly defined on  $C_1$ ) and that the function  $f_2(z)$  is analytic exterior to  $C_2$  and continuous in the corresponding closed region, even at infinity, when it is properly defined on  $C_2$  and at infinity. By virtue of Theorem III† the function  $f_1(z)$  can be expanded uniformly in the closed interior of  $C_1$  by polynomials in  $z$ , and the function  $f_2(z)$  can be expanded uniformly in the closed exterior of  $C_2$  by polynomials in  $1/z$ . Equation (1) holds on the curves  $C_1$  and  $C_2$ , for the functions  $f(z)$ ,  $f_1(z)$ ,  $f_2(z)$  are all continuous in  $\bar{R}$ , so our theorem is completely established in Case I.

In this particular case the given function  $f(z)$  can be expressed in  $\bar{R}$  as the sum of two functions, which in  $\bar{R}$  can be respectively represented as a series of polynomials in  $z$  and as a series of polynomials in  $1/z$ . This simple result is not to be expected in the more general case.

**Case II.** *The curves  $C_1$  and  $C_2$  are tangent at  $z=\alpha$  and circles can be drawn tangent to them at  $z=\alpha$  which contain respectively no point of the interior of  $C_1$  and no point of the exterior of  $C_2$ .* Here there can be drawn a Jordan curve  $C_1'$  with a continuously turning tangent and continuous curvature which is tangent to  $C_1$  at  $z=\alpha$  and contains all other points of  $C_1$  in its interior. There can likewise be drawn a Jordan curve  $C_2'$  which has a continuously turning tangent and continuous curvature which is tangent to  $C_2$  at  $z=\alpha$ , whose exterior contains all other points of  $C_2$ , and which contains the origin in its interior.‡ We shall show how there may be defined a function  $\phi(z)$  which differs from  $f(z)$  in  $\bar{R}$  by an amount less than any preassigned positive quantity  $\epsilon$  and which is analytic in the region  $R'$  bounded by  $C_1'$  and  $C_2'$ , contin-

\* Monatshefte für Mathematik und Physik, vol. 19 (1908), pp. 205-210.

† In connection with  $f_2(z)$  we need to make a reciprocal transformation  $w=1/z$  or to apply the theorem given by the present writer, *Mathematische Annalen*, vol. 96 (1926), p. 435.

‡ The curves  $C_1'$  and  $C_2'$  are readily drawn, say by using arcs of the two circles mentioned in the hypothesis on  $C_1$  and  $C_2$ , by using broken lines in conjunction with those arcs to construct the curve desired, finally by smoothing out the corners. The latter process is readily accomplished, so far as concerns the intersection of two straight lines, by the use of arches of the curve  $y=a \sin bx$ ,  $0 \leq x \leq \pi/b$ , where  $a$  and  $b$  are suitably chosen. The same curve (but not an entire arch) can also be used to smooth out the intersection of a straight line and a circle.

uous in the corresponding closed region, and which satisfies a Lipschitz condition on the curves  $C'_1$  and  $C'_2$ . Transform the plane by means of the transformation  $w = z^{1/2}$ , so that the regions  $R$  and  $R'$  are transformed into regions  $S$  and  $S'$  respectively. In reality each of the regions  $R$  and  $R'$  is of course transformed into *two* regions in the  $w$ -plane, but we fasten our attention merely on one pair of such regions, resulting from transformation by a single branch of the function  $w$ . It follows from Theorem III that when  $\epsilon > 0$  is given, there exists a function  $\Phi(w)$  analytic in  $\bar{S}'$  and which differs in  $\bar{S}$  from  $f(w^2)$  by less than  $\epsilon$ . This function can be chosen, by Theorem X, so that the two values  $\Phi(+\alpha^{1/2})$  and  $\Phi(-\alpha^{1/2})$  are equal, in fact both equal to  $f(a)$ . We transform back to the  $z$ -plane by the transformation  $z = w^2$ , and consider the  $z$ -plane cut along a curve from the origin to the point  $z = \alpha$ , the curve having no point in common with (the open region)  $R'$ . The function  $\Phi(w)$  is transformed into a function  $\phi(z) = \Phi(z^{1/2})$ , where  $z^{1/2}$  is a properly determined branch of that function, and this function  $\phi(z)$  is single-valued in  $\bar{R}'$  and has in  $\bar{R}'$  the properties mentioned in the requirements for Case I. We notice that  $\phi(z)$  is analytic in every point of  $\bar{R}'$  except  $z = \alpha$  and continuous in that closed region, but we omit the details of the proof that  $\phi(z)$  satisfies a Lipschitz condition on  $C'_1$  and on  $C'_2$ . We have Case I, however, and the function  $\phi(z)$  can be uniformly approximated in  $\bar{R}'$  and hence in  $\bar{R}$  as closely as desired by means of a polynomial in  $z$  and  $1/z$ ; therefore the function  $f(z)$ , which differs from  $\phi(z)$  in  $\bar{R}$  by less than  $\epsilon$ , can in  $\bar{R}$  be uniformly approximated as closely as desired by such a polynomial, and Case II of the theorem is established.

**Case III. *The general case.*** By a series of transformations we proceed to bring the general case under Case II. We assume, as we may do with no loss of generality, that  $|z - \alpha| < 1$  for all  $z$  in  $\bar{R}$ . The first transformation to be used is

$$z - \alpha = e^{1/(w-\alpha)}$$

which transforms  $R$  into an infinite number of regions in the  $w$ -plane, of which we consider a single region  $R_1$ , necessarily of like connectivity to  $R$ . That is,  $R_1$  is bounded by two Jordan curves with the single point  $w = \alpha$  in common. The effect of this transformation is perhaps best seen by breaking it up into the two transformations  $z - \alpha = e^{z'}$ ,  $z' = 1/(w - \alpha)$ . The former transformation carries  $z = \alpha$  into  $z' = \infty$ , but any transform of  $R$  onto the  $z'$ -plane lies in the half-plane to the left of the axis of imaginaries, for  $|z - \alpha| < 1$  in  $R$ . The transformation  $z' = 1/(w - \alpha)$  carries the axis of imaginaries in the  $z'$ -plane into a straight line through  $w = \alpha$ , so the new transform of  $R$  (i.e. in the  $w$ -plane) is a region  $R_1$  of like connectivity to  $R$ , as stated. More-

over the region  $R_1$  is finite and lies in a half-plane bounded by a line through  $w = \alpha$ .

We now use again this same transformation, setting

$$w - \alpha = e^{1/(u-\alpha)}.$$

Under the first part of this transformation,  $w - \alpha = e^{w'}$ , the region  $R_1$  is transformed into a region  $\Sigma$  of the  $w'$  plane (we consider only one of the infinitely many transforms of  $R_1$ ) which lies not merely in a half-plane bounded by a parallel to the axis of imaginaries, but also in a strip of width  $\pi$  bounded by two parallels to the axis of reals. Then the transformation  $w' = 1/(u-\alpha)$  carries the region  $\Sigma$  into a region  $R_2$  which lies between two circles which are tangent at the point  $u = \alpha$ . The region  $R_2$  is of course finite, for  $u = \infty$  corresponds to  $w - \alpha = 1$ ,  $z - \alpha = e$ , which is impossible in  $\bar{R}$ . The region  $R_2$  lies, in the neighborhood of  $u = \alpha$ , entirely on one side of the common perpendicular to these circles at  $u = \alpha$ . Suppose  $R_2$  to be bounded by the Jordan curves  $\Gamma_1$  and  $\Gamma_2$ , with  $\Gamma_2$  interior, except for the single common point  $u = \alpha$ , to the curve  $\Gamma_1$ .

