ON THE REGULAR POINTS OF A CONTINUUM*

BY

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I. Introduction

1. We consider a compact, connected metric space M which we shall call the continuum M. A point p of M is said to be a *regular point* if for each $\epsilon>0$ there exists a neighborhood U_p of p (i.e., an open subset of M containing p) such that $d(U_p)<\epsilon$ and $F(U_p)$ consists of a finite number of points. The point p is said to be a *point of order* α if (1) for each $\epsilon>0$ there exists a neighborhood U_p such that $d(U_p)<\epsilon$ and the cardinal number of $F(U_p)\leqq\alpha$, and (2) α is the smallest cardinal number for which (1) is true. Regular points of no finite order are called points of order ω. Let M^α denote the set of all points of M of order α. Then

$$M = M^1 + M^2 + \cdots + M^\omega + M^{\aleph_0} + M^c.$$

These definitions were introduced by Urysohn and Menger† several years ago and since that time have been studied in a number of papers.

One of the interesting studies in this theory is that of the distribution and structure of the various sets M^α. It is of course quite obvious that M may be composed entirely of points of order 2 or entirely of points of order c. Urysohn‡ has given examples to show that M may be composed entirely of points of order ω or entirely of points of order $\aleph_0$. Except for these four orders this is not possible, a consequence of a theorem proved independently by Urysohn,§ G. T. Whyburn‖ and H. Künneth¶ that if all points of M are of order $\leqq n$, then the points of order $\leqq\frac{1}{2}n+1$ are dense in M.

Since $M\neq M^n (n\neq 2)$, it would be interesting to know more of the distribution of the points of M^n. Whyburn** has shown that M^n is punctiform and has raised the question as to whether it is of dimension zero. In the present

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† P. Urysohn, Comptes Rendus, vol. 175 (1922), pp. 481–483; and K. Menger, Monatshefte für Mathematik und Physik, vol. 33 (1923), pp. 148–160.

‡ *Mémoire sur les multiplicités cantoriennes*, 2ème Partie, Verhandelingen, Akademie van Wetenschappen, Amsterdam, vol. 13 (1927), No. 4, pp. 109–115.

§ Ibid., pp. 105–9.

‖ *On regular points of continua* etc., Bulletin of the American Mathematical Society, vol. 35 (1929), pp. 218–224.

¶ *Ein Theorem der Kurventheorie*, Monatshefte für Mathematik und Physik, vol. 36 (1929), pp. 149–152.

** Loc. cit.

paper we will answer this question in the affirmative as a special case of our theorem that $M^n+M^{n+1}+\cdots+M^{2n-3}$ is a set of dimension zero. As corollaries of this theorem we obtain most of the previously known results concerning the distribution of the points of finite orders. In the last section of the paper we complete our study of the structure of the sets of various finite orders by examining the set M^2. Here we prove that M^2 is composed of a set of dimension zero plus a countable set of arcs.

2. **Notation.** Capitals will denote sets, lower case letters denote individual elements which are either points or numbers. The usual notation of the theory of sets will be employed. Below we will list some special notation which, while not new, is not universally employed by writers in this field and thus needs definition.

$p \mathcal{E} N \equiv p$ is an element of the set N.
p non-$\mathcal{E} N \equiv p$ is not an element of the set N.
$\rho(p, q) \equiv$ distance between the points p and q.
$\rho(M, N) \equiv$ greatest lower bound of $\rho(p, q)$ for $p \mathcal{E} M$ and $q \mathcal{E} N$.
$d(M) \equiv$ diameter of $M \equiv$ least upper bound of $\rho(p, q)$ for $p+q \subset M$.
$S(p, \eta) \equiv$ set of all points q such that $\rho(p, q)<\eta$.
$\mathcal{C}(p, N) \equiv$ component of set N containing the point p.
$F(N) \equiv \overline{N} \cdot (M-N)+\overline{M-N} \cdot N \equiv$ frontier or boundary of N.
$C(N) \equiv M-N \equiv$ complement of N.
$\dim_p E$, $\text{order}_p E \equiv$ dimension of set E at p, order of E at p.

