CONCERNING NON-DENSE PLANE CONTINUA*

BY

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It has been shown by Menger† that a necessary and sufficient condition that a plane continuum M contains no domain is that for each point P of M and each positive number ϵ there exists a simple closed curve J of diameter less than ϵ which encloses P, and such that $M \cdot J$ is totally disconnected.

In the present paper it is shown that if M is a continuum which contains no domain then there exists a set G of simple closed curves filling the whole plane and indeed topologically equivalent to the set of all polygons, such that the common part of M and any curve of the set G is vacuous or totally disconnected. Additional results are obtained for the special case where M is a continuous curve.

I wish to acknowledge my indebtedness to Professor R. L. Moore, and to thank him. Credit is due him for the suggestion of most of the theorems of this paper, and for many helpful criticisms of the proofs.

Part I

In this paper I make frequent use of the notion of a *double ruling*.‡ If, on the simple closed curve $ABCDA$, $X_1, X_2, \cdots, X_n, Y_1, Y_2, \cdots, Y_m, X_n', X_{n-1}', \cdots, X_1', Y_m', Y_{m-1}', \cdots, Y_1'$ are points in the order $AX_1X_2 \cdots X_nBY_1Y_2 \cdots Y_mCX_n'X_{n-1}' \cdots X_1'DY_m'Y_{m-1}' \cdots Y_1'A$, and $X_1X_1', X_2X_2', \cdots, X_nX_n'$, and $Y_1Y_1', Y_2Y_2', \cdots, Y_mY_m'$ are arcs which, except for their end points, lie entirely within $ABCDA$, and finally, for every $i, j (1 \leqq i \leqq n, 1 \leqq j \leqq m)$ X_iX_i' has just one point in common with Y_jY_j' and no point in common with X_jX_j' (unless $i=j$), then these two sets of arcs are said to constitute a *double ruling* of the interior of $ABCDA$ (or merely a *double ruling* of $ABCDA$). The arcs $X_1X_1', X_2X_2', \cdots, X_nX_n'$ are said to be *parallel* to BC and to AD, and the arcs $Y_1Y_1', Y_2Y_2', \cdots, Y_mY_m'$ are said

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† *Über die Dimensionalität von Punktmengen*, Monatshefte für Mathematik und Physik, vol. 33 (1923), pp. 148–160. In his paper *Sur les multiplicités Cantoriennes*, Fundamenta Mathematicae, vol. 7 (1925), pp. 30–137, Urysohn obtained the slightly weaker result that for each point P of M and each positive number ϵ there exists a totally disconnected closed subset T of M such that $M-T$ is the sum of two mutually separated sets M_1 and M_2, such that M_1 contains P and is of diameter less than ϵ.

‡ See R. L. Moore, *Concerning a set of postulates for plane analysis situs*, these Transactions, vol. 20 (1919), p. 172.

to be *parallel* to AB and CD. Each of these two sets of arcs is a *single ruling* of $ABCDA$. Two sets of arcs H and K are said to constitute a *complete double ruling* of $ABCDA$ if (1) every finite subset of the set of arcs $H+K$ is a single or double ruling of $ABCDA$,(2) through each point within $ABCDA$ there is an arc of H and an arc of K, and (3) each point of $ABCDA$ (except A, B, C and D) is an end point of some arc of H or of some arc of K. If P is a point with both coördinates rational, then P is said to be a rational point. By a rational line is meant a line whose equation is $x=r$ or $y=r$ where r is some rational number.

As an obvious corollary of a theorem due to Fréchet* I state the following

THEOREM I. *If M is a point set containing no domain then there exists a continuous transformation T of the plane S into itself such that if P is any point of M then $T(P)$ is not a rational point.*

Schoenflies has proved the following theorem:†

If T_1 is a continuous one-to-one correspondence between the points of two simple closed curves J_1 and J_2 such that $T_1(J_1)=J_2$, then there exists a continuous one-to-one correspondence T_2 between J_1 plus its interior and J_2 plus its interior such that $T_2(P)=T_1(P)$ for every point P on J_1.

Therefore for every simple closed curve J there exists a continuous transformation T which throws J plus its interior into a square plus its interior. The truth of the following lemma is apparent in view of this fact and Theorem I.

LEMMA I. *Suppose M is a point set containing no rational point, and A, B, C and D are rational points which are the vertices of a rectangle, and AB is a horizontal interval. Suppose α is a double ruling of $ABCD$ with arcs $h_1, h_2, \cdots, h_n$ parallel to BA and arcs $v_1, v_2, \cdots, v_m$ parallel to BC such that* (1) *for each i $[j]$ $(i\leqq n, j\leqq m)$ the end points of $h_i[v_j]$ have the same rational y-coördinate [x-coördinate],* (2) *for every i and j $(i\leqq n, j\leqq m)$ the point common to h_i and v_j is rational, and* (3) *the rational points are everywhere dense on h_i and on v_j $(i\leqq n, j\leqq m)$. Then there exists a continuous transformation T throwing $ABCD$ plus its interior into itself and such that* (1) *T reduces to the identity transformation on the rectangle $ABCD$,* (2) *$T(P)$ is not rational for any point P of M, and* (3) *for every $i[j]$ $(i\leqq n, j\leqq m)$ the arc $T(h_i)$ $[T(v_j)]$ is an interval of some horizontal [vertical] rational line.*

* Mathematische Annalen, vol. 68 (1910), p. 159. See also Urysohn, loc. cit., p. 83.

† *Beiträge zur Theorie der Punktmengen*, Mathematische Annalen, vol. 62 (1906), pp. 286–328. See also J. R. Kline, *A new proof of a theorem due to Schoenflies*, Proceedings of the National Academy of Sciences, vol. 6 (1920), pp. 529–531.

Theorem II. *If M is the sum of a countable number of closed point sets containing no domain and lying in a euclidean plane S, then there exists a continuous transformation T of S into itself such that if L denotes any straight line of S the point set $L \cdot T(M)$ is either totally disconnected or vacuous, and no point of $T(M)$ is rational.*

I will assume that M contains no rational point. In view of Theorem I this is no essential restriction. Let A, B, C, D and E be points with coördinates

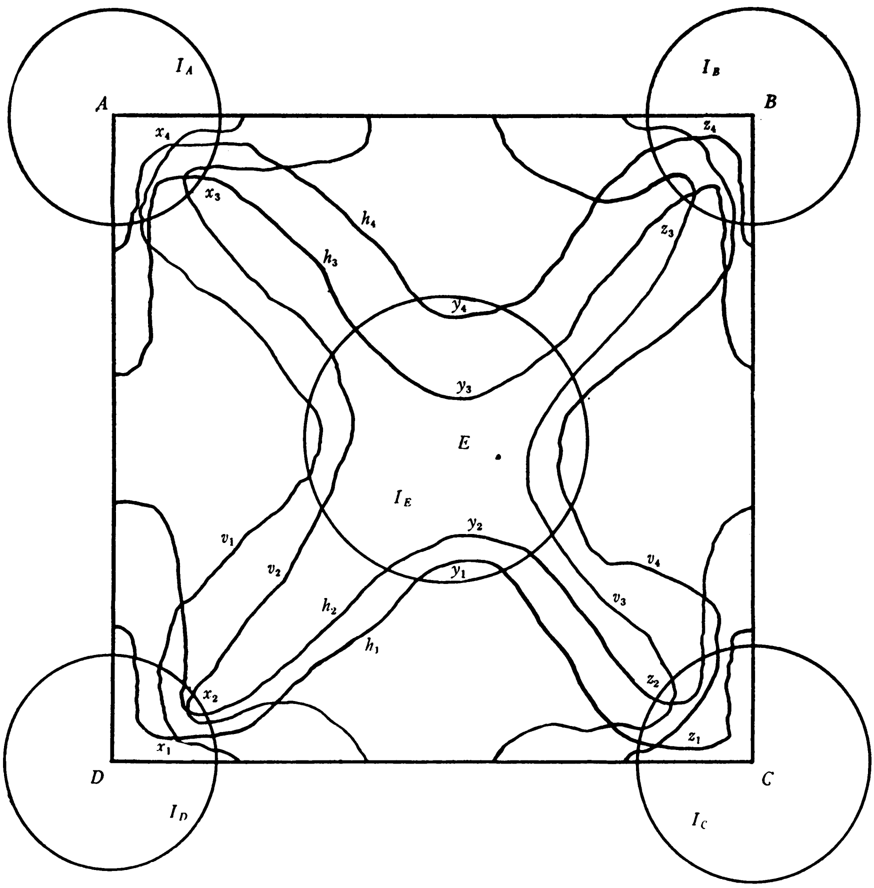

Fig. 1

(0,1), (1,1), (1,0), (0,0) and (1/2, 1/2) respectively, and let R denote the interior of the square $ABCD$. Suppose $M = M_1 + M_2 + M_3 + \cdots$, where for every n the set M_n is closed. There exist five circles with centers A, B, C, D and E respectively, such that their interiors I_A, I_B, I_C, I_D and I_E are mutually exclusive, and no one of them contains a point of the closed set M_1.