We transform now to the  $t$ -plane by setting

$$\frac{t - \alpha}{t} = \left( \frac{u - \alpha}{u - \beta} \right)^{1/2},$$

where  $\beta$  is an arbitrarily chosen point interior to  $\Gamma_2$ . If the  $u$ -plane is considered cut along a Jordan arc joining  $\alpha$  and  $\beta$  and which contains no point of  $\bar{R}_2$  other than  $\alpha$ , the region  $R_2$  is transformed into two regions of the  $t$ -plane, of which we consider a single finite region  $T$ . The region  $T$  is bounded by two Jordan curves  $C'_1$  and  $C'_2$ , so that  $C'_2$  is interior to  $C'_1$  except for the single point  $t = \alpha$  which the two curves have in common. Moreover a circle can be drawn tangent to  $C'_1$  and  $C'_2$  at  $t = \alpha$  which contains no point of the interior of  $C'_1$ , and a circle can be drawn tangent to  $C'_1$  and  $C'_2$  at  $t = \alpha$  which contains no point of the exterior of  $C'_2$ . In fact the transformation just used is, except for linear transformation of the two planes which transforms circles into circles, of the form

$$y = x^{1/2}.$$

A circle in the  $x$ -plane tangent to the axis of reals at the origin is transformed into a curve\* one of whose branches in the neighborhood of the origin lies between that circle and the axis of reals. For under the transformation

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\* A lemniscate as a matter of fact, but for our present purpose we do not need such detailed information.

$y = x^{1/2}$ , the argument of the complex quantity  $x$  is halved, the modulus raised to the power one-half.

The transformation  $x = (u - \alpha)/(u - \beta)$  is here used to transform  $R_2$  into a region  $R'_2$  of slightly different type. If a point tracing either  $\Gamma_1$  or  $\Gamma_2$  leaves the curve at  $u = \alpha$  and traces the tangent without reversing the sense of motion, the point moves *exterior* to  $\Gamma_1$  and  $\Gamma_2$ . If a point traces either bounding curve of  $R'_2$ , leaves the curve at  $x = 0$  and traces the tangent without reversing the sense of motion, the point moves *interior* to those curves. Such a transformation as the one used is necessary to ensure that  $R'_2$  is the latter of these two types of regions instead of the former, giving us the proper region  $T$ .

The region  $T$  satisfies the conditions imposed in Case II, except that we do not know that the origin  $t = 0$  lies interior to  $C'_2$ . Hence the transform of  $f(z)$ , which is analytic in  $T$  and continuous in  $\bar{T}$ , can be uniformly developed in  $\bar{T}$  in a series of polynomials in  $t$  and  $1/(t - \gamma)$ , where  $\gamma$  is a suitably chosen point interior to  $C'_2$ . But in the  $u$ -plane, the functions  $t$  and  $1/(t - \gamma)$  (i.e., suitably chosen branches) are analytic in a region bounded by  $\Gamma_2$  and an arbitrary Jordan curve containing  $\Gamma_1$ , and are continuous in the corresponding closed region. Hence in that closed region, which contains  $\bar{R}_2$ , the functions  $t$  and  $1/(t - \gamma)$  can each be developed into a uniformly convergent series of polynomials in  $u$  and  $1/(u - \beta)$ . Thus the transform of  $f(z)$  can be developed uniformly in such a series of polynomials.

In our transformation from the  $z$ -plane to the  $u$ -plane it is convenient to think of the  $z$ -plane as cut along a Jordan curve from the point  $z = \alpha$  to infinity; we suppose that this curve has no point other than  $z = \alpha$  in common with  $\bar{R}$ . Then when  $u$  is properly chosen as a branch of the function of  $z$  defined by

$$z - \alpha = e^{1/(w - \alpha)}, \quad w - \alpha = e^{1/(u - \alpha)},$$

the functions  $u$  and  $1/(u - \beta)$  are analytic in an annular region which is bounded by  $C_1$  and by a second Jordan curve interior to  $C_2$  and are continuous in the corresponding closed region. This annular region is thus chosen so that it contains  $R$ , but is of course so chosen that it does not contain the transform of  $u = \beta$ . In the closed annular region the functions  $u$  and  $1/(u - \beta)$  can be uniformly expanded in series of polynomials in  $z$  and  $1/z$ ; this expansion is valid likewise in  $\bar{R}$ , so  $f(z)$  itself can in  $\bar{R}$  be uniformly expanded in a series of polynomials in  $z$  and  $1/z$ , and the theorem is completely established.

4. **Regions bounded by several Jordan curves.** Theorem XII is to be used in proving a more general result:

**THEOREM XIII.** *If the region  $R$  is bounded by a finite number of Jordan curves, then an arbitrary function  $f(z)$  analytic in  $R$  and continuous in  $\bar{R}$  can be uniformly expanded in  $\bar{R}$  in a series of rational functions of  $z$ .*

Our proof here is by induction. We suppose the theorem true for a region bounded by  $n$  Jordan curves and shall prove the theorem true for a region bounded by  $n+1$  such curves. Any region bounded by two Jordan curves is included either under Theorem IV or Theorem XII, so the induction is started.

Suppose  $R$  to be the region interior to a Jordan curve  $C_0$  and exterior to the Jordan curves  $C_1, C_2, \dots, C_n$ . If none of these latter curves has a point of intersection with  $C_0$ , we proceed as in Theorem II. Cauchy's Integral enables us to express the given function  $f(z)$  as the sum of two functions, the one analytic interior to  $C_0$  and continuous in the corresponding closed region, the other function analytic exterior to  $C_1, C_2, \dots, C_n$  and continuous in the corresponding closed region. Each of these functions can be expanded in a series of rational functions uniformly convergent in the closed regions indicated, so the same is true of  $f(z)$  in  $\bar{R}$ .

We assume, then, as we may do with no loss of generality, that say  $C_1$  has a point  $z=\alpha$  in common with  $C_0$ . Let  $\beta$  be a point interior to  $C_1$ , consider the  $z$ -plane cut from  $\beta$  to infinity along a Jordan arc which has no point other than  $\alpha$  in common with  $\bar{R}$ , and transform  $\bar{R}$  by means of the transformation  $z-\beta=w^2$ . A properly chosen branch of the function  $w$  transforms  $\bar{R}$  into a region  $\bar{R}'$  and  $f(z)$  into a function  $\phi(w)$  analytic interior to  $R'$  and continuous in  $\bar{R}'$ . The region  $R'$  is bounded by the transforms  $C'_2, C'_3, \dots, C'_n$  of the curves  $C_2, C_3, \dots, C_n$ , and by the Jordan curve  $C'_0+C'_1$ , the transform of the curves  $C_0$  and  $C_1$ . If an arbitrary  $\epsilon>0$  be given, then by our theorem as assumed for a region bounded by  $n$  Jordan curves, there exists a rational function  $\Phi(w)$  which is analytic in  $\bar{R}'$ , which in  $\bar{R}'$  differs from  $\phi(w)$  by a function in absolute value not greater than  $\epsilon$ , and such that  $\Phi+(\alpha-\beta)^{1/2}=\Phi(-(\alpha-\beta)^{1/2})$ ; this last fact results from Theorem X.

The function  $\Phi(w)$  is therefore analytic in a region  $\bar{S}'$  bounded by  $C'_0+C'_1$  and by certain Jordan curves  $\Gamma'_2, \Gamma'_3, \dots, \Gamma'_n$  which are *interior* respectively to  $C'_2, C'_3, \dots, C'_n$ . We transform now back to the  $z$ -plane; the function  $\Phi(w)$  goes into a function  $F(z)$  which differs from  $f(z)$  in  $\bar{R}$  by at most  $\epsilon$ , and this function  $F(z)$  is analytic interior to a region (which includes  $R$ ) bounded by  $C_0, C_1$ , and by the transforms  $\Gamma_2, \Gamma_3, \dots, \Gamma_n$  of  $\Gamma'_2, \Gamma'_3, \dots, \Gamma'_n$  respectively. The function  $F(z)$  is continuous in the corresponding closed region. Cauchy's Integral enables us as in Theorem II to express  $F(z)$  in  $\bar{R}$  as the sum of  $n-1$  functions analytic respectively in

the closed regions exterior to  $\Gamma_2, \Gamma_3, \dots, \Gamma_n$  and analytic in the region bounded by  $C_0$  and  $C_1$ , continuous in the corresponding closed region. Each of these functions forming the sum is expressible in the closed region corresponding to it as a uniformly convergent series of rational functions of  $z$ . Hence  $F(z)$  and  $f(z)$  are so expressible in  $\bar{R}$ .