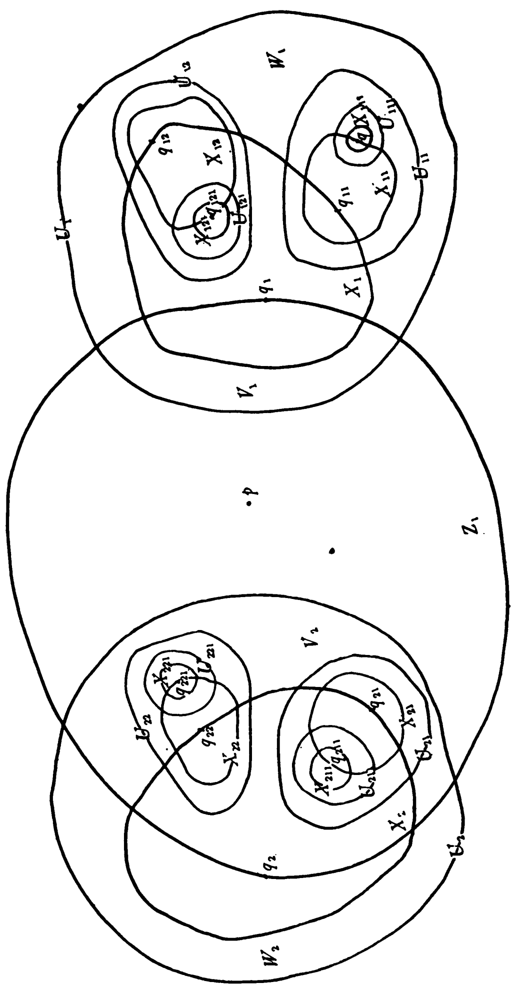
II. The structure of M^n, $n \neq 2$

3. Theorem. *For any integer $n>2$, the set of all points p of a continuum M such that $n \leqq \text{order}_p M \leqq 2n-3$ is a zero-dimensional set (or vacuous).**

Let E denote the set of points p such that $n \leqq \text{order}_p M \leqq 2n-3$. Given $\epsilon>0$ and $p \mathcal{E} E$, we shall show the existence of a neighborhood $U_p \subset S(p, \epsilon)$ such that $F(U_p) \cdot E=0$, i.e., $\dim E=0$. If p is a point of order m, by an *order neighborhood* of p we shall mean a neighborhood of p whose boundary consists of exactly m points.

Let $Z_1 \subset S(p, \frac{1}{2}\epsilon)$ be an order neighborhood of p. If $E \cdot F(Z_1)=0$, our proof is complete. If not, let $E \cdot F(Z_1)=q_1+q_2+\cdots+q_{s_1}$ $(s_1 \leqq 2n-3)$. Let U_k be an order neighborhood of $q_k (1 \leqq k \leqq s_1)$ such that $\overline{U}_k \subset S(q_k, r_k)$, where r_k is the smaller of the numbers $\epsilon/4$ and $\frac{1}{2}\rho(q_k, F(Z_1)+p-q_k)$. We see that $\overline{U}_i \cdot \overline{U}_j=0$ if $i \neq j$. Let $V_k=Z_1 \cdot U_k$ and $W_k=C(\overline{Z}_1) \cdot U_k$. Since $q_k \mathcal{E} E$, $F(U_k)$

* Professor K. Menger has called attention to the fact that no use is made of the condition that M be connected and compact in the proof of this theorem. Hence the theorem is true for *any* metric space M.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consists of $\leqq 2n-3$ points. And since $F(V_k)+F(W_k)=F(U_k)+q_k$ and $F(V_k)\cdot F(W_k)=q_k$, either $F(V_k)$ or $F(W_k)$ consists of $\leqq n-1$ points. Let I_1 be the set of all integers i such that $F(W_i)$ consists of $\leqq n-1$ points, and let J_1 be the set of all integers $1\leqq j\leqq s_1$ not in I_1. Let X_k be an order neighborhood of q_k such that $\overline{X}_k\subset U_k$. Now let

$$Z_2 = Z_1 + \sum X_i - \sum \overline{X}_j,$$

where the first summation extends over I_1 and the second over J_1.

If $F(Z_2)\cdot E=0$, then Z_2 is the desired neighborhood U_p. If not, we have $F(Z_2)\cdot E\subset\sum_k F(X_k)$. Let $F(X_k)\cdot F(Z_2)\cdot E=q_{k1}+q_{k2}+\cdots+q_{ks_{2}k}$. For each point $q_{im}(i\,\mathcal{E}\,I_1)$, let U_{im} be an order neighborhood of q_{im} such that $\overline{U}_{im}\subset W_i\cdot S(q_{im},\ r_{im})$, where r_{im} is the smaller of the numbers $\epsilon/8$ and $\frac{1}{2}\rho(q_{im},\ F(X_i)-q_{im})$. For each point $q_{jm}(j\,\mathcal{E}\,J_1)$, let U_{jm} be an order neighborhood of q_{jm} such that $\overline{U}_{jm}\subset V_{jm}\cdot S(q_{jm},\ r_{jm})$, where r_{jm} is the smaller of the numbers $\epsilon/8$ and $\frac{1}{2}\rho(q_{jm},\ F(X_j)-q_{jm})$. We see that $\overline{U}_{k_1m_1}\cdot\overline{U}_{k_2m_2}=0$ unless $k_1=k_2$, $m_1=m_2$. Let $V_{km}=Z_2\cdot U_{km}$ and $W_{km}=C(\overline{Z}_2)\cdot U_{km}$. Since $q_{km}\,\mathcal{E}\,E$, $F(U_{km})$ consists of $\leqq 2n-3$ points. Then either $F(W_{km})$ or $F(V_{km})$ consists of $\leqq n-1$ points. Let I_2 be the set of all pairs $(k,\ m)$ such that $F(W_{km})$ consists of $\leqq n-1$ points. Let J_2 be the set of all pairs $(k,\ m)$ for which U_{km} is defined that are not in I_2. Let X_{km} be an order neighborhood of q_{km} such that $\overline{X}_{km}\subset U_{km}$. Now let