There exists a double ruling α_1 (see Fig. 1) of R consisting of four arcs

$h_1, \cdots, h_4$ parallel to AB and four arcs $v_1, \cdots, v_4$ parallel to BC such that (1) the end points of the arcs $h_1, \cdots, h_4, v_1, \cdots, v_4$ divide each side of the square $ABCD$ into five *equal* parts, (2) for each $i(i \leqq 4)$ the arc h_i contains three points, x_i, y_i and z_i, in the order $x_i y_i z_i$ from AD to BC which belong to the domains I_D, I_E, and I_C respectively, for $i = 1, 2$, and to the domains I_A, I_E, and I_B respectively for $i = 3, 4$, (3) the points x_1, x_2, x_3, x_4, z_1, z_2, z_3, z_4 are the points $h_1 \cdot v_1$, $h_2 \cdot v_2$, $h_3 \cdot v_2$, $h_4 \cdot v_1$, $h_1 \cdot v_4$, $h_2 \cdot v_3$, $h_3 \cdot v_3$, $h_4 \cdot v_4$, respectively, (4) for each i $(i \leqq 4)$ v_i contains a point of I_E between the arcs h_2 and h_3, and (5) for each i the rational points are dense on h_i and on v_i and every point $h_i \cdot v_j$ $(i \leqq 4, j \leqq 4)$ is rational. Since M does not contain any rational

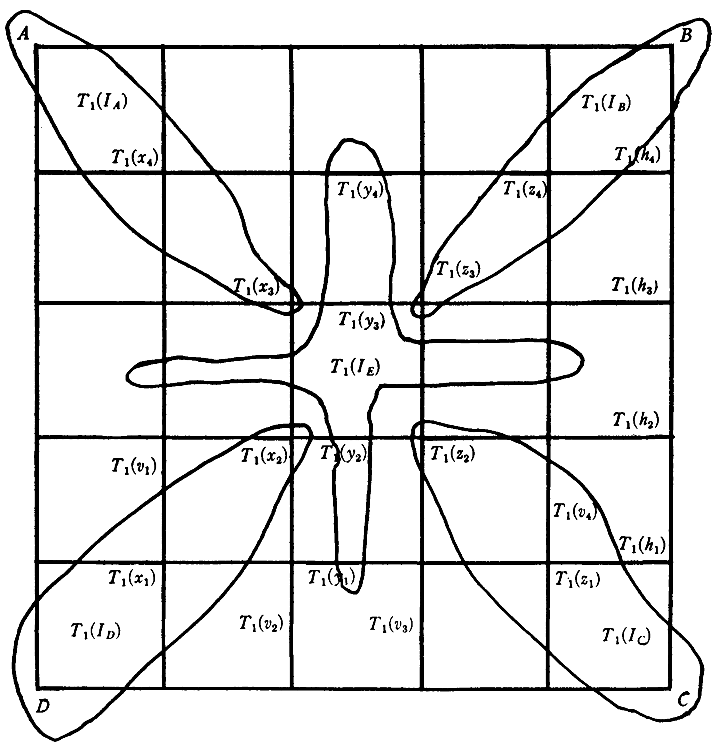

FIG. 2

point it is easily seen that the hypotheses of Lemma I are satisfied by M and the double ruling α_1. Let T_1 denote a transformation satisfying the conclusion of Lemma I. It is easily seen that every straight line which contains two points of the square $ABCD$ contains a point not belonging to $T_1(M_1)$. (See Fig. 2.)

Let t_1 be the set of all arcs k such that $k = T_1(g)$, where g is an arc of the double ruling α_1. Obviously t_1 is a double ruling of R, the arcs of which are horizontal and vertical rational intervals. There exists a positive number $\epsilon_1(\epsilon_1<1/2)$ such that if P and Q are points of $\overline{R}$ and $\delta(P, Q)$† $\geqq 1/2$, then $\delta[T_1(P), T_1(Q)]>\epsilon_1$. Let $P_1, P_2, P_3, \cdots$ denote the rational points on the arcs AB and BC. Let β_1 denote a double ruling of R consisting of a finite set of rational horizontal and vertical intervals such that (1) β_1 contains an interval through the point P_1, and contains‡ t_1 and (2) every component§ of $\overline{R}$ minus the sum of the arcs of the ruling β_1 is of diameter less than ϵ_1.

Let F_1 denote the collection of all components of R minus the sum of the arcs of the ruling β_1 and let G denote any element of F_1. The domain G is the interior of a rectangle $A_GB_GC_GD_G$, where A_GB_G is an interval of a horizontal rational line, and all of the points A_G, B_G, C_G, and D_G are rational. Let E_G denote the center of the rectangle and let $I_{A_G}, I_{B_G}, I_{C_G}, I_{D_G}$, and I_{E_G} be mutually exclusive circular domains which contain no point of the closed set $T_1(M_1+M_2)$ and whose centers are the points A_G, B_G, C_G, D_G, and E_G. Clearly there exists a double ruling α_{2_G} of G having with respect to $\overline{G}$, $T_1(M_1+M_2)$, $T_1(M)$ and the above mentioned circular domains the same properties which the double ruling α_1 has with respect to $\overline{R}$, M_1, M and the domains I_A, I_B, I_C, I_D and I_E. Let N_1 be the sum of all arcs k where k belongs to some double ruling α_{2_G} for some domain G of F_1. The point set N_1 is the sum of a finite number of arcs which constitute a double ruling γ_1 of R. Let α_2 be the double ruling consisting of all arcs of the rulings β_1 and γ_1. Since $T_1(M)$ contains no rational point the hypotheses of Lemma I are satisfied by $T_1(M)$ and the double ruling α_2. There exists a transformation T_2 satisfying the conclusion of Lemma I and reducing to the identity transformation on the square $ABCD$ and on the arcs of the ruling β_1. Let t_2 denote the double ruling consisting of all arcs k such that $k=T_2(g)$, where g is an arc of the ruling α_2.

Let r denote an interval of length $2\epsilon_1$ which is a subset of R. Clearly if r is a subset of some arc of the ruling β_1 then it contains a point not belonging to $T_2T_1(M_1+M_2)$. If r is not a subarc of any arc of β_1 then there exists a rectangle which is a subset of the sum of the arcs of β_1 and which contains exactly two points of r. With the assistance of Fig. 2 it can readily be seen that r contains a point not belonging to the set $T_2T_1(M_1+M_2)$. Hence if T is any continuous transformation of $\overline{R}$ into itself such that $T(P)=P$ for every point P on $ABCD$ or on some arc of the ruling t_2, then every interval in $\overline{R}$

† If P and Q are points, $\delta(P, Q)$ will denote the distance from P to Q.

‡ A double ruling β is said to *contain* a double ruling t if every arc of t is an arc of β.

§ By a *component* of a point set G is meant a maximal connected subset of G.

which is of length greater than or equal to $2\epsilon_1$ contains a point not belonging to the set $TT_2T_1(M_1+M_2)$. There exists a positive number ϵ_2 $(\epsilon_2<1/3)$ such that if $\delta(U, V)\geqq 1/3$ and U and V belong to $\overline{R}$, then $\delta[T_2T_1(U), T_2T_1(V)]>\epsilon_2$. Let β_2 denote a double ruling of R consisting of rational horizontal and vertical intervals such that (1) some interval of β_2 contains P_2 (the second rational point on $ABCD$), (2) β_2 contains t_2, (3) every component of R minus the sum of the arcs of the ruling β_2 is of diameter less than ϵ_2.

Let F_2 denote the collection of all components of R minus the sum of the arcs of the ruling β_2 and let G denote any element of F_2. The domain G is the interior of a rectangle composed of subsets of rational lines. There exists a double ruling α_{3_G} of G having with respect to $\overline{G}$, $T_2T_1(M_1+M_2+M_3)$ and $T_2T_1(M)$ the properties which the double ruling α_1 has with respect to $\overline{R}$, M_1 and M. Let N_2 be the sum of all arcs k, where k belongs to some double ruling α_{3_G} for some domain G of F_2. The point set N_2 is the sum of a finite number of arcs which constitute a double ruling γ_2 of R. Let α_3 be the double ruling consisting of all arcs of the rulings β_2 and γ_2. Since $T_2T_1(M)$ contains no rational point the hypotheses of Lemma I are satisfied by $T_2T_1(M)$ and the double ruling α_3. Hence there exists a transformation T_3 satisfying the conclusion of Lemma I and reducing to the identity transformation on $ABCD$ and on the arcs of the double ruling β_2. Let t_3 denote the double ruling consisting of all arcs k such that $k=T_3(g)$, where g is an arc of the ruling α_3. Each component of $\overline{R}$ minus the sum of the arcs of the ruling β_2 is of diameter less than ϵ_2. Hence it follows that if T is a continuous transformation of $\overline{R}$ into itself which reduces to the identity transformation on $ABCD$ and on every arc of the ruling t_3 then every interval which is a subset of $\overline{R}$ and is of diameter greater than or equal to $2\epsilon_2$ contains a point not belonging to the set $TT_3T_2T_1(M_1+M_2+M_3)$.