5. **Two regions with common points.** Somewhat similar to Theorem XIII is

**THEOREM XIV.** *Let the point set  $R$  be composed of the interiors of two Jordan curves  $C_1$  and  $C_2$  which lie exterior to each other except for  $n$  points which they have in common. Then an arbitrary function  $f(z)$  analytic in  $R$  and continuous in  $\bar{R}$  is uniformly developable in  $\bar{R}$  in a series of rational functions of  $z$ .*

The case  $n=1$  is essentially the case of Theorem XI. The proof for the general case is by induction, assuming the result true for  $n$  common points and proving the result for  $n+1$  common points. The proof is moreover so similar to the proof of Theorem XIII that it will be merely sketched. The point set  $R$  may be considered interior to the bounding Jordan curve  $D_0$  and exterior to the bounding Jordan curves  $D_1, D_2, \dots, D_n$ ; each of these curves is composed of an arc of  $C_1$  and an arc of  $C_2$ . Assume  $D_1$  to have a point  $z=\alpha$  in common with  $D_0$  and choose a point  $\beta$  interior to  $D_1$ . Cut the  $z$ -plane as before and transform by setting  $z-\beta=w^2$ . The present theorem for  $n$  common points yields a function differing only slightly from  $f(z)$ , analytic in the transform of the point set  $R$ , taking on the same values in the two points  $+(\alpha-\beta)^{1/2}$  and  $-(\alpha-\beta)^{1/2}$ . Transformation back to the  $z$ -plane gives a function differing in  $\bar{R}$  only slightly from  $f(z)$ , and analytic in  $\bar{R}$  except possibly at  $\alpha$ . This new function is analytic interior to a region bounded by  $D_0$  and  $D_1$ , and by curves  $\Delta_2, \Delta_3, \dots, \Delta_n$  respectively interior to  $D_2, D_3, \dots, D_n$ , and is continuous in the corresponding closed region. This function is therefore the sum of  $n$  functions, analytic in the regions bounded by  $D_0$  and  $D_1$ , and by  $\Delta_2, \Delta_3, \dots, \Delta_n$  respectively, and continuous in the corresponding closed regions. Each of these  $n$  functions can be expanded in  $\bar{R}$  in a series of rational functions of  $z$ , so that is true also of  $f(z)$ .

6. **Approximation in two regions with common points.** Theorem XIV leads almost at once to

**THEOREM XV.** *Let the two Jordan curves  $C_1$  and  $C_2$  lie exterior to each other except for the  $k$  points  $\alpha_1, \alpha_2, \dots, \alpha_k$  which they have in common. Let functions  $f_1(z)$  and  $f_2(z)$  defined in  $\bar{C}_1$  and  $\bar{C}_2$  respectively or more generally on point sets contained in  $\bar{C}_1$  and  $\bar{C}_2$  be uniformly developable in  $\bar{C}_1$  and  $\bar{C}_2$  re-*

spectively (or on those point sets) in rational functions of  $z$ . Let moreover both  $f_1(\alpha_i)$  and  $f_2(\alpha_i)$  be defined and let the equations

$$f_1(\alpha_i) = f_2(\alpha_i) \quad (i = 1, 2, \dots, k)$$

hold. Then there exists a series of rational functions of  $z$  converging to the sum  $f_1(z)$  uniformly in  $\bar{C}_1$  (or on the sub-set) and to the sum  $f_2(z)$  uniformly in  $\bar{C}_2$  (or on the sub-set).

We write the given sequences in the form

$$f_1(z) = \lim_{n \rightarrow \infty} (f'_n(z) + \phi'_n(z)), \text{ uniformly on } \bar{C}_1 \text{ or sub-set,}$$

$$f_2(z) = \lim_{n \rightarrow \infty} (f''_n(z) + \phi''_n(z)), \text{ uniformly on } \bar{C}_2 \text{ or sub-set,}$$

where the poles of  $f'_n(z)$  and  $f''_n(z)$  lie in  $\bar{C}_1$  and  $\bar{C}_2$  respectively, and the poles of  $\phi'_n(z)$  and  $\phi''_n(z)$  lie outside of  $\bar{C}_1$  and  $\bar{C}_2$  respectively. The two functions

$$\phi'_n(z) - f''_n(z), \quad \phi''_n(z) - f'_n(z)$$

are analytic in  $\bar{C}_1$  and  $\bar{C}_2$  respectively, and their difference approaches the value zero at each of the points  $z = \alpha_i$ . Let  $\epsilon_n$  be the largest

$$| [\phi'_n(\alpha_i) - f''_n(\alpha_i)] - [\phi''_n(\alpha_i) - f'_n(\alpha_i)] |, \quad \text{for } i = 1, 2, \dots, k,$$

so that  $\epsilon_n$  approaches zero as  $n$  becomes infinite. Then there exists\* a rational function  $\Phi_n(z)$ ,  $n = 1, 2, \dots$ , analytic in  $\bar{C}_2$  such that

$$\Phi_n(\alpha_i) = \phi'_n(\alpha_i) - f''_n(\alpha_i),$$

$$| \Phi_n(z) - [\phi''_n(z) - f'_n(z)] | < N\epsilon_n,$$

uniformly for  $z$  in  $\bar{C}_2$ , where  $N$  is a suitable number independent of  $n$ .

The function

$$\begin{cases} \phi'_n(z) - f''_n(z), & z \text{ in } \bar{C}_1, \\ \Phi_n(z), & z \text{ in } \bar{C}_2, \end{cases}$$

is analytic interior to  $C_1$  and interior to  $C_2$  and continuous on the entire closed point set  $\bar{C}_1 + \bar{C}_2$ , so by Theorem XIV there exists a rational function  $r_n(z)$  such that we have

$$| r_n(z) - [\phi'_n(z) - f''_n(z)] | \leq \epsilon_n, \quad z \text{ in } \bar{C}_1,$$

$$| r_n(z) - \Phi_n(z) | \leq \epsilon_n, \quad z \text{ in } \bar{C}_2.$$

Then the rational function

$$R_n(z) = r_n(z) + f'_n(z) + f''_n(z)$$

satisfies the equation

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\* See the proof of Theorem X, loc. cit.

$$R_n(z) - [f'_n(z) + \phi'_n(z)] = r_n(z) - [\phi'_n(z) - f'_n(z)];$$

the right-hand member is in absolute value not greater than  $\epsilon_n$  on  $\bar{C}_1$ , so that  $R_n(z)$  approaches  $f_1(z)$  as  $n$  becomes infinite, uniformly on  $\bar{C}_1$  or on the sub-set of  $\bar{C}_1$  on which  $f_1(z)$  is defined. We have similarly

$$\begin{aligned} R_n(z) - [f''_n(z) + \phi''_n(z)] &= r_n(z) - [\phi''_n(z) - f''_n(z)] \\ &= [r_n(z) - \Phi_n(z)] + \{\Phi_n(z) - [\phi''_n(z) - f''_n(z)]\}, \end{aligned}$$

which is in absolute value not greater than  $(N+1)\epsilon_n$  on  $\bar{C}_2$ , so that  $R_n(z)$  approaches  $f_2(z)$  as  $n$  becomes infinite, uniformly on  $\bar{C}_2$  or on the sub-set on which  $f_2(z)$  is defined. This completes the proof.

It will be noticed that we have actually proved more than is stated formally in the theorem, for we have shown

$$\begin{aligned} R_n(z) - [f'_n(z) + \phi'_n(z)] &\rightarrow 0 \quad \text{uniformly for } z \text{ in } \bar{C}_1, \\ R_n(z) - [f''_n(z) + \phi''_n(z)] &\rightarrow 0 \quad \text{uniformly for } z \text{ in } \bar{C}_2. \end{aligned}$$

If the two Jordan curves  $C_1$  and  $C_2$  lie exterior to each other, having no point in common, and if  $f_1(z)$  and  $f_2(z)$  defined on  $\bar{C}_1$  and  $\bar{C}_2$  or on sub-sets thereof are uniformly developable on those point sets in series of rational functions, then there exists a sequence of rational functions which approaches  $f_1(z)$  uniformly on  $\bar{C}_1$  and  $f_2(z)$  uniformly on  $\bar{C}_2$  or on the point sets thereof. This can be proved easily by the methods of Runge, or indeed by means of Theorem XV.