$$Z_3 = Z_2 + \sum X_{km} - \sum \overline{X}_{km},$$

where in the first summation $(k,\ m)\,\mathcal{E}\,I_2$ and in the second $(k,\ m)\,\mathcal{E}\,J_2$.

If $F(Z_3)\cdot E=0$, Z_3 is the desired neighborhood U_p. If not, we repeat this process on the points of $F(Z_3)\cdot E=F(Z_3)\cdot E\cdot\sum_k\sum_m F(X_{km})$. Continuing this process, at some stage we reach a neighborhood Z_i such that $F(Z_i)\cdot E=0$ or the process continues indefinitely.

In case the process continues indefinitely, we define a monotonic increasing sequence of neighborhoods of p as follows:

$$Y_1 = Z_1 - \sum_{k=1}^{s_1}\overline{U}_k,\quad Y_2 = Z_2 - \sum_{k=1}^{s_1}\sum_{m=1}^{s_{2k}}\overline{U}_{km},$$

and similarly we define Y_t for each positive integer t. Now let

$$U_p = \sum_{t=1}^{\infty} Y_t.$$

Since $U_p\subset Z_1+\sum_k U_k\subset S(p,\ \epsilon)$, then U_p is the desired neighborhood if $F(U_p)\cdot E=0$. Every point of $F(Z_t)$ not belonging to E is a point of $F(Y_t)$ and of every $F(Y_s)$ for $s\geqq t$. Then the points of $F(U_p)$ are of two classes:

(a) points which belong to $F(Y_s)$ for every s greater than some fixed integer, and (b) $\Pi_{t=1}^{\infty}(F(U_p)-F(Y_t))$. Just above it was seen that no point of the class (a) is a point of E. Now as

$$F(U_p) - F(Y_t) \subset \sum U_{k_1 k_2 \cdots k_t},$$

we have that every point of $F(U_p)$ of class (b) belongs to the set

$$H = \prod_{t=1}^{\infty} \sum U_{k_1 k_2 \cdots k_t}.$$

Now consider any point y of H. Each neighborhood U_{km} is a subset of V_k or W_k according as $k \in I_1$ or J_1, i.e. U_{km} is a subset of a neighborhood of the first stage whose boundary contains $\leqq n-1$ points. Similarly for the neighborhoods $U_{k_1 k_2 \cdots k_t}$ of any stage. Then at each stage y belongs to a neighborhood whose boundary contains $\leqq n-1$ points. And as the diameters of the neighborhoods approach zero, it follows that $\text{order}_y\ M \leqq n-1$. Hence y non-$\in E$. Then U_p is the desired neighborhood of p as

$$U_p \subset S(p, \epsilon) \text{ and } F(U_p) \cdot E = 0.$$

4. This section proves corollaries to the preceding theorem.

Corollary 1. *For each positive integer $n \neq 2$, the set of all points of M of order n is zero-dimensional (or vacuous).*

Since a subset of a vacuous or zero-dimensional set is necessarily of the same type, for $n>2$ this follows from our theorem. That the set M^1 is of this type has been shown by Menger and Urysohn.*

Corollary 2.† *There exists no continuum all of whose points are of order $n \neq 2$.*

If M is a continuum, dim $M \geqq 1$. Hence $M - M^n \neq 0$ for any $n \neq 2$.

Corollary 3. *The simple closed curve is the only (compact) continuum all of whose points are of the same finite order.*

Corollary 4. *If the order of every point of the continuum M is $\leqq m$, then the points of order $\leqq \frac{1}{2}m+1$ are dense in M.*

From our theorem it follows that the set of all points p such that $\frac{1}{2}m+1 < \text{order}_p\ M \leqq m$ is zero-dimensional. Hence it contains no open subset and the remaining points are dense in M.