Proceeding in this way one can see that there exists a countable infinity of double rulings $t_1, t_2, \cdots$, and a countable infinity of continuous transformations $T_1, T_2, T_3, \cdots$ of $\overline{R}$ into itself such that for every positive integer n the following properties obtain: (1) t_{n+1} contains t_n, (2) t_n is composed of horizontal and vertical *rational* intervals and every component of $\overline{R}$ minus the sum of the arcs of the ruling t_{n+1} is of diameter less than $1/n$, (3) if U^m denotes the point $T_mT_{m-1}\cdots T_1(U)$ and U and V are points of $\overline{R}$ such that $\delta(U, V)\geqq 1/n$, then for every integer $m(m>n)$ the points U^m and V^m are separated in $\overline{R}$ by some arc of the ruling t_n, and, for every n, $\delta(U^{n+1}, U^{n+k})<2/n$ $(k=1, 2, \cdots)$, (4) the transformation T_{n+1} reduces to the identity transformation on $ABCD$ and on all arcs of the ruling t_n, (5) some arc of t_n contains P_n (the nth rational point on $ABCD$), (6) no point belonging to two arcs of t_n belongs to the set $T_nT_{n-1}\cdots T_1(M)$, and (7) if T is any continuous transformation of $\overline{R}$ into itself which reduces to the identity transformation on $ABCD$ and on the arcs of the ruling t_{n+1} then every interval which is a subset of $\overline{R}$ and is of length greater than or equal to $2/n$ contains a point not belonging to the set $TT_{n+1}T_n \cdots T_1(M_1+M_2+ \cdots +M_{n+1})$.

From property (3) it clearly follows that for every point U of $\overline{R}$ the sequence $U, U^1, U^2, \cdots$ has a sequential limit point. Let T be the transformation which carries U into this sequential limit point. In particular, if for some integer n the point U^n belongs to some arc of t_n or to $ABCD$, then $U^n = U^{n+k}$ $(k=1, 2, \cdots)$ and $T(U)$ is merely U^n. I shall now show that T is a continuous one-to-one transformation of $\overline{R}$ into itself which satisfies the conclusion of Theorem II with respect to $\overline{R}$.

First, T is a one-to-one transformation of $\overline{R}$ into itself which reduces to the identity transformation on $ABCD$. Let U and V denote distinct points of $\overline{R}$. We see at once from property (3) that there exist two mutually exclusive intervals g_1 and g_2 which belong to some double ruling t_n and each of which separates U^m from V^m in $\overline{R}$ for every integer $m(m>n)$. Then the sequences $U^1, U^2, U^3, \cdots$ and $V^1, V^2, V^3, \cdots$ have distinct sequential limit points. Hence for each point X of $\overline{R}$ there is not more than one point U_X such that $X=T(U_X)$. That for each X there is at least one point U_X follows from properties (2) and (3).

Second, the transformation T is continuous. Let U be a point not belonging to a point set W. If U is not a limit point of W then there exists an integer m such that any point of $T_nT_{n-1} \cdots T_1(W)$ is separated from $T(U)$ by some arc of the ruling t_n for every n $(n>m)$. Hence $T(U)$ is not a limit point of $T(W)$. If however U is a limit point of the set W then for every ϵ there is a point V_ϵ of W and an integer n_ϵ such that $\delta(U^m, V_\epsilon^m)<\epsilon$ for every integer m greater than n_ϵ. Hence $T(U)$ is a limit point of $T(W)$.

It follows from properties (2), (5) and (6) that if U is a point such that $T(U)$ is rational, then U does not belong to M.

Let L denote any interval which is a subset of $\overline{R}$, and suppose that $L\cdot T(M)$ contains a nondegenerate† connected set. Since no continuum is the sum of a countable number of totally disconnected closed sets it follows that there exists an integer n such that L contains a nondegenerate connected subset of $T(M_n)$. Let r denote some interval which is a subset of $L\cdot T(M_n)$ and let ϵ be its length. Let n_1 be an integer such that $2/n_1<\epsilon$ and $n_1>n$. Now the transformation T reduces to the transformation $T_{n_1+1}T_{n_1} \cdots T_1$ on $ABCD$ and on every arc of t_{n_1} (property 4). Hence if T^*

† A point set containing but a single point is said to be degenerate.

denotes the transformation such that $T(U)=T^*T_{n_1+1}T_{n_1}\cdots T_1(U)$ then T^* reduces to the identity transformation on $ABCD$ and on every arc of t_{n_1}. Hence by property (7) the interval r contains a point not belonging to the set $T^*T_{n_1+1}T_{n_1}\cdots T_1(M_1+\cdots+M_{n_1+1})$. But this set is exactly $T(M_1+\cdots+M_{n_1+1})$. Hence we have a contradiction which means that no interval which is a subset of $\overline{R}$ contains a nondegenerate connected subset of $T(M)$.

I have now shown that T is a continuous one-to-one transformation of $\overline{R}$ into itself such that if L is any straight line then the point set $L\cdot T(M)\cdot\overline{R}$ is either vacuous or totally disconnected, and no point which beongs to $T(M)$ is rational. Hence T satisfies the conclusion of Theorem II with respect to the rectangle $ABCD$ plus its interior. The extension to the whole plane is obvious.

Part II

In his paper† *Grundzüge einer Theorie der Kurven*, Karl Menger proves that if M is a bounded continuum containing no domain then a necessary and sufficient‡ condition that M be a continuous curve [regular curve§] is that, for each positive number ϵ, M is the sum of a finite number of continua all of diameter less than ϵ, such that the common part of any two of these continua is vacuous or totally disconnected [finite]. The necessity of this condition follows from Theorem III for the case where M is a continuous curve, and from Theorem VI for the case where M is a regular curve.

Theorem III. *If M is any continuous curve which contains no domain then there exists a continuous transformation T of the plane S into itself such that* (1) *no straight line contains a nondegenerate connected subset of $T(M)$ and no point of $T(M)$ is rational, and* (2) *if R is the interior of a rectangle composed of intervals of lines with equations of the form $x=r$ and $y=r$, where r is a rational number, then the point set $R\cdot T(M)$ is the sum of a finite number of connected sets and every point of M on the boundary of R is a limit point of $R\cdot T(M)$.*

To help establish Theorem III I will first prove the following lemma.

† Mathematische Annalen, vol. 95 (1925), pp. 277–306.

‡ For the case of a continuous curve the sufficiency of the given condition was proved by W. Sierpinski, *Sur une condition pour qu'un continu soit une courbe jordanienne*, Fundamenta Mathematicae, vol. 1 (1920), pp. 44–60.

§ A continuum M is said to be a *regular curve* if for each point P of M and each positive number ϵ there exists a domain of diameter less than ϵ which contains P and whose boundary contains only a finite number of points of M. See K. Menger, loc. cit.

LEMMA II. *If M is a continuous curve containing no domain and J is a simple closed curve $EKFGLHE$ such that the arcs EKF and GLH of J contain no point of M, then there exists a simple continuous arc KL lying, except for K and L, within J, and such that* (1) *the arc KL contains no nondegenerate connected subset of M,* (2) *every point of M on the arc KL is a limit point of M from both sides*† *of KL, and* (3) *the point set $M - M \cdot KL$ is the sum of a finite number of connected sets.*

Proof of Lemma II. Let I denote the interior of J and let N be the point set $M \cdot I + EH + FG$. If N is not connected there exists an arc KL which contains no point of M. I will therefore suppose that N is connected. Then N is a continuous curve. The set of junction‡ points of M is countable.‡ With the help of this fact and Theorem I it is easily seen that there exists an arc KZL which except for K and L is within J and which contains no junction point of M and no nondegenerate connected subset of M. Let S_E and $\overline{S}_F$ denote the components of $N - N \cdot KZL$ containing E and F respectively. Let S_E^* denote the component of $N - \overline{S}_F$ which contains E. Let S_F^* denote $N - \overline{S}_E^*$.

Since N is a continuous curve, S_E^* and S_F^* are mutually separated sets. The common part of the two sets $\overline{S}_E^*$ and $\overline{S}_F^*$ is a subset of $N \cdot KL$ and is therefore totally disconnected. Now S_E^* is connected, by definition. I will show that S_F^* is also connected.

Suppose that Q is a point of S_F and P is any point of $S_F^* - S_F^* \cdot \overline{S}_F$. There exists§ in N a simple continuous arc PQ. Let P_1 be the first point on this arc from P to Q which belongs to the set $\overline{S}_E^* + \overline{S}_F$. The point P_1 belongs to $\overline{S}_F$, for otherwise the subarc PP_1 of PQ belongs to S_E^*, which is contrary to the supposition that P is in S_F^*. If P_1 is not in $\overline{S}_E^*$ then it is in $\overline{S}_F$ and the arc PP_1 is connected to Q by an arc in S_F^*.

Suppose then that P_1 belongs to both of the sets $\overline{S}_E^*$ and $\overline{S}_F$. Then P_1 is a

† If M is a point set and P is an interior point of an arc AB then P is said to be a limit point of M from both sides of AB if there exists a simple closed curve J containing AB such that, I_J denoting the interior of J, P is a limit point of $M \cdot I_J$ and of $M \cdot (S - \bar{I}_J)$.