We remark incidentally that Theorem XV enables us to prove Theorem VI by means of Theorem V; this remark is of some interest in virtue of the fact that the original proof of Theorem V is by means of Theorem VI.

#### 7. Proof of Theorem VIII. As a matter of convenience we state

**THEOREM XVI.** *Let the regions  $R_1, R_2, \dots, R_n$ , each bounded by a finite number of Jordan curves, be mutually exclusive, and let no two of the corresponding closed regions have more than a finite number of points in common. Then if the function  $f_i(z)$  defined in  $\bar{R}_i$  or more generally on a sub-set of  $\bar{R}_i$  which contains the points common to  $\bar{R}_i$  and the other regions  $\bar{R}_k$ , is in  $\bar{R}_i$  (or on the sub-set) uniformly developable in a series of rational functions of  $z$ , for  $i=1, 2, \dots, n$ , and if the function*

$$f(z) = f_i(z), \quad z \text{ in } \bar{R}_i \text{ or on the prescribed sub-set of } \bar{R}_i,$$

*is uniquely defined and continuous on  $\bar{R}_1 + \bar{R}_2 + \dots + \bar{R}_n$  or on the corresponding sub-set, then the function  $f(z)$  is developable in a uniformly convergent series of rational functions of  $z$  on  $\bar{R}_1 + \bar{R}_2 + \dots + \bar{R}_n$  or on the sub-set of this latter set on which  $f(z)$  is defined.*

The proof is by induction; the case  $n=2$  follows immediately from Theorem XV if  $\bar{R}_1$  and  $\bar{R}_2$  have common points, and from the method of Runge in the contrary case. Assume the theorem true for  $n-1$  regions  $R_i$ ; we shall prove it true for  $n$  such regions.

Let  $C$  be any Jordan curve which bounds completely or partially a region  $R_i$  and which separates two of the regions  $R_1, R_2, \dots, R_n$ . Then there exists another Jordan curve  $C'$ , which has no points in common with (the interior points of)  $R_1+R_2+\dots+R_n$ , which has not more than a finite number of points in common with  $C$ , each of these a point of definition of  $f(z)$ , and such that no region  $R_i$  lies between  $C$  and  $C'$ . To the given regions interior to  $C$  (or exterior to  $C$ , if the interior of  $C$  contains all but a finite number of points of  $C'$ ) we apply the present theorem, for the number of regions concerned is less than  $n$ , and we proceed similarly for the given regions interior to  $C'$  (or exterior to  $C'$ , if the interior of  $C'$  contains all but a finite number of points of  $C$ ). Then we apply Theorem XV to the interior of  $C$  and the interior of  $C'$  (or to the exteriors in the respective cases described), the sub-sets here being the given regions or their sub-sets, which completes the proof in case  $C$  and  $C'$  actually possess common points. If  $C$  and  $C'$  do not possess common points, the method of Runge instead of Theorem XV yields the conclusion desired.

Theorem VIII now follows from Theorem XVI. The point set  $M$  consists of a finite number of closed regions, each bounded by a finite number of Jordan curves, and of a finite number of Jordan arcs  $A$  exterior to those regions; if  $M$  is composed partly of Jordan curves not bounding regions which belong to  $M$ , we consider each of these curves broken up into a finite number of Jordan arcs. The entire set of Jordan arcs may be considered, by a suitable change of notation if necessary, to have no points other than end points common to any two of them, or common to a Jordan arc and any one of the Jordan curves which bound regions belonging to  $M$ . Then each Jordan arc  $A$  can be enclosed in a new region bounded by a new Jordan curve whose interior (on the sphere) contains the interior points of  $A$  and no other points of  $M$ , and so that the new curve has no points in common with  $M$  or with other such new curves except such end points of  $A$  as belong likewise to other Jordan arcs or curves belonging to  $M$ .

These new Jordan regions, together with the regions originally belonging to  $M$ , form a set of regions to which Theorem XVI can be applied immediately; the given function  $f(z)$  is of course not defined on the entire closed point set consisting of these regions, only on the sub-set  $M$ , but the possibility of approximating  $f(z)$  on  $M$  follows at once.

The second part of Theorem VIII can be obtained by the following

reasoning. Any rational function continuous on  $M$  has its poles in the  $k$  regions into which  $M$  divides the plane, and can be separated into partial fractions each of which has merely a single pole. If  $R_1$  denotes one of those  $k$  regions and  $z_1$  the chosen point of  $R_1$ , any rational function with no poles outside of  $R_1$  may, by Runge's theorem (i.e., Theorem I, after transformation  $z' = 1/(z - z_1)$  if necessary), be approximated on  $M$  as closely as desired by a rational function with but a single pole and that in  $z_1$ . Thus all component parts of the rational functions approximating to  $f(z)$  can be approximated on  $M$  by rational functions with poles in the preassigned points, so that fact is true of  $f(z)$  itself. In particular if  $k = 1$  and  $M$  does not contain the point at infinity,  $f(z)$  can be approximated on  $M$  as closely as desired by a polynomial in  $z$ . This remark (second part of Theorem VIII) is indeed entirely general in its application, and refers to practically all the theorems of the present paper.

If the function  $f(z)$  to be approximated is even or odd, and if the point set  $M$  is symmetric in the origin, then the function  $f(z)$  can be uniformly approximated as closely as desired by even or odd rational functions. This remark is entirely general; we prove a broader result, that if  $f(z)$  satisfies the functional equation

$$f(\omega z) \equiv \lambda \cdot f(z), \quad \text{where } \omega^n = \lambda^n = 1,$$

and if  $\omega z$  belongs to  $M$  whenever  $z$  belongs to  $M$ , then  $f(z)$  can be uniformly approximated on  $M$  by rational functions which satisfy this same functional equation. Let us assume

$$|r(z) - f(z)| < \epsilon, \quad z \text{ on } M,$$

where  $r(z)$  is rational. Then we have also

$$|r(\omega z) - f(\omega z)| < \epsilon, \quad z \text{ on } M,$$

that is,

$$|r(\omega z) - \lambda \cdot f(z)| < \epsilon, \quad z \text{ on } M.$$

By continued replacing of  $z$  by  $\omega z$  we obtain

$$\begin{aligned} |r(\omega^2 z) - \lambda^2 f(z)| &< \epsilon, & z \text{ on } M, \\ \cdot &\cdot & \cdot &\cdot & \cdot &\cdot & \cdot &\cdot \\ |r(\omega^{n-1} z) - \lambda^{n-1} f(z)| &< \epsilon, & z \text{ on } M. \end{aligned}$$

We define  $\rho(z)$  by means of the equation

$$n\lambda^{n-1}\rho(z) \equiv \lambda^{n-1}r(z) + \lambda^{n-2}r(\omega z) + \dots + \lambda \cdot r(\omega^{n-2}z) + r(\omega^{n-1}z);$$

it is seen by direct substitution that  $\rho(z)$  satisfies the functional equation

$$\rho(\omega z) \equiv \lambda \cdot \rho(z).$$

Moreover we find from the inequalities already considered, by addition after multiplication by suitable powers of  $\lambda$ ,

$$|n \cdot \lambda^{n-1} \rho(z) - n \cdot \lambda^{n-1} f(z)| < n\epsilon, \quad z \text{ on } M,$$

that is,

$$|\rho(z) - f(z)| < \epsilon, \quad z \text{ on } M,$$

so that the rational function  $\rho(z)$  has the properties required.