5. **Remarks.** It may be noticed in the proof of the preceding theorem

* K. Menger, Mathematische Annalen, vol. 95 (1925), p. 293; and P. Urysohn, Second reference, p. 79.

† The corollaries 2, 3 and 4 are all known results. See the papers cited in the Introduction.

that, while at each stage the neighborhoods have only finite boundaries, the final neighborhood U_p may have an infinite boundary. This raises the question as to whether it is possible to select U_p with only a finite boundary. This is not always true. If we join two of the Sierpinski triangle curves* at the vertices we have a continuum containing only points of orders 3 and 4. Now if we take $n=4$ the only neighborhoods U_p such that $F(U_p)\cdot E=0$ have infinite boundaries.

We have seen that dim $M^n=0$ for $n>2$. It would be interesting now to determine the conditions under which dim $\sum_{n>2} M^n=0$.

Urysohn has constructed very interesting examples of continua containing points of orders n and $2n-2$ only for any $n>2$. This should lead to a study of continua containing points of orders m and n only $(m>n>2)$. From our theorem or Corollary 4, it follows that such a continuum can exist only if $m\geqq 2n-2$. It would be interesting to determine whether it can exist if $m\neq k(n-1)$. Also it seems likely that in such a continuum the points of order m are countable.

III. The structure of M^2

6. Lemma. *If p is a point of M^2 and a cut point im kleinen of M and* $\dim_p M^2>0$, *then M contains an arc one of whose end points is p such that the arc-segment*† *is an open subset of M.*

As p is a cut point im kleinen of M, there is a neighborhood Z_p such that $\mathcal{C}(p,\ \overline{Z}_p)=R+S$, where R and S are continua and $R\cdot S=p$. Then either $\dim_p R\cdot M^2>0$ or $\dim_p S\cdot M^2>0$ and we suppose the former. There exists a neighborhood U_p such that if the neighborhood $V_p\subset U_p$, then

$$F(V_p)\cdot M^2\cdot R\neq 0.$$

As $p\in M^2$ there is a neighborhood W_p such that $\overline{W}_p\subset U_p$ and $F(W_p)\cdot R$ is a single point r and $R-\overline{W}_p\neq 0$. Let $G=\overline{W}_p\cdot R$. As $r\in M^2$ and $R-\overline{W}_p\neq 0$, the point $r\in G^1$, i.e. the points of G of order 1 with respect to G. As $p\in G^1$ and $M^2\cdot G=G^2+p+r$,we will complete our proof by establishing the next lemma.

7. Lemma. *If G is a continuum and* (a) *p and r are end points of G, and* (b) *for any neighborhood U_p such that r non-$\in \overline{U}_p$, the set $F(U_p)\cdot G^2\neq 0$, then G is an arc with end points p and r.*

Let Q denote the set consisting of $p+r+$those points of G^2 which separate p and r in G. We shall prove that Q is a closed set. Let $[q_i]$ be a sequence of

* Comptes Rendus, vol. 160 (1915), pp. 302–5; and Prace Matematyczno-Fizyczne, vol. 27 (1915), pp. 77–86.

† By the *arc-segment* is meant the arc minus its end points.

distinct points of Q such that $\lim q_i = q$. We shall show that $q \mathcal{E} Q$. This is true if $p = q$ or $r = q$. If $p \neq q \neq r$, there are two cases:

Case I. There is an infinite subsequence $[q_{i_k}]$ such that

$$G - q_{i_k} = G_{pk} + G_{rk}, \ G_{pk} \cdot \overline{G}_{rk} + \overline{G}_{pk} \cdot G_{rk} = 0,$$
$$p \mathcal{E} G_{pk}, \ q + q_{i_{k+1}} + r \subset G_{rk}.$$

Since $q_{i_k} \mathcal{E} G^2$, G_{pk} and G_{rk} are connected and $G_{pk} \subset G_{pk+1}$. Let

$$X_p = \sum_{k=1}^{\infty} G_{pk}.$$

The set X_p is a neighborhood of p* and q non-$\mathcal{E} X_p$. Suppose $F(X_p) \supset t \mathcal{E} G^2 + r$, $t \neq q$. Let $[t_n]$ be a sequence such that $t_n \mathcal{E} G_{pn}$, $\lim t_n = t$. As $t \mathcal{E} \Pi G_{rk}$ it follows that if N is any subcontinuum of G containing t and t_n, then $q_{i_j} \mathcal{E} N$ for $j \geqq n$. Then N contains q. For this reason G is not locally connected at t. But $t \mathcal{E} G^2 + r$ and a continuum is locally connected at every point of finite order. Hence t cannot exist. By condition (b), $F(X_p) \cdot G^2 \neq 0$. Then $q \mathcal{E} G^2$.