‡ Cf. R. L. Moore, *Concerning triods in the plane and the junction points of plane continua*, Proceedings of the National Academy of Sciences, vol. 14 (1928). If P is a point of a continuous curve N, and K is a domain containing P such that P is a cut point of the component of $N \cdot K$ which contains P, and furthermore there exist three arcs PA_1, PA_2, and PA_3 which lie in N and have only the point P in common, then P is said to be a *junction point* of N. The continuum $PA_1 + PA_2 + PA_3$ is called a *triod* and the point P is its *emanation point*.

§ R. L. Moore, *A theorem concerning continuous curves*, Bulletin of the American Mathematical Society, vol. 23 (1917), pp. 233–236. See also Mazurkiewicz, *Sur les lignes de Jordan*, Fundamenta Mathematicae, vol. 1 (1920), pp. 166–209, and H. Tietze, *Ueber stetige Kurven, Jordansche Kurvenbogen, und geschlossene Jordansche Kurven*, Mathematische Zeitschrift, vol. 5 (1919), pp. 284–291.

limit point of each of the mutually separated connected sets S_E^* and S_F. Therefore† there exists an arc R_1P_1 in $S_E^*+P_1$, and an arc T_1P_1 in S_F+P_1, where R_1 and T_1 are on the arcs EH and FG, respectively. (See Fig. 3.) Since the point P_1 is an emanation point of the triod of N composed of the three arcs R_1P_1, PP_1, and T_1P_1, and is not a junction point of M, it is not a

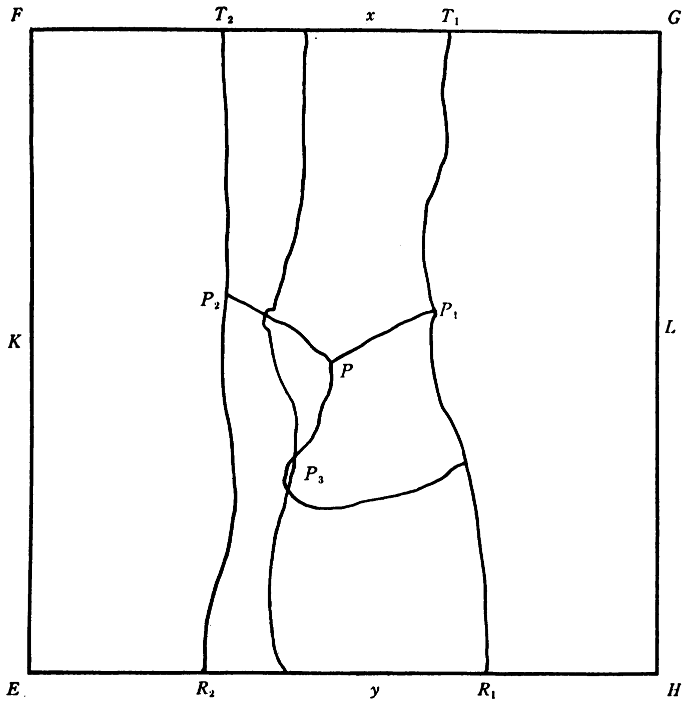

FIG. 3

cut point of N. Therefore there exists an arc from P to Q which does not contain the point P_1. Let P_2 denote the first point of this arc in the order from P to Q which belongs to the set $\overline{S}_E^*+\overline{S}_F$. Then as before either P and Q can be connected by an arc in S_F^* or P_2 belongs to both $\overline{S}_E^*$ and $\overline{S}_F$. Suppose the latter is true. Then P_2 does not belong to any of the arcs PP_1, R_1P_1 or T_1P_1. In S_F+P_2 there exists an arc P_2T_2, where T_2 is either on the arc FG or on the arc P_1T_1 and no other point of the arc P_2T_2 belongs to either FG or P_1T_1. In $S_E^*+P_2$ there exists an arc P_2R_2, where R_2 is on EH or on R_1P_1, and no other point of P_2R_2 is on EH or R_1P_1. Suppose for definiteness that T_2 and R_2 are on the arcs FG and EH, respectively. Let X and Y be points

† See R. L. Wilder, *Concerning continuous curves*, Fundamenta Mathematicae, vol. 7 (1925), pp. 340–377.

on FG and EH between T_1 and T_2, and R_1 and R_2, respectively. Then the simple closed curve h ($h=T_1XT_2P_2R_2YR_1P_1T_1$) encloses the segments PP_1 and PP_2. Let R be a domain which contains the arc PP_1 but does not contain the point P_2. The component of $N\cdot R$ which contains P_1 contains subarcs of PP_1, R_1P_1, and T_1P_1. Since P_1 is not a junction point of N it is not a cut point of this component. Hence there exists within $R\cdot(N-P_1)$ a simple continuous arc PW, where W belongs to one of the segments R_1P_1 and T_1P_1. Let P_3 be the first point in the order from P to W which the arc PW has in common with the set $\overline{S}_E^*+\overline{S}_F$. Assume that P_3 belongs to both $\overline{S}_E^*$ and $\overline{S}_F$. Clearly P_3 is within the simple closed curve h. There exists a segment of an arc within h with P_1 and P_2 as end points which contains no point of $\overline{S}_E^*+\overline{S}_F$. This segment P_1P_2 divides the interior of h into two connected domains, one of which contains P_3. Now there exist arcs from P_3 to T_1 and from P_3 to R_1 which lie in the sets S_F+P_3 and $S_E^*+P_3$, respectively. Since the simple closed curve $P_1R_1YR_2P_2P_1$ contains no point of S_F+P_3, and the simple closed curve $P_1T_1XT_2P_2P_1$ contains no point of $S_E^*+P_3$ it is clear that we have reached a contradiction. Hence every point P of $S_F^*-S_F^*\cdot\overline{S}_F$ lies in the component of S_F^* which contains S_F, which means that S_F^* is connected.

We now have $N=\overline{S}_E^*+\overline{S}_F^*$ where S_E^* and S_F^* are connected and have no point in common, and $\overline{S}_E^*\cdot\overline{S}_F^*$ is totally disconnected. Let X_E and X_F be the two continua obtained by adding to $\overline{S}_E^*$ and $\overline{S}_F^*$, respectively, all of their bounded complementary domains. Now no point is in a bounded complementary domain of both $\overline{S}_E^*$ and $\overline{S}_F^*$. Therefore the point set $X_E\cdot X_F$ is the same as the set $\overline{S}_E^*\cdot\overline{S}_F^*$. Call this set T. Then X_E-T is connected, and neither X_E nor X_F separates the plane. As a result of a theorem of R. L. Moore† it follows that there exists a simple closed curve k enclosing X_E-T, containing T, and not containing or enclosing any point of X_F-T. Clearly k contains an arc whose end points lie on the segments EF and GH, respectively, but which otherwise lies within J. This arc can be modified so as to have K and L for end points and retain the property of separating X_E-T and X_F-T as above.‡ I will show that this arc satisfies the conclusion of Lemma II.

(1) Clearly the arc KL contains no nondegenerate connected subset of M, for it contains no point of M not belonging to the totally disconnected set T. (2) Every point of M on the arc KL is a limit point of each of the sets S_E^* and S_F^* and is therefore a limit point of M from both sides of KL.

† *Concerning the separation of point sets by curves*, Proceedings of the National Academy of Sciences, vol. 11 (1925), pp. 469–476.

‡ This follows readily from the fact that the segments EKF and GLH do not contain any point of N.

(3) Every component of $M-M\cdot KL$ contains a point either on the arc EH or on the arc FG because of the fact that $(M-T)+EH+FG$ is the sum of two or less connected sets. Since not infinitely many components of $M-T$ have a point on EH or FG and also have a limit point on KL it follows that only a finite number of such components can have a limit point on KL. But *every* component of $M-T$ has a limit point on KL. Hence the number of such components is finite. This completes the proof of Lemma II.

Proof of Theorom III. As in Theorem II there is no loss of generality in assuming that no point with both coördinates rational belongs to M. Clearly the plane S can be regarded as the sum of a countable infinity of rectangles plus their interiors, the rectangles being subsets of *irrational* horizontal and vertical lines. Let R_1, R_2, R_3, $\cdots$ denote the mutually exclusive interiors of such a set of rectangles.