Likewise if the function  $f(z)$  to be approximated satisfies the functional equation  $f(\bar{z}) \equiv \bar{f}(z)$ , and if the point set  $M$  is symmetric in the axis of reals, the approximating rational functions can be chosen to satisfy this same functional equation. If we have

$$|f(z) - r(z)| < \epsilon, \quad z \text{ on } M,$$

where  $r(z)$  is rational, then we have also

$$|f(\bar{z}) - r(\bar{z})| < \epsilon, \quad z \text{ on } M,$$

and, by virtue of the equation  $\bar{f}(\bar{z}) \equiv f(z)$ ,

$$|f(z) - \bar{r}(\bar{z})| < \epsilon, \quad z \text{ on } M.$$

From the first and last of these inequalities we obtain

$$\left| f(z) - \frac{r(z) + \bar{r}(\bar{z})}{2} \right| < \epsilon, \quad z \text{ on } M;$$

the rational function  $R(z) \equiv \frac{1}{2}[r(z) + \bar{r}(\bar{z})]$  satisfies the functional equation  $\bar{R}(z) \equiv R(\bar{z})$ . This determination of the new function  $R(z)$  is independent of the previous functional equation considered; if we have  $r(\omega z) \equiv \lambda r(z)$ , then we have also  $R(\omega z) \equiv \lambda R(z)$ .

Theorem VIII (and also Theorem XVI) can obviously be slightly extended, to include functions  $f(z)$  not continuous on  $M$  but meromorphic in the interior points of  $M$  and even with properly restricted infinite values on the boundary of  $M$ .

Theorem VIII has applications to conformal mapping analogous to results considered in detail for the polynomial case.\* We remark too that Theorem X in conjunction with Theorem VIII gives a more precise result than Theorem VIII itself.

\* Walsh, these Transactions, vol. 30 (1928), pp. 472-482, §7.

These remarks all have application as well to the results of the two succeeding paragraphs.

8. **An infinity of Jordan arcs.** The methods already used apply, with some modification, to certain point sets composed of an infinity of Jordan arcs, curves, and regions. We shall use the term *Jordan element*  $E$  to denote a single point, a Jordan arc, or a region bounded by a finite number of Jordan curves no two of which have more than a finite number of points in common. A *Jordan configuration*  $K$  is a point set consisting of a denumerable or non-denumerable infinity of Jordan elements of which no two have more than a finite number of points in common. The *limit configuration*  $L$  of a Jordan configuration  $K$  is the point set, necessarily closed, such that in each neighborhood of an arbitrary point of  $L$  lie points of an infinity of different Jordan elements  $E$  belonging to  $K$ . It is of course true that the limit configuration is not always uniquely defined by the Jordan configuration itself, but to some extent by the notation. We now prove

**THEOREM XVII.** *Let  $M$  be an arbitrary closed point set and  $N$  a Jordan configuration which consists of a set of points and of Jordan arcs whose limit configuration is contained in  $M$ . Then an arbitrary function  $f(z)$  continuous on  $M+N$  and which on  $M$  is uniformly developable in a series of rational functions of  $z$ , is likewise uniformly developable on  $M+N$  in such a series.*

*In particular if  $N$  is an arbitrary Jordan configuration which is a closed point set consisting of a set of points and of Jordan arcs whose limit configuration (or whose  $n$ th limit configuration) consists of a finite number of points and Jordan arcs which do not divide the plane into more than a finite number of regions, then an arbitrary function  $f(z)$  continuous on  $N$  can be developed uniformly on  $N$  in a series of rational functions of  $z$ .*

We use here the idea of  *$n$ th limit configuration*. The second limit configuration of  $N$  is defined as the limit configuration of the (first) limit configuration, and the other limit configurations are likewise found by iteration. With this understanding, the second part of Theorem XVII is by virtue of Theorem IX contained in the first part. We proceed to the proof of the first part.

We find it convenient for purposes of exposition to consider  $N$  to consist, except for possible points belonging to  $M$ , of a set of Jordan arcs. This is always possible, for a point  $P$  of  $N$  not a point of  $M$  nor a point of a Jordan arc belonging to  $N$  is isolated, and there can be drawn through  $P$  a Jordan arc having no point of  $M$  nor of  $N$  other than  $P$  itself in common with  $M$  or  $N$ . This can be accomplished, moreover, for every requisite point  $P$  without altering the limit configuration of  $N$ . The definition of the function

$f(z)$  can be extended over these new Jordan arcs so that the new function is continuous over the entire new point set.\*

Let an arbitrary positive  $\epsilon$  be given; there exists a rational function  $r(z)$  which on  $M$  differs from  $f(z)$  by less than  $\epsilon$ . The function  $r(z)$  has no poles on  $M$ , hence is continuous and uniformly continuous on a closed point set  $M'$  which contains  $M$  in its interior and which is bounded by a finite number of Jordan curves. The function  $f(z)$  is also uniformly continuous on  $M+N$ . Suppose for convenience in exposition that  $M+N$  does not contain the point at infinity, and suppose  $\delta > 0$  so chosen that  $|z' - z''| \leq \delta$  implies

$$|r(z') - r(z'')| < \epsilon, \quad z', z'' \text{ in } M',$$

$$|f(z') - f(z'')| < \epsilon, \quad z', z'' \text{ on } M + N.$$

Choose a point set  $M''$  consisting of a finite number of regions, each bounded by a finite number of Jordan curves of which no two have a common point, so that  $M''$  contains  $M$  in its interior but is contained in  $M'$ , and such that no point of  $M''$  is at a distance of more than  $\delta$  from some point of  $M$ . At most a finite number of the Jordan arcs  $A$  composing  $N$  have points exterior to  $M''$ ; otherwise we should have infinitely many points belonging to different Jordan arcs of  $N$  with a limit point exterior to  $M''$ , hence not in  $M$ . It is conceivable that one or more Jordan arcs belonging to  $N$  should have infinitely many points belonging to the boundary of  $M''$ , but we can and do avoid this possibility by diminishing the size of  $M''$  if necessary.

We now define a function  $f_1(z)$  continuous on  $M''+N$  which differs but little from  $f(z)$  on  $M+N$ . The function  $f_1(z)$  shall be equal to  $r(z)$  on  $M''$ . Let  $A$  be an arbitrary Jordan arc of  $N$  which has one or more points in common with  $M''$  but which does not lie wholly within  $M''$ ; we change the notation if necessary so that  $A$  has precisely one point, an end point  $P$ , in common with  $M''$ . Let  $Q$  be a point of  $A$  not a point of  $M''$  at a distance from  $P$  not greater than  $\delta$ , and such that no point of  $A$  on the arc  $PQ$  of  $A$  is at a distance from  $P$  greater than  $\delta$ . Define the function  $f_1(z)$  equal to  $r(z)$  at  $P$ , equal to  $f(z)$  at  $Q$ , and varying monotonically† along  $A$  from  $P$  to  $Q$ . The function  $f_1(z)$  differs from  $f(z)$  at most by  $3\epsilon$  in  $P$ , for there exists a point  $R$  of  $M$  at a distance not greater than  $\delta$  from  $P$ , and we have  $f_1(P) = r(P)$ ,

\* Lebesgue, Rendiconti del Circolo Matematico di Palermo, vol. 24 (1907), pp. 371-402; p. 379.

† That is, the function  $w=f_1(z)$  maps the Jordan arc  $PQ$  one-to-one and continuously on a line segment in the  $w$ -plane.

$$\begin{aligned} |r(P) - r(R)| &< \epsilon, \\ |r(R) - f(R)| &< \epsilon, \\ |f(R) - f(P)| &< \epsilon. \end{aligned}$$

Hence the function  $f_1(z)$  differs from  $f(z)$  at most by  $5\epsilon$  along the arc  $PQ$ . The function  $f_1(z)$  is defined equal to  $f(z)$  on the portions of arcs  $A$  exterior to  $M''$  but not arcs  $PQ$ . The function  $f_1(z)$  is likewise defined equal to  $f(z)$  on the Jordan arcs which belong to  $N$  but have no points in common with  $M''$ .

The function  $f_1(z)$  is continuous on  $M'' + N$  and hence by Theorem VIII or Theorem XVI can be approximated on  $M'' + N$  by a rational function of  $z$  with an error not greater than  $\epsilon$ , so  $f(z)$  can be approximated on  $M + N$  by a rational function of  $z$  with an error not greater than  $6\epsilon$ , and the theorem is established.