Suppose there exists a neighborhood U_q of q such that both p and r belong to one component L of $G - U_q$. Then $L \supset \sum q_{i_k}$. But as $\lim q_{i_k} = q$, for k sufficiently large $q_{i_k} \subset U_q$. But this is absurd as $L \cdot U_q = 0$. Hence, for any neighborhood U_q such that $p + r \subset G - U_q$, p and r belong to different components of $G - U_q$. Now let $[U_n]$ be a set of neighborhoods of q such that $\overline{U}_{n+1} \subset U_n$, $d(U_n) < 1/n$, $F(U_n)$ consists of two points, $G - U_1 \supset p + r$. It is easily seen that

$$G - U_n = \mathcal{C}(p, G - U_n) + \mathcal{C}(r, G - U_n);$$

and thus

$$\mathcal{C}(p, G - U_{n+1}) \supset \mathcal{C}(p, G - U_n),$$
$$\mathcal{C}(r, G - U_{n+1}) \supset \mathcal{C}(r, G - U_n).$$

Then

$$G - q = \sum_n \mathcal{C}(p, G - U_n) + \sum_n \mathcal{C}(r, G - U_n).$$

Hence $q \mathcal{E} Q$.

Case II. There is an infinite subsequence $[q_{i_k}]$ such that

$$G - q_{i_k} = G_{pk} + G_{rk}, \ \overline{G}_{pk} \cdot G_{rk} + G_{pk} \cdot \overline{G}_{rk} = 0,$$
$$p + q + q_{i_{k+1}} \subset G_{pk}, \ r \mathcal{E} G_{rk}.$$

Let

$$P = \sum_{k=1}^{\infty} G_{rk}.$$

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* For this lemma we consider G as our space. Relative to G alone, the set X_p is open and hence is a neighborhood.

Exactly as in Case I we may show that if $F(P) \supset t \mathcal{E} G^2 + p$, $t \neq q$, then G is not locally connected at t. We have thus

$$(1) \qquad F(P) \cdot (G^2 - q + p) = 0.$$

The set $G - \overline{P}$ is a neighborhood of p, and r non-$\mathcal{E} F(G - \overline{P})$. We have $F(G - \overline{P}) \subset F(P)$. From (1) then,

$$F(G - \overline{P}) \cdot (G^2 - q) = 0;$$

and, from condition (b), we must have $q \mathcal{E} G^2$. As in Case I we may show that q separates p and r in G.

We have shown that the set Q is closed. Let H be a subcontinuum of G irreducible between p and r. Suppose there exists a point $x \mathcal{E} H - Q$, and let $G_x = \mathcal{C}(x, H - Q)$. Let $y \mathcal{E} \overline{G}_x \cdot Q$. Since $p \mathcal{E} G^1$, there is a neighborhood Z_p such that $F(Z_p) = u$ (a single point) and $(x + r) \cdot \overline{Z}_p = 0$. The point u separates p and r, and $u \mathcal{E} Q$ from condition (b). Then $G_x \subset G - \overline{Z}_p$ and hence $y \neq p$. Similarly $y \neq r$. Let U_y be a neighborhood of y such that $(x + p + r) \cdot \overline{U}_y = 0$. As $y \mathcal{E} G^2$ there is a neighborhood $V_y \subset U_y$ such that $F(V_y) = u_1 + u_2$. From the fact that H is an irreducible continuum it follows that both u_1 and u_2 separate p and r in G. Then there exists a neighborhood Y_{pi} $(i = 1, 2)$ of p such that $F(Y_{pi}) = u_i$ and thus $u_i \mathcal{E} Q$ from (b). The connected set G_x contains a point in V_y and a point x non-$\mathcal{E} V_y$, so either u_1 or u_2 belongs to G_x. But $G_x \cdot Q = 0$ and $u_i \mathcal{E} Q$. This is a contradiction and hence $H - Q = 0$.

From this it follows that $H = Q = G$. Then $p + r = G^1$ and $G - p - r = G^2$. By a result due to Urysohn and Menger* the set G is an arc with end points p and r.