If ϵ is any positive number and if we have any finite double ruling α of R_i (i being any positive integer) such that no arc of α contains a nondegenerate connected subset of M, then we can obtain a finite double ruling β which contains the arcs of α and is such that every component of R_i minus the sum of the arcs of β is of diameter less than ϵ, and every arc of β which does not belong to α has properties (1), (2), and (3) of Lemma II, and such that no point common to two such arcs belongs to M. Now the arcs of the double ruling β_n defined in the proof of Theorem II were taken to be *rational* lines so that no nondegenerate connected subset of $T_nT_{n-1}\cdots T_1(M)$ would be a subset of an arc of β_n. In view of Theorem II, however, it can be seen that transformations T_1, T_2, T_3, $\cdots$ can now be chosen so that no nondegenerate connected subset of $T_nT_{n-1}\cdots T_1(M)$ is a subset of *any* straight line. Hence some of the arcs of the rulings β_n can be taken as *irrational* horizontal and vertical intervals. By obvious modification of the argument given in the proof of Theorem II it can be seen that there exists a countable infinity of double rulings t_1, t_2, t_3, $\cdots$ of R_i, and a countable infinity of continuous transformations T_1, T_2, T_3, $\cdots$ of $\overline{R}_i$ into itself which, except for (6) and a modification of (2) to allow t_n to contain intervals of *irrational* horizontal and vertical lines, have properties (1)-(7) as stated in the proof of Theorem II, and the additional property that the double ruling t_n contains a double ruling r_n every arc of which has properties (1), (2), and (3) of Lemma II and P_n (the nth rational point on the boundary of R_i) is an end point of some arc of the ruling r_n. Let W_i be the transformation corresponding to the transformation T as defined in the proof of Theorem II. Let T be the transformation of the plane S into itself which for every i reduces to W_i over $\overline{R}_i$.

Clearly then no straight line contains a nondegenerate connected subset of $T(M)$, and no point of $T(M)$ is rational. The second conclusion of Theorem III follows readily from the fact that every rational horizontal and vertical interval with rational end points has properties (1), (2), and (3) of Lemma II with respect to the continuous curve $T(M)$. Hence Theorem III is established.

The following is an example of a regular curve which contains an uncountable set H of points such that no arc containing a point of H has property (3) of Lemma II. It therefore follows that Theorem III would be false if the stipulation that the boundary of R is composed of intervals of *rational* horizontal and vertical lines were omitted, or if the word *rational* were replaced by the word *irrational*.

Example 1. (See Fig. 4.) Let H denote a nondense perfect point set on the interval $0\leqq x\leqq 1$, and let K denote any acyclic† continuous curve such that H is the set of end points† of K. Let $G_1, G_2, G_3, \cdots$ denote a

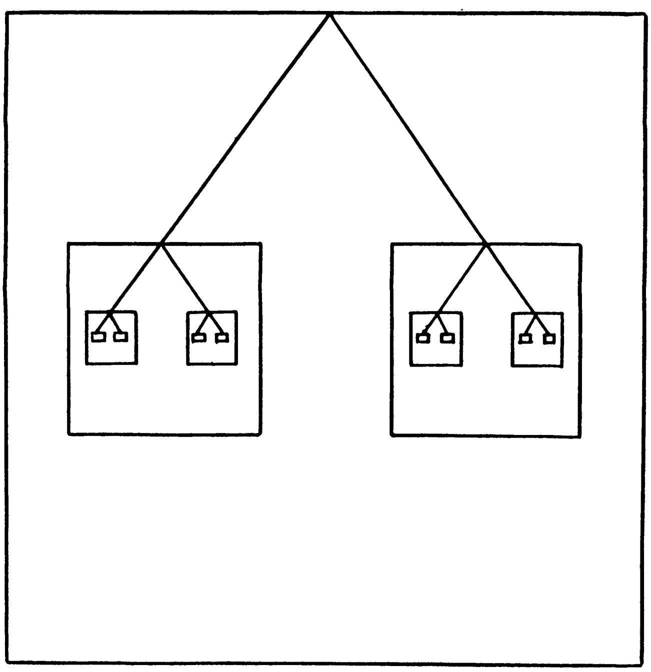

Fig. 4

† A continuous curve is said to be *acyclic* if it contains no simple closed curve. See H. M. Gehman, *Concerning acyclic continuous curves*, these Transactions, vol. 29 (1927), pp. 553–568. An end point of an acyclic continuous curve is a point which is not an interior point of any arc of that curve. See R. L. Wilder, loc. cit., or H. M. Gehman, loc. cit.

contracting sequence† of mutually exclusive simple closed curves such that every point of H is enclosed by infinitely many curves of the sequence $G_1, G_2, G_3, \cdots$, and for every n the curve G_n has just one point in common with K, and this point does not belong to H. Let M be the continuous curve $K+G_1+G_2+G_3+\cdots$. Let P denote a point of $K-H$. Then if Q is any point of H there is one and only one arc PQ from P to Q in M. Let J denote a simple closed curve containing Q but not containing a non-degenerate connected subset of M. Either (1) $PQ-PQ\cdot J$ is the sum of infinitely many maximal connected subsets, in which case $M-M\cdot J$ is not the sum of a finite number of connected sets, or (2) there exists a point X on the arc PQ distinct from Q such that the arc XQ has only the point Q on J. Suppose for definiteness that X is within J. There exist infinitely many simple closed curves of M enclosing Q, having points within J and points without J, and having only one point on the arc XQ of PQ. Since no maximal connected subset of $M-M\cdot J$ which lies in the exterior of J can contain a point of the arc XQ it follows that the number of components of $M-M\cdot J$ is infinite.

In his paper‡ *Concerning irreducible cuttings of continua*, G. T. Whyburn raises the question as to whether or not every open§ subset of a plane continuous curve M contains an irreducible cutting of M. This question is answered by the following theorem which is an application of Theorem III.

THEOREM IV. *Every open subset of a plane continuous curve M contains an irreducible cutting of M.*

Let G denote an open subset of a continuous curve M. Clearly if G contains a domain then it contains a circle which is an irreducible cutting of M. If G contains no domain let R denote the interior of a circle such that R contains a point of G but $\overline{R}$ does not contain a point of $M-G$. Let M_1 denote any maximal connected subset of $M\cdot\overline{R}$ which contains more than one point. Then‖ M_1 is a continuous curve which contains no domain. From Theorem III it readily follows that there exists a simple closed curve J which encloses some point of M_1 but does not contain or enclose any point

† If H is a sequence of point sets and for each positive number ϵ only a finite number of point sets of the set H are of diameter greater than ϵ then H is said to be a *contracting sequence* of point sets. See R. L. Moore, *Concerning upper semi-continuous collections*, Monatshefte für Mathematik und Physik, vol. 36 (1929), pp. 81–88.

‡ Fundamenta Mathematicae, vol. 13, pp. 42–57.

§ An open subset of a continuum M is a set such that its complement with respect to M is closed. An irreducible cutting of a continuum M is a point set K of M such that $M-K$ is not connected, but such that if G is any proper subset of K then $M-G$ is connected. See G. T. Whyburn, ibid.

‖ H. M. Gehman, *Concerning the subsets of a plane continuous curve*, Annals of Mathematics, vol. 27, p. 34.

of $M-M_1$, and such that the point set $M\cdot J$ is totally disconnected and separates M_1 into a finite number (greater than 1) of connected sets. Clearly then $M\cdot J$ has the same properties with respect to M. Then† the set $M\cdot J$ contains a subset which is an irreducible cutting of M.

THEOREM V. *A necessary and sufficient condition that a continuum M (not the whole plane) be a regular curve is that if R is a connected domain containing two distinct points A and B not belonging to M then in R there exists a simple continuous arc from A to B which contains only a finite number of points of M.*

The condition is necessary. Suppose M is a regular curve and R is a connected domain containing two points A and B not belonging to M. Let AB denote any simple continuous arc from A to B which lies in R, and let A' and B' be points in the order $AA'B'B$ such that no point of M is on the arc AA' or the arc BB' of AB. Enclosing each point of the arc $A'B'$ there exists a simple closed curve containing only a finite number of points of M and not containing or enclosing A or B or any point not in the domain R. There exists a finite set of such curves whose interiors cover the arc $A'B'$. Call the curves of such a set $J_1, J_2, \cdots, J_n$. If H denotes the continuous curve $AA'+BB'+J_1+J_2+\cdots+J_n$ then H contains only a finite number of points of M. Let AXB denote an arc from A to B which is a subset of H. Obviously this arc contains only a finite number of points of M.

The condition is sufficient. Clearly M cannot contain a domain. Suppose P is any point of M and ϵ is any positive number. Let J_1 and J_2 denote two circles with P as center and radii $\epsilon/2$ and $\epsilon/3$, respectively. Let P_1 and P_2 denote the extremities of a diameter of J_1. Let D denote the domain bounded by J_1+J_2, and let A and B denote two points not belonging to M and lying in D on different sides of the diameter P_1PP_2. Let D_1 and D_2 be the connected domains $D-D\cdot PP_1$ and $D-D\cdot PP_2$, respectively. Let AX_1B and AX_2B denote arcs lying in D_1 and D_2, respectively, and containing only a finite number of points of M. The continuous curve AX_1B+AX_2B contains a simple closed curve which encloses P, contains only a finite number of points of M, and is of diameter less than ϵ. Hence the point P is a regular point and M is a regular curve.

THEOREM VI. *If M is a regular curve (not necessarily bounded) in a euclidean plane S, then there exists a continuous transformation T of S into itself such that* (1) *no straight line contains a nondegenerate connected subset of $T(M)$ and no point of $T(M)$ is rational, and* (2) *each rational horizontal or vertical line has in common with $T(M)$ a point set which has no limit point.*

† See G. T. Whyburn, loc. cit.

The only essential difference between the proof of this theorem and that of Theorem III is that here we require that the arcs of the double ruling r_n shall have only a finite number of points in common with M, instead of requiring that they have properties (1), (2) and (3) of Lemma II. In case M is a *bounded* regular curve the second conclusion of Theorem VI is equivalent to the statement that no rational line contains more than a finite number of points of M.