Many point sets more general than Jordan curves, or indeed more general than boundaries of regions, are included in the hypothesis of the theorem, so we have (compare §1) a new generalization of Weierstrass's theorem on approximation by trigonometric polynomials. For instance we may consider the region bounded by the point set

$$\begin{aligned} y &= \sin \frac{1}{x}, & -\frac{1}{\pi} &\leq x \leq \frac{1}{\pi}; \\ x &= -\frac{1}{\pi}, & -2 &\leq y \leq 0; \\ x &= \frac{1}{\pi}, & -2 &\leq y \leq 0; \\ x &= 0, & -1 &\leq y \leq 1; \\ y &= -2, & -\frac{1}{\pi} &\leq x \leq \frac{1}{\pi}. \end{aligned}$$

Any function continuous on this closed point set (i.e., boundary of the region) can be uniformly approximated on this point set as closely as desired by polynomials in  $z$  and  $1/(z+3i/2)$ .

We may consider likewise a region consisting of a strip closed at one end which lies exterior to the unit circle, and which approaches this circle by winding about it infinitely many times. An arbitrary function continuous on the boundary of this region can be uniformly approximated on that boundary by rational functions of  $z$ , with poles only in the origin, at infinity, and at an arbitrarily chosen point interior to the region.

Similarly consider the point set  $x=n, y=n$ , where  $n$  takes all integral values, positive, negative, or zero. An arbitrary function continuous (in the extended plane) on this point set can be uniformly approximated on the point set by rational functions of  $z$ .

9. An infinity of Jordan or more general regions. The extension of Theorem VIII to include infinitely many regions is not so simple:

**THEOREM XVIII.** *Let infinitely many regions  $R$  form a Jordan configuration whose limit configuration  $L$  is contained in a closed point set  $M$ . Suppose moreover that the totality of regions  $R$  not belonging to  $M$  have at most a finite number of distinct points in common with  $M$ , and suppose that if an arbitrary region  $Q$  be considered which contains  $M$  in its interior, then each of those regions  $R$  not lying wholly in  $Q$  has at most a finite number of its points in common with other regions  $R$ , whether these other regions  $R$  lie wholly in  $Q$  or not.*

*If  $f(z)$  is a function which can be uniformly approximated on  $M$  by a rational function of  $z$ ; if  $f(z)$  is analytic interior to each region  $R$  or more generally if in each  $R$  or on a point set contained in each  $R$  but which includes all points common to this region  $R$  and either to  $M$  or to the other regions  $R$ —if on such a point set the function  $f(z)$  can be uniformly approximated by a rational function of  $z$ ; and if  $f(z)$  is continuous on the entire closed point set consisting of  $M$  and the regions  $R$  (or the sub-sets), then on this closed point set the function  $f(z)$  can be uniformly approximated as closely as desired by a rational function of  $z$ .*

*In particular if  $M=L$  satisfies the condition given and consists of a finite number of points and of Jordan arcs which do not divide the plane into infinitely many regions, then an arbitrary function  $f(z)$  analytic interior to each  $R$  and continuous on the entire point set  $L + \{R\}$  can be uniformly approximated on that point set as closely as desired by a rational function of  $z$ .*

The latter part of the theorem is an immediate consequence of the former part, in conjunction with Theorem IX. We proceed to consider the former part of the theorem.

The proof here is quite similar to the proof of Theorem XVII. Let an arbitrary positive  $\epsilon$  be given, and let  $r(z)$  be a rational function which differs from  $f(z)$  on  $M$  by less than  $\epsilon$ , and which coincides with  $f(z)$  in the points common to  $M$  and the regions  $R$  (compare Theorem X). Assume neither  $M$  nor the regions  $R$  to contain the point at infinity, choose  $M'$  a closed point set containing  $M$  in its interior but containing on or within it no point of discontinuity of  $r(z)$ , and choose  $\delta > 0$  so that  $|z' - z''| \leq \delta$  implies

$$\begin{aligned} |r(z') - r(z'')| &< \epsilon, & z', z'' \text{ in } M', \\ |f(z') - f(z'')| &< \epsilon, & z', z'' \text{ on } L + \{R\}. \end{aligned}$$

Choose a point set  $M''$  containing  $M$  but contained in  $M'$ , bounded by a finite number of Jordan curves of which no two have more than a finite number of common points, and such that no point of  $M''$  is at a distance greater than  $\delta$  from some point of  $M$ . It follows that for points of  $M''$  where  $f(z)$  is defined, the functions  $r(z)$  and  $f(z)$  differ by less than  $3\epsilon$ . We now modify  $M''$ , cutting away part of this point set and giving us a new point set  $M''$ , as follows. At most a finite number of regions  $R$  lie partly or wholly outside  $M''$ . A region  $R$  that lies partly outside  $M''$  but has no points in common with  $M$  has Jordan arcs drawn, exterior to  $R$ , not meeting  $M$  nor other regions  $R$ , except that these Jordan arcs may have such points in common with  $R$  as  $R$  itself has with other regions  $R$ , modifying  $M''$  so that the new  $M''$  shall be bounded partly by these new Jordan arcs, and so that  $R$  and the new  $M''$  shall have at most a finite number of common points, namely those points already explicitly mentioned. We have thus pared off a portion of  $M''$  by means of Jordan arcs, so that except possibly for a finite number of points,  $R$  is isolated from  $M''$ . For the regions  $R$  which lie partly outside  $M''$  but have points in common with  $M$ , we proceed similarly, cutting out parts of  $M''$  adjacent to these regions by means of Jordan arcs, so that  $R$  and the new  $M''$  have no common points other than those common to  $R$  and  $M$  and those common to  $R$  and other regions  $R$ .\* The regions cut out of  $M''$  are open regions, finite in number, each bounded by a finite number of Jordan arcs, containing in their interiors no point of  $M$  nor of a region  $R$ , and whose boundaries have at most a finite number of points in common with  $M$ , namely those points common to  $M$  and the regions  $R$ .

The point set consisting of the new  $M''$  plus the regions  $R$  is then of precisely the kind considered in Theorem XVI; the function to be developed (or approximated) is  $r(z)$  on  $M''$  and  $f(z)$  in the regions  $R$  or on the sub-sets, which is continuous on the entire point set considered. Theorem XVIII follows at once.

Iteration of this theorem gives (as in Theorem XVII) much more general results than the theorem itself.

We give merely a few of the very simplest examples to illustrate this theorem. In each case we give the open point set. A function analytic on the point set given and continuous on the corresponding closed point set can be

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\* One essential point is omitted here, namely the demonstration that if a closed Jordan region  $R$  has but a finite number of points  $P$  in common with an arbitrary closed point set  $M$ , then there exists a second Jordan region  $R'$  whose boundary has only the points  $P$  in common with the boundary of  $R$  and precisely the same points in common with  $M$ , and whose interior (on the sphere) contains the interior of  $R$  but no points of  $M$ . There is no difficulty about the proof if  $M$  is a Jordan region; in any case the proof can be given with comparative ease and is left to the reader.

uniformly approximated on the closed point set as closely as desired by a rational function of  $z$ .

$$(i) \quad \left(x - \frac{1}{3^k}\right)^2 + y^2 < \frac{1}{4 \cdot 3^{2k}} \quad (k = 0, 1, 2, \dots);$$

$$(ii) \quad x^2 + 2^{4k+2} \left(y - \frac{1}{2^{2k}}\right)^2 < 1 \quad (k = 0, 1, 2, \dots);$$

$$(iii) \quad 3\pi/2^k < \arg z < 5\pi/2^k \quad (k = 2, 3, 4, \dots);$$

$$(iv) \quad 3/2^k < |z| < 5/2^k \quad (k = 0, 1, 2, \dots).$$

It is not to be supposed, however, that an arbitrary function analytic on an *arbitrary* point set and continuous on the corresponding closed point set can be uniformly approximated as closely as desired on that closed point set by means of rational functions of  $z$ . In fact there exists a function  $f(z)$  not a constant but continuous over the entire plane, and analytic everywhere (even at infinity) except on the points of a Jordan arc  $C$ .\* Let  $K$  be a circle which contains  $C$  in its interior. Then  $f(z)$  is analytic in the interior points of  $K$  not points of  $C$  and is continuous on the corresponding closed point set, but on this closed point set cannot be uniformly approximated as closely as desired by a rational function of  $z$ . For an approximating rational function must be continuous in  $\bar{K}$ , must have its poles exterior to  $\bar{K}$ , hence leads directly to uniform approximation by a polynomial in  $z$ . Such approximation leads in turn to a series of polynomials convergent to the sum  $f(z)$  uniformly on and within  $K$ , which means that  $f(z)$  is analytic throughout the interior of  $K$ , so that  $f(z)$  is a constant, which is a contradiction.