8. LEMMA. *If $p \mathcal{E} M^2$ and $\dim_p M^2 > 0$, then p is a cut point im kleinen of M.*

There exists a neighborhood U_p such that if $V_p \subset U_p$ then

$$(2) \qquad F(V_p) \cdot M^2 \neq 0,$$

and

$$(3) \qquad F(V_p) \supset \text{at least two points}.$$

There exists a sequence of neighborhoods $V_1, V_2, \cdots$ of p such that $V_i \subset U_p$, $\overline{V}_{i+1} \subset V_i$, $d(V_i) < 1/i$, $F(V_i) = u_i + v_i$. Consider the set $\overline{V}_{i-1} - V_i$. Obviously the set $u_{i-1} + v_{i-1} + u_i + v_i = G \subset \overline{V}_{i-1} - V_i$. Let $x \mathcal{E} \overline{V}_{i-1} - V_i$. Then $\mathcal{C}(x, \overline{V}_{i-1} - V_i) \cdot G \neq 0$, for otherwise there would be a separation of $\overline{V}_{i-1} - V_i$ into mutually separated sets containing $\mathcal{C}(x, \overline{V}_{i-1} - V_i)$ and G respectively. But this separation would effect a separation of M, which was connected. Then $\overline{V}_{i-1} - V_i$ contains at most four components. But $\mathcal{C}(u_i, \overline{V}_{i-1} - V_i)$ must con-

* P. Urysohn, loc. cit., and K. Menger, loc. cit., p. 303.

tain u_{i-1} or v_{i-1} for otherwise $V_i+\mathcal{C}(u_i, \overline{V}_{i-1}-V_i)$ is a neighborhood $V_p\subset U_p$, but $F(V_p)=v_i$ contrary to (3). Similarly for $\mathcal{C}(v_i, \overline{V}_{i-1}-V_i)$. Also $\mathcal{C}(u_{i-1}, \overline{V}_{i-1}-V_i)\cdot(u_i+v_i)\neq 0$, for otherwise $V_{i-1}-\mathcal{C}(u_{i-1}, \overline{V}_{i-1}-V_i)$ is a neighborhood V_p and (3) is not true. Similarly for $\mathcal{C}(v_{i-1}, \overline{V}_{i-1}-V_i)$. Thus $\overline{V}_{i-1}-V_i$ has at most two components and each contains a point of u_i+v_i and a point of $u_{i-1}+v_{i-1}$. From (2) we have that $u_i\mathcal{E}M^2$ or $v_i\mathcal{E}M^2$. Further at least one of these points is a point of M^2 and is such that dim $M^2\cdot(\overline{V}_{i-1}-V_i)>0$ at the point. For suppose $u_i+v_i\subset M^2$, dim $M^2\cdot(\overline{V}_{i-1}-V_i)=0$ at both points. There exists a neighborhood $U_{u_i}\subset V_{i-1}$ such that $F(U_{u_i})\cdot M^2\cdot(\overline{V}_{i-1}-V_i)=0$, and similarly a neighborhood $U_{v_i}\subset V_{i-1}$. Then $W_p=V_i+U_{u_i}+U_{v_i}$ is a neighborhood of p such that $W_p\subset U_p$ and $F(W_p)\cdot M^2=0$ contrary to (2). In case $u_i\mathcal{E}M^2$, v_i non-$\mathcal{E}M^2$, $\dim_{u_i} M^2\cdot(\overline{V}_{i-1}-V_i)=0$, we define $W_p=V_i+U_{u_i}$. The only other possibilities are the interchange of u_i and v_i.