It follows that if R is the interior of a rectangle whose sides are intervals of rational horizontal and vertical lines, then the point set $T(M)\cdot(\overline{R}-R)$ contains only a finite number of points of M. However the following example shows that it does not follow that the set $R\cdot T(M)$ is the sum of a finite number of connected sets.

EXAMPLE 2. (See Fig. 5.) For each pair of positive integers n and $k(k\leqq 2^n)$ let I_{kn} denote the interval with end points $[(k-1)/2^n, 0]$ and $[k/2^n, 0]$, and let C_{kn} denote the semicircle above the x-axis with I_{kn} as diameter. Let H denote the continuum which is the sum of the interval $I(0\leqq x\leqq 1)$ and all semicircles $I_{kn}(k\leqq 2^n,\ n=1,\ 2,\ 3,\cdots)$. Let P_1, P_2, $P_3, \cdots$ denote the points of the x-axis which are extremities of diameters of semicircles belonging

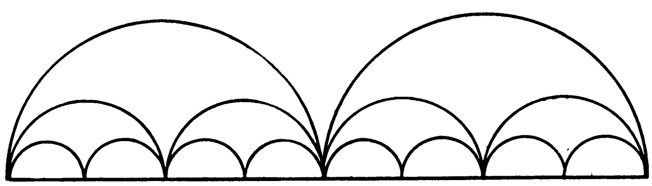

FIG. 5.

to H, and for each n let a_{1n}, a_{2n}, $a_{3n}, \cdots$ denote a contracting sequence of arcs all of diameter less than $1/n$, such that for each m the arc a_{mn} contains the point P_n but no other point of H and no other point of the arc a_{kn} $(k\neq m)$. Let M be the continuum $H+\sum_{i=1}^{\infty}\sum_{n=1}^{\infty}a_{in}$. Then M is a regular curve. Now any arc which lies between the lines $x=0$ and $x=1$ and has a point above and a point below the x-axis either contains the point P_n for some integer n or it contains infinitely many points of M. In the first case it cuts M into infinitely many components.

THEOREM VII. *If every point of a bounded regular curve M in a euclidean plane S is of finite order*† *then there exists a continuous transformation T of S into itself such that* (1) *no straight line contains a nondegenerate connected subset*

† If P is a point of a regular curve M and there exists an integer n such that for every positive number ϵ there is a domain of diameter less than ϵ which contains P and whose boundary has not more than n points in common with M then the point P is said to be of *finite order*. Cf. K. Menger, loc. cit.

of $T(M)$ and no point of $T(M)$ is rational, (2) *the rational lines contain only a finite number of points of $T(M)$, and* (3) *if R is the interior of a rectangle composed of intervals of rational lines, then $R \cdot T(M)$ is the sum of a finite number of connected sets.*

Let T be a transformation satisfying the conclusion of Theorem VI. Since each point of M on a rational line L is of finite order it is not a limit point of infinitely many components of $T(M)-L$. In view of this, and the additional fact that the set of points of M on any rational line is finite, it is clear that the transformation T satisfies the conclusion of Theorem VII.

THEOREM VIII. *If M is a bounded regular curve which contains only a finite number of simple closed curves then there exists a transformation T satisfying the conclusion of Theorem* VII.

To help prove Theorem VIII I will establish the following lemma.

LEMMA III. *If M is a bounded regular curve which contains only a finite number of simple closed curves, and R is a connected domain containing two points A and B not belonging to M, then there exists a simple continuous arc AB which lies within R, contains only a finite number of points of M, and is such that $M-M\cdot AB$ is the sum of a finite number of connected sets.*

Proof of Lemma III. Let H denote the set of junction points of M. Let J be a simple closed curve enclosing A and B and lying in R, and let g denote a simple continuous arc from A to B which lies within J, contains no non-degenerate connected subset of M, and no point of H. But the outer boundary of every bounded complementary domain of a continuous curve is† a simple closed curve and‡ no two bounded complementary domains of a continuous curve have the same outer boundary. Hence since M contains only a finite number of simple closed curves it follows that only a finite number of complementary domains of $M+J$ have boundary points on the arc g. If one of these domains contains both A and B the lemma is obviously established. If not let P_1 be the last point of g in the order from A to B which is on the boundary of that complementary domain of $M+J$ which contains A. Then§ there exists an arc AP_1 which lies wholly in this domain except for the point P_1. Since P_1 is a limit point of the points of $S-M$ on the

† R. L. Moore, *Concerning continuous curves in the plane*, Mathematische Zeitschrift, vol. 15 (1922), Theorem 4 and p. 259.

‡ R. L. Moore, *Concerning paths that do not separate a given continuous curve*, Proceedings of the National Academy of Sciences, vol. 12 (1926), Theorem 1.

§ Schoenflies, *Die Entwickelung der Lehre von den Punktmannigfaltigkeiten*, zweiter Teil, Jahresbericht der Deutschen Mathematiker-Vereinigung, Ergänzungsbände, vol. 2 (1908).

arc P_1B of g it follows that P_1 is a boundary point of some complementary domain of $M+J$ which contains points on the arc P_1B. Let P_2 be the last point on the arc P_1B belonging to the boundary of a complementary domain of $M+J$ which also has P_1 on its boundary. If P_2 is the same as P_1 then the arc P_1B contains no point of $M+J$. In either case there exists a simple continuous arc with P_1 and P_2 as end points which contains no point of $M+J$ except P_1 and P_2. Continuing this process a finite number of times one obtains a simple continuous arc AB which contains only a finite number of points of M and no junction point of M. Clearly this arc satisfies the conclusion of the lemma.

A proof of Theorem VIII can now be given which is closely analogous to the proof of Theorem III. The essential difference is that the arcs of the double ruling r_n are here to be chosen so as to have the properties stated in Lemma III rather than those stated in Lemma II.

THEOREM IX. *If A and B are distinct points of a continuous curve M then M contains a simple continuous arc from A to B every subarc of which contains a subarc which either lies on the boundary of some complementary domain of M or lies in some domain which belongs to M.*

(1) Suppose M contains no domain. Let AXB denote a simple continuous arc from A to B such that the common part of AXB and M is totally disconnected. Let T denote the set $M\cdot AXB$. Let $D_1, D_2, \cdots$ denote the complementary domains of M which contain limit points on the arc AXB and for each i let J_i denote the boundary of D_i. Let K be the point set $T+J_1+J_2+J_3+\cdots$. Since $J_1, J_2, J_3, \cdots$ is a contracting sequence of continuous curves all containing points on the arc AXB it is readily seen that the set K is closed. If P is an interior point of the arc AXB which does not belong to K then there exists a connected subset of K containing the last point of T which precedes P on the arc AXB and the first point of T which follows P on this arc. Therefore K is connected. With the use of the fact that the boundary of every complementary domain of a continuous curve is itself a continuous curve† it readily follows that K is connected im kleinen. Hence K is a continuous curve. Let AB denote any arc which lies in K‡ and let EF denote any subarc of AB. Since T is totally disconnected the arc EF contains a subarc $E'F'$ which contains no point of T. The arc $E'F'$ is a subset of $J_1+J_2+J_3+\cdots$, and hence is equal to $J_1\cdot E'F'+J_2\cdot E'F'+J_3\cdot E'F'+\cdots$. But the sum of a countable number of totally dis-

† R. L. Moore, *Concerning continuous curves in the plane*, Mathematische Zeitschrift, vol. 15 (1922), p. 259.

‡ See third footnote on p. 14.

connected closed point sets is not connected. Hence there exists at least one integer i such that the set $J_i \cdot E'F'$ contains an arc. Thus every subarc of AB contains an arc belonging to the boundary of some complementary domain of M.

(2) If A and B are distinct points of a continuous curve M which contains a domain let M_1 be a continuous curve containing A and B which is obtained by taking from M the interiors $I_1, I_2, I_3, \cdots$ of a contracting sequence of circles such that (1) for every domain D which is a subset of M there is an integer n such that I_n contains at least one point of D, (2) for every n, $\bar{I}_n$ is a subset of some domain belonging to M and (3) $\bar{I}_k \bar{I}_n = 0 (k \neq n)$. Let AB denote an arc satisfying the conclusion of the theorem with respect to M_1. Since the boundary of a complementary domain of M_1 which is not a complementary domain of M belongs in a domain lying in M it is obvious that AB satisfies the conclusion of the theorem with respect to M.

THEOREM X. *If M is a bounded continuous curve which contains no domain then there exists a continuous transformation T of the plane S into itself such that* (1) *if AB is an arc such that $T(AB)$ is a subset of a rational line then $AB \cdot M = c_1 + c_2 + \cdots + c_n$ where for each i $(i \leqq n)$ c_i is an arc or a point and if c_i is an arc then every subarc of c_i contains a subarc lying on the boundary of some complementary domain of M and* (2) *if AB is an arc such that $T(AB)$ is a subset of an irrational horizontal or vertical line then $AB \cdot M$ is vacuous or totally disconnected.*

To help establish Theorem X, I will prove several lemmas. To avoid repetition I will say that an arc AB has *property c_ϵ with respect to M*, or merely that it has *property c_ϵ* if the common part of M and AB is the sum of a finite number of connected sets such that each of these sets which is an arc is of diameter less than ϵ and has the property that every subarc of it contains a subarc lying on the boundary of some complementary domain of M.