10. **Non-uniform expansion.** We devote a few lines to the very simplest applications of our results to the expansion, not necessarily uniform, of arbitrary functions in terms of rational functions. The following theorem is essentially contained in Runge's classical results:

*Let  $M$  be an arbitrary open point set. Then an arbitrary function  $f(z)$  meromorphic in each point of  $M$  can be expanded on  $M$  in a series of rational functions of  $z$ . This series converges uniformly on any closed point set belonging to  $M$ , and all poles of the functions of the series, except the poles of  $f(z)$  itself, can be chosen to lie on the boundary of  $M$ , or indeed in arbitrarily chosen points, at least one in each of the regions or other point sets into which  $M$  divides the plane.*

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\* This arc  $C$  is to be chosen so as to have positive area but is otherwise arbitrary. See Pompeiu, *American Journal of Mathematics*, vol. 32 (1910), pp. 327-332.

Consider, in fact, a sequence of closed point sets  $M_1, M_2, \dots$ , constructed perhaps by the method of continued subdivision of the plane by a network of squares, such that each point set  $M_i$  is composed of a finite number of non-intersecting closed regions each bounded by a finite number of non-intersecting Jordan curves, such that each point set  $M_i$  lies interior to its successors and to  $M$ , and such that every point of  $M$  lies in some  $M_i$ . There exists a rational function  $r_k(z)$  such that we have

$$|f(z) - r_k(z)| < \frac{1}{k}, \quad z \text{ on } M_k;$$

all the poles of  $r_k(z)$  except the poles of  $f(z)$  may be chosen on the boundary of  $M$ . For each of the finite number of regions into which  $M_k$  separates the plane either contains a boundary point of  $M$  (compare Theorem VIII) or can itself be adjoined to  $M_k$ . The sequence  $\{r_k(z)\}$  leads directly to the required series  $r_1(z) + [r_2(z) - r_1(z)] + \dots$ . It is an essential part of the proof to notice that if  $M$  is the entire plane, the function  $f(z)$  itself must be rational, and all the functions  $r_k(z)$  can be chosen equal to  $f(z)$ .

The method of proof just used yields also the following:

*Let the point set  $M$  be the sum of the open sets  $M_1, M_2, \dots$ , no two of which have a common point, and suppose the function  $f_i(z)$  can be expanded on the point set  $M_i$  or on a subset thereof in a sequence of rational functions  $\{r_{ij}(z)\}$ . Then these functions  $f_i(z)$  can be expanded on  $M$  or on the corresponding subset in a single sequence of rational functions  $\{r_j(z)\}$ ; if  $M'$  is a closed set belonging to  $M$ , and if on the subset of  $M'$  in each  $M_i$  the sequence  $\{r_{ij}(z)\}$  converges uniformly, then the sequence  $\{r_j(z)\}$  converges uniformly on  $M'$ .*

*In particular some or all of these point sets  $M_i$  may be chosen as regions of arbitrary connectivity, and the functions  $f_i(z)$  corresponding as arbitrary functions meromorphic in those regions; the sequence  $\{r_j(z)\}$  converges uniformly on any closed point set interior to the totality of such  $M_i$ .*

Let the point set  $M_i$  be expressed as the sum of the point sets  $M_{ij}$ , so that  $M_{ij}$  consists of a finite number of non-intersecting closed regions each bounded by a finite number of non-intersecting Jordan curves, so that  $M_{ij}$  lies interior to  $M_i$  and to  $M_{i,j+1}$ , and so that each point of  $M_i$  lies in some  $M_{ij}$ . The sequence  $\{r_j(z)\}$  can be defined so that we have

$$|r_{ij}(z) - r_j(z)| < \frac{1}{j}, \quad z \text{ on } M_{1j} + M_{2,j-1} + \dots + M_{j1}.$$

The uniformity of the convergence as stated follows from the fact that

no closed set belonging to  $M$  can contain points of an infinity of the point sets  $M_i$ .

We establish next a theorem concerning Jordan arcs, more specific than a similar theorem on expansion in series of polynomials recently proved by Hartogs and Rosenthal.\*

*Let the point set  $C$  consist of an infinity of Jordan arcs  $C_1, C_2, \dots$ , of which no two have more than a finite number of points in common. Then an arbitrary function  $f(z)$  continuous on  $C$  can be expanded on  $C$  in a series of rational functions of  $z$ , and the series converges uniformly on each arc  $C_k$ .*

The proof is entirely obvious; the series is formed from the sequence  $\{r_k(z)\}$ , where  $r_k(z)$  is so determined that we have

$$|f(z) - r_k(z)| < \frac{1}{k}, \quad z \text{ on } C_1 + C_2 + \dots + C_k;$$

this choice of  $r_k(z)$  is possible by Theorem IX. In particular if no finite set of arcs  $C_1 + C_2 + \dots + C_k$  divides the plane, and if  $C$  does not contain the point at infinity, it is clear by the second part of Theorem VIII that the rational function  $r_k(z)$  can be chosen as a polynomial in  $z$ .

We establish now a result more general than those just considered; a somewhat less general result has recently been given by Carathéodory.†

*Let the point set  $M$  be composed of the mutually exclusive sets  $D_0, D_1, D_2, \dots$ , of which the first is open (and may be empty) and the others closed, and suppose the function  $f_i(z)$  can be expanded on  $D_i$  or on a subset thereof in a sequence of rational functions  $\{r_{ij}(z)\}$ . Then those functions  $f_i(z)$  can be expanded on  $M$  or on the corresponding subset in a single sequence of rational functions  $\{r_k(z)\}$ .*

*The sequence  $\{r_k(z)\}$  converges uniformly on any closed set belonging to  $D_0$  on which the sequence  $\{r_{0j}(z)\}$  converges uniformly, and on any subset of  $D_i$ ,  $i > 0$ , on which  $\{r_{ij}(z)\}$  converges uniformly.*

In particular some or all of these point sets  $D_i$ ,  $i > 0$ , may be chosen as closed regions of arbitrary connectivity, and the functions  $f_i(z)$  corresponding as arbitrary functions meromorphic in the closed regions. Some or all of the  $D_i$ ,  $i > 0$ , may be chosen as boundaries of regions composing  $D_0$ , or as boundaries of other regions, and the corresponding functions  $f_i(z)$  as arbitrary functions meromorphic on those point sets; the sequence  $\{r_k(z)\}$  can be chosen uniformly convergent on each such  $D_i$ . If some of the sets  $D_i$  are

\* *Mathematische Annalen*, vol. 100 (1928), pp. 212–263.

† *Bulletin of the American Mathematical Society*, vol. 34 (1928), pp. 721–725.

composed of a finite number of Jordan arcs, no two having more than a finite number of common points, the corresponding functions  $f_j(z)$  can be chosen as arbitrary continuous functions and the sequence  $\{r_k(z)\}$  can be made to converge uniformly on each such  $D_j$ . If a point set  $D_j$  is composed of Jordan arcs, the function  $f_j(z)$  may be chosen as an arbitrary function of Baire's class 0 or 1; in this case the convergence on  $D_j$  of the sequence  $\{r_k(z)\}$  is not necessarily uniform.