Case I. Suppose (a) $u_i\mathcal{E}M^2$, (b) $\dim_{u_i} M^2\cdot(\overline{V}_{i-1}-V_i)>0$, (c) v_i non-$\mathcal{E}M^2$. Since $u_i\mathcal{E}M^2$ there is a neighborhood $U_{u_i}\subset V_{i-1}-\overline{V}_{i+1}-v_i$ such that $F(U_{u_i})=x+y$. Since there is a component of $\overline{V}_i-V_{i+1}$ containing u_i and a point of $u_{i+1}+v_{i+1}$, either x or y, let us suppose y, belongs to $V_i-\overline{V}_{i+1}$. Then $x\mathcal{E}M^2$, since $\dim_{u_i} M^2\cdot(\overline{V}_{i-1}-V_i)>0$. And $\mathcal{C}(x, \overline{U}_{u_i}\cdot(\overline{V}_{i-1}-V_i))=H$ is a continuum such that $u_i+x\subset H^1$. Now if W_{u_i} is any neighborhood of u_i such that x non-$\mathcal{E}\overline{W}_{u_i}$, then $W_{u_i}\cdot U_{u_i}$ is a neighborhood such that $F(W_{u_i}\cdot U_{u_i})\cdot(\overline{V}_{i-1}-V_i)\subset F(W_{u_i})\cdot H$. Then from (b) we have that $F(W_{u_i})\cdot H\cdot M^2\neq 0$. Thus by the lemma of §7 the continuum H is an arc from x to u_i. Now let N_i denote u_i plus all points of $\overline{V}_{i-1}-V_i$ that can be joined to u_i by an *open* arc-segment of $\overline{V}_{i-1}-V_i$, i.e., a point $z\mathcal{E}\overline{V}_{i-1}-V_i$ is a point of N_i if there is an arc $A\subset\overline{V}_{i-1}-V_i$ with end points z and u_i such that $M-A+u_i+z$ is closed. Evidently $H\subset N_i$. Since u_i is a point of order 1 of $\overline{V}_{i-1}-V_i$, we see that N_i is an arc or homeomorphic with an arc minus one end point. Consider the second possibility. As V_i+N_i is a neighborhood of p contained in U_p, it follows from (2) and (c) that one point of $\overline{N}_i-N_i$ is a point $q\mathcal{E}M^2$. Since M is locally connected at q we have that q is the only point of $\overline{N}_i-N_i$. Then N_i+q is an arc and q may be joined to u_i by the open arc-segment N_i-u_i. Thus $q\mathcal{E}N_i$ which is a contradiction. Hence N_i is an arc and let u_i and q be its end points. As V_i+N_i-q is a neighborhood of p, $q\mathcal{E}M^2$ from (2) and (c). If q non-$\mathcal{E}u_{i-1}+v_{i-1}$, there exists a neighborhood $U_q\subset V_{i-1}-\overline{V}_i$ such that $F(U_q)$ consists of just two points. Then just as was the case with H, we may show that $\overline{U}_q-N_i+q$ is an arc by using the lemma of §7. Then any point of this new arc belongs to N_i by definition, a contradiction.

Hence $q\mathcal{E}u_{i-1}+v_{i-1}$, say $q=u_{i-1}$. As $q\mathcal{E}M^2$ and q is a limit point of $M-\overline{V}_{i-1}$, $q=u_{i-1}$ is a point of order 1 of $\overline{V}_{i-1}-V_i$. Hence N_i is a component of $\overline{V}_{i-1}-V_i$ and $\overline{V}_{i-1}-V_i-N_i$ is closed. Then

$$\overline{V}_{i-1} - V_i = N_i + M_i,$$

where M_i is a continuum containing v_i and v_{i-1}.

Case II. Suppose (a) $u_i + v_i \subset M^2$, (b) $\dim_{u_i} M^2 \cdot (\overline{V}_{i-1} - V_i) > 0$, (c) $\dim_{v_i} M^2 \cdot (\overline{V}_{i-1} - V_i) = 0$. From (c) there exists a neighborhood U_{v_i} such that $\overline{U}_{v_i} \subset V_{i-1} - \overline{V}_{i+1} - u_i$ and $F(U_{v_i}) \cdot (\overline{V}_{i-1} - V_i) \cdot M^2 = 0$. The proof in Case II is exactly the same as Case I except that where a neighborhood of p is formed by taking V_i plus some open set, we take $V_i + U_{v_i}$ plus the open set.

Case III. Suppose (a) $u_i + v_i \subset M^2$, (b) $\dim_{u_i} M^2 \cdot (\overline{V}_{i-1} - V_i) > 0$, (c) $\dim_{v_i} M^2 \cdot (\overline{V}_{i-1} - V_i) > 0$. Let N_{u_i} and N_{v_i} denote the sets consisting of u_i and v_i respectively together with all points of $\overline{V}_{i-1} - V_i$ that can be joined to u_i or v_i, as the case may be, by an arc of $\overline{V}_{i-1} - V_i$ such that the arc-segment is an open subset of M. The sets N_{u_i} and N_{v_i} are either arcs or homeomorphic with an arc minus one end point. Either N_{u_i} or N_{v_i} is an arc, for otherwise $V_i + N_{u_i} + N_{v_i}$ is a neighborhood of p and we have a contradiction exactly as in Case I. Also we may prove, similar to the proof of Case I, that one of these must be an arc with u_{i-1} or v_{i-1} as one end point and this arc is a component of $\overline{V}_{i-1} - V_i$ and also an open subset of it. Hence

$$\overline{V}_{i-1} - V_i = N_i + M_i, \quad N_i \cdot M_i = 0,$$

where each is a continuum joining a point of $u_i + v_i$ to a point of $u_{i-1} + v_{i-1}$.

Thus we have seen in any case that each set $\overline{V}_{i-1} - V_i$ consists of two components N_i and M_i, and suppose the components are so lettered that $N_i \cdot N_{i+1} \neq 0 \neq M_i \cdot M_{i+1}$. Then

$$\overline{V}_1 = \left(p + \sum_{i=2}^{\infty} N_i\right) + \left(p + \sum_{i=2}^{\infty} M_i\right)$$

is the sum of two continua having only p in common. Hence p is a cut point im kleinen of M.