LEMMA IV. *If I is the interior of a simple closed curve and M is a bounded continuous curve containing no domain and A and B are distinct points lying in I and ϵ is any positive number, then there exists a simple continuous arc from A to B which is a subset of I and which has property c_ϵ.*

With the help of Theorem II it can readily be seen that there exists a simple closed curve J_1 lying in I, enclosing A and B, and such that (1) $J_1 \cdot M$ is totally disconnected, and (2) if I_1 denotes the interior of J_1 then no component of $M \cdot \bar{I}_1$ is of diameter greater than or equal to ϵ. Let AXB denote any simple continuous arc from A to B which lies in I_1 and let $s_1, s_2, \cdots, s_n$ denote the components of $M \cdot \bar{I}_1$ which have points on AXB. For each i

$(i \leqq n)$ s_i is† a continuous curve. In view of this fact and Theorem IX it follows that there exists an arc AB in $\bar{I}_1$ such that $M \cdot AB$ is a subset of $s_1+s_2+\cdots+s_n$ and is the sum of n or less connected sets such that each of these sets which is an arc has the properties of the arc of Theorem IX with respect to that one of the continuous curves $s_1, s_2, \cdots, s_n$ to which it belongs. Let EF denote an arc belonging to $M \cdot AB$. Since EF contains a subarc lying wholly within J it can easily be shown that EF has the properties stated in Theorem IX with respect to M. Since in addition EF is of diameter less than ϵ the lemma is proved.

LEMMA V. *If J is a simple closed curve and KL is a simple continuous arc which lies within J except that K and L are on J and KL is on the boundary of a complementary domain D of M, then there exists a simple continuous arc AB which lies within J such that* (1) *the common part of AB and KL is a single point,* (2) *KL separates A from B within J, and* (3) *$AB \cdot M$ is either an arc or a point, and if it is an arc it has property c_ϵ.*

Let C denote the interior of a circle which lies within J and encloses a point of KL. There exists a point A in $C \cdot D$ and a subarc $E'F'$ of EF such that for every point P of $E'F'$ there exists an arc AP which lies in $C \cdot D$ except for the point P. Let O denote some interior point of $E'F'$ and let C_1 denote the interior of a circle J_1 which lies in C such that C_1 contains O but contains no point of $KL-E'F'$. Let A' and B' denote points in C_1 lying respectively on the A side and the non A side of KL and let $B'A'$ denote an arc having property c_ϵ and lying in C_1. Let Q denote the first point of $B'A'$ on KL in the order from B' to A'. Let AQ denote an arc lying in $C \cdot D$ except for the point Q. Let QB denote a subarc of QB' such that $QB \cdot M$ is connected. The sum of the arcs AQ and QB gives an arc AB which satisfies the conclusion of the lemma.

LEMMA VI. *If M is a continuous curve containing no domain and lying within a simple closed curve J whose interior is R, ϵ_1 and ϵ_2 are any positive numbers and α is a double ruling of R such that every arc of α has property c_{ϵ_1}, then there exists a double ruling β of R such that every arc of α is also an arc of β, every arc of β which is not an arc of α has property c_{ϵ_2} and every component of $\bar{R}$ minus the sum of the arcs of the ruling β is of diameter less than ϵ_2.*

With the help of a theorem of Schoenflies‡ it is easily seen that there exists a continuous transformation T_1 of the plane into itself which throws J into a square $ABCD$ and the arcs of α into horizontal and vertical inter-

† See H. M. Gehman, loc. cit.

‡ Loc. cit.

vals. Let α_1 be the double ruling of $T(R)$ which is composed of all arcs $T(g)$ where g is an arc of α. Since T_1 is continuous it obviously follows that there exist two finite sets of rectangles $h_1, h_2, \cdots, h_n, v_1, v_2, \cdots, v_n$ such that (1) for each i $(i \leqq n)$, $h_i[v_i]$ has no point in common with any arc of α_1 which is parallel to AB $[BC]$ and no point in common with $h_j[v_j]$ $(j \leqq n, j \neq i)$ but contains at least one point of AD and one point of BC $[AB$ and $CD]$, and (2) if α_2 is any double ruling of R which contains all of the arcs of α_1 and in addition contains arcs a_i and b_i such that $T(a_i)$ lies in h_i plus its interior and $T(b_i)$ lies in v_i plus its interior $(i=1, 2, \cdots, n)$ then every component of $\overline{R}$ minus the sum of the arcs of α_2 is of diameter less than ϵ_2. It is easily shown with the help of Lemmas IV and V that a particular such ruling β can be obtained such that the arcs of β which do not belong to α have property c_{ϵ_2}.

LEMMA VII. *Suppose M is a continuous curve which contains no domain and lies in the interior R of a square $ABCD$. Let α be any double ruling of R and let P be a point of R not belonging to any arc of α. Then there exists an integer k such that if $n>k$ and β is any double ruling of R such that* (1) *β contains α,* (2) *no arc of β contains P,* (3) *every component of $\overline{R}$ minus the arcs of β is of diameter less than $1/n$, and* (4) *every arc of β which does not belong to α has property $c_{1/n}$, then if E denotes the component, containing P, of $\overline{R}$ minus the arcs of β which are parallel to AB $[BC]$, there exists in E an arc a_E with property $c_{1/(n+1)}$ which together with the arcs of β forms a double ruling of R and such that no component of $M \cdot (E-a_E)$ contains points in more than two components of $\overline{R}$ minus the arcs of α.* (See Fig. 6.)

Let ϵ_1 be a positive number such that if a_1 and b_1 denote any arcs of α which have no point in common then the distance from any point of a_1 to any point of b_1 is greater than ϵ_1. Let ϵ_2 be a positive number such that a circle with P as center and ϵ_2 as radius neither contains nor encloses any point of any arc of α or of $ABCD$. Let k be any integer greater than both $1/\epsilon_1$ and $1/\epsilon_2$. Suppose β is a double ruling of R with properties (1), (2), (3), and (4) as given above. Let E denote the component, containing P, of $\overline{R}$ minus the arcs of β which are parallel to AB (for example). In view of property (3) and the additional fact that $1/n<\epsilon_2$ it is obvious that a_1 and a_2, the arcs of β on the boundary of E, do not belong to α. From (4) it follows that both a_1 and a_2 have property $c_{1/n}$. Since $1/n<\epsilon_1$ it follows that between each two distinct arcs of β parallel to BC the arc $a_i (i=1, 2)$ contains a point not belonging to M. Hence if $m+1$ denotes the number of arcs of β which are parallel to BC it is easily seen that there exist $2m$ circles lying in R with interiors $C_{11}, C_{12}, \cdots, C_{1m}, C_{21}, C_{22}, \cdots, C_{2m}$ such that (1) $C_{ik} (i=1, 2; k \leqq m)$

contains no point of M and no point of any arc of α, and (2) for each two adjacent arcs b_1 and b_2 of α which are parallel to BC there exist integers i and j such that C_{1i} and C_{2j} contain points of a_1 and a_2, respectively, which lie between b_1 and b_2. With the help of Lemmas IV and V it is seen that there exists in E an arc a_E with property $c_{1/(n+1)}$ which together with the arcs of β forms a double ruling of R, and in addition contains a point in $C_{ik}(i=1, 2; k=1, 2, \cdots, m)$. Obviously no component of $M\cdot(E-a_E)$ contains points in more than two components of $\overline{R}$ minus the arcs of α.