We proceed to the proof of the theorem. We construct, perhaps by the same method previously employed of continued subdivision of the plane into squares, a double sequence of closed point sets  $D_{ij}$ . The point sets  $D_{ij}$  shall each consist of a finite number of mutually exclusive closed regions each bounded by a finite number of non-intersecting Jordan curves. The set  $D_{0k}$  shall lie completely interior to  $D_{0,k+1}$  and to  $D_0$ , and each point of  $D_0$  shall lie interior to some set  $D_{0k}$ . The point sets  $D_{1k}$  and  $D_{1,k+1}$  shall lie interior to  $D_{1k}$ , and no point not in  $D_1$  shall lie in all the  $D_{1k}$ . Moreover,  $D_{1k}$  is to be chosen so as to have no point in common with  $D_{0,k+1}$ . The remaining point sets are to be similarly constructed; the sets  $D_i$  and  $D_{i,k+1}$  shall lie interior to  $D_{ik}$ , and no point not in  $D_i$  shall lie in all the  $D_{ik}$ . Moreover,  $D_{ik}$  is to be chosen so as to have no point in common with  $D_{0,i+k}$ ,  $D_{1,i+k-1}$ ,  $\dots$ ,  $D_{i-1,k+1}$ . The sequence of rational functions  $\{r_n(z)\}$  required may now be determined by the conditions

$$\begin{aligned} |r_n(z) - r_{0n}(z)| &< 1/n, & z \text{ on } D_{0n}, \\ |r_n(z) - r_{1,n-1}(z)| &< 1/n, & z \text{ on } D_{1,n-1}, \\ \cdot & \cdot \\ |r_n(z) - r_{n-1,1}(z)| &< 1/n, & z \text{ on } D_{n-1,1}. \end{aligned}$$

The paper by Hartogs and Rosenthal to which reference has already been made considers the general problem of *uniform or non-uniform* expansion of arbitrary functions, and by means of polynomials instead of by more general rational functions. There is thus comparatively little overlapping between that paper and the present one. We mention, however, the following theorem, proved by Hartogs and Rosenthal for the more restricted case of polynomials, which can be proved in its present form by a combination of the methods given by them and those of the present paper:

*If  $M$  is an arbitrary closed point set and if  $f(z)$  is an arbitrary function continuous on  $M$  such that on each component of  $M$  the function  $f(z)$  can be approximated as closely as desired by a rational function of  $z$ , then  $f(z)$  can be approximated as closely as desired on the entire point set  $M$ , by a rational function of  $z$ .*

The word *component* (i.e. *Stück*) here means a continuum (or single point) belonging to  $M$  but not contained in any other continuum belonging to  $M$ .

11. **New proof of Theorem XI.** It is perhaps of interest to remark that Theorem XI can be proved by the methods of the present paper. Inspection of the proof about to be given and of the proof of Theorem XII shows that it is sufficient in our present proof if we take only the special case that two circles exist externally tangent to each other at  $z=\alpha$ , neither of which contains a point of  $\bar{C}_1$  or  $\bar{C}_2$  in its interior. We suppose too that in the neighborhood of  $\alpha$  the curves  $C_1$  and  $C_2$  lie on opposite sides of the line of centers of these two circles. Then there exists a Jordan curve  $C'_1$ , with continuously turning tangent and continuous curvature except at  $z=\alpha$  and at a second point  $z=\beta_1$ , which contains all points interior to  $C_1$  but no point interior to  $C_2$  in its interior, which has no point other than  $\alpha$  in common with  $\bar{C}_2$ , which in the neighborhood of  $\alpha$  consists of an arc of each of the two circles mentioned, and which is of such a nature that at  $\beta_1$  a Jordan arc  $A_1$  can be joined on exterior to  $C'_1$  so that each of the two arcs of  $C'_1$  from  $\alpha$  to  $\beta_1$  form with  $A_1$  an arc with continuously turning tangent and continuous curvature. We find too a second Jordan curve  $C'_2$  which has analogous properties with respect to  $C_2$ , and we join the two arcs  $A_1$  and  $A_2$  so that they together form a Jordan arc  $A$  which has a continuously turning tangent and continuous curvature, and which has no point other than  $\beta_1$  and  $\beta_2$  in common with  $C'_1$  and  $C'_2$ .

By using such a transformation as  $w^2=z-\beta$ , we find as in the proof of Theorem XII a function  $f_1(z)$  analytic in  $\bar{C}'_1$  except at  $z=\alpha$ , and differing from  $f(z)$  in  $\bar{C}_1$  by less than an arbitrary preassigned positive  $\epsilon$ . We find likewise a function  $f_2(z)$  analytic in  $\bar{C}'_2$  except at  $z=\alpha$  and differing from  $f(z)$  in  $\bar{C}_2$  by less than  $\epsilon$ , and can assume  $f_1(\alpha)=f_2(\alpha)=f(\alpha)$ . Denote by  $\Gamma_1$  and  $\Gamma_2$  two Jordan curves which have in common the point  $z=\alpha$  and the arc  $A$ , and which are each composed (in addition to  $A$ ) of an arc of  $C'_1$  and an arc of  $C'_2$ . We assume  $\bar{C}_1$  and  $\bar{C}_2$  to lie interior to  $\Gamma_1$  and exterior to  $\Gamma_2$ . Extend the definition of the function\*

$$F(z) = \begin{cases} f_1(z), & z \text{ on } \bar{C}'_1, \\ f_2(z), & z \text{ on } \bar{C}'_2, \end{cases}$$

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\* The function  $F(z)$  is easily found in the  $w$ -plane,  $w^2=z-\beta$ , where  $\beta$  is interior to  $\Gamma_1$  and  $\Gamma_2$ . Approximate by a polynomial in  $w$  the function  $f_1(z)$  in the transform of the region  $\bar{C}_1$ , and the function  $f_2(z)$  in the transform of the region  $\bar{C}_2$ , so that this polynomial takes on the same value in the two points  $w=\pm(\alpha-\beta)^{1/2}$ . The transform in the  $z$ -plane of this polynomial in  $w$  may be used as the function  $F(z)$ .

so that  $F(z)$  is defined over  $\Gamma_1$  and  $\Gamma_2$  and satisfies a Lipschitz condition on both of these curves.

Cauchy's Integral for the function  $F(z)$  (and for  $z$  in  $C_1$  or  $C_2$  this integral formula is valid) may be written

$$\begin{aligned} F(z) &= \frac{1}{2\pi i} \int_{C_1} \frac{F(t)dt}{t-z} + \frac{1}{2\pi i} \int_{C_2} \frac{F(t)dt}{t-z} \\ &= \frac{1}{2\pi i} \int_{r_1} \frac{F(t)dt}{t-z} - \frac{1}{2\pi i} \int_{r_2} \frac{F(t)dt}{t-z}, \end{aligned}$$

where the integrals are taken in the positive (i.e., counterclockwise) sense over the curves indicated. By Plemelj's theorem, the first integral in the right-hand member represents a function  $\phi_1(z)$  analytic in  $\Gamma_1$  and continuous in the corresponding closed region, and the second integral represents a function  $\phi_2(z)$  analytic exterior to  $\Gamma_2$  and continuous in the corresponding closed region.

There exists a Jordan region  $J_1$  within which  $\phi_1(z)$  is analytic and which contains every interior point of  $C_1$  and of  $C_2$  in its interior and such that  $\phi_1(z)$  is continuous in the closed region  $\bar{J}_1$ . In  $\bar{J}_1$  the function  $\phi_1(z)$  can be approximated uniformly by a polynomial in  $z$ . There exists likewise a Jordan region  $J_2$  within which  $\phi_2(z)$  is analytic and which contains every interior point of  $C_1$  and of  $C_2$  in its interior and such that  $\phi_2(z)$  is continuous in the closed region  $\bar{J}_2$ . In  $\bar{J}_2$  the function  $\phi_2(z)$  can be uniformly approximated by a polynomial in  $z$ . Hence in  $\bar{C}_1 + \bar{C}_2$  the function  $F(z)$  can be uniformly approximated as closely as desired by a polynomial in  $z$ , so the same is true of the function  $f(z)$ , and Theorem XI is established.

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