9. Theorem. *If M is any continuum, then $M^2 = H + K$, where* (a) *H is vacuous or* $\dim H = 0$, (b) *if $p \in H$ then* $\dim_p M^2 = 0$, (c) *K is vacuous or a countable set of arcs A_i,* (d) *each arc-segment A_i is an open subset of M,* (e) *$A_i \cdot A_j = 0$, $A_i \subset A_j$, or $A_j \subset A_i$.*

Let K denote the set of all points p such that $p \in M^2$ and $\dim_p M^2 > 0$. Then if $q \in M^2 - K$, $\dim_q M^2 = 0$. Then as $H = M^2 - K \subset M^2$, $\dim_q H = 0$ for each $q \in H$. Now let $p \in K$. By the lemmas of §§6 and 8, there is an arc B_p one of whose end points is p such that $B_p \subset M^2$ and $\langle B_p \rangle$, i.e. B_p minus its end points, is an open subset of M. And the arcs B_p may be chosen so that if x is any point of an open arc-segment of M, there is some point $p \in K$

such that $x\varepsilon\langle B_p\rangle$. Then the set $[\langle B_p\rangle]$ for all points $p\varepsilon K$ is a set of open subsets of M covering all points of open arc-segments of M. By the Lindelöf property there is a countable subset, $\langle B_1\rangle$, $\langle B_2\rangle, \cdots$, which covers the same set. Since each $B_i\subset M^2$ it follows that if the sum of any finite number of the B_i's is connected, then it is an arc. From this we find that

$$N = \sum_i B_i$$

consists of a countable number of maximal connected subsets, $N_1, N_2, \cdots$, each of which is homeomorphic with a closed, half-open, or an open interval. Take the interval $(0, 1)$ and let Φ_i be the homeomorphism which carries this interval (closed, half-open, or open) into N_i. There are three cases.

Case I. N_i is homeomorphic with the closed interval. In this case N_i is an arc and let $A_{i1}=N_i$, $A_{ij}=0$ for $j>1$.

Case II. N_i is homeomorphic with the half-open interval $(0, 1\rangle$. In case $\lim_{n\to\infty} \Phi_i(n/(n+1))$ exists and is a point $x_i\varepsilon M^2$, then N_i+x_i is an arc $\subset M^2$ and we define $A_{i1}=N_i+x_i$, $A_{ij}=0$ for $j>1$. In case the limit does not exist or x_i non-εM^2, we define $A_{ij}=\Phi_i(I_j)$, where I_j is the closed interval $(0, j/(j+1))$.

Case III. N_i is homeomorphic with the open interval $\langle 0, 1\rangle$. Suppose (a) $\lim_{n\to\infty} \Phi_i(1/n)$ exists and is a point $x_i\varepsilon M^2$, (b) $\lim_{n\to\infty} \Phi_i(n/(n+1))$ exists and is a point $y_i\varepsilon M^2$. Then $N_i+x_i+y_i$ is an arc $\subset M^2$ and we define $A_{i1}=N_i+x_i+y_i$, $A_{ij}=0$ for $j>1$. If (a) is true and (b) false, we define $A_{ij}=x_i+\Phi_i(I_j)$, where I_j is the half-open interval $\langle 0, j/(j+1))$. If (b) is true and (a) false, we define $A_{ij}=y_i+\Phi_i(I_j)$, where I_j is the half-open interval $(1/(j+1), 1\rangle$. If both (a) and (b) are false, we define $A_{ij}=\Phi_i(I_j)$, where I_j is the closed interval $(1/(j+2), (j+1)/(j+2))$.

We shall show now that $[A_{ij}]$ is the required countable set of arcs, i.e., for every i and j, $A_{ij}\subset K$, and if $p\varepsilon K$, then $p\varepsilon A_{ij}$ for some i and j. The first part is obvious from the definition of A_{ij}. Now if $p\varepsilon K$, there is an arc $B_p\subset M^2$ with p as one end point such that $\langle B_p\rangle$ is an open subset of M. Now as $[B_i]$ covers all such open subsets, there exists an integer i such that $\langle B_p\rangle\subset N_i$. If $p\varepsilon N_i$, then $p\varepsilon A_{ij}$ for some j. If p non-εN_i, since p is an end point of B_p, it follows that either $\lim_{n\to\infty} \Phi_i(1/n)$ or $\lim_{n\to\infty} \Phi_i(n/(n+1))$ exists and is the point p. In this case $p\varepsilon A_{i1}$.

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