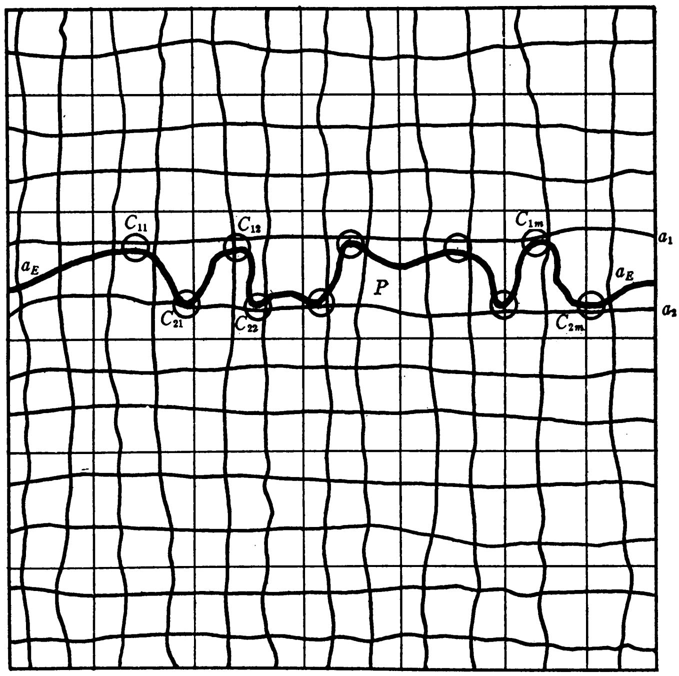

Fig. 6

Proof of Theorem X. Suppose M lies in the interior R of a square $ABCD$. With the help of Lemmas IV, V, and VI one can readily see that there exists a double ruling β_1 of $ABCD$ such that (1) every component of $\overline{R}$ minus the arcs of β_1 is of diameter less than 1, (2) every arc of β_1 has property c_1, and (3) between each two adjacent arcs of β_1 parallel to BC there exists on each arc of β_1 parallel to AB a point not belonging to M. Let a_1 and a_2 denote any two adjacent arcs of β_1 parallel to AB (or a_1 or a_2 may be AB or CD) and let E denote the set of all points of $\overline{R}$ which lie between a_1 and a_2. If $m+1$ denotes the number of arcs of β_1 which are parallel to BC it is clear that there exist $2m$ circles lying in R with interiors $C_{11}, C_{12}, \cdots, C_{1m}, C_{21}, C_{22}, \cdots, C_{2m}$ such that (1) $C_{ik} (i=1, 2; k \leqq m)$ contains no point of M and no point of any arc of β_1 which is parallel to BC, and (2) for each two adjacent arcs b_1 and b_2 of β_1 parallel to BC there exist integers j and k such that C_{1j} and C_{2k} contain points of a_1 and a_2, respectively, which lie between b_1 and b_2. With the help of Lemmas IV and V it is seen that there exists in E an arc a_E which together with the arcs of β_1 forms a double ruling of R and such that (1) a_E contains points in the set $C_{ik} (i=1, 2; k=1, 2, \cdots, m)$, and (2) a_E has property $c_{1/2}$. Obviously no component of $M \cdot (E - a_E)$ contains points in more than two components of $\overline{R}$ minus the arcs of the ruling β_1. Let α_1 be the double ruling obtained by adding to β_1 the arc a_E for every component E of $\overline{R}$ minus the arcs of β_1 which are parallel to AB. Let β_2 be a double ruling which contains α_1 and is such that (1) every component of $\overline{R}$ minus the arcs of β_2 is of diameter less than $1/2$ and (2) the arcs of β_2 which do not belong to β_1 have property $c_{1/2}$.

Now let E denote the set of all points of $\overline{R}$ which lie between two adjacent arcs of β_2 which are parallel to BC. If there exists in E an arc a^* which together with the arcs of β_2 forms a double ruling of R and such that (1) no component of $M \cdot (E - a^*)$ has points in more than two components of $\overline{R}$ minus the arcs of β_2, and (2) the arc a^* has property $c_{1/3}$, then let a_E be such an arc a^*. If no such arc exists, but there does exist an arc a^{**} having the above properties except that in (1) the symbol β_1 replaces the symbol β_2, then let a_E denote such an arc a^{**}. If neither a^* nor a^{**} exists let a_E be any arc in E which together with the arcs of β_2 forms a double ruling of R and which has property $c_{1/3}$. Let α_2 be the double ruling obtained by adding to β_2 the arc a_E for every component E of $\overline{R}$ minus the arcs of β_2 which are parallel to BC. Let β_3 be a double ruling which contains α_2 and is such that (1) every component of $\overline{R}$ minus the arcs of β_3 is of diameter less than $1/3$, and (2) the arcs of β_3 which do not belong to β_2 have property $c_{1/3}$.

Proceeding in this way one can show that there exists an infinite sequence of double rulings $\beta_1, \beta_2, \beta_3, \cdots$, of R such that for every n the following properties obtain: (1) β_{n+1} contains β_n, (2) every component of $\overline{R}$ minus the arcs of β_n is of diameter less than $1/n$, (3) every arc of β_{n+1} which does not belong to β_n has property $c_{1/(n+1)}$, and (4) if a_{1n} and a_{2n} are adjacent arcs of β_n which are parallel to AB for n odd and parallel to BC for n even, and E is the set of all points of $\overline{R}$ which lie between a_{1n} and a_{2n}, then if there exists a positive integer $j (j \leqq n-1)$ and an arc a_{3E} which lies in E such that (a) the arc a_{3E} together with the arcs of β_n forms a double ruling of R, and (b) no component of $M \cdot (E - a_{3E})$ contains points in more than two components of $\overline{R}$ minus the arcs of β_j, then, k_E denoting the largest such integer j, β_{n+1}

contains an arc having the properties stated above for the arc a_{3E} with j replaced by k_E.

Let P denote a point of R not belonging to any arc of any of the double rulings $\beta_1, \beta_2, \beta_3, \cdots$. For each positive integer n let $E_{n,P}$ denote the component, containing P, or $\bar{R}$ minus the arcs of β_n which are parallel to AB (for example). In view of properties (1), (2), and (3), and Lemma VII, it can be seen that the integer n can be taken large enough so that the integer j as qualified in property (4) does exist for $E_{n,P}$, and furthermore $k_{E_{n,P}}$ increases indefinitely as n increases indefinitely. Hence it follows that for every connected subset L of M there exists an integer n_L such that some arc of β_{n_L} parallel to $AB[BC]$ has a point in common with L. In view of properties (1) and (2) it follows by methods employed in proving previous theorems that there exists a continuous transformation T_1 of $\bar{R}$ into itself such that (1) for every n, T_1 throws the arcs of β_n into intervals of rational lines, and (2) if L is any rational line then there exists an integer n and an arc g of β_n such that $T_1(g)$ is a subset of L. Obviously there exists a continuous transformation T of the plane S into itself which reduces to T_1 for points of $\bar{R}$. Such a transformation satisfies the conclusion of the theorem.

Now as shown in the proof of Theorem IX a continuous curve M which contains a domain contains a continuous curve M_1 such that M_1 contains no domain but does contain every boundary point of M, and such that if D is a complementary domain of M_1 which is not a complementary domain of M then $\bar{D}$ lies in a domain of M. In view of this fact and the previous theorem the following corollaries may be easily established.

COROLLARY 1. *If M is a bounded continuous curve then there exists a continuous transformation T of the plane into itself such that if AB is an arc and $T(AB)$ is an interval of some rational line then* (1) *$AB \cdot M$ is the sum of a finite number of connected sets, and* (2) *every arc which is a subset of $AB \cdot M$ contains a subarc which either lies on the boundary of a complementary domain of M or lies in a domain which belongs to M.*

COROLLARY 2. *If M is a bounded continuous curve and P is a point of M which is not in a domain belonging to M then there exists a continuous transformation T of S into itself such that if APB is any arc such that $T(APB)$ is a subset of a horizontal line, then the component of $M \cdot APB$ which contains P is P, and if AB is an arc such that $T(AB)$ is a subset of a rational horizontal line then the number of components of $AB \cdot M$ is finite.*

Let A be any point and for each n let C_n be a circle of radius $1/n$ and center A. Let AB be a unit interval and let M be the continuous curve

$AB+C_1+C_2+\cdots$. This example shows that it is not true that if M is any continuous curve then there exists a continuous transformation T of the plane S into itself such that if AB is an arc and $T(AB)$ is a horizontal interval then $AB\cdot M$ is the sum of a finite number of connected sets.

THEOREM XI. *If P is a point of a bounded continuous curve M then there exists an upper semi-continuous collection*† *G of subcontinua of M which fills up M such that P is an element of G and G is a regular curve with respect to its elements.*

Suppose first that P is a point which does not belong to a domain which belongs to M and let T denote a transformation satisfying the conclusion of Corollary 2. For each point x of M let g_x be the greatest continuum containing x such that $T(g_x)$ is a subset of some horizontal line, and let G denote the collection of continua g_x for all points x of M. Clearly $g_P=P$, and the collection G is upper semi-continuous. Now if M_1 is any continuum such that the common part of any rational horizontal line and M_1 is a finite point set, and the common part of any horizontal line and M_1 is totally disconnected, then M_1 is a regular curve. Hence G is a regular curve with respect to its elements.

Suppose P is a point lying in a domain D of M. There exists a set K of mutually exclusive simple closed curves lying in D, all enclosing P, no two having a point in common, and such that every point of $D-P$ belongs to some curve of the set K. Let G be the upper semi-continuous collection of continua consisting of the curves of the collection K and the continua $M-D$ and P. The collection G is an arc with respect to its elements and one of its elements is P.

† See R. L. Moore, *Concerning upper semi-continuous collections of continua*, these Transactions, vol. 27 (1925), pp. 416–428. A collection G of continua is said to be an *upper semi-continuous* collection if for each element g of the collection G and each positive number e there exists a positive number d such that if x is any element of G at a lower distance from g less than d then the upper distance of x from g is less than e. If M is a point set and P is a point, then by $l(PM)$ is meant the lower bound of the distances from P to all the different points of M. If M and N are two point sets, then by $l(MN)$ is meant the lower bound of the values $[l(PN)]$ for all points P of M, while by $u(MN)$ is meant the upper bound of these values for all points P of M. The point set M is said to be at the upper distance $u(MN)$ from the point set N and is said to be at the lower distance $l(MN)$ from N.